Beyond CES: Three Alternative Classes of Flexible Homothetic Demand Systems[1]

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Abstract

We characterize three classes of demand systems, all of which are defined non-parametrically: homothetic demand systems with a single aggregator (HSA), those with direct implicit additivity (HDIA), and those with indirect implicit additivity (HIIA). For any number of goods, all the cross-price effects are summarized by one price aggregator in the HSA class and by two price aggregators in the HDIA and HIIA classes. Each of these three classes contains CES as a special case. Yet, they are *pairwise disjoint* with the sole exception of CES. Thus, these classes of homothetic demand systems offer us three *alternative* ways of departing from CES.

Keywords: Homothetic preferences, CRS production functions, Demand systems with a single aggregator, direct implicit additivity, indirect implicit additivity, CES, translog, gross complements and gross substitutes, essential (indispensable) and inessential (dispensable) goods.

JEL Classification: D11, D21

1. Introduction

Across many fields of applied general equilibrium, such as economic growth and development, macroeconomics, international trade, and public finance, it is routinely assumed that consumers have *homothetic preferences* and that production technologies have *constant returns to scale* (CRS). The homotheticity plays an important role in these areas for a variety of reasons. First, the identical homothetic preference allows for the aggregation across households with different income and expenditure levels, which makes it possible to derive the aggregate consumption behavior as the outcome of the utility maximization of the hypothetical representative consumer. Second, the ubiquitous assumption of perfect competition is valid only when the industry has CRS technologies.[4] Third, these assumptions imply that, holding the prices constant, the budget share of each good (or factor) is independent of the household expenditure (or the scale of operation by industries). This allows us to focus on the role of relative prices in the allocation of resources. Furthermore, in multi-sector growth models, it is necessary to make these assumptions in order to ensure the existence of a balanced growth path.

In practice, however, most models assume not only that preferences (technologies) are homothetic (CRS), but also that they satisfy the stronger property of the *constant-elasticity-of-substitution* (CES), particularly when there are more than two goods (or factors). While CES buys a great deal of tractability, it is highly restrictive. First, CES implies that the price elasticity of demand for each good and factor is not only constant but also identical across goods and factors. Second, it also implies that the relative demand for any two goods (or factors) is independent of the prices of any other goods (or factors). Third, under CES, the marginal rate of substitution between any two goods is independent of the consumption of any other goods. Fourth, under CES, gross substitutes (complements) automatically imply that all goods are inessential (essential) goods. In particular, CES cannot capture a demand pattern in which some goods are essential while others are inessential.[5] Fifth, in a monopolistically competitive setting, the CES demand for differentiated products implies that each firm sells its product at a constant markup, which is independent of pricing behavior of other firms or the number of competing firms. Although there have been some isolated attempts to replace CES by different parametric homothetic utility

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[4] Recall that, even when each firm has a U-shaped average cost curve, a competitive industry, consisting of many identical firms, has CRS technologies, because its scale can change at the extensive margin.

[5] We provide formal definitions of essential/inessential goods later. Intuitively, water is essential (indispensable) because zero consumption of water makes marginal utility of all other goods zero. This distinction should not be confused with that of necessities/luxuries, which are about the goods whose income elasticities are smaller/greater than one. Under homothetic preferences, there are no necessities/luxuries.

functions and CRS production functions, a vast majority of studies continue to use the CES due to the lack of better options.

In this paper, we characterize three alternative classes of flexible homothetic demand systems. Each of these classes contains CES as a special case. Yet, they offer three *alternative* ways of departing from CES, because non-CES homothetic demand systems in these three classes do not overlap. Furthermore, they are *flexible* in the sense that each of them is defined non-parametrically.

Here is a roadmap of the present paper. After a quick review of the basic properties of general homothetic demand systems in Section 2, we define and characterize *homothetic demand systems with a single aggregator* (HSA) in Section 3. This class of demand systems is identified by the property that the budget share of each good is a function of its own price divided by a single price aggregator (linear homogeneous in all prices), which is *common* across all goods. For each set of the budget-share functions, this common price aggregator is defined so that the budget constraint of the consumer is satisfied.[6] The defining feature of HSA is that a single price aggregator captures the cross-price effects on demands for *all* goods.[7] We find the necessary and sufficient conditions on a set of the budget-share functions under which the corresponding HSA demand system is generated by a well-defined, monotone, convex, homothetic preference. Furthermore, we establish the one-to-one mapping between the preference and the budget-share functions that satisfy these conditions whenever there are more than two goods. This means that we can treat the budget-share functions as *the primitives of the model.* In addition, we derive the functional relationship between the common price aggregator, which summarizes all the cross-price effects on demand, and the ideal price index of the preference, which summarizes the welfare effect of prices. Remarkably, these two price aggregators coincide if and only if the preference is CES when there are more than two goods. We also discuss some parametric examples of HSA, including *separable translog*, for which the common price aggregator is the weighted geometric

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[6] Alternatively, this class of demand systems can be identified by the property that the budget share of each good is a function of its own *quantity* divided by a common *quantity* aggregator, and for each set of the budget-share functions, this common quantity aggregator is defined so that the budget constraint of the consumer is satisfied. We will show the equivalence of these two definitions.

[7] The HSA demand systems is an intersection of homothetic demand systems and what Pollak (1972) called generalized additively separable (GAS) demand systems. (We thank Peter Neary for bringing our attention to Pollak 1972). However, it makes little sense to call it "homothetic generalized additively separable," because Pollak's class of demand systems is "generalized additively separable," in that it nests both directly explicitly additive and indirectly explicitly additive demand systems, which are all *nonhomothetic* with the sole exception of CES. Furthermore, this class of demand systems nests neither directly implicitly additive nor indirectly implicitly additive demand systems. Instead, we prefer to call it "homothetic demand systems with a single aggregator" (HSA), because it describes the exact nature of the restrictions imposed.

mean of prices and *linear expenditure shares*, for which the common price aggregator is the weighted arithmetic mean of prices. We also offer an example of HSA, which is a hybrid of Cobb-Douglas and CES, such that all goods are gross substitutes, and yet only some of them are essential.

Then, in Section 4, we turn to two other classes of homothetic demand systems, both of which are generated by *implicitly additive* preferences;[8] one with *direct implicit additivity* (HDIA) in Section 4.1, and one with *indirect implicit additivity* (HIIA) in Section 4.2. We show that, in each of these two classes of homothetic demand systems, the cross-price effects are characterized by *two* price aggregators, one of which is the ideal price index. Furthermore, it is shown that these two price aggregators are proportional to each other and these homothetic demand systems become HSA if and only if the preference is CES. From these results, it is a short step to demonstrate in Section 4.3 that these three classes of homothetic demand systems, HSA, HDIA, and HIIA, are *pairwise disjoint* with the sole exception of CES.[9]

Before proceeding, it should be pointed out that, although we discuss homothetic demand systems in terms of consumer preferences, they could be equally interpreted as input demand systems by the producers with a CRS technology. With this interpretation, the prices of goods become the prices of factors; the indifference sets of a homothetic preference becomes the isoquants of a CRS technology, and the ideal price index becomes the unit cost function.

2. Homothetic demand systems: a quick review

Let us first recall some general properties of homothetic demand systems. In what follows, we use the following standard notation. We assume an n-dimensional consumption space $\mathbb{R}^n_+$. Consumption bundles and price vectors are denoted, respectively, by $\mathbf{x} \in \mathbb{R}^n_+$ and $\mathbf{p} \in \mathbb{R}^n_+$, where n is the number of available goods, which is exogenously fixed.

Recall that a rational preference $\succsim$ over $\mathbb{R}^n_+$ is called *homothetic* if any two indifference sets generated by $\succsim$ can be mapped one into another by means of a homothetic transformation, or a uniform rescaling, which means a proportional expansion from or contraction towards the origin. A homothetic preference can be represented by an increasing, linear homogeneous, and concave direct utility function, $u(\mathbf{x})$, and its indirect utility function $V(\mathbf{p}, h)$ can be represented as:

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[8] We do not consider exploring explicit additivity, direct or indirect, for a simple reason: non-CES homothetic demand systems are incompatible with explicit additivity. See Berndt and Christensen (1973, Theorem 6).
[9] To be precise, these results hold true when there are more than two goods.

$$V(\mathbf{p}, h) \equiv \max_{\mathbf{x} \in \mathbb{R}_+^n}\{u(\mathbf{x}) \mid \mathbf{px} \le h\} = \frac{h}{P(\mathbf{p})}, \tag{1}$$

where h is consumer's income while $P(\mathbf{p})$ is an increasing, linear homogeneous, and concave function of $\mathbf{p}$. This function, often called an *ideal price index* or a *cost-of-living index*, fully characterizes a homothetic preference.

Note that both the direct utility function $u(\mathbf{x})$ and the ideal price index $P(\mathbf{p})$ of a homothetic preference $\succsim$ are defined up to an arbitrary positive coefficient, meaning that $u(\mathbf{x})$ and $u(\mathbf{x})/c$ or $P(\mathbf{p})$ and $cP(\mathbf{p})$ correspond to the same preference for any $c > 0$.

In this paper, we will focus on such homothetic preferences for which the price index $P(\mathbf{p})$ is almost everywhere twice continuously differentiable in $\mathbf{p}$. In this case, we can use Roy's identity to obtain the demand system associated with (1):

$$x_i = \frac{h}{p_i}\mathcal{E}_{p_i}(P), \tag{2}$$

where $\mathcal{E}_{p_i}(P)$ is the elasticity of the ideal price index with respect to the price:

$$\mathcal{E}_{p_i}(P) \equiv \frac{\partial P}{\partial p_i}\frac{p_i}{P}.$$

The elasticity $\mathcal{E}_{p_i}(P)$ captures the budget-share of good i. As implied by the Euler's Theorem on linear homogeneity, these budget-shares sum up to one:

$$\sum_{i=1}^{n} \mathcal{E}_{p_i}(P) = 1.$$

Likewise, the inverse demand system associated with (1) can be represented as:

$$p_i = \frac{h}{x_i}\mathcal{E}_{x_i}(u), \tag{2*}$$

where $\mathcal{E}_{x_i}(u)$ is the elasticity of the direct utility function with respect to the quantity:

$$\mathcal{E}_{x_i}(u) \equiv \frac{\partial u}{\partial x_i}\frac{x_i}{u},$$

which is also the budget-share of good i. Again, as implied by the Euler's Theorem on linear homogeneity, these budget-shares sum up to one:

$$\sum_{i=1}^{n} \mathcal{E}_{x_i}(u) = 1.$$

3. Homothetic demands with a single aggregator (HSA)

We now introduce *homothetic demand systems with a single aggregator* (henceforth HSA), the first of the three classes we study. A HSA demand system is given by:

$$x_i = \frac{h}{p_i} s_i\left(\frac{p_i}{A(\mathbf{p})}\right), \tag{3}$$

where $s_i: \mathbb{R}_+ \to \mathbb{R}_+$ are functions, while $A(\mathbf{p})$ is a common price aggregator, which is a linear homogenous function of $\mathbf{p}$ defined as a solution to

$$\sum_{i=1}^{n} s_i\left(\frac{p_i}{A}\right) = 1. \tag{4}$$

Clearly, any demand system described by (3) – (4) satisfies the budget constraint:

$$\sum_{i=1}^{n} p_i x_i = h. \tag{5}$$

HSA demand systems are the homothetic restriction on what Pollak (1972) refers to as generalized additively separable (GAS) demand systems. To the best of our knowledge, there was no attempt to provide sufficient conditions for the "candidate" demand system described by (3) – (4) to actually be a demand system generated by some well-behaved convex homothetic preference.[10] This is what we do in Proposition 1 below.

Before providing a general characterization of HSA demand systems, however, we start with the two most familiar examples to gain some intuition.

Example 1: Cobb-Douglas. When $s_i(z_i) = \alpha_i$, where $\alpha_1, \dots, \alpha_n$ are positive constants such that $\sum_{i=1}^{n} \alpha_i = 1$, (3) – (4) boils down to the Cobb-Douglas demand system. Note that in this case, the ideal price index is $P(\mathbf{p}) = c \prod_{i=1}^{n} p_i^{\alpha_i}$, while the common price aggregator $A(\mathbf{p})$ is indeterminate, since (4) holds identically for any $A(\mathbf{p})$.

Example 2: CES. We obtain the CES demand system if we set $s_i(z_i) = \beta_i z_i^{1-\sigma}$ in (3) – (4), where the coefficients β_i are such that

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[10] For many properties of GAS, Pollack (1972) referred to the unpublished manuscript by W. Gorman, which was subsequently published as Gorman (1996), in which he wrote the disclaimer on pp.202-203: "Throughout this paper I have talked as if my claims were definitely proven. Of course, this is not so: my arguments are far from rigorous... Finally, I have said nothing at all about convexity."

$$\beta_i > 0, \qquad \sum_{k=1}^{n} \beta_k = 1,$$

while $\sigma > 0$ is the constant elasticity of substitution. Note that $s_i(z_i)$ are all increasing when $0 < \sigma < 1$ (i.e. when the goods are gross complements), all decreasing when $\sigma > 1$ (i.e. when the goods are gross substitutes) and all constant when $\sigma = 1$ (i.e. the Cobb-Douglass case). The corresponding CES utility function is given by

$$u(\mathbf{x}) = \left(\sum_{i=1}^{n} \gamma_i x_i^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}},$$

where $\gamma_i \equiv \beta_i^{\frac{1}{\sigma}}$. As implied by (4), the common price aggregator $A(\mathbf{p})$ is proportional to the standard CES price index:

$$A(\mathbf{p}) = \left(\sum_{i=1}^{n} \beta_i p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}} = cP(\mathbf{p}).$$

However, it is worth pointing out that in general $A(\mathbf{p})$ need not be an ideal price index associated with a homothetic preference generating (3) – (4) as a Marshallian demand system. In fact, when there are at least three goods, CES is *the only case where* $A(\mathbf{p}) = cP(\mathbf{p})$ (see Proposition 1 below).

Before proceeding, another restrictive feature of CES demands is worth mentioning. The goods are gross substitutes (i.e., the budget-share functions, $s_i(z) = \beta_i z^{1-\sigma}$, are decreasing) only if $\sigma > 1$. Thus, the budget share of each good converges to zero when $z \to \infty$. And each good is *inessential* (or *dispensable*), i.e., $\lim_{p_i \to \infty} P(\mathbf{p}) < \infty$ and $u(\mathbf{x})$ generates monotone preferences over $\mathbf{x} \in \mathbb{R}_+^n \cap \{x_i = 0\}$ for each i. On the other hand, if $\sigma \leq 1$, the budget share of each good, $s_i(z) = \beta_i z^{1-\sigma}$, is bounded away from zero when $z \to \infty$. And each good is *essential* (or *indispensable*), i.e., $\lim_{p_i \to \infty} P(\mathbf{p}) = \infty$ and $u(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}_+^n \cap \{x_i = 0\}$ for each i. Thus, each good is *essential* (or *indispensable*) in the sense that, at zero consumption of that good, the utility is at its minimum level and the marginal utility of all other goods is zero. Thus, CES implies that, if one good is essential, all goods must be essential; the CES cannot capture situations when some goods are essential, and others are not. Furthermore, under the CES, the very concept of a good being essential or inessential becomes somewhat redundant: gross complements (respectively, substitutes) are always essential (inessential) goods.

A question which is both natural and legitimate at this point is as follows: what restrictions are to be imposed on the functions $s_i(\cdot)$ for a demand system given by (3) – (4) to be compatible with rational consumer behavior? The answer to this question is given by the following Proposition.

Proposition 1. *Consider a mapping* $\mathbf{s}(\mathbf{z}) = \left(s_1(z_1), \dots, s_n(z_n)\right)^{\mathrm{T}}$ *from* $\mathbb{R}_+^n$ *to* $\mathbb{R}_+^n$, *which is differentiable almost everywhere, normalized by*

$$\sum_{k=1}^{n} s_k(1) = 1, \tag{6}$$

and satisfies the following conditions:

$$s_i'(z_i) \cdot z_i < s_i(z_i), \quad i = 1, \dots n, \tag{7}$$

$$s_i'(z_i) s_j'(z_j) \geq 0, \quad i, j = 1, \dots, n, \tag{8}$$

for all $\mathbf{z}$ *such that* $\sum_{k=1}^{n} s_k(z_k) = 1$. *Then*:

(i) For any such mapping, there exists a unique monotone, convex, continuous and homothetic rational preference $\succsim$ *over* $\mathbb{R}_+^n$, *such that the demand system described by* (3) – (4) *is generated by* $\succsim$. *Furthermore, when* $n \geq 3$ *and* $\succsim$ *is HSA, the mapping* $\mathbf{s}(\mathbf{z})$ *is uniquely defined over the set* $\{\mathbf{z} \in \mathbb{R}_+^n : \sum_{k=1}^{n} s_k(z_k) = 1\}$.

(ii) This homothetic preference is described by an ideal price index which is defined by

$$\ln P(\mathbf{p}) = \ln A(\mathbf{p}) + \sum_{i=1}^{n} \int_{c_1}^{p_i/A(\mathbf{p})} \frac{s_i(\xi)}{\xi} d\xi, \tag{9}$$

where $c_1 > 0$ *is a constant.*

(iii) When $n \geq 3$, $A(\mathbf{p}) = cP(\mathbf{p})$ *iff* $\succsim$ *is a CES.*

Proof. See the Appendix. ■

Without loss of generality, Eq. (6) normalizes the common price aggregator $A(\mathbf{p})$ so that $A(\mathbf{1}) = 1$, where $\mathbf{1} \equiv (1,1,\dots,1)$. Eq. (7) requires that the elasticities of the budget-share functions $s_i(z_i)$, whenever well-defined, do not exceed 1. Eq.(8) states that, for (3) – (4) to be generated by a well-defined preference for all $\mathbf{p} \in \mathbb{R}_+^n$, the budget-share functions cannot be increasing for some goods while decreasing for others at some $\mathbf{z}$ such that $\sum_{k=1}^{n} s_k(z_k) = 1$. They can be all increasing (in which case the goods are gross complements), all decreasing (in which case the goods are gross substitutes), or all constant (which corresponds to the Cobb-Douglas case). They can be also increasing (decreasing) for some goods and constant for others.

Part (i) of Proposition 1 establishes a one-to-one correspondence between the set of homothetic preferences generating a demand system with a single aggregator and the mappings

$\mathbf{s}(\mathbf{z})$ for $n \geq 3$. Hence, the budget-share mapping $\mathbf{s}(\mathbf{z})$ can be treated as a *primitive of the model*.[11]

Part (ii) of Proposition 1 provides an explicit dual description of a homothetic preference associated with the budget-share mapping $\mathbf{s}(\mathbf{z})$. In addition, equation (9) clearly shows the difference and the relation between the ideal price index $P(\mathbf{p})$, which captures the welfare effects of prices, and the common price aggregator $A(\mathbf{p})$, which summarizes the cross-price effects on demand for all goods. This difference is captured by the "residual term":

$$\sum_{i=1}^{n} \int_{c_1}^{p_i/A(\mathbf{p})} \frac{s_i(\xi)}{\xi} d\xi.$$

Finally, part (iii) of Proposition 1 stresses the exceptional nature of the CES, which has been emphasized in many different contexts.[12]

Three remarks are in order.

Remark 1. Note that $A(\mathbf{p})$ itself cannot serve as a primitive. To see this, for any budget-share mapping, $\mathbf{s}(\mathbf{z}) = \left(s_1(z_1), \dots, s_n(z_n)\right)^{\mathrm{T}}$, which satisfies the conditions (6) – (8), consider another budget-share mapping given by

$$\mathbf{s}^{\varepsilon}(\mathbf{z}) \equiv (1 - \varepsilon)\mathbf{s}(\mathbf{z}) + \varepsilon\boldsymbol{\alpha}, \tag{10}$$

where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^{\mathrm{T}}$ is a positive vector, satisfying $\sum_{i=1}^{n} \alpha_i = 1$, while $\varepsilon \in (0,1)$. Thus, $\mathbf{s}^{\varepsilon}(\mathbf{z})$ is a convex combination of the original demand system and a Cobb-Douglas demand system (Example 1). Clearly, $\mathbf{s}^{\varepsilon}(\mathbf{z})$ also satisfies (6) – (8). Furthermore,

$$\sum_{i=1}^{n} s_i^{\varepsilon}\left(\frac{p_i}{A}\right) = \sum_{i=1}^{n}(1 - \varepsilon)s_i\left(\frac{p_i}{A}\right) + \sum_{i=1}^{n} \varepsilon\alpha_i = (1 - \varepsilon)\sum_{i=1}^{n} s_i\left(\frac{p_i}{A}\right) + \varepsilon = 1$$

is equivalent to $\sum_{i=1}^{n} s_i\left(\frac{p_i}{A}\right) = 1$. Thus, as implied by (4), $\mathbf{s}^{\varepsilon}(\mathbf{z})$ gives rise to the same price aggregator with $\mathbf{s}(\mathbf{z})$, $A(\mathbf{p})$, and yet they generate different demand systems.

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[11] A fully precise formulation would be "the restriction of the mapping $\mathbf{s}(\mathbf{z})$ to $\mathbf{s}^{-1}(\Delta_{n-1})$ can be treated as a primitive of the model", where Δ_{n-1} denotes the standard $(n-1)$-dimensional simplex.

[12] For example, in one of the first applications of the duality principle in consumption theory, Samuelson (1965) showed that the CES is the only preference which is both directly and indirectly explicitly additive. Berndt and Christensen (1973, Theorem 6) showed that the CES is the only homothetic preference which is (either directly or indirectly) explicitly additive. Parenti et al. (2017) have extended these results to the case of a continuum of goods by showing that the three classes of symmetric preferences – directly explicitly additive, indirectly explicitly additive, and homothetic – are pairwise disjoint, except for the CES.

Remark 2. When $n = 2$, the "one-to-one" property stated in part (i) of Proposition 1 fails to hold. In fact, when $n = 2$, any homothetic preference is HSA. Furthermore, any increasing linear homogeneous function can be used as $A(\mathbf{p})$. To see why this is so, consider an arbitrary homothetic demand system over $\mathbb{R}_+^2$:

$$x_i = \frac{h}{p_i} g_i \left(\frac{p_j}{p_i} \right),$$

where $i, j = 1, 2$, $i \neq j$, while g_i are budget-share functions. Fix an arbitrary increasing, continuous and linear-homogeneous function $F(p_1, p_2)$ and let $\tilde{p}_i(p_j, A)$, $i, j = 1, 2$, $i \neq j$, denote the solution to the equation $F(p_1, p_2) = A$ with respect to each price. Clearly, $\tilde{p}_i(p_j, A)$ are linear homogeneous. Setting $s_i(z_i) \equiv g_i\big(\tilde{p}_j(1, 1/z_i)\big)$ yields representation (3) – (4) with $A(\mathbf{p}) = F(p_1, p_2)$. However, because the function $F(p_1, p_2)$ is arbitrary, $s_i(z_i)$ can be defined in many ways for the same preference $\succsim$ over $\mathbb{R}_+^2$. This should not be surprising, since the question of whether the cross-price effects can be summarized in a single number is vacuous when there are only two goods.

Remark 3. The reader may wonder if it might be possible to define another class of HSA demand systems, in which the budget share of each good is a function of its own *quantity* divided be a common *quantity* aggregator. More specifically, consider an inverse demand system given by:

$$p_i = \frac{h}{x_i} s_i^* \left(\frac{x_i}{A^*(\mathbf{x})} \right), \tag{3^*}$$

where $s_i^*: \mathbb{R}_+ \to \mathbb{R}_+$ are functions, while $A^*(\mathbf{x})$ is a common quantity aggregator, which is a linear homogenous function of $\mathbf{x}$ defined as a solution to

$$\sum_{i=1}^n s_i^* \left(\frac{x_i}{A^*} \right) = 1. \tag{4^*}$$

In fact, the class of HSA demand systems, (3^*) – (4^*), characterized by the budget-share mappings $\mathbf{s}^*(\mathbf{y}) = \big(s_1^*(y_1), \dots, s_n^*(y_n) \big)^{\mathrm{T}}$ satisfying the conditions:

$$s_i^{*\prime}(y_i) \cdot y_i < s_i^*(y_i), \quad i = 1, \dots n, \tag{7^*}$$

$$s_i^{*\prime}(y_i) \cdot s_j^{*\prime}(y_j) \geq 0, \quad i, j = 1, \dots n, \tag{8^*}$$

are equivalent to the class of HSA demand systems, (3) – (4), characterized by the budget-share mappings $\mathbf{s}(\mathbf{z}) = \big(s_1(z_1), \dots, s_n(z_n) \big)^{\mathrm{T}}$ satisfying the conditions (7) and (8), with the one-to-one correspondence between $\mathbf{s}(\mathbf{z})$ and $\mathbf{s}^*(\mathbf{y})$ given by

$$s_i^* = s_i(s_i^*/y_i), \qquad i = 1, \ldots, n,$$

or, equivalently, by

$$s_i = s_i^*(s_i/z_i), \qquad i = 1, \ldots, n. \text{ [13]}$$

In short, these two classes of HSA are *self-dual* to each other. See the Appendix for the proof. Furthermore, using the budget-share mapping $\mathbf{s}^*(\mathbf{y})$ as a primitive, the direct utility function can be obtained as follows:

$$\ln u(\mathbf{x}) = \ln A^*(\mathbf{x}) + \sum_{i=1}^{n} \int_{c_2}^{x_i/A^*(\mathbf{x})} \frac{s_i^*(\xi)}{\xi} d\xi. \qquad (9^*)$$

The derivation of (9^*) is similar to that of (9), hence omitted.

We consider a few more examples to demonstrate the flexibility of HSA demand systems given by $(3) - (4)$ or, equivalently, by $(3^*) - (4^*)$. The following examples also help to gain some intuition behind the difference between $A(\mathbf{p})$ and $P(\mathbf{p})$.

Example 3a: Separable translog. The translog expenditure function (Christensen et al., 1973, 1975) is characterized by the following ideal price index:

$$\ln P(\mathbf{p}) = \sum_{i=1}^{n} \delta_i \ln p_i - \frac{1}{2} \sum_{i,j=1}^{n} \gamma_{ij} \ln p_i \ln p_j - \ln c,$$

where δ_i are all positive while the matrix $(\gamma_{ij})_{i,j=1,\ldots,n}$ is symmetric and positive semidefinite, and the following normalizations hold:

$$\sum_{j=1}^{n} \delta_j = 1, \qquad \sum_{j=1}^{n} \gamma_{ij} = 0.$$

for all $i = 1, \ldots, n$. Using (2), the corresponding demand system is given by

$$x_i = \frac{h}{p_i} \left(\delta_i - \gamma_{ii} \ln p_i - \sum_{j \neq i}^{n} \gamma_{ij} \ln p_j \right).$$

In general, this demand system is not HSA. To single out a subclass of translog demand systems which have the HSA property, assume additionally that $\gamma_{ij} = -\gamma \beta_i \beta_j$ for all $i \neq j$, where $\gamma \geq 0$ while β_i are all positive and such that $\sum_{i=1}^{n} \beta_i = 1$. This implies $\gamma_{ii} = \gamma \beta_i (1 - \beta_i)$. We refer to

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[13] Note that $s_i^* = s_i(s_i^*/y_i)$ uniquely defines $s_i^*(y_i)$ when $s_i(z_i)$ satisfies (7), while $s_i = s_i^*(s_i/z_i)$ uniquely defines $s_i(z_i)$ when $s_i^*(y_i)$ satisfies (7^*).

translog demand systems satisfying these restrictions as *separable translog*.[14] In this case, the demand system becomes

$$x_i = \frac{h}{p_i}\left(\delta_i - \gamma\beta_i \ln\frac{p_i}{A(\mathbf{p})}\right), \tag{11}$$

where the common price aggregator $A(\mathbf{p})$ is the weighted geometric mean of prices:

$$\ln A(\mathbf{p}) = \sum_{i=1}^{n} \beta_i \ln p_i. \tag{12}$$

By setting $s_i(z_i) = \delta_i - \gamma\beta_i \ln z_i$, the demand system (11) – (12) can be obtained as a special case of (3) – (4).[15] Notice that the weight on the price of good i in (11) is β_i, the relative responsiveness of the budget share of good i to its "relative price", $p_i/A(\mathbf{p})$. In contrast, δ_i affects the level of the budget share of good i, but not its responsiveness to the relative price. This explains why $A(\mathbf{p})$ is independent of δ_i.

The ideal price index associated with separable translog preferences is given by

$$\ln P(\mathbf{p}) = \sum_{i=1}^{n} \delta_i \ln p_i - \frac{\gamma}{2}\left[\sum_{i=1}^{n}\beta_i(\ln p_i)^2 - \left(\sum_{i=1}^{n}\beta_i \ln p_i\right)^2\right] - \ln c \neq \ln A(\mathbf{p}),$$

which is confirmed by the general formula (9). Notice that, unlike in the CES case, here $A(\mathbf{p}) \neq cP(\mathbf{p})$. In particular, δ_i, $i = 1,\ldots n$, appear in the ideal price index, $P(\mathbf{p})$, despite they do not appear in the common price aggregator, $A(\mathbf{p})$.

It is also worth noting that that Cobb-Douglas is a special case of (11) – (12) where $\gamma = 0$.

In example 3a, γ must be non-negative, since otherwise we have

$$\frac{z_i s_i'(z_i)}{s_i(z_i)} = \frac{(-\gamma)\beta_i}{\delta_i + (-\gamma)\beta_i \ln z_i},$$

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[14] With these restrictions, the Allen-Uzawa elasticity of substitution between good i and good $j \neq i$ is multiplicatively separable, as $\sigma_{ij} = \gamma(\beta_i/s_i)(\beta_j/s_j)$, which implies, among other things, that σ_{ik}/σ_{jk} is independent of $k \neq i,j$. Note also that *symmetric translog preferences*, proposed by Feenstra (2003) in the context of monopolistic competition, satisfy these restrictions with $\beta_i = 1/n$.

[15] Strictly speaking, this is only true when prices are such that all the goods are consumed in positive volumes. This price domain is defined by the following system of inequalities:

$$\ln p_i - \sum_{k=1}^{n}\beta_k \ln p_k < \frac{\delta_i}{\gamma\beta_i}$$

for all $i = 1,\ldots,n$, which means that the dispersion of log-prices should be sufficiently low. When these conditions are not satisfied, the expression for $A(\mathbf{p})$ becomes more complex than (12). However, the separable translog demand system can still be obtained as a special case of (3) – (4) by setting $s_i(z_i) = \max\{\delta_i - \gamma\beta_i \ln z_i, 0\}$.

and assumption (7) will be violated whenever

$$\exp\left(-\frac{\delta_i}{(-\gamma)\beta_i}\right) < z_i < \exp\left(1 - \frac{\delta_i}{(-\gamma)\beta_i}\right).$$

In other words, unlike the CES, this example is *not compatible with gross complementarity*.

Example 3b: Modified translog. To overcome the incompatibility of Example 3a with gross complements, we consider the following modification:

$$s_i(z_i) = \max\{\delta_i + \gamma\beta_i \ln z_i , \gamma\beta_i\} \tag{13}$$

where δ_i and β_i are all positive and such that

$$\sum_{k=1}^{n} \beta_k = \sum_{k=1}^{n} \delta_k = 1, \ \ 0 < \gamma < \min_{k=1,\dots,n}\left\{\frac{\delta_k}{\beta_k}\right\}.$$

It can be easily verified that $\mathbf{s}(\mathbf{z})$ defined by (13) is a budget-share mapping which satisfies all the assumptions of Proposition 1. The corresponding HSA demand system is given by

$$x_i = \frac{h}{p_i} \max\left\{\delta_i + \gamma\beta_i \ln\frac{p_i}{A(\mathbf{p})} , \gamma\beta_i\right\}, \tag{14}$$

where the price aggregator $A(\mathbf{p})$ is again defined by (12), while the ideal price index is given by

$$\ln P(\mathbf{p}) = \sum_{i=1}^{n} \delta_i \ln p_i + \frac{\gamma}{2}\left[\sum_{i=1}^{n} \beta_i(\ln p_i)^2 - \left(\sum_{i=1}^{n} \beta_i \ln p_i\right)^2\right] - \ln c,$$

which is confirmed by the general formula (9).[16]

Example 4: Linear expenditure shares. Another natural extension of Cobb-Douglas is a demand system with *linear expenditure shares*:

$$s_i(z_i) = \max\{(1 - \delta)\alpha_i + \delta\beta_i z_i, 0\},$$

where $\alpha_i > 0$ and $\beta_i > 0$ are such that

$$\sum_{i=1}^{n} \alpha_i = 1, \qquad \sum_{i=1}^{n} \beta_i = 1,$$

while $\delta < 1$. The goods are gross complements when $0 < \delta < 1$ and gross substitutes when $\delta < 0$. The Cobb-Douglas is a borderline case when $\delta = 0$.

In this case, the common price aggregator $A(\mathbf{p})$ is the weighted arithmetic mean of prices:

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[16] Note that remarks similar to that in footnote 11 do not apply here, since the demand system (14) does not have, unlike the standard translog, finite reservation prices.

$$A(\mathbf{p}) = \sum_{i=1}^{n} \beta_i p_i,$$

with the weight β_i given by the relative responsiveness of the budget share to its relative price. In contrast, the ideal price index is given by

$$P(\mathbf{p}) = c[A(\mathbf{p})]^{\delta} \left(\prod_{i=1}^{n} p_i^{\alpha_i} \right)^{1-\delta}.$$

Notice that α_i, which affect the level of the budget share, appear in the ideal price index, $P(\mathbf{p})$, but not the common price aggregator, $A(\mathbf{p})$.

Examples 3a, 3b, and 4 illustrate the crucial difference between $A(\mathbf{p})$ and $P(\mathbf{p})$. The prices of goods whose budget shares are more responsive to the relative price $p_i/A(\mathbf{p})$, carry more weight in the common price aggregator $A(\mathbf{p})$, while the prices of goods which account for larger shares in the budget carry more weight in the ideal price index $P(\mathbf{p})$. Indeed, this holds generally. To see this, recall that, from (2), the weight on p_i in $P(\mathbf{p})$, or the elasticity of $P(\mathbf{p})$ with respect to p_i, is given by

$$\mathcal{E}_{p_i}(P) \equiv \frac{\partial P}{\partial p_i} \frac{p_i}{P} = s_i \left(\frac{p_i}{A} \right),$$

while the weight of on p_i in $A(\mathbf{p})$, or the elasticity of $A(\mathbf{p})$ with respect to p_i, can be calculated, from (4)

$$\mathcal{E}_{p_i}(A) \equiv \frac{\partial A}{\partial p_i} \frac{p_i}{A} = \frac{s_i'\left(\frac{p_i}{A}\right)\frac{p_i}{A}}{\sum_k s_k'\left(\frac{p_k}{A}\right)\frac{p_k}{A}} = \frac{\mathcal{E}_{p_i}(s_i) \cdot s_i\left(\frac{p_i}{A}\right)}{\sum_{k=1}^{n} \mathcal{E}_{p_k}(s_k) \cdot s_k\left(\frac{p_k}{A}\right)}.$$

Thus, the weight of good i in $P(\mathbf{p})$ is equal to its budget share itself, while the weight of good i in $A(\mathbf{p})$ is proportional to the elasticity of its budget share, $\mathcal{E}_{p_i}(s_i)$, multiplied by the budget share. This also explains why $A(\mathbf{p}) = cP(\mathbf{p})$ holds for the CES: with the isoelastic budget share functions, $s_i(z) = \beta_i z^{1-\sigma}$, $\mathcal{E}_{p_i}(s_i)$ is independent of i, so that $\mathcal{E}_{p_i}(A) = s_i = \mathcal{E}_{p_i}(P)$.

Example 5: A Hybrid of Cobb-Douglas and CES: HSA is broad enough to include demand systems that allow for goods to be gross substitutes and at the same time essential. Furthermore, only a subset of goods can be essential. To see this, we apply the trick used in the Remark 1, Eq. (10), to the CES budget-share functions, which yields $s_i(z) = \varepsilon \alpha_i + (1-\varepsilon)\beta_i z^{1-\sigma}$ for $0 < \varepsilon < 1$, where α_i and β_i are such that $\alpha_i \geq 0$, $\sum_{k=1}^{n} \alpha_k = 1$, $\beta_i > 0$,

$\sum_{k=1}^{n} \beta_k = 1$. The resulting demand system is a convex combination of a Cobb-Douglas and a CES, except α_i can be zero for some i. Using the same reasoning as in Remark 1, it follows from (4) that, the common price aggregator is equal to $A(\mathbf{p}) = (\sum_{i=1}^{n} \beta_i p_i^{1-\sigma})^{\frac{1}{1-\sigma}}$, which is independent of ε. Hence, the demand system can be written as follows:

$$x_i = \frac{h}{p_i}\left[\varepsilon\alpha_i + (1-\varepsilon)\frac{\beta_i p_i^{1-\sigma}}{\sum_{k=1}^{n}\beta_k p_k^{1-\sigma}}\right]. \tag{15}$$

while the corresponding ideal price index is given by

$$P(\mathbf{p}) = c\left(\prod_{i=1}^{n} p_i^{\alpha_i}\right)^{\varepsilon}\left(\sum_{i=1}^{n}\beta_i p_i^{1-\sigma}\right)^{\frac{1-\varepsilon}{1-\sigma}} = c\left(\prod_{i=1}^{n} p_i^{\alpha_i}\right)^{\varepsilon}\left(A(\mathbf{p})\right)^{1-\varepsilon},$$

as implied by Proposition 1(ii). For $\sigma > 1$, the goods are gross substitutes. Furthermore, for a small $\varepsilon > 0$, eq. (15) is practically indistinguishable from a CES demand system. Yet, for any $\varepsilon > 0$, $\lim_{p_i \to \infty} P(\mathbf{p}) = \infty$ and $u(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbb{R}_+^n \cap \{x_i = 0\}$ for each i such that $\alpha_i > 0$, but not for i such that $\alpha_i = 0$. In other words, good i is essential and indispensable if and only if $\alpha_i > 0$.[17]

4. HDIA and HIIA: Two Cases of Homothetic Implicit Additivity

We now turn to two classes of *implicitly additive* homothetic preferences. Hanoch (1975) introduces the general notion of implicitly additive preferences. Unlike explicit additivity, under which homotheticity is equivalent to CES, implicit additivity imposes a priori no functional relation between the income elasticity of each good and the cross-price elasticity of substitution across goods. Hence, implicit additivity allows for both *nonhomothetic CES*, used by Comin, Lashkari, and Mestieri (2015) and Matsuyama (2015), and *homothetic non-CES*, which is our focus here. Following Hanoch (1975), we distinguish between *direct* and *indirect* implicit additivity.

4.1   HDIA preferences.

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[17]To understand the implications of this example, consider international trade between two countries, and suppose that some of the essential goods (say, water, food, fuels, electricity, medical supplies, etc) can be produced only in one country. With a small $\varepsilon > 0$, the demand system can be approximated by CES, and generates a high estimate of the trade elasticity, $\sigma > 1$. If we were to assume that the demand system is CES (i.e., $\varepsilon = 0$), we would come up with a relatively small welfare loss due to autarky, even though the welfare loss of autarky is infinity (if measured in the ideal price index) for the country which cannot produce such essential goods.

A preference $\succsim$ over $\mathbb{R}^n_+$ is said to be *homothetic with direct implicit additivity* (henceforth HDIA) if $\succsim$ can be described by a linear homogeneous and strictly quasi-concave direct utility function $u(\mathbf{x})$ which is implicitly defined as a solution to

$$\sum_{i=1}^{n} \phi_i\left(\frac{x_i}{u}\right) = 1, \tag{16}$$

where the functions $\phi_i: \mathbb{R}_+ \to \mathbb{R}$ are either strictly increasing and strictly concave (in which case the goods are gross substitutes) or strictly decreasing and strictly convex (in which case the goods are gross complements). Moreover, $\phi_i(\cdot)$ are assumed to be twice continuously differentiable functions with the following normalization:

$$\sum_{i=1}^{n} \phi_i(1) = 1, \tag{17}$$

which implies $u(\mathbf{1}) = 1$. The CES demand system is a special case, where $\phi_i(z) = \gamma_i z^{1-1/\sigma}$, with $\gamma_i > 0$, $\sum_{i=1}^{n} \gamma_i = 1$, and $\sigma > 0$. These functions $\phi_i(z)$ are all strictly decreasing and strictly convex for $0 < \sigma < 1$ (the case of gross complements), and are all strictly increasing and strictly concave for $\sigma > 1$ (the case of gross substitutes).[18]

The following Proposition summarizes the properties of HDIA demand systems generated by (16).

Proposition 2. *Assume that the functions $\phi_i(\cdot)$ are either strictly increasing and strictly concave or strictly decreasing and strictly convex, are twice continuously differentiable functions, and satisfy the normalization* (17). *Then*:

(i) *the HDIA preference $\succsim$ described by* (16) *generates the Marshallian demand system of the following structure*:

$$x_i = h\frac{(\phi_i')^{-1}\left(\frac{p_i}{B(\mathbf{p})}\right)}{P(\mathbf{p})}, \tag{18}$$

where $P(\mathbf{p})$ is the HDIA ideal price index given by

$$P(\mathbf{p}) = \sum_{k=1}^{n} p_k (\phi_k')^{-1}\left(\frac{p_k}{B(\mathbf{p})}\right), \tag{19}$$

while $B(\mathbf{p})$ is another price aggregator defined as the solution to

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[18] A special case of (16) where there is a continuum of goods, while $\phi_i(\cdot)$ are strictly increasing and strictly concave (the case of gross substitutes) and are the same for all i, is known as *Kimball's flexible aggregator* (Kimball, 1995).

$$\sum_{k=1}^{n} \phi_k \left((\phi_k{}')^{-1} \left(\frac{p_k}{B} \right) \right) = 1. \tag{20}$$

(ii) *When $n \geq 3$, $B(\mathbf{p}) = cP(\mathbf{p})$ iff $\succsim$ is a CES.*

Proof. See the Appendix. ∎

4.2   HIIA preferences.

A preference $\succsim$ over $\mathbb{R}_+^n$ is said to be *homothetic with indirect implicit additivity* (henceforth HIIA) if the ideal price index $P(\mathbf{p})$ associated with $\succsim$ is defined as a solution to

$$\sum_{i=1}^{n} \theta_i \left(\frac{p_i}{P} \right) = 1, \tag{21}$$

where the functions $\theta_i \colon \mathbb{R}_+ \to \mathbb{R}$ are either strictly decreasing and strictly convex (in which case the goods are gross substitutes) or strictly increasing and strictly concave (in which case the goods are gross complements). Moreover, $\theta_i(\cdot)$ are assumed to be twice continuously differentiable functions satisfying the following normalization:

$$\sum_{i=1}^{n} \theta_i(1) = 1. \tag{22}$$

which implies $P(\mathbf{1}) = 1$. The CES demand system is a special case, where $\theta_i(z) = \beta_i z^{1-\sigma}$, with $\beta_i > 0$, $\sum_{i=1}^{n} \beta_i = 1$, which are all strictly increasing and strictly concave for $0 < \sigma < 1$ (the case of gross complements), and all strictly decreasing and strictly convex for $\sigma > 1$ (the case of gross substitutes).

The following Proposition summarizes the properties of the HIIA demand system generated by (21).

Proposition 3. *Assume that the functions $\theta_i(\cdot)$ are either strictly decreasing and strictly convex or strictly increasing and strictly concave, are twice continuously differentiable, and satisfy the normalization* (22). *Then*:

(i) The Marshallian demand system generated by HIIA preferences must satisfy

$$x_i = h \frac{\theta_i' \left(\frac{p_i}{P(\mathbf{p})} \right)}{C(\mathbf{p})}, \tag{23}$$

where $C(\mathbf{p})$ is a price aggregator defined by

$$C(\mathbf{p}) \equiv \sum_{k=1}^{n} p_k \theta_k' \left(\frac{p_k}{P(\mathbf{p})} \right). \tag{24}$$

(ii) When $n \geq 3$, $C(\mathbf{p}) = cP(\mathbf{p})$ only holds in the CES case.

Proof. See the Appendix. ■

4.3   Comparing HSA, HDIA, and HIIA

A quick glance at Eqs. (18) and (23) suggests that both HDIA and HIIA are in general incompatible with HSA. Indeed, in both cases, the demand for each good depends on the own price and *two price aggregators*. For HDIA, they are $P(\mathbf{p})$ and $B(\mathbf{p})$, which do not move together except the CES case, as shown by Proposition 2(ii). For HIIA, they are $P(\mathbf{p})$ and $C(\mathbf{p})$, which do not move together except the CES case, as shown by Proposition 3(ii). We confirm this conjecture in the next proposition.

Proposition 4. *Assume that $n \geq 3$. Then*:

(i) HDIA ∩ HSA = CES;

(ii) HIIA ∩ HSA = CES;

(iii) HDIA ∩ HIIA = CES.

Proof. See the Appendix. ■

Proposition 4 states that, apart from the CES, the three classes of homothetic preferences under consideration – HSA, HDIA and HIIA – are *pairwise disjoint*. Figure 1 visualizes this result.[19]

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[19]Notice an analogy between this result and the well-known result that the three classes of preferences, – homothetic, directly explicitly additive, and indirectly explicitly additive – are *pairwise disjoint*, apart from the CES. Compare our Figure 1 with Figure 1 in Thisse and Ushchev (2016).
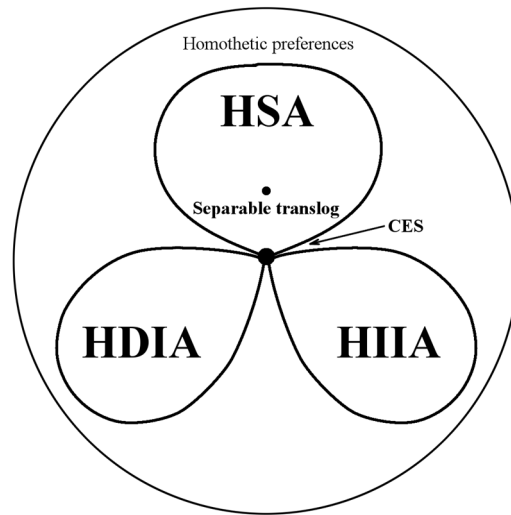
Figure 1. Three alternative ways of departure from CES

In other words, these three classes offer us three *alternative* (i.e., nonoverlapping) ways of departing from CES. Another implication of this Proposition is that a separable translog (Example 3a), as a member of HSA demand systems, does not belong to HDIA nor HIIA.

5. Concluding Remarks

In modeling applied general equilibrium, working with homothetic preferences and CRS production technologies is necessary to justify the assumption of the representative consumers and perfect competition, and helps to isolate the role of relative prices in the market allocation of resources. In practice, however, most models assume that the utility and production functions satisfy not only the homotheticity/CRS but also the CES property, which imposes many additional restrictions. For those who want to investigate the implications of the homotheticity without the straightjacket of CES, the three classes of homothetic demand systems studied here – HSA, HDIA, and HIIA, – should be promising. Considering that these three classes of non-CES homothetic demand systems are defined non-parametrically and do not overlap, they should offer a plenty of new opportunities to explore.

The reader may ask, legitimately, which one of these three classes should be explored *first.* To this, our recommendation is HSA, as it promises to be more tractable than HDIA and HIIA. To see this, compare the form of HSA demand systems, given by (3) – (4), with those of HDIA demand systems, given by (16) – (18) and of HIIA demand systems, given by (19), (21), and (22). For non-

CES HDIA, the budget share of good i, $p_i x_i / h$, depends on *two* relative prices, $p_i / B(\mathbf{p})$ and $p_i / P(\mathbf{p})$. For non-CES HIIA, it also depends on *two* relative prices, $p_i / C(\mathbf{p})$ and $p_i / P(\mathbf{p})$. In both HDIA and HIIA, all the cross-price effects are summarized in two price aggregators. In contrast, the budget share of good i, $p_i \mathsf{x}_i / \mathsf{h}$, depends solely on *one* relative price, $p_i / A(\mathbf{p})$, for HSA, because all the cross-price effects are summarized in one price aggregator. Indeed, this is precisely the reason why we call this class *H*omothetic demand systems with a *S*ingle *A*ggregator. This makes it easier to solve for the equilibrium allocations and prices under HSA than under HDIA and HIIA. This is not to say that the ideal price index, $P(\mathbf{p})$, is unimportant under HSA. It is important for evaluating the welfare implications of the market outcome. It does suggest, however, that the ideal price index, $P(\mathbf{p})$, is not needed for computing the market outcome and conducing the comparative static exercises, which is a great advantage of HSA.[20]

Needless to say, we are *not* arguing that HSA are superior to HDIA or HIIA. We are merely suggesting that one might want to explore HSA *before* HDIA and HIIA. All the cross-price effects in HDIA and HIIA are summarized in *only two* price aggregators, which buys some tractability compared to the general class of homothetic preferences.

Finally, our characterization of the three classes is also relevant for those interested in applying non-CES homothetic demand systems in the Dixit-Stiglitz (1977) type horizontally differentiated monopolistic competition framework.[21] For such applications, one should focus on a special case of symmetric preferences with gross substitutes ($\alpha'(z) < 0$ for HSA, $\phi'(z) > 0$ and $\phi''(z) < 0$ for HDIA, and $\Theta'(z) < 0$ and $\Theta''(z) > 0$ for HIIA). In addition, two modifications become necessary. First, to justify the assumption that the market power of each monopolistically competitive firm has no effect on the price aggregates, the set of differentiated goods (and of firms selling them) has to be a continuum, instead of a finite set. With some caveats on technical issues, this could be taken care of by replacing all the summation signs by the integral signs in the equations above. Second, to allow for the number (or, more precisely, the mass) of goods offered on the market to be endogenized through entry-exit of firms, we need to ensure the continuity of preferences by imposing an additional boundary conditions: $\alpha(\infty) = 0$ for HSA; $\phi(0) = 0$ for HDIA, and $\Theta(\infty) = 0$ for HIIA. Despite all these additional considerations, our results would still

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[20] It should also be pointed out that the presence of two price aggregators in HDIA and HIIA systems makes them neither more general nor more flexible than HSA, because the two price aggregators are subject to the functional relationship, given by (17) in the case of HDIA and by (22) in the case of HIIA.

[21] Benassy (1996), Feenstra (2003) or Kimball (1995) are among some earlier examples of non-CES homothetic demand systems in monopolistic competition.

go through. That is, HSA, HDIA, and HIIA offer three alternative (i.e., pairwise disjoint) classes of non-CES homothetic demand systems, and the cross-price effects are summarized in one price aggregator in the case of HSA and in two price aggregators in the case of HDIA and HIIA.

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Appendix

Proof of Proposition 1

Proof of part (i). Recall that the *Slutsky matrix* $(\mathfrak{s}_{ij})$ associated with a demand system is defined by

$$\mathfrak{s}_{ij} \equiv \frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial h}. \tag{A1}$$

We proceed by proving the following two lemmas.

Lemma 1. *The Slutsky matrix associated with the demand system* (3) – (4) *is symmetric*:

$$\mathfrak{s}_{ij} = \mathfrak{s}_{ji}.$$

Proof. Combining (3) and (A1), we obtain for $i \neq j$:

$$\mathfrak{s}_{ij} = -\frac{h}{A^2} s_i' \left(\frac{p_i}{A}\right) \frac{\partial A}{\partial p_j} + \frac{h}{p_i p_j} s_i \left(\frac{p_i}{A}\right) s_j \left(\frac{p_j}{A}\right). \tag{A2}$$

Differentiating (3) with respect to p_j, we obtain after simplifications:

$$\frac{\partial A}{\partial p_j} = \frac{s_j' \left(\frac{p_j}{A}\right)}{\sum_{k=1}^n s_k' \left(\frac{p_k}{A}\right) \frac{p_k}{A}}.$$

Plugging this expression into (A2) yields

$$\mathsf{s}_{ij} = -\frac{1}{\sum_{k=1}^{n} s'_k\left(\frac{p_k}{A}\right)\frac{p_k}{A}} \frac{h}{A^2} s_i{}'\left(\frac{p_i}{A}\right) s_j{}'\left(\frac{p_j}{A}\right) + \frac{h}{p_i p_j} s_i\left(\frac{p_i}{A}\right) s_j\left(\frac{p_j}{A}\right). \tag{A3}$$

Equation (A3) immediately implies Slutsky symmetry. This proves Lemma 1. ∎

Lemma 2. *The Slutsky matrix associated with the demand system* (3) − (4) *is negative semidefinite.*

Proof. The diagonal entries s_{ii} of the Slustky matrix are given by

$$\mathsf{s}_{ii} = -\frac{h}{p_i^2}\left[s_i\left(\frac{p_i}{A}\right) - s_i^2\left(\frac{p_i}{A}\right)\right] + \frac{h}{p_i} s'_i\left(\frac{p_i}{A}\right)\frac{1}{A} - \frac{h}{A^2} s'_i\left(\frac{p_i}{A}\right)\frac{\partial A}{\partial p_i},$$

reorganizing which slightly, we obtain

$$\mathsf{s}_{ii} = -\frac{h}{p_i^2} s_i\left(\frac{p_i}{A}\right)\left[1 - s_i\left(\frac{p_i}{A}\right)\right] + \frac{h}{A^2}\frac{A}{p_i} s'_i\left(\frac{p_i}{A}\right)\left(1 - \frac{\partial A}{\partial p_i}\frac{p_i}{A}\right).$$

The quadratic form induced by the Slustky matrix (s_{ij}) of the demand system (3) − (4) is then given by

$$\sum_{i,j} \mathsf{s}_{ij} t_i t_j = -\sum_i \frac{h}{p_i^2} s_i\left(\frac{p_i}{A}\right)\left[1 - s_i\left(\frac{p_i}{A}\right)\right] t_i^2 + \frac{h}{A^2}\sum_i \frac{A}{p_i} s'_i\left(\frac{p_i}{A}\right)\left(1 - \frac{\partial A}{\partial p_i}\frac{p_i}{A}\right) t_i^2$$

$$+ \sum_{i,j:i\neq j}\left[\frac{h}{p_i p_j} s_i\left(\frac{p_i}{A}\right) s_j\left(\frac{p_j}{A}\right) - \frac{h}{A^2} s'_i\left(\frac{p_i}{A}\right)\frac{\partial A}{\partial p_j}\right] t_i t_j.$$

Here $t_1, \dots, t_n$ are independent variables which take on arbitrary real values. Define

$$\tau_i \equiv \frac{\sqrt{h}}{p_i} t_i.$$

Clearly, the new variables $\tau_1, \dots, \tau_n$ also attain arbitrary real values, whence the quadtraic form associated with (s_{ij}) becomes

$$\sum_{i,j} \mathsf{s}_{ij} t_i t_j = -\left\{\sum_{i=1}^{n} s_i\left(\frac{p_i}{A}\right)\left[1 - s_i\left(\frac{p_i}{A}\right)\right]\tau_i^2 - \sum_{i,j:i\neq j} s_i\left(\frac{p_i}{A}\right) s_j\left(\frac{p_j}{A}\right)\tau_i \tau_j\right\}$$

$$+ \sum_k s'_k\left(\frac{p_k}{A}\right)\frac{p_k}{A}\cdot\left\{\sum_i \mathcal{E}_{p_i}(A)\left[1 - \mathcal{E}_{p_i}(A)\right]\tau_i^2 - \sum_{i,j:i\neq j} \mathcal{E}_{p_i}(A)\mathcal{E}_{p_j}(A)\tau_i \tau_j\right\}$$

where $\mathcal{E}_{p_i}(A)$ is the elasticity of the common price aggregator $A(\mathbf{p})$ with respect to p_i:

$$\mathcal{E}_{p_i}(A) \equiv \frac{\partial A}{\partial p_i}\frac{p_i}{A} = \frac{s'_i\left(\frac{p_i}{A}\right)\frac{p_i}{A}}{\sum_k s'_k\left(\frac{p_k}{A}\right)\frac{p_k}{A}}.$$

Introduce the following notation:

$$a_i \equiv s_i\left(\frac{p_i}{A}\right), \qquad b_i \equiv \mathcal{E}_{p_i}(A), \qquad \gamma \equiv \sum_k s_k'\left(\frac{p_k}{A}\right)\frac{p_k}{A}.$$

With this notation, assumption (7) takes the form $a_i > \gamma b_i$, while the quadratic form associated with the Slutsky matrix can be expressed as follows:

$$\sum_{k=1}^{n} \mathrm{s}_{ij} t_i t_j = -\boldsymbol{\tau}^{\mathrm{T}}(\mathbf{A} - \gamma \mathbf{B})\boldsymbol{\tau}.$$

Here $\mathbf{A}$ and $\mathbf{B}$ are $(n \times n)$-matrices having the following structure:

$$\mathbf{A} \equiv \mathrm{diag}\{a_1, \dots, a_n\} - \mathbf{a}\mathbf{a}^{\mathrm{T}}, \qquad \mathbf{B} \equiv \mathrm{diag}\{b_1, \dots, b_n\} - \mathbf{b}\mathbf{b}^{\mathrm{T}},$$

where $\mathbf{a} \equiv (a_1, \dots, a_n)^{\mathrm{T}}$ and $\mathbf{b} \equiv (b_1, \dots, b_n)^{\mathrm{T}}$.

To show that the matrices $\mathbf{A}$ and $\mathbf{B}$ are both positive semidefinite, observe that, since both $\mathbf{a}$ and $\mathbf{b}$ belong to the standard $(n-1)$-dimensional simplex Δ_{n-1}, one can view them as probability distributions. The quadratic forms induced by $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\boldsymbol{\tau}^{\mathrm{T}}\mathbf{A}\boldsymbol{\tau} \equiv \sum_{i=1}^{n} a_i \tau_i^2 - \left(\sum_{i=1}^{n} a_i \tau_i\right)^2 = \mathrm{var}(\mathrm{T}_a),$$

$$\boldsymbol{\tau}^{\mathrm{T}}\mathbf{B}\boldsymbol{\tau} \equiv \sum_{i=1}^{n} b_i \tau_i^2 - \left(\sum_{i=1}^{n} b_i \tau_i\right)^2 = \mathrm{var}(\mathrm{T}_b),$$

where $\mathrm{var}(\cdot)$ is the variance operator, while T_a and T_b are discrete random variables with the following probability distributions:

$$\mathrm{Prob}\{\mathrm{T}_a = \tau_i\} = a_i, \qquad \mathrm{Prob}\{\mathrm{T}_b = \tau_i\} = b_i.$$

The fact that $\mathbf{A}$ and $\mathbf{B}$ are positive semidefinite proves Lemma 2 for the case when $\gamma \leq 0$. To handle the case when $\gamma > 0$, note that $\boldsymbol{\tau}$ annihilates $\boldsymbol{\tau}^{\mathrm{T}}\mathbf{B}\boldsymbol{\tau} = \mathrm{var}(\mathrm{T}_b)$ if and only if the distribution of T_b is degenerate, i.e. when $\boldsymbol{\tau} = c\mathbf{1}$. Furthermore, $\boldsymbol{\tau} = c\mathbf{1}$ also annihilates the Slutsky matrix. Therefore, it remains to prove that $\inf_{\boldsymbol{\tau} \neq c\mathbf{1}} \boldsymbol{\tau}^{\mathrm{T}}(\mathbf{A} - \gamma \mathbf{B})\boldsymbol{\tau} \geq 0$. Furthermore, since $\boldsymbol{\tau}^{\mathrm{T}}(\mathbf{A} - \gamma \mathbf{B})\boldsymbol{\tau}$ is positive homogeneous of degree 2, we can focus on such vectors $\boldsymbol{\tau}$ for which $\boldsymbol{\tau}^{\mathrm{T}}\mathbf{B}\boldsymbol{\tau} = 1$. In other words, we need to show that γ cannot exceed the minimum value of the objective function in the following quadratic optimization problem:

$$\min_{\boldsymbol{\tau}} \boldsymbol{\tau}^{\mathrm{T}}\mathbf{A}\boldsymbol{\tau}$$

$$\text{s.t.}$$

$$\boldsymbol{\tau}^{\mathrm{T}}\mathbf{B}\boldsymbol{\tau} = 1.$$

Using the standard Lagrangian technique, we find that the minimizer $\boldsymbol{\tau}^*$ must satisfy the FOC:

$$\mathbf{A}\boldsymbol{\tau}^* = \lambda^* \mathbf{B}\boldsymbol{\tau}^*,$$

where λ^* is the minimum value of the objective function. In the coordinate form, the FOCs become

$$a_i[\tau_i^* - \mathbb{E}(\mathsf{T}_a^*)] = \lambda^* b_i[\tau_i^* - \mathbb{E}(\mathsf{T}_b^*)] \tag{A4}$$

for all $i = 1, \dots, n$, where $\mathbb{E}(\cdot)$ stands for the expectation operator, so that

$$\mathbb{E}(\mathsf{T}_a^*) \equiv \sum_{k=1}^{n} a_k \tau_k^*, \qquad \mathbb{E}(\mathsf{T}_b^*) \equiv \sum_{k=1}^{n} b_k \tau_k^*.$$

It remains to show that $\lambda^* > \gamma$. We argue by contradiction. Assume that $\lambda^* \leq \gamma$. Then, due to $a_i > \gamma b_i$, we also have $a_i > \lambda^* b_i$. Define

$$\tau_{\max} \equiv \max_i \tau_i^*, \qquad \tau_{\min} \equiv \min_i \tau_i^*.$$

Two cases may arise.

Case 1: $\mathbb{E}(\mathsf{T}_a^*) \leq \mathbb{E}(\mathsf{T}_b^*)$. As the random variables T_a^* and T_b^* both have non-degenerate distributions, we have

$$\tau_{\max} - \mathbb{E}(\mathsf{T}_a^*) \geq \tau_{\max} - \mathbb{E}(\mathsf{T}_b^*) > 0.$$

Combining this with $a_i > \lambda^* b_i$, we find that (A4) fails to hold for $i \in \text{Arg}\max_k \tau_k^*$.

Case 2: $\mathbb{E}(\mathsf{T}_a^*) > \mathbb{E}(\mathsf{T}_b^*)$. In this case, we have

$$\tau_{\min} - \mathbb{E}(\mathsf{T}_a^*) < \tau_{\min} - \mathbb{E}(\mathsf{T}_b^*) < 0,$$

which, taken together with $a_i > \lambda^* b_i$, implies that (A4) fails to hold for $i \in \text{Arg}\min_k \tau_k^*$.

This yields a contradiction, and proves Lemma 2. ∎

Combining Lemma 1, Lemma 2 and the budget balancedness property (5) implied by (3) and (4), we find that the demand system (3) – (4) satisfies all the conditions of *Antonelli's integrability theorem* (Antonelli, 1886; Hurwicz and Uzawa, 1971; see also Ch. 3 in Mas-Colell et al., 1995, and Ch. 2 in Jehle and Reny, 2012). Hence, there exists a unique monotone, convex and continuous rational preference generating the demand system (3) – (4). Furthermore, as the budget shares $s_i \equiv p_i x_i / h$ of this demand system are independent of the income h, the underlying preference must also be homothetic.

It remains to prove that the budget-share mapping $\mathbf{s}(\mathbf{z})$ over $\mathbf{s}^{-1}(\Delta_{n-1})$ can be uniquely recovered from an HSA demand system (3) – (4). Consider two different budget-share mappings $\mathbf{s}(\mathbf{z})$ and $\tilde{\mathbf{s}}(\mathbf{z})$ satisfying (6) – (8) and generating the same HSA demand system: $\mathbf{x}(\mathbf{p}, h) = \tilde{\mathbf{x}}(\mathbf{p}, h)$ for all $(\mathbf{p}, h) \in \mathbb{R}_+^{n+1}$. Then, it must be that the price aggregators $A(\mathbf{p})$ and $\tilde{A}(\mathbf{p})$

implied by $\mathbf{s}(\mathbf{z})$ and $\tilde{\mathbf{s}}(\mathbf{z})$ are different. Fix p_1 and vary $\mathbf{p}_{-1}$ so that $A(\mathbf{p})$ remains unchanged while $\tilde{A}(\mathbf{p})$ changes (which is possible since $n \geq 3$). Then, we have $x_1(h, \mathbf{p}) \neq \tilde{x}_1(h, \mathbf{p})$. Hence, the demand systems $\mathbf{x}(\mathbf{p}, h)$ and $\tilde{\mathbf{x}}(\mathbf{p}, h)$ are not identical. This yields a contradiction, which completes the proof of part (i). ∎

Proof of part (ii). Equations (2) and (3) imply the following identities:

$$\mathcal{E}_{p_i}(P) = s_i \left(\frac{p_i}{A(\mathbf{p})} \right), \quad i = 1, \dots, n.$$

Hence, the ideal price index $P(\mathbf{p})$ associated with a HSA demand system (3) – (4) must be a solution to the following system of first-order linear PDEs:

$$\frac{\partial P(\mathbf{p}) / \partial p_i}{P(\mathbf{p})} = \frac{1}{p_i} s_i \left(\frac{p_i}{A(\mathbf{p})} \right). \tag{A5}$$

Because $P(\mathbf{p})$ is linear homogeneous while $\partial P(\mathbf{p}) / \partial p_i$ are homogeneous of degree zero, equation (A5) can be recast as follows:

$$\frac{\partial P(\mathbf{z}) / \partial z_i}{P(\mathbf{z})} = \frac{s_i(z_i)}{z_i}, \tag{A6}$$

where $\mathbf{z} \equiv \mathbf{p}/A(\mathbf{p})$. Integrating (A6), we get

$$\ln P(\mathbf{z}) = \sum_{i=1}^{n} \int_{c_1}^{z_i} \frac{s_i(\xi)}{\xi} d\xi,$$

which yields (9) after replacing $\mathbf{z}$ with $\mathbf{p}/A(\mathbf{p})$. This completes the proof of part (ii). ∎

Proof of part (iii). Assume that $P(\mathbf{p}) = cA(\mathbf{p})$. Then, it follows immediately from (9) that

$$\sum_{i=1}^{n} \int_{c}^{z_i} \frac{s_i(\xi)}{\xi} d\xi = \text{const}, \tag{A7}$$

where $\mathbf{z} \equiv \mathbf{p}/A(\mathbf{p})$. One can also rewrite equation (4) as follows:

$$\sum_{i=1}^{n} s_i(z_i) = 1. \tag{A8}$$

Hence, the gradients of the two $(n-1)$-dimensional surfaces described, respectively, by (A7) and (A8), must be collinear:

$$\left(s_1'(z_1), \dots, s_1'(z_n) \right) \parallel \left(\frac{s_1(z_1)}{z_1}, \dots, \frac{s_n(z_n)}{z_n} \right).$$

This, in turn, implies

$$\frac{s_i{}'(z_i)z_i}{s_i(z_i)} = \frac{s_j{}'(z_j)z_j}{s_j(z_j)} < 1$$

for all $i, j = 1, 2, \dots, n$. Hence, the elasticities of $s_i(\cdot)$ must be independent of i. Furthermore, when $n \geq 3$, these elasticities must also be constant, which can be shown by fixing z_i and varying $\mathbf{z}_{-i}$ along the surface (A7). We conclude that $s_i(\cdot)$ must have the form $s_i(z_i) = \beta_i z_i^{1-\sigma}$, where $\beta_i > 0$ and $\sigma > 0$. As shown in Section 2 (see Example 2), this corresponds to the CES case. This completes the proof of part (iii) and proves the whole Proposition 1. ∎

Proof of the Self-Duality of HSA Demand Systems

First, the one-to-one correspondence between $\mathbf{s}(\mathbf{z})$ and $\mathbf{s}^*(\mathbf{y})$ implies $s_i^*(y_i) \equiv s_i(s_i^*(y_i)/y_i)$, or equivalently $s_i(z_i) \equiv s_i^*(s_i(z_i)/z_i)$. Differentiating either of these two identities yields

$$\left(1 - \frac{s_i{}'(z_i)z_i}{s_i(z_i)}\right)\left(1 - \frac{s_i^*{}'(y_i)y_i}{s_i^*(y_i)}\right) = 1.$$

Hence, we have:

$$0 < \frac{s_i{}'(z_i)z_i}{s_i(z_i)} < 1 \Longleftrightarrow \frac{s_i^*{}'(y_i)y_i}{s_i^*(y_i)} < 0;$$

$$\frac{s_i{}'(z_i)z_i}{s_i(z_i)} < 0 \Longleftrightarrow 0 < \frac{s_i^*{}'(y_i)y_i}{s_i^*(y_i)} < 1;$$

and

$$\frac{s_i{}'(z_i)z_i}{s_i(z_i)} = 0 \Longleftrightarrow \frac{s_i^*{}'(y_i)y_i}{s_i^*(y_i)} = 0.$$

Thus, the conditions (7) – (8) are equivalent to the conditions (7^*) – (8^*).

Next, to see that (3) – (4) imply (3^*) – (4^*), rewrite (3) as:

$$\frac{p_i x_i}{h} = s_i\left(\frac{p_i}{A(\mathbf{p})}\right) = s_i\left(\frac{p_i x_i}{h}\bigg/\frac{A(\mathbf{p})x_i}{h}\right) \Rightarrow s_i\left(\frac{p_i}{A(\mathbf{p})}\right) = \frac{p_i x_i}{h} = s_i^*\left(\frac{A(\mathbf{p})x_i}{h}\right).$$

Let $\nabla u(\mathbf{x}) \equiv \left(u_{x_1}(\mathbf{x}), u_{x_2}(\mathbf{x}), \dots, u_{x_n}(\mathbf{x})\right)$ denote the gradient of $u(\mathbf{x})$. Then, the inverse demand system (2^*) generated by $u(\mathbf{x})$ can be written as $\mathbf{p} = (h/u(\mathbf{x}))\nabla u(\mathbf{x})$, so that:

$$s_i\left(\frac{p_i}{A(\mathbf{p})}\right) = \frac{p_i x_i}{h} = s_i^*\left(\frac{A(\mathbf{p})x_i}{h}\right) = s_i^*\left(\frac{A(\nabla u(\mathbf{x}))x_i}{u(\mathbf{x})}\right) = s_i^*\left(\frac{x_i}{A^*(\mathbf{x})}\right),$$

by setting $A^*(\mathbf{x}) \equiv u(\mathbf{x})/A(\nabla u(\mathbf{x}))$, from which (3^*) and (4^*) follow. Likewise, rewrite (3^*) as:

$$\frac{p_i x_i}{h} = s_i^* \left(\frac{x_i}{A^*(\mathbf{x})}\right) = s_i^* \left(\frac{p_i x_i}{h} / \frac{A^*(\mathbf{x}) p_i}{h}\right) \Rightarrow s_i^* \left(\frac{x_i}{A^*(\mathbf{x})}\right) = \frac{p_i x_i}{h} = s_i \left(\frac{A^*(\mathbf{x}) p_i}{h}\right).$$

Let $\nabla P(\mathbf{p}) \equiv \left(P_{p_1}(\mathbf{p}), P_{p_2}(\mathbf{p}), \ldots, P_{p_n}(\mathbf{p})\right)$ denote the gradient of $P(\mathbf{p})$. Then, the demand system (2) generated by $P(\mathbf{p})$ can be written as $\mathbf{x} = (h/P(\mathbf{p}))\nabla P(\mathbf{p})$, so that:

$$s_i^* \left(\frac{x_i}{A^*(\mathbf{x})}\right) = \frac{p_i x_i}{h} = s_i \left(\frac{A^*(\mathbf{x}) p_i}{h}\right) = s_i \left(\frac{A^*(\nabla P(\mathbf{p})) p_i}{P(\mathbf{p})}\right) = s_i \left(\frac{p_i}{A(\mathbf{p})}\right),$$

by setting $A(\mathbf{p}) \equiv P(\mathbf{p})/A^*(\nabla P(\mathbf{p}))$, from which (3) and (4) follow. This completes the proof. ∎

Proof of Proposition 2

Proof of part (i). The expenditure minimization problem for a HDIA preference can be stated as follows:

$$\min_{\mathbf{x}} \sum_{k=1}^{n} p_k x_k$$

$$\text{s. t.}$$

$$\sum_{k=1}^{n} \phi_k \left(\frac{x_k}{u}\right) = 1,$$

The FOCs of this problem are given by

$$p_k = \lambda \phi_k'(z_k),$$

where $\mathbf{z} \equiv \mathbf{x}/u$, while λ is the Lagrange multiplier. Solving this FOCs for z_k, we get

$$z_k = (\phi_k')^{-1} \left(\frac{p_k}{\lambda}\right),$$

Plugging this into the constraint $\sum_{k=1}^{n} \phi_k(z_k) = 1$, we find that $\lambda = B(\mathbf{p})$, where $B(\mathbf{p})$ is a solution to (20). Plugging $z_k = (\phi_k')^{-1}(p_k/B(\mathbf{p}))$ into the expression $P(\mathbf{p}) = \sum_{k=1}^{n} p_k z_k$ for the price index yields (19). Finally, combining $x_k = u z_k$ with $P(\mathbf{p}) = \sum_{k=1}^{n} p_k z_k$ yields $P(\mathbf{p})u = \sum_{k=1}^{n} p_k x_k = h$, from which $x_k = u z_k = h z_k/P(\mathbf{p})$. Plugging $z_k = (\phi_k')^{-1}(p_k/B(\mathbf{p}))$ into this expression yields (18). ∎

Proof of part (ii). Sufficiency is straightforward. To prove the necessity, assume that $B(\mathbf{p}) = cP(\mathbf{p})$. Then, it follows immediately from (19) that

$$\sum_{k=1}^{n} \phi_k'(z_k) z_k = \text{const}, \tag{A9}$$

where $z_k = (\phi_k')^{-1}(p_k/B(\mathbf{p}))$, which also satisfies

$$\sum_{k=1}^{n} \phi_k(z_k) = 1. \tag{A10}$$

Hence, the gradients of the two $(n-1)$-dimensional surfaces described by (A9) and (A10) must be collinear to each other:

$$\left(\phi_1'(z_1), \ldots, \phi_n'(z_n)\right) \parallel \left(\phi_1'(z_1) + z_1\phi_1''(z_1), \ldots, \phi_n'(z_n) + z_n\phi_n''(z_n)\right).$$

Put it differently, the coordinates of these two gradients are proportional to each other, which implies

$$\frac{z_i\phi_i''(z_i)}{\phi_i'(z_i)} = \frac{z_j\phi_j''(z_j)}{\phi_j'(z_j)}$$

for all $i, j = 1, 2, \ldots, n$. Hence, the elasticities of $\phi_i'(\bullet)$ must be independent of i. Furthermore, for $n \geq 3$, these elasticities must be also constant, which can be shown by fixing z_i and varying $\mathbf{z}_{-i}$ along the surface (A9). This implies $\phi_i(z) = \gamma_i z^{1-1/\sigma} + a_i$. Taking into account (15) and the requirement that $\phi_i(\bullet)$ are all strictly increasing and strictly concave or strictly decreasing and strictly convex, we conclude that the underlying utility u is CES. ∎

Proof of Proposition 3.

Proof of part (i). Part (i) follows immediately from (2) and (21).

Proof of part (ii). Sufficiency is straightforward. To prove the necessity, assume that $C(\mathbf{p}) = cP(\mathbf{p})$. Then, it follows immediately from (19) that

$$\sum_{k=1}^{n} \theta_k'(z_k)z_k = const, \tag{A11}$$

where $z_k = p_k/P(\mathbf{p})$, which also satisfies, from (19),

$$\sum_{k=1}^{n} \theta_k(z_k) = 1. \tag{A12}$$

Using equations (A11) – (A12), one could show the underlying utility u is CES, along the same lines as the part of the proof of Proposition 2(ii) which followed (A9) and (A10). ∎

Proof of Proposition 4

Proof of part (i). Sufficiency is straightforward. To prove necessity, assume that there exists a homothetic demand system which is both HSA and HDIA. Then, as implied by (3) and (18), the expenditure share of good i must satisfy

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$$\frac{p_i x_i}{h} = s_i\left(\frac{p_i}{A(\mathbf{p})}\right) = \frac{p_i}{P(\mathbf{p})}(\phi_i')^{-1}\left(\frac{p_i}{B(\mathbf{p})}\right).$$

Setting $\mathbf{z} \equiv \mathbf{p}/A(\mathbf{p})$ and using linear homogeneity of $B(\mathbf{p})$ and $P(\mathbf{p})$, we get:

$$s_i(z_i) = \frac{z_i}{P(\mathbf{z})}(\phi_i')^{-1}\left(\frac{z_i}{B(\mathbf{z})}\right). \tag{A13}$$

Assume that $P(\mathbf{z}) \neq cB(\mathbf{z})$. Then, because $n \geq 3$, we can perturb $\mathbf{z}_{-i}$ holding z_i fixed so that $P(\mathbf{z})$ changes while $B(\mathbf{z})$ remains unchanged. This contradicts (A13). Hence it must be that $P(\mathbf{z}) = cB(\mathbf{z})$ for all $\mathbf{z} \in \mathbf{s}^{-1}(\Delta_{n-1})$. Because $B(\mathbf{p})$ and $P(\mathbf{p})$ are linear homogeneous, this implies $P(\mathbf{p}) = cB(\mathbf{p})$. Thus, from Proposition 2(ii), the demand system must be CES. ■

Proof of part (ii). The proof goes along the same lines as that of part (i), except that the proportionality of $P(\mathbf{p})$ to $B(\mathbf{p})$ is replaced by that of $P(\mathbf{p})$ to $C(\mathbf{p})$.

Proof of part (iii). Sufficiency is straightforward. To prove the necessity, assume there is a demand system which is both HDIA and HIIA. Then, it must be that

$$\frac{p_i x_i}{h} = \frac{p_i}{P(\mathbf{p})}(\phi_i')^{-1}\left(\frac{p_i}{B(\mathbf{p})}\right) = \frac{p_i}{C(\mathbf{p})}\theta_i'\left(\frac{p_i}{P(\mathbf{p})}\right).$$

Setting $\boldsymbol{\xi} \equiv \mathbf{p}/C(\mathbf{p})$, we get

$$(\phi_i')^{-1}\left(\frac{\xi_i}{B(\boldsymbol{\xi})}\right) = P(\boldsymbol{\xi})\theta_i'\left(\frac{\xi_i}{P(\boldsymbol{\xi})}\right).$$

Hence, using the same argument as in the proof of part (i), it is readily verified that

$$P(\mathbf{p}) = cB(\mathbf{p}).$$

Proposition 2(ii) then implies that the demand system must be CES.

This completes the proof of Proposition 4. ■