On a two-sector endogenous growth model with quasi-geometric discounting

Ryoji Hiraguchi*†

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Abstract

This article studies a two-sector endogenous growth model with quasi-geometric discounting. We first specialize to log utility and derive a recursive equilibrium path and a planner's solution path. We find that if the discount factor is close to 1, the planner's solution welfare-dominates the equilibrium path. Our result contrasts with the results in Krusell, Kuruşçu, and Smith (Journal of Economic Theory, 2002) who study the Ramsey model to show that the competitive economy always outperforms the planned economy. We next show that if the utility function demonstrates constant relative risk aversion, there can be multiple balanced growth paths.

Keywords: endogenous growth; quasi-geometric discounting; time consistency

JEL classification: E5;

*Faculty of Economics, Ritsumeikan University, 1-1-1, Noji-higashi, Kusatsu, Shiga, Japan. Tel: 81-77-561-4837. Fax: 81-77-561-4837.

†Email: rhira@fc.ritsumei.ac.jp

1 Introduction

Experimental evidence suggests that discounting of future rewards is not geometric (Thaler, 1981; Benzion et al., 1989). Economic models of time-inconsistent preferences are initially studied by Strotz (1956), Phelps and Pollak (1968), and Pollak (1968). Laibson (1997) and Barro (1999) reformulate these models by adopting quasi-geometric (quasi-hyperbolic) discounting. Although some authors (Rubinstein, 2003; Schwarz and Sheshinski, 2007) argue that quasi-geometric discounting is controversial from an empirical viewpoint, Salois and Moss (2011) use market asset data and obtain a statistically significant quasi-geometric parameter.

An important study by Krusell, Kuruşçu, and Smith (2002) (henceforth KKS) introduces quasi-geometric discounting into the standard discrete-time neoclassical growth model. KKS solve an individual's problem as a game between current-self and future-self and focus on Markov equilibria in which only current state variables affect an individual's behavior. Surprisingly, they show that the recursive competitive equilibrium path always welfare-dominates the social planner's solution path. Moreover, they perform a numerical analysis to find that the competitive equilibrium is uniquely determined for a general class of utility functions (KKS, p.48, line 28).

Their results, however, are based on the assumption of only one sector in the economy. This paper examines whether their results hold in a two-sector economy. The set-up is almost the same as in KKS. The only difference is that production of the final good requires human as well as physical capital. The individual has a unit time endowment that is allocated between working (i.e., final goods production) and studying (i.e., human capital accumulation).

First we derive the recursive competitive equilibrium path and the planner's solution path explicitly. Following KKS, we specialize to log period utility and assume that physical and human capitals depreciate completely. We find that the two paths converge to different balanced growth paths. We then compare the competitive equilibrium path with the planner's solution path in terms of social welfare and show that when the discount factor is sufficiently close to 1, the planner's solution path welfare-dominates the equilibrium path. This result differs from KKS.

The crucial difference between our model and KKS is that the welfare function (i.e., the intertemporal utility at date 0) in our model depends both on the savings rate and working time, whereas in the one-sector economy of KKS, welfare depends only on the latter and working time is fixed. (It is shown that these two variables are constant for both the competitive economy and the planned economy.) As in KKS, the savings rate in the competitive economy is closer to the welfare-maximizing savings rate than in the planned economy. Thus the competitive equilibrium would welfare-dominate the planner's solution path if working time is the same in both economies. However, working time in the competitive economy is actually *farther* from the welfare-maximizing working time than in the planned economy. Therefore, the planned economy performs better than the competitive economy for some parameter values.

When preferences are quasi-geometric, the solution path chosen by the social planner generally differs from the competitive equilibrium path. Moreover, even the benevolent planner cannot choose the savings rate and working time that maximize social welfare, because such a strategy is shown to be time inconsistent. This is the reason why the planned economy may perform better or worse than the competitive economy.

Next we use a general constant relative risk aversion (CRRA) utility function and characterize the balanced growth path. Here we focus on the stationary competitive equilibrium path because, as KKS point out, deriving the optimal path becomes analytically difficult when we abandon the log preferences. We find that when the coefficient of relative risk aversion is less than 1, two balanced growth equilibria are possible. This result contrasts with results in KKS. The multiplicity result is also obtained by Krusell and Smith (2003), who show that the neoclassical growth model with quasi-geometric discounting has a continuum of solution paths. One difference between Krusell and Smith (2003) and our paper is that they investigate the planned economy whereas we study the competitive economy.

The quasi-geometric discounting function is now applied to many fields of economics, particularly macroeconomics. Laibson (2001) argues that quasi-geometric discounting leads to under-saving, whereas Salanié and Treich (2006) show that the result can be opposite for some classes of utility functions. Diamond and Kőszegi (2003) show that workers with quasi-geometric discounting retire early. Schwarz and Sheshinski (2007) investigate social security issues. Maliar and Maliar (2006) find indeterminate results. However, these papers are based on either partial equilibrium models or on standard one-sector models. To our knowledge, no previous paper investigates quasi-geometric discounting in a multi-sector economy.

There has been a long debate about whether human capital accumulation enhances economic growth. Whereas earlier empirical works argue that increases in education are not significantly connected with economic growth (Mankiw, Romer, and Weil, 1992; Bils and Klenow, 2000), more recent literature finds a link. For example, Cohen and Soto (2007) construct a comparative multinational dataset for years of schooling and show that schooling contributes to economic growth even after controlling for physical capital accumulation. Ciccone and Papaioannou (2009) obtain similar results. These findings suggest the importance of studying models in which human capital accumulation is explicitly described.

Two-sector endogenous growth models were initiated by Uzawa (1965) and developed by Lucas (1988). Numerous extensions have followed these pioneering works. Caballé and Santos (1993) investigate the transitional dynamics. Benhabib and Perli (1994) incorporate human capital externalities into the model and find that the optimal path can be locally indeterminate. Bond, Wang, and Yip (1996) investigate the case where physical capital is needed in both final goods production and human capital accumulation. Ladron-de-Guevara, Ortigueira, and Santos (1996) introduce labor-leisure choice into the model and find multiple balanced growth paths. Jones, Manuelli, and Rossi (1993); Garcia-Castrillo and Sanso (2000); Goméz (2002); and García-Belenguer (2007) study fiscal policy issues. However, each of these papers uses geometric discounting.

This paper is organized as follows. Section 2 describes the model. Section 3 studies an equilibrium path. Section 4 examines a planner's solution. Section 5 studies the equilibrium multiplicity. Section 6 concludes. The Appendix presents proofs of the propositions.

2 The environment

The environment in this paper extends that of KKS to allow for human capital accumulation. Time is discrete and ranges from 0 to $+\infty$. Preferences of the representative individual at the beginning of period 0 are given by $U_0 = u_0 + \beta(\delta u_1 + \delta^2 u_2 + ...)$, where u_t is the utility in period t, $\delta \in (0,1)$ is the discount factor, and $\beta > 0$ shows the degree of time inconsistency. If $\beta < 1$ ($\beta > 1$), the individual is present biased (future biased). Following KKS, we treat the individual's problem as a game between the current self and the future self and focus on Markov equilibria in which only current state variables matter. For the time being, we assume that period utility takes the simple form $u(c) = \ln(c)$. Intertemporal utility reduces to $U_0 = \ln c_0 + \beta \sum_{t=0}^{\infty} \delta^t \ln c_t$.

The individual has a unit time endowment. He spends n_t units producing final goods and $1 - n_t$ units on human capital accumulation. The feasible allocation must satisfy $n_t \in [0,1]$. In the following, we simply call n_t "working time." The production function is Cobb-Douglas, and the resource constraint is given by

$$k_{t+1} = Ak_t^{\alpha}(n_t h_t)^{1-\alpha} - c_t \quad \text{for } t \geq 0, \tag{1}$$

where k_t is the stock of physical capital in period t, $A > 0$ is the productivity parameter, $\alpha \in (0,1)$ is the physical capital share, h_t is the stock of human capital in period t, and c_t is consumption at time t. Physical capital is fully depreciated. The term $n_t h_t$ shows efficiency per unit of labor. Evolution of human capital is governed by

$$h_{t+1} = B(1 - n_t)h_t \quad \text{for } t \geq 0, \tag{2}$$

where the parameter $B > 0$ shows the efficiency of human capital accumulation. Moreover, the human capital is fully depreciated.

Factor markets are perfectly competitive, and the real interest rate and the wage rate in period t are respectively given by $r_t = \alpha A k_t^{\alpha-1}(n_t h_t)^{1-\alpha}$ and $w_t = (1-\alpha)Ak_t^{\alpha}(n_t h_t)^{-\alpha}$.

3 Recursive competitive equilibrium

In this section, we obtain the recursive competitive equilibrium path. Let $\boldsymbol{x} \equiv (k, h) \in \mathbb{R}^2_+$ denote a vector of the state variables. Variables with primes show next-period values. For example, k' denotes the value of physical capital in the next period.

3.1 Definition of the equilibrium

The individual chooses his future state $\boldsymbol{x}'$ by assuming that factor prices $r(\bar{\boldsymbol{x}})$ and $w(\bar{\boldsymbol{x}})$ depend on the aggregate state $\bar{\boldsymbol{x}} = (\bar{k}, \bar{h})$ and that the process of $\bar{\boldsymbol{x}}$, $\bar{\boldsymbol{x}}' = G(\bar{\boldsymbol{x}}) = (G_1(\bar{\boldsymbol{x}}), G_2(\bar{\boldsymbol{x}}))$ and his future decision rule $\boldsymbol{x}' = g(\boldsymbol{x}, \bar{\boldsymbol{x}}) = (g_1(\boldsymbol{x}, \bar{\boldsymbol{x}}), g_2(\boldsymbol{x}, \bar{\boldsymbol{x}}))$ are given. Here G_1 (G_2) is the next period's total physical (human) capital, and g_1 (g_2) denotes the individual's physical (human) capital in the next period. (Note that $g,\ G : \mathbb{R}^2_+ \to \mathbb{R}^2_+$.)

The individual's budget constraint is $k' = r(\bar{\boldsymbol{x}})k + w(\bar{\boldsymbol{x}})nh - c$, and human capital accumulation is according to $h' = B(1-n)h$, which is re-written as $n = 1 - h'/(Bh)$. The set-up follows García-Belenguer (2007). The rule g determines his consumption $c(\boldsymbol{x}, \bar{\boldsymbol{x}})$ and working time $n(\boldsymbol{x}, \bar{\boldsymbol{x}})$, that depend on his state $\boldsymbol{x}$ and the aggregate state $\bar{\boldsymbol{x}}$:

$$
\begin{aligned}
c(\boldsymbol{x}, \bar{\boldsymbol{x}}) &= r(\bar{\boldsymbol{x}})k + w(\bar{\boldsymbol{x}}) \left\{ 1 - \frac{g_2(\boldsymbol{x}, \bar{\boldsymbol{x}})}{Bh} \right\} h - g_1(\boldsymbol{x}, \bar{\boldsymbol{x}}), && (3)\\
n(\boldsymbol{x}, \bar{\boldsymbol{x}}) &= 1 - \frac{g_2(\boldsymbol{x}, \bar{\boldsymbol{x}})}{Bh}. && (4)
\end{aligned}
$$

The current self's problem, say P_1, is

$$
\begin{aligned}
(P_1)\ \ V_0^e(\boldsymbol{x}, \bar{\boldsymbol{x}}) &= \max_{c \geq 0, n \in [0,1]} \left[\ln c + \beta\delta V^e(\boldsymbol{x}', \bar{\boldsymbol{x}}') \right], && (5)\\
\text{s.t. } k' &= r(\bar{\boldsymbol{x}})k + w(\bar{\boldsymbol{x}})nh - c, && (6)\\
h' &= B(1-n)h. && (7)
\end{aligned}
$$

The value function $V^e(\boldsymbol{x}, \bar{\boldsymbol{x}})$ satisfies the functional equation (hereafter FE)

$$
\text{FE: } V^e(\boldsymbol{x}, \bar{\boldsymbol{x}}) = \ln(c(\boldsymbol{x}, \bar{\boldsymbol{x}})) + \delta V^e(g(\boldsymbol{x}, \bar{\boldsymbol{x}}), G(\bar{\boldsymbol{x}})), \tag{8}
$$

and the terminal condition (hereafter TC)

$$\text{TC: } \lim_{T\to\infty} \delta^T V^e(\boldsymbol{x}_T, \bar{\boldsymbol{x}}_T) = 0, \tag{9}$$

where the sequence $\boldsymbol{x}_t = (k_t, h_t)$ is defined as $\boldsymbol{x}_0 = (k, h)$ and $\boldsymbol{x}_t = g(\boldsymbol{x}_{t-1}, \bar{\boldsymbol{x}}_{t-1})$ for $t \geq 1$.

Before proceeding, we comment on the terminal condition (9) that KKS do not impose. Iterations of Eq. (8) yield

$$V^e(\boldsymbol{x}, \bar{\boldsymbol{x}}) = \sum_{t=0}^{T} \delta^t \ln(c(\boldsymbol{x}_t, \bar{\boldsymbol{x}}_t)) + \delta^{T+1} V^e(\boldsymbol{x}_{T+1}, \bar{\boldsymbol{x}}_{T+1}),$$

for any $T \geq 0$. To assure that V^e is the correct value function, the second term on the right-hand side (RHS) of the above equation must go to 0 as t goes to infinity as Stokey, Lucas, and Prescott (1989) point out. Thus, we need Eq. (9). Its satisfaction is not obvious because the economy grows in our model and $V(\boldsymbol{x}_t)$ diverges as t goes to infinity.

We formally define the recursive competitive equilibrium.

Definition 1: A *recursive competitive equilibrium* consists of a decision rule $g(\boldsymbol{x}, \bar{\boldsymbol{x}})$, a value function $V(\boldsymbol{x}, \bar{\boldsymbol{x}})$, factor prices $r(\bar{\boldsymbol{x}})$ and $w(\bar{\boldsymbol{x}})$, and a difference equation on the aggregate state $\bar{\boldsymbol{x}}' = G(\bar{\boldsymbol{x}})$ such that

1. given $V(\boldsymbol{x}, \bar{\boldsymbol{x}})$, the decision rule $g(\boldsymbol{x}, \bar{\boldsymbol{x}})$ solves the problem P_1;

2. given $g(\boldsymbol{x}, \bar{\boldsymbol{x}})$, the value function $V(\boldsymbol{x}, \bar{\boldsymbol{x}})$ satisfies the FE (8) and the TC (9);

3. $r(\bar{\boldsymbol{x}}) = \alpha A \bar{k}^{\alpha-1}(n(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}})\bar{h})^{1-\alpha}$ and $w(\bar{\boldsymbol{x}}) = (1-\alpha)A\bar{k}^{\alpha}(n(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}})\bar{h})^{-\alpha}$; and

4. $G(\bar{\boldsymbol{x}}) = g(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}})$.

With regard to the first condition of Definition 1, the original problem P_1 is to choose the current control variables c and n, not the future state variables g. However, we can easily confirm there is a one-to-one relationship between the current control variables (i.e., c and n) the future state variables (i.e., k' and h). Thus, when $c(\boldsymbol{x}, \bar{\boldsymbol{x}})$ and $n(\boldsymbol{x}, \bar{\boldsymbol{x}})$ solve P_1, we say that the rule $g(\boldsymbol{x}, \bar{\boldsymbol{x}})$ solves P_1. The fourth condition implies that aggregate capital accumulation is consistent with the individual's capital accumulation.

3.2 Characterization

To obtain the equilibrium path, we first substitute Eq. (7) into Eq. (6) and obtain a consolidated budget constraint $k' + (w(\bar{\boldsymbol{x}})/B)h' = r(\bar{\boldsymbol{x}})k + w(\bar{\boldsymbol{x}})h - c$. It shows that the rate of return on physical capital in period t is $r(\bar{\boldsymbol{x}}')$ and the return on human capital is $Bw(\bar{\boldsymbol{x}}')/w(\bar{\boldsymbol{x}})$. To guarantee existence of the equilibrium, the non-arbitrage condition between physical and human capital must hold. [1] Thus we have

$$r(\bar{\boldsymbol{x}}_{t+1})w(\bar{\boldsymbol{x}}_t) = Bw(\bar{\boldsymbol{x}}_{t+1}) \text{ for all } t \geq 0. \tag{10}$$

Under Eq. (10), the choice of n is irrelevant to the individual's utility, and the consolidated budget constraint is simplified as $y(\boldsymbol{x}', \bar{\boldsymbol{x}}') = r(\bar{\boldsymbol{x}}')\{y(\boldsymbol{x}, \bar{\boldsymbol{x}}) - c\}$, where $y(\boldsymbol{x}, \bar{\boldsymbol{x}}) = r(\bar{\boldsymbol{x}})k + w(\bar{\boldsymbol{x}})h$ is the market value of the individual's total personal capital. We have

Proposition 1 *The recursive equilibrium is given by*

1. $G(\bar{\boldsymbol{x}}) = [s^e A\bar{k}^\alpha(n^e\bar{h})^{1-\alpha}, B(1-n^e)\bar{h}]$, *where* $s^e = \alpha\delta\beta/(1-\delta+\delta\beta)$ *is the equilibrium savings rate and* $n^e = (1-\delta)/(1-\delta+\delta\beta)$ *is the equilibrium working time;*

2. $g(\boldsymbol{x}, \bar{\boldsymbol{x}}) = [(1-n^e)r(\bar{\boldsymbol{x}})k, (1-n^e)Bh]$; *and*

3. $V^e(\boldsymbol{x}, \bar{\boldsymbol{x}}) = p_1 \ln(r(\bar{\boldsymbol{x}})k + w(\bar{\boldsymbol{x}})h) + p_2 \ln r(\bar{\boldsymbol{x}}) + p_3$, *where* $p_1 = 1/(1-\delta)$, $p_2 = \alpha\delta/\{(1-\delta)(1-\alpha\delta)\}$ *and* $p_3 = \{\ln n^e + \delta p_1 \ln(1-n^e) + \delta(p_1+p_2)\ln(B^{1-\alpha})\}/(1-\delta)$.

Proof. See the Appendix. ■

Let $\{k_t^e, h_t^e, c_t^e\}_{t=0}^\infty$ denote the equilibrium allocation where k_t^e, h_t^e and c_t^e respectively denote physical capital, human capital, and consumption in period t. It satisfies $k_{t+1}^e = s^e A k_t^\alpha (n^e h_t)^{1-\alpha}$, $h_{t+1}^e = B(1-n^e)h_t^e$ and $c_t^e = (1-s^e)Ak_t^\alpha(n^e h_t)^{1-\alpha}$. Thus, the human capital growth rate $\Gamma^e = \delta\beta B/(1-\delta+\delta\beta)$ is constant, and as t goes to infinity, the growth rates of physical capital and output converge to Γ^e. The balanced growth rate Γ^e decreases with β, that is, the more future-biased the individual is, the less time he spends

[1] If $r(\bar{\boldsymbol{x}}') < Bw(\bar{\boldsymbol{x}}')/w(\bar{\boldsymbol{x}})$, each individual spends all his time on human capital accumulation (i.e., $n = 0$). In this case, total output in period t is zero, and positive consumption is impossible because physical capital is fully depreciated. On the other hand, if $r(\bar{\boldsymbol{x}}') > Bw(\bar{\boldsymbol{x}}')/w(\bar{\boldsymbol{x}})$, the individual chooses $n = 1$, and human capital (and output) will be zero tomorrow.

producing final goods and the more the economy grows.

4 Planning problem

Now, we investigate the problem of benevolent government. Following KKS, we assume that the preferences of the planner are the same as those of the current self, and we look for a time-consistent solution path to the planning problem. Suppose the current self perceives its future decision on physical and human capital is embodied by $g^*(\boldsymbol{x}) = (g_1^*(\boldsymbol{x}), g_2^*(\boldsymbol{x}))$, where g_1^* (g_2^*) denotes the next period's physical (human) capital. The rule g^* uniquely determines the decision rules for consumption and working time as

$$c^*(\boldsymbol{x}) = Ak^\alpha\{h - B^{-1}g_2^*(\boldsymbol{x})\}^{1-\alpha} - g_1^*(\boldsymbol{x}), \tag{11}$$

$$n^*(\boldsymbol{x}) = 1 - (Bh)^{-1}g_2^*(\boldsymbol{x}). \tag{12}$$

Let P_2 denote the problem of the planner's current self. It is given by

$$(P_2) \quad V_0^*(\boldsymbol{x}) = \max_{c,n} \left[\ln(c) + \beta\delta V^*(\boldsymbol{x}')\right],$$
$$\text{s.t. } k' = Ak^\alpha(nh)^{1-\alpha} - c \text{ and } h' = B(1-n)h.$$

The value function V^* satisfies

$$\text{FE} \ : \ V^*(\boldsymbol{x}) = \ln[c^*(\boldsymbol{x})] + \delta V^*(g^*(\boldsymbol{x})), \tag{13}$$

$$\text{TC} \ : \ \lim_{t\to\infty} \delta^t V(g^{*(t)}(\boldsymbol{x})) = 0, \tag{14}$$

where a sequence of the function $\{g^{*(t)}(\boldsymbol{x})\}_{t=0}^\infty$ is defined by $g^{*(t+1)}(\boldsymbol{x}) = g[g^{*(t)}(\boldsymbol{x})]$ for $t \geq 0$ and $g^{*(0)}(\boldsymbol{x}) = \boldsymbol{x}$. We formally define the planner's solution.

Definition 2: A solution to the planner's problem consists of a decision rule $g^*(\boldsymbol{x})$ ($= (g_1^*(\boldsymbol{x}), g_2^*(\boldsymbol{x}))$) and a value function $V^*(\boldsymbol{x})$ such that

1. given $V^*(\boldsymbol{x})$, the rule $g^*(\boldsymbol{x})$ solves the problem (P_2); and

2. given $g^*(\boldsymbol{x})$, the value function $V^*(\boldsymbol{x})$ satisfies the FE (13) and the TC (14).

We have

Proposition 2 *A solution to the planner's problem is given by*

1. $g^*(\boldsymbol{x}) = [s^* A k^\alpha (n^* h)^{1-\alpha}, B(1-n^*)h]$, *where* $s^* = \beta\delta\alpha/\{1-\alpha\delta+\beta\delta\alpha\}$ *is the savings rate and* $n^* = (1-\alpha\delta+\beta\delta\alpha)(1-\delta)/\{(1-\alpha\delta+\beta\delta\alpha)(1-\delta)+\beta\delta\}$ *is working time; and*

2. $V^*(\boldsymbol{x}) = \rho_1 \ln k + \rho_2 \ln h + \rho_3$ *where* $\rho_1 = \alpha/(1-\alpha\delta)$, $\rho_2 = (1-\alpha)/\{(1-\delta)(1-\alpha\delta)\}$ *and* $\rho_3 = \{(1+\delta\rho_1)\ln(An^{*(1-\alpha)}) + \ln(1-s^*) + \delta\rho_1 \ln s^* + \delta\rho_2 \ln(B(1-n^*))\}/(1-\delta)$.

Proof. See the Appendix. ■

Let $\{k_t^*, h_t^*, c_t^*\}_{t=0}^\infty$ denote a solution path to the planner's problem where k_t^*, h_t^*, and c_t^* are period-t physical capital, human capital, and consumption, respectively. It satisfies $k_{t+1}^* = s^* A (k_t^*)^\alpha (n^* h_t^*)^{1-\alpha}$, $h_{t+1}^* = B(1-n^*)h_t^*$ and $c_t^* = (1-s^*)A(k_t^*)^\alpha (n^* h_t^*)^{1-\alpha}$. Thus, human capital growth rate $\Gamma^* = B(1-n^*)$ is constant, and growth rates of physical capital and output converge to the same value Γ^* as time goes to infinity.

The dynamics of the planner's solution path is very similar to the dynamics of the equilibrium path. The two paths differ only in their savings rates and working time. The next lemma investigates the differences.

Lemma 1 *If* $\beta < 1$, *then* $s^e > s^*$ *and* $n^e > n^*$. *The orders are reversed if* $\beta > 1$.

Proof. See the Appendix. ■

Thus the individual is present-biased (i.e., $\beta < 1$), and the balanced growth rate of the planned economy, $\Gamma^* = B(1-n^*)$, is higher than for the competitive economy $\Gamma^e = B(1-n^e)$.

When discounting is geometric ($\beta = 1$), there is no time inconsistency, and the competitive equilibrium allocation is the same as the planner's solution. We can easily check that $s^e = s^* = \alpha\delta$, $n^e = n^* = 1-\delta$ and $\Gamma^e = \Gamma^* = \delta B$ if $\beta = 1$. This coincides with the findings of Bethmann (2007) who obtains a closed-form solution path for the discrete-time two-sector endogenous growth model with geometric discounting (Bethmann, 2007, p. 96, Eq (22)).

5 Utility comparison

Here, we compare the equilibrium path with the planner's solution path in regard to social welfare (i.e., utility of the current self). As the previous section made clear, the savings rate s and working time n are constant for the two paths. The next lemma provides a simple expression for welfare when the savings rate and working time are fixed.

Lemma 2 *Define $\theta \equiv (1-\delta)(1-\alpha\delta)$. If the initial state is $\boldsymbol{x}$, the savings rate is s and working time is n, the utility $U_0 = u_0 + \delta(\beta u_1 + \beta^2 u_2 + ...)$ is expressed as*

$$U_0(\boldsymbol{x}) = \theta^{-1} f(s,n) + \psi_1 \ln k + \psi_2 \ln h, \tag{15}$$

where $\psi_1 = \alpha(1-\beta) + \alpha\beta/(1-\alpha\delta)$ and $\psi_2 = (1-\alpha)(1-\beta+\theta^{-1}\beta)$. The function f is given by

$$f(s,n) = \phi_1 \ln s + \phi_2 \ln(1-s) + \phi_3 \ln n + \phi_4 \ln(1-n) + \phi_5,$$

where $\phi_1 = \alpha\delta\beta$, $\phi_2 = (1-\alpha\delta)(1-\delta+\delta\beta)$, $\phi_3 = \theta\psi_2$, $\phi_4 = \beta\delta(1-\alpha)/(1-\delta)$, and $\phi_5 = \phi_3/(1-\alpha)\ln A + \phi_4 \ln B$.

Proof. See the Appendix. ■

Thus the difference in welfare between the competitive equilibrium and the planner's solution, $V_0^*(\boldsymbol{x}) - V_0^e(\boldsymbol{x}) = \theta^{-1}\{f(s^*,n^*) - f(s^e,n^e)\}$ is independent of the capital level, and the planned economy welfare-dominates the competitive economy if and only if

$$\Delta V \equiv f(s^*,n^*) - f(s^e,n^e) > 0. \tag{16}$$

Investigations of Eq. (16) are not simple. However, we can show that the planned economy outperforms the competitive economy if the individual is sufficiently patient.

Proposition 3 *If the discount factor δ is close to 1 and $\beta \neq 1$, the solution to the planning problem welfare-dominates the competitive equilibrium.*

Proof. See the Appendix. ■

Although there exists a savings rate $s^f = \phi_1/(\phi_1 + \phi_2) = \alpha\delta\beta/\{\beta + \theta(1-\beta)\}$ and a working time $n^f = \phi_3/(\phi_3 + \phi_4) = \{(1-\delta)\theta(1-\beta) + \beta - \beta\delta\}/\{(1-\delta)\theta(1-\beta) + \beta\}$ which uniquely maximize the welfare function $f(s, n)$, the planner cannot choose a path along which $(s, n) = (s^f, n^f)$ because of the time inconsistency. We can check that $s^f > s^e > s^*$ and $n^e > n^* > n^f$ if $\beta < 1$ and that the order is reversed if $\beta > 1$. [2] This implies that the first-best savings rate s^f is closer to the equilibrium savings rate s^e than the savings rate of the planner s^*, whereas the first-best working time n^f is *further* from the equilibrium working time n^e than the planner's working time n^*. In regard to the savings rate, the competitive equilibrium performs better than the planner's solution. This is consistent with the findings of KKS (See p. 68). In KKS, welfare depends only on the savings rate, and the competitive economy always works better than the planned economy. However, the planned economy performs better with regard to working time. This is why the planner's solution can welfare-dominate the equilibrium path in our model.

We now turn to the numerical analysis. We follow Krusell et al. (2009) and Henriksen and Kydland (2010) and define the welfare difference as a consumption-equivalent λ that satisfies

$$u(c_0^*) + \beta \sum_{t=1}^{\infty} \delta^t u(c_t^*) = u((1+\lambda)c_0^e) + \beta \sum_{t=1}^{\infty} \delta^t u((1+\lambda)c_t^e),$$

so that the planner's solution performs better than the competitive equilibrium if and only if $\lambda > 0$. Because we are using the log utility, the welfare difference and λ satisfy $V_0^* - V_0^e = \{1 + \beta/(1-\delta)\} \ln(1+\lambda)$. Figure 1 shows λ as a function of the degree of time inconsistency β. Following Coibion, Gorodnichenko, and Wieland (2011), we assume that capital share is 0.4 and the discount factor is 0.99.

The figure shows that the welfare difference is positive when $\beta \neq 1$. When the discounting is geometric (i.e., $\beta = 1$), the competitive economy coincides with the planned economy, so there is no welfare difference.

[2] If $\beta < 1$, $s^f/s^e = \{\beta + (1-\delta)(1-\beta)\}/\{\beta + \theta(1-\beta)\} > 1$. Moreover, $(1-n^f)/(1-n^*) = \{\theta(1-\beta) + \beta\}/\{(1-\delta)\theta(1-\beta) + \beta\} > 1$.

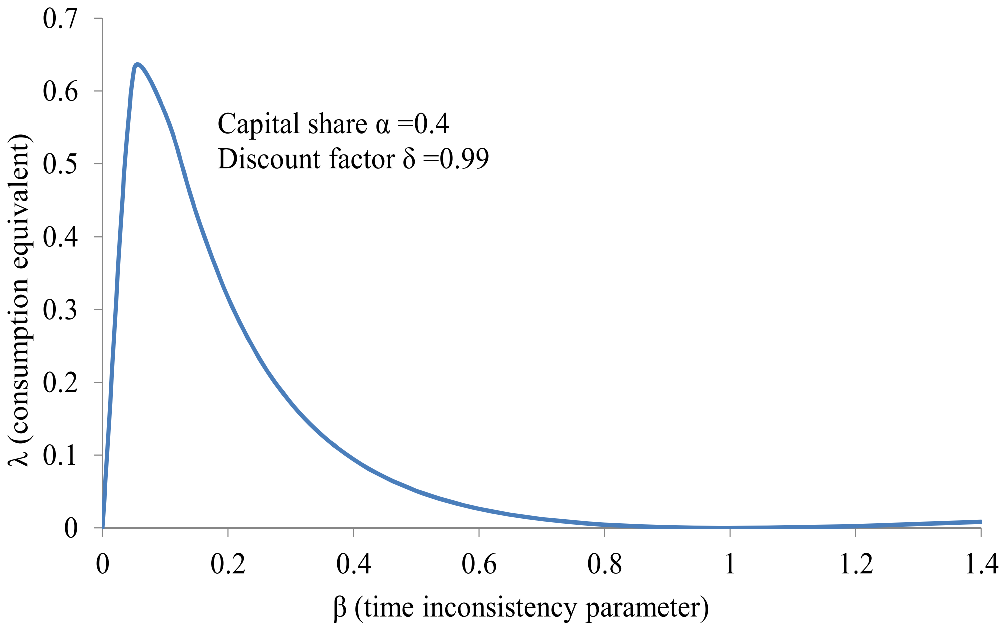

Fig 1. Welfare dominance of the planning economy over the competitive economy

6 Multiple balanced growth paths

So far we have assumed simple log preferences. In this section, we use a general CRRA utility function $u(c) = c^{1-\sigma}/(1-\sigma)$ with $\sigma \leq 1$, and we focus on the balanced growth equilibrium (BGE).

Definition 3: BGE is the recursive competitive equilibrium along which the growth rates of consumption, output, physical capital, and human capital are the same constant.

Along the BGE, working time is constant because the growth rate of human capital is constant. Moreover, the interest rate and wage rate are constant because of the constant returns to scale. Thus the non-arbitrage condition (10) requires that $r = B$.

We first characterize the partial equilibrium in which the real interest rate r is equal to B and the wage w is constant. The individual maximizes his utility by taking his future decision rule $\boldsymbol{x}' = \hat{g}(\boldsymbol{x}) = (\hat{g}_1(\boldsymbol{x}), \hat{g}_2(\boldsymbol{x}))$ as exogenous. His future consumption $\hat{c}(\boldsymbol{x})$ and working time $\hat{n}(\boldsymbol{x})$ are determined by $\hat{g}$:

$$\hat{c}(\boldsymbol{x}) = Bk + w\left\{h - \frac{\hat{g}_2(\boldsymbol{x})}{B}\right\} - \hat{g}_1(\boldsymbol{x}) \quad \text{and} \quad \hat{n}(\boldsymbol{x}) = 1 - \frac{\hat{g}_2(\boldsymbol{x})}{Bh}. \tag{17}$$

The current self's problem is

$$(P_3)\ \hat{V}_0(\boldsymbol{x}) = \max_{c,n}\left[u(c) + \beta\delta\hat{V}(\boldsymbol{x}')\right], \tag{18}$$

$$\text{s.t. } k' = Bk + wnh - c \text{ and } h' = B(1-n)h. \tag{19}$$

The value function $\hat{V}(\boldsymbol{x})$ satisfies

$$\text{FE:} \qquad \hat{V}(\boldsymbol{x}) = u(\hat{c}(\boldsymbol{x})) + \delta\hat{V}(\boldsymbol{x}'), \tag{20}$$

$$\text{TC:} \qquad \lim_{t\to\infty}\delta^t\hat{V}(\hat{g}^{(t)}(\boldsymbol{x})) = 0, \tag{21}$$

where $\hat{g}^{(t)}(\boldsymbol{x})$ is defined as $\hat{g}^{(0)}(\boldsymbol{x}) = \boldsymbol{x}$ and $\hat{g}^{(t+1)}(\boldsymbol{x}) = \hat{g}[\hat{g}^{(t)}(\boldsymbol{x})]$ for $t \geq 0$.

Definition 4: Given constant factor prices $r = B$ and w, *a recursive partial equilibrium* consists of a rule $\hat{g}(\boldsymbol{x}) = (\hat{g}_1(\boldsymbol{x}), \hat{g}_2(\boldsymbol{x}))$ and a value function $\hat{V}(\boldsymbol{x})$ such that

1. given $\hat{V}(\boldsymbol{x})$, the rule $\hat{g}(\boldsymbol{x})$ solves the problem (P_3); and

2. given $g(\boldsymbol{x})$, the function $V(\boldsymbol{x})$ satisfies Eqs (20) and (21).

The next lemma characterizes the partial equilibrium.

Lemma 3 *Let q denote a constant that solves*

$$m(q) = B^{\sigma-1} \text{ with } m(q) \equiv \frac{q + \beta^{1/\sigma-1}\delta^{1/\sigma}}{(q + \beta^{1/\sigma}\delta^{1/\sigma})^{1-\sigma}}, \tag{22}$$

There is a recursive partial equilibrium such that the decision rule is $\hat{g}(\boldsymbol{x}) = [(1-\hat{n})Bk,\ B(1-\hat{n})h]$ where $\hat{n} = q/(q + \beta^{1/\sigma}\delta^{1/\sigma})$ is working time and the value function is $\hat{V}(\boldsymbol{x}) = q^{-\sigma}(k + B^{-1}wh)^{1-\sigma}/(1-\sigma)$. The growth rate of the economy is $B(1-\hat{n})$.

Proof. See the Appendix. ■

This lemma implies there are multiple recursive partial equilibria if Eq. (22) has multiple solutions. When there is no time inconsistency (i.e., $\beta = 1$), the function $m(q) = (q + \delta^{1/\sigma})^\sigma$ is monotonically increasing and the solution to Eq. (22) is unique if it exists. We have the following lemma on the shape of m when $\beta \neq 1$.

Lemma 4 *The function m satisfies $m(0) = \delta$ and $m(+\infty) = +\infty$. If $1 - \sigma > \beta$, m is U-shaped and is minimized when $q = q^*$ with $q^* \equiv \beta^{-1}\sigma^{-1}(1 - \sigma - \beta)(\beta\delta)^{1/\sigma} > 0$. If $1 - \sigma \leq \beta$, then $m'(q) > 0$ for all $q > 0$.*

Proof. See the Appendix. ■

Lemma 2 implies that if $B^{\sigma-1} \geq \delta(= m(0))$ then the solution to Eq. (22) is unique. On the other hand, there are two solutions if the parameters satisfy $\beta < 1 - \sigma$ and

$$(m(q^*) =) \ \frac{\beta^{1-\sigma}(1-\beta)^\sigma \delta}{\sigma^\sigma (1-\sigma)^{1-\sigma}} < B^{\sigma-1} < \delta. \tag{23}$$

Otherwise, the solution does not exist.

We have

Proposition 4 *For each q satisfying Eq. (22), there exists a BGE such that*

1. *the initial physical and human capital satisfies*

$$\frac{h_0}{k_0} = \frac{1}{\hat{n}} \left(\frac{\alpha A}{B} \right)^{1/(\alpha-1)} \quad \text{with } \hat{n} = \frac{q}{q + (\beta\delta)^{1/\sigma}}; \tag{24}$$

2. *the wage rate $w(\bar{\boldsymbol{x}})$ is constant, and the interest rate r is B;*
3. *the decision rule is $g(\boldsymbol{x}, \bar{\boldsymbol{x}}) = [(1-\hat{n})Bk, \ B(1-\hat{n})h]$;*
4. *the law of motion for the aggregate state is $G(\bar{\boldsymbol{x}}) = [(1-\hat{n})B\bar{k}, \ B(1-\hat{n})\bar{h}]$;*
5. *the value function is $V(\boldsymbol{x}, \bar{\boldsymbol{x}}) = q^{-\sigma}(k + B^{-1}w(\bar{\boldsymbol{x}})h)^{1-\sigma}/(1-\sigma)$; and*
6. *the balanced growth rate is equal to $B(1-\hat{n})$.*

If Eq. (23) holds and $\beta < 1 - \sigma$, two BGEs with different growth rates exist.

Proof. See the Appendix. ■

Note that the BGEs are multiple only if the individual is present-biased (i.e., $\beta < 1$) and the coefficient of the relative risk aversion σ is strictly less than 1. Otherwise, the inequality $\beta < 1-\sigma$ is not satisfied. For example, when the utility function is logarithmic and $\sigma = 1$, the inequality is violated for any β and the BGE is unique. If $\sigma < 1$, then for any $\delta > 0$, we can easily check that there exists parameters β and B satisfying the two inequalities $\beta < 1 - \sigma$ and Eq. (23). [3]

Endogenous growth models without externality usually have only one BGE. (One of the few exceptions is Ladron-de-Guevara, Ortigueira, and Santos (1996)). However, in some sense, multiplicity of BGEs is not surprising because the competitive equilibrium is defined as the Nash equilibrium between current self and future self.

Some authors have already shown that models with quasi-geometric discounting can have multiple solutions. For example, Krusell and Smith (2003) find that the neoclassical growth model with quasi-geometric discounting has a continuum of solution paths converging to different steady states. One difference between Krusell and Smith (2003) and our paper is that they investigate the planned economy whereas we study the competitive economy. As KKS point out, the one-sector model always has a unique stationary competitive equilibrium even if the discounting is quasi-geometric. Here we find that stationary equilibria can be multiple when the model has multiple sectors.

7 Conclusion

This paper has studied a two-sector model of endogenous growth with time-inconsistent preferences. We first assumed log preferences and found that the planner's solution path can welfare-dominate the equilibrium path. This result differs from KKS. The consideration of human capital accumulation substantially changes the welfare property of models with quasi-geometric discounting. In addition, we showed there can be multiple balanced growth paths when the utility function is of the CRRA type. In future studies, we wish

[3] First of all, take B such that $B > \delta^{1/(1-\sigma)}$. Next, for such B, take β such that $\beta < 1 - \sigma$ and $\beta^{1-\sigma}(1-\beta)^{\sigma}\delta < \sigma^{\sigma}(1-\sigma)^{1-\sigma}B^{\sigma-1}$. The two inequalities hold if β is sufficiently close to zero.

to investigate a time-consistent tax policy. KKS assume that government can tax income and investment proportionally and obtain the time-consistent policy equilibrium. They find that the optimal investment tax is positive when the individual is excessively present-biased. It will be interesting to see whether their result pertains in our two-sector economy. In KKS, the policy equilibrium reproduces the allocation that solves the planner's problem. This implies that time-consistent taxes always lower equilibrium welfare. However, in our model, the planned economy can welfare-dominate the competitive economy, and we may surmise that time-consistent tax policy may improve the equilibrium allocation.

Appendix

A Proof of Proposition 1

The proof is divided to five steps for clarity. For simplicity, we write $w(\bar{\boldsymbol{x}})$, $w(\bar{\boldsymbol{x}})$, $r(\bar{\boldsymbol{x}}')$, and $w(\bar{\boldsymbol{x}}')$ as $\bar{r}$, $\bar{w}$, $\bar{r}'$, and $\bar{w}'$, respectively.

Step 1. [*Given G, the non-arbitrage condition* (10) *holds*]: the rule G implies $B\bar{k}'/\bar{h}' = A\alpha\bar{k}^\alpha(n^e\bar{h})^{1-\alpha}/\bar{h}$. This yields

$$\frac{\bar{r}'\bar{w}}{B\bar{w}'} = \frac{A\alpha(\bar{k}')^{\alpha-1}(n^e\bar{h}')^{1-\alpha}}{B(\bar{k}')^\alpha(n^e\bar{h}')^{-\alpha}}\bar{k}^\alpha(n^e\bar{h})^{-\alpha} = \frac{A\alpha n^e}{B(\bar{k}'/\bar{h}')}\bar{k}^\alpha(n^e\bar{h})^{-\alpha} = 1.$$

Therefore Eq. (10) holds.

Step 2. [*Given g and G, V^e solves the FE* (8)]: it is straight forward to show that the rule g implies $n(\boldsymbol{x}, \bar{\boldsymbol{x}}) = n^e$ and $c(\boldsymbol{x}, \bar{\boldsymbol{x}}) = n^e(\bar{r}k + \bar{w}h)$. Thus Eq. (8) holds if and only if

$$\begin{aligned} p_1\ln(\bar{r}k + \bar{w}h) + p_2\ln\bar{r} + p_3 \;&=\; \ln n^e + \ln(\bar{r}k + \bar{w}h) \\ &\quad +\delta[p_1\ln(\bar{r}'k' + \bar{w}'h') + p_2\ln\bar{r}' + p_3]. \end{aligned} \tag{25}$$

Recall that given G, the non-arbitrage condition holds. The consecutive period physical capital is $k' = \bar{r}k + \bar{w}n^eh - c(\boldsymbol{x}, \bar{\boldsymbol{x}}) = (1 - n^e)\bar{r}k$. Thus we get

$$\bar{r}'k' + \bar{w}'h' = \bar{r}'\left(k' + \frac{\bar{w}}{B}h'\right) = \bar{r}'(1 - n^e)\{\bar{r}k + \bar{w}h\}. \tag{26}$$

Because $n(\boldsymbol{x}, \bar{\boldsymbol{x}})$ is time-independent, we have $(\bar{r}'/\bar{r})^\alpha = \{(\bar{k}'/\bar{h}')/(\bar{k}/\bar{h})\}^{\alpha(\alpha-1)} = (\bar{w}'/\bar{w})^{\alpha-1}$. Inserting this equation into Eq. (10), we have $(\bar{r}'/\bar{r})^\alpha = (\bar{r}'/B)^{1-\alpha}$ or equivalently

$$\ln\bar{r}' = \ln(B^{1-\alpha}) + \alpha\ln\bar{r}. \tag{27}$$

Substitution of Eqs. (26) and (27) into Eq. (25) gives

$$\begin{aligned} p_1 \ln(\bar{r}k + \bar{w}h) + p_2 \ln \bar{r} + p_3 &= (1 + \delta p_1) \ln(\bar{r}k + \bar{w}h) + \delta\alpha(p_1 + p_2) \ln \bar{r} + \ln n^e \\ &\quad + \delta p_1 \ln(1 - n^e) + \delta(p_1 + p_2) \ln(B^{1-\alpha}) + \delta p_3. \end{aligned}$$

This equality holds if and only if $p_1 = 1 + \delta p_1$, $p_2 = \alpha\delta(p_1 + p_2)$, and $(1 - \delta)p_3 = \ln n^e + \delta p_1 \ln(1 - n^e) + \delta(p_1 + p_2) \ln(B^{1-\alpha})$. The solutions are $p_1 = 1/(1 - \delta)$, $p_2 = \alpha\delta/\{(1 - \delta)(1 - \alpha\delta)\}$, and $p_3 = \{\ln n^e + \delta p_1 \ln(1 - n^e) + \delta(p_1 + p_2) \ln(B^{1-\alpha})\}/(1 - \delta)$.

Step 3. [*Given g and G, V^e satisfies the TC* (9)]: Eq. (9) holds if $\lim_{t\to\infty} \delta^t \ln r(\bar{\boldsymbol{x}}_t) = 0$ and $\lim_{t\to\infty} \delta^t \ln y(\boldsymbol{x}_t, \bar{\boldsymbol{x}}_t) = 0$. The first condition holds since $\lim_{t\to\infty} \ln r(\bar{\boldsymbol{x}}_t) = \ln B$ by Eq. (27). To show the second condition, note that

$$\ln y(\boldsymbol{x}', \bar{\boldsymbol{x}}') = \ln(1 - n^e) + \ln \bar{r}' + \ln y(\boldsymbol{x}, \bar{\boldsymbol{x}})$$

by Eq. (26). Therefore, we get $z_{t+1} = \delta^{t+1}\{\ln(1-n^e)+\ln \bar{r}'\}+\delta z_t$ where $z_t = \delta^t \ln y(\boldsymbol{x}_t, \bar{\boldsymbol{x}}_t)$. Because $\lim_{t\to\infty} \delta^{t+1}\{\ln(1 - n^e) + \ln \bar{r}'\} = 0$, we get $\lim_{t\to\infty} z_t = 0$.

Step 4. [*Given V^e and G, the decision rule g^e solves P_1*]: the non-arbitrage condition (10) implies that $\bar{r}'k' + \bar{w}'h' = \bar{r}'(\bar{r}k + \bar{w}h - c)$. Therefore, given the functional form of V^e, the problem P_1 reduces to

$$\max_{c\geq 0, n\in[0,1]} \left[\ln c + \frac{\beta\delta}{1 - \delta} \ln(\bar{r}k + \bar{w}h - c)\right].$$

If we denote by $(c^{\max}, n^{\max})$ a solution to the problem above, we get $c^{\max} = n^e\{\bar{r}k + \bar{w}h\}$ since $n^e = 1/(1+\beta\delta/(1-\delta))$. By Eq. (10), any choice of n is optimal and then we can set $n^{\max} = n^e$. In this case, next period state variables are given by $k' = \bar{r}k + \bar{w}n^e h - c^{\max} = (1 - n^e)\bar{r}k$ and $h' = (1 - n^e)Bh$. These coincide with the decision rule g.

Step 5. [*The rules g and G are consistent*]: we have $g_1(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}}) = (1 - n^e)\bar{r}\bar{k} = (1 - n^e)A\alpha\bar{k}^\alpha(n^e\bar{h})^{1-\alpha}$ and $g_2(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}}) = (1 - n^e)B\bar{h}$. Thus $G(\bar{\boldsymbol{x}}) = g(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}})$.

Thus, there exists a recursive competitive equilibrium with a decision rule g, a value function V, and a difference equation on the aggregate state $\bar{\boldsymbol{x}}' = G(\bar{\boldsymbol{x}})$. ■

B Proof of Proposition 2

The proof consists of three steps.

Step 1. [*Given* g^**, the function* $V^*(\boldsymbol{x})$ *solves Eq.* (13)]: the rule g^* implies that the consumption level is $c^* = (1-s^*)Ak^{*\alpha}(n^*h^*)^{1-\alpha}$. Thus V^* satisfies Eq. (13) if and only if

$$\begin{aligned}\rho_1 \ln k + \rho_2 \ln h + \rho_3 &= \ln\{(1-s^*)Ak^\alpha(n^*h)^{1-\alpha}\} \\ &\quad +\delta\{\rho_1 \ln\{s^*Ak^\alpha(n^*h)^{1-\alpha}\} + \rho_2 \ln(B(1-n^*)h) + \rho_3\}.\end{aligned}$$

The RHS of the equation above is simplified as

$$\begin{aligned}RHS &= (1+\delta\rho_1)\alpha \ln k + \{(1+\delta\rho_1)(1-\alpha) + \delta\rho_2\} \ln h \\ &\quad + \ln(1-s^*) + (1+\delta\rho_1)\ln A(n^*)^{1-\alpha} + \delta\rho_1 \ln s^* + \delta\rho_2 \ln(B(1-n^*)) + \delta\rho_3.\end{aligned}$$

Eq. (13) holds for any k and h if and only if $\rho_1 = (1+\delta\rho_1)\alpha$, $\rho_2 = (1+\delta\rho_1)(1-\alpha) + \delta\rho_2$, and $(1-\delta)\rho_3 = (1+\delta\rho_1)\ln A(n^*)^{1-\alpha} + \ln(1-s^*) + \delta\rho_1 \ln s^* + \delta\rho_2 \ln(B(1-n^*))$. Thus $\rho_1 = \alpha/(1-\alpha\delta)$ and $\rho_2 = (1-\alpha)/\{(1-\delta)(1-\alpha\delta)\}$.

Step 2. [*Given* g^**,* $V^*(\boldsymbol{x})$ *satisfies Eq.* (14)]: the asymptotic growth rate of human and physical capitals, $B(1-n^*)$ is constant. Thus $\lim_{t\to\infty}\delta^t \ln k_t^* = \lim_{t\to\infty}\delta^t \ln h_t^* = 0$. Since $V^*(\boldsymbol{x}) = \rho_1 \ln k + \rho_2 \ln h$ plus constant, we get (14).

Step 3. [*Given* V^**, the decision rule of the current self is* g^*]: the current self maximizes $\ln c + \beta\delta\{\rho_1 \ln(Ak^\alpha(nh)^{1-\alpha} - c) + \rho_2 \ln(B(1-n)h)\}$ by choosing c and n. A solution to the maximization problem above, say $(\tilde{c}, \tilde{n})$, is given by

$$\tilde{c} = \frac{Ak^\alpha(\tilde{n}h)^{1-\alpha}}{1+\beta\delta\rho_1} = (1-s^*)Ak^\alpha(\tilde{n}h)^{1-\alpha}, \tag{28}$$

$$\tilde{n} = \frac{1+\beta\delta\rho_1}{1+\beta\delta\rho_1 + \frac{\beta\delta\rho_2}{1-\alpha}} = \frac{1-\alpha\delta(1-\beta)}{1-\alpha\delta(1-\beta) + \frac{\beta\delta}{1-\delta}} = n^*. \tag{29}$$

Thus the consecutive period aggregate state (k', h') is $k' = Ak^\alpha(n^*h)^{1-\alpha} - c = s^*Ak^\alpha(n^*h)^{1-\alpha}$ and $h' = B(1-n^*)$. It coincides with the decision rule g^*.

Thus a planner's solution exists with a decision rule g^* and a value function V^*. ■

C Proof of Lemma 1

First, with respect to the savings rates, one has

$$\frac{s^e}{s^*} = \frac{1-\alpha\delta+\beta\delta\alpha}{1-\delta+\delta\beta} = \frac{\beta+(1-\beta)(1-\alpha\delta)}{\beta+(1-\beta)(1-\delta)}.$$

Because $0 < 1-\delta < 1-\alpha\delta < 1$, s^e/s^* is greater than one if and only if $\beta < 1$.

Next, concerning working time, one has $1-n^e = \beta\delta/(1-\delta+\delta\beta)$ and $1-n^* = \delta\beta/\{(1-\alpha\delta+\beta\delta\alpha)(1-\delta)+\beta\delta\}$. Thus

$$\frac{1-n^e}{1-n^*} = \frac{(1-\alpha\delta(1-\beta))(1-\delta)+\beta\delta}{1-\delta+\delta\beta} = \frac{\beta+(1-\delta)(1-\beta)(1-\alpha\delta)}{\beta+(1-\delta)(1-\beta)}.$$

Because $1-\alpha\delta<1$, $1-n^e < 1-n^*$ (or equivalently $n^e > n^*$) if and only if $\beta < 1$. ■

D Proof of Lemma 2

We have $k_{t+1} = sak_t^{\alpha}h_t^{1-\alpha}$ and $h_{t+1} = bh_t$, where $a = An^{1-\alpha}$ and $b = B(1-n)$. If we let $z_t = k_t/h_t$ denote the physical/human capital ratio, the utility U_0 is re-expressed as $U_0 = (1-\beta)u_0 + \beta\sum_{t=0}^{\infty}\delta^t u_t$ where $u_t = \ln\{a(1-s)z_t^{\alpha}h_t\}$. Thus

$$\sum_{t=0}^{\infty}\delta^t u_t = \alpha\sum_{t=0}^{\infty}\delta^t\ln z_t + \sum_{t=0}^{\infty}\delta^t\ln h_t + \frac{\ln(1-s)}{1-\delta} + \frac{\ln a}{1-\delta}. \tag{30}$$

We have $(1-\alpha\delta)\sum_{t=0}^{\infty}\delta^t\ln z_t = \ln z_0 + \delta\sum_{t=0}^{\infty}\delta^t(\ln z_{t+1} - \alpha\ln z_t)$. Since $\ln z_{t+1} = \ln(sa/b) + \alpha\ln z_t$ and $\ln z_0 = \ln k - \ln h$, we have

$$\alpha\sum_{t=0}^{\infty}\delta^t\ln z_t = \frac{\alpha}{1-\alpha\delta}(\ln k - \ln h) + \frac{\alpha\delta}{\theta}(\ln s + \ln a - \ln b), \tag{31}$$

Because $\ln h_t = t\ln b + \ln h$ and $\sum_{t=0}^{\infty} t\delta^t = \delta/(1-\delta)^2$, the second term of the RHS of Eq. (30) is

$$\sum_{t=0}^{\infty}\delta^t\ln h_t = (1-\delta)^{-2}\delta\ln b + (1-\delta)^{-1}\ln h. \tag{32}$$

Substitution of Eqs. (31) and (32) into Eq. (30) gives

$$\sum_{t=0}^{\infty} \delta^t u_t = \frac{\alpha\delta}{\theta} \ln s + \frac{\ln(1-s)}{1-\delta} + \frac{1}{\theta} \ln a + \frac{\delta}{\theta} \frac{1-\alpha}{1-\delta} \ln b + \frac{\alpha \ln k}{1-\alpha\delta} + \frac{1-\alpha}{\theta} \ln h. \tag{33}$$

In Eq. (33), the coefficient on $\ln b$ is $(\delta/\theta)\{(1-\alpha)/(1-\delta)\}$ because $(1-\delta)^{-2}\delta - \theta^{-1}\alpha\delta = \theta^{-1}\delta\{(1-\alpha\delta)/(1-\delta) - \alpha\}$. Substitution of Eq. (33) and the initial utility $u_0 = \ln(1-s) + \ln a + \alpha \ln k + (1-\alpha)\ln h$ into $U_0 = (1-\beta)u_0 + \beta \sum_{t=0}^{\infty} \delta^t u_t$ gives

$$\begin{aligned} U_0 \quad = \quad & \frac{\alpha\beta\delta}{\theta} \ln s + \frac{1-\delta+\beta\delta}{1-\delta} \ln(1-s) + \left(1-\beta+\frac{\beta}{\theta}\right) \ln a + \frac{\delta\beta}{\theta} \frac{1-\alpha}{1-\delta} \ln b \\ & + \alpha\left(1-\beta+\frac{\beta}{1-\alpha\delta}\right) \ln k + (1-\alpha)\left(1-\beta+\frac{\beta}{\theta}\right) \ln h. \end{aligned}$$

This yields Eq. (15) since $\ln a = \ln A + (1-\alpha)\ln n$ and $\ln b = \ln B + \ln(1-n)$. This completes the proof of Lemma 1. ■

E Proof of Proposition 3

We would like to show that $\lim_{\delta\to 1} \Delta V > 0$. The welfare difference ΔV is expressed as

$$\Delta V = \phi_1 \ln \frac{s^*}{s^e} + \phi_2 \ln \frac{1-s^*}{1-s^e} + \phi_3 \ln \frac{n^*}{n^e} + \phi_4 \ln \frac{1-n^*}{1-n^e}. \tag{34}$$

As $\delta \to 1$, $\phi_1 \to \alpha\beta$, $\phi_2 \to \beta(1-\alpha)$, $\phi_3 \to \beta(1-\alpha)$, $\phi_4 \to +\infty$, $s^* \to \beta\alpha/(1-\alpha+\alpha\beta)$, $s^e \to \alpha$, $n^* \to 0$ and $n^e \to 0$. If we let ΔV_i be the i $(i = 1,2,3,4)$ th term of the RHS of Eq. (34), by definition, we have $\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3 + \Delta V_4$. In what follows, we calculate $\lim_{\delta\to 1} \Delta V_i$ $(i = 1,2,3,4)$ separately.

First, $\Delta V_1 = \phi_1 \ln(s^*/s^e)$ satisfies

$$\lim_{\delta\to 1} \Delta V_1 = \alpha\beta \ln \frac{\beta}{\beta + (1-\alpha)(1-\beta)} = -\alpha\beta \ln(1 + \eta(1-\alpha)), \tag{35}$$

where $\eta = 1/\beta - 1$. Next, $\Delta V_2 = \phi_2 \ln\{(1-s^*)/(1-s^e)\}$ satisfies

$$\lim_{\delta\to 1} \Delta V_2 = \beta(1-\alpha)\ln\frac{1-\frac{\beta\alpha}{1-\alpha+\alpha\beta}}{1-\alpha} = -\beta(1-\alpha)\ln(1-\alpha+\alpha\beta). \tag{36}$$

Third, since

$$\lim_{\delta\to 1}(n^*/n^e) = \lim_{\delta\to 1}\frac{(1-\delta+\delta\beta)(1-\alpha\delta+\beta\delta\alpha)}{(1-\alpha\delta+\beta\delta\alpha)(1-\delta)+\beta\delta} = 1-\alpha+\alpha\beta,$$

we get

$$\lim_{\delta\to 1}\Delta V_3 = \beta(1-\alpha)\ln(1-\alpha+\alpha\beta). \tag{37}$$

Finally we study $\Delta V_4 = \phi_4 \ln\{(1-n^*)/(1-n^e)\}$. With some algebra, we get $\beta\delta/(1-n^*) = \beta\{1+(1-\alpha\delta)(1-\delta)\eta\}$ and $\beta\delta/(1-n^e) = \beta\{1+(1-\delta)\eta\}$. Thus

$$\frac{1}{1-\delta}\ln\left(\frac{1-n^*}{1-n^e}\right) = \frac{\ln\{1+(1-\delta)\eta\}}{1-\delta} - \frac{\ln\{1+(1-\alpha\delta)(1-\delta)\eta\}}{1-\delta}.$$

The first term on the RHS of this equation satisfies $\lim_{\delta\to 1}[\ln\{1+(1-\delta)\eta\}/(1-\delta)] = \eta\lim_{\epsilon\to 0}[\ln(1+\epsilon)/\epsilon] = \eta$ where $\epsilon = (1-\delta)\eta$. Similarly, the second term satisfies

$$\lim_{\delta\to 1}\frac{\ln\{1+(1-\alpha\delta)(1-\delta)\eta\}}{1-\delta} = \eta\lim_{\delta\to 1,\varepsilon\to 0}\left[(1-\alpha\delta)\frac{\ln(1+\varepsilon)}{\varepsilon}\right] = \eta(1-\alpha).$$

where $\varepsilon = (1-\alpha\delta)(1-\delta)\eta$. Thus as $\delta \to 1$, $\ln\{(1-n^*)/(1-n^e)\}/(1-\delta) \to \eta\alpha$. Since $\lim_{\delta\to 1}(1-\delta)\phi_4 = \beta(1-\alpha)$, one has

$$\lim_{\delta\to 1}\Delta V_4 = \lim_{\delta\to 1}\left\{(1-\delta)\phi_4\cdot\frac{1}{1-\delta}\ln\frac{1-n^*}{1-n^e}\right\} = \alpha\beta(1-\alpha)\eta. \tag{38}$$

From Eqs. (35), (36), (37) and (38), we get

$$\frac{1}{\alpha\beta}\left(\lim_{\delta\to 1}\Delta V\right) = (1-\alpha)\eta - \ln\{1+(1-\alpha)\eta\} > 0,$$

as long as $\eta \neq 0$ or equivalently $\beta \neq 1$. The last inequality holds since $x > \ln(1+x)$ for any $x > 0$. Thus if δ is sufficiently close to one, one has $\Delta V > 0$. ■

F Proof of Lemma 3

We first show that given $\hat{g}$, the function $\hat{V}$ solves the FE (37). The rule $\hat{g}(\boldsymbol{x}) = [(1-\hat{n})Bk,\ B(1-\hat{n})h]$ implies that $n = \hat{n}$ and $c = \hat{n}(B\hat{k} + w\hat{h})$. Hence $\hat{V}$ satisfies Eq.(37) if and only if

$$q^{-\sigma}B^{\sigma-1}\frac{(Bk+wh)^{1-\sigma}}{1-\sigma} = \frac{(\hat{n}(Bk+wh))^{1-\sigma}}{1-\sigma} + \delta q^{-\sigma}\frac{(\hat{g}_1(\boldsymbol{x}) + B^{-1}w\hat{g}_2(\boldsymbol{x}))^{1-\sigma}}{1-\sigma}. \tag{39}$$

Because $\hat{g}_1(\boldsymbol{x}) + B^{-1}w\hat{g}_2(\boldsymbol{x}) = (1-\hat{n})(Bk+wh)$, Eq. (39) holds if and only if $B^{\sigma-1} = q^{\sigma}\hat{n}^{1-\sigma} + \delta(1-\hat{n})^{1-\sigma}$. The RHS of this equation coincides with $m(q)$ because $\hat{n} = q/\{q + (\beta\delta)^{1/\sigma}\}$. Therefore, it coincides with Eq. (22). Thus Eq. (39) always holds.

We next show that given $\hat{g}$, $\hat{V}$ solves the TC (21). The growth rates of consumption and human and physical capital are all equal to $B(1-\hat{n}) = B(\beta\delta)^{1/\sigma}/\{q + (\beta\delta)^{1/\sigma}\}$. Therefore, the growth rate of the variable $\delta^t\hat{V}(\boldsymbol{x}_t)$, $\delta\hat{V}(\boldsymbol{x}_{t+1})/\hat{V}(\boldsymbol{x}_t) = \delta\{B(1-\hat{n})\}^{1-\sigma}$ is also constant and is strictly less than one since

$$\frac{\delta\hat{V}(\boldsymbol{x}_{t+1})}{\hat{V}(\boldsymbol{x}_t)} = \delta B^{1-\sigma}\frac{(\beta\delta)^{1/\sigma-1}}{\{q + (\beta\delta)^{1/\sigma}\}^{1-\sigma}} = \frac{\beta^{1/\sigma-1}\delta^{1/\sigma}}{q + \beta^{1/\sigma-1}\delta^{1/\sigma}} < 1.$$

Here the second equality holds because $B^{1-\sigma} = 1/m(q)$. Thus $\delta^t\hat{V}(\boldsymbol{x}_t)$ converges to zero as t goes to infinity and Eq. (21) always hold.

We finally show that given $\hat{V}$, the rule $\hat{g}$ solves P_3. When the current level of consumption is c, $k' + B^{-1}wh' = Bk + wh - c$. Thus the problem P_3 is simplified as

$$\max_{c,n\in[0,1]}\left[\frac{c^{1-\sigma}}{1-\sigma} + \beta\delta q^{-\sigma}\frac{(Bk+wh-c)^{1-\sigma}}{1-\sigma}\right].$$

The FOC is $c^{-\sigma} = \beta\delta q^{-\sigma}(Bk+wh-c)^{-\sigma}$ and then the optimal consumption is $q(Bk + wh)/(q + (\beta\delta)^{1/\sigma}) = \hat{n}(Bk+wh)$. This coincides the consumption rule $\hat{c}$ determined by $\hat{g}$. The choice of n is arbitrary because of the non-arbitrage condition. Hence $\hat{g}$ is a solution to the problem P_3.

Thus there exists an equilibrium with the decision rule $\hat{g}$ and the value function $\hat{V}$. ■

G Proof of Lemma 4

One has

$$m'(q) = \frac{q + \beta^{1/\sigma}\delta^{1/\sigma} - (q + \beta^{1/\sigma-1}\delta^{1/\sigma})(1-\sigma)}{(q + \beta^{1/\sigma}\delta^{1/\sigma})^{2-\sigma}} = \sigma \frac{q - q^*}{(q + \beta^{1/\sigma}\delta^{1/\sigma})^{2-\sigma}}.$$

Suppose $1 - \sigma - \beta > 0$. Then we have $q^* > 0$, $m'(q) < 0$ if $q < q^*$, $m'(q) > 0$ if $q > q^*$ and $m'(q^*) = 0$. ■

H Proof of Proposition 4

We first prove the existence of the equilibrium. First, when the law of motion for aggregate state $G(\bar{\boldsymbol{x}})$ is given, human and physical capital growth rate $(1-\hat{n})B$ and working time $\hat{n}$ are constant. In addition, the wage rate and the real interest rate are constant. Moreover, when the initial state variables satisfy Eq. (24), $r_t = r_0 = \alpha A(\hat{n}h_0/k_0)^{\alpha-1} = B$. Next, as Lemma 2 shows, when the wage rate is constant and the real interest rate r is equal to B, the individual decision rule $g(\boldsymbol{x}, \bar{\boldsymbol{x}}) = \hat{g}(\boldsymbol{x})$ and the value function $V(\boldsymbol{x}, \bar{\boldsymbol{x}}) = \hat{V}(\boldsymbol{x})$ solves the problem of the current self. Finally, the individual decision rule and the law of motion for the aggregate state G are consistent.

We next show the equilibrium multiplicity. We have already shown that for any q satisfying the above equation, we can construct a recursive competitive equilibrium. Therefore, if $1 - \sigma > \beta$ and Eq. (23) holds, there are multiple equilibria. Moreover, $\hat{n} = q/\{q + (\beta\delta)^{1/\sigma}\}$ is an increasing function of q and the equilibrium with higher q has lower growth rate. ■

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