# Exchange Rates and Fundamentals: A Generalization

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*Abstract*

Exchange rates have raised the ire of economists for more than 20 years. The problem is that few, if any, exchange rate models are known to systematically beat a naive random walk in out of sample forecasts. Engel and West (2005) show that these failures can be explained by the standard-present value model (PVM) because it predicts random walk exchange rate dynamics if the discount factor approaches one and fundamentals have a unit root. This paper generalizes the Engel and West (EW) hypothesis. The EW hypothesis is shown to hold for a canonical open economy dynamic stochastic general equilibrium (DSGE) model. We show that all the predictions of the standard-PVM carry over to the DSGE-PVM. The DSGE-PVM also yields an unobserved components (UC) models that we estimate using Bayesian methods and a quarterly Canadian–U.S. sample. Bayesian model evaluation reveals that the data support a UC model that calibrates the discount factor to one implying the Canadian dollar–U.S. dollar exchange rate is a random walk dominated by permanent cross-country monetary and productivity shocks. Thus, our results generalize the EW hypothesis to the larger class of open economy DSGE models which presents new challenges for future research.

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*Key Words*: Exchange rates; present-value model and fundamentals; random walk; DSGE model; unobserved components model; model comparison.

# 1. Introduction

The search for satisfactory exchange rate models continues to be elusive. Since the seminal work of Meese and Rogoff (1983a, 1983b), a train of models have been tried in vain to improve on naive random walk forecasts of exchange rates. These include linear rational expectations models examined by Meese (1986) and nonlinear models proposed by Diebold and Nason (1990), Engel and Hamilton (1990), Meese and Rose (1991), Gençay (1999), and Kilian and Taylor (2003).

This paper steps back from the exchange rate forecasting problem to study a workhorse theory of currency market equilibrium determination, the present-value model (PVM) of exchange rates.[1]

Our approach is in the spirit of Engel and West (2005). Starting with the PVM and using uncontroversial assumptions about fundamentals and the discount factor, Engel and West (EW) hypothesize that the PVM generates an approximate random walk in exchange rates if the PVM discount approaches one and fundamentals are $I(1)$. An important implication of the EW hypothesis is that fundamentals have no power to forecast future exchange rates although the PVM dictates equilibrium in the currency market. EW support their hypothesis with a key theorem and empirical and simulation evidence.

This paper complements Engel and West (2005) by generalizing their main hypothesis. We generalize the EW hypothesis using a canonical two-country monetary dynamic stochastic general equilibrium (DSGE) model. The linearized uncovered interest parity (UIP) and money demand equations yield the DSGE-PVM. We show the standard- and DSGE-PVMs are equivalent up to definitions of the PVM discount factor. Table 1 reviews the key elements of the standard- and DSGE-PVMs.

The EW hypothesis is also generalized with five proposition summarized in table 2. The propositions are consistent with the standard- and DSGE-PVMs of the exchange rate. Thus, the EW hypothesis is generalized to a wider class of macro models. The propositions are: (1) the exchange rate and funda-

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[1] The JOURNAL OF INTERNATIONAL ECONOMICS volume edited by Engel, Rogers, and Rose (2003) indicates that there has been a split between theoretical exchange rate models and what is considered a useful forecasting model. For example, Kilian and Taylor (2003) argue that there are specific nonlinear forecasting models that can vie with a naive random walk of exchange rates. This approach maybe useful to obtain candidates for a forecast competition. Nonetheless, there are limits because, as Diebold and Nason (1990) note, the class of nonlinear exchange rate models might be infinite.

mental cointegrate [Campbell and Shiller (1987)], (2) the PVM yields an error correction model (ECM) for currency returns in which the lagged cointegrating relation is the only regressor, (3) the PVM predicts a limiting economy (*i.e.*, the PVM discount factor approaches one from below) in which the exchange rate is a martingale, (4) given fundamental growth depends only on the lagged cointegrating relation, the exchange rate and fundamental have a common trend-common cycle decomposition [Vahid and Engle (1993)], and (5) the EW hypothesis is also satisfied when the exchange rate and fundamental share a common feature and the PVM discount factor approaches one. A corollary to (5) is that the exchange rate is unpredictable when the PVM discount factor goes to one.

We report evidence from vector autoregression (VARs) about the propositions using floating rate Canadian–, Japanese–, and U.K.–U.S. samples. The VAR evidence rejects cointegration and reveals substantial serial correlation for the exchange rate and the fundamental. There is also evidence that a common feature exists between the Canadian dollar–, Yen–, and Pound–U.S. dollar exchange rates and the relevant fundamentals. Nonetheless, the VAR approach is unable to address the EW hypothesis question of whether the PVM discount factor approaches one.[2]

The DSGE-PVM possesses a deep structure tied to the primitives of the underlying open economy unlike the standard-PVM. Rather than rely on the entire set of DSGE optimality and equilibrium condition, we give empirical content to the DSGE-PVM by placing restrictions on its fundamentals which are cross-country money and consumption. We restrict these fundamentals with permanent-transitory decompositions. This decomposition allows us to cast the DSGE-PVM as a tri-variate unobserved components (UC) model in the exchange rate and observed fundamentals. The UC model also incorporates DSGE-PVM cross-equation restrictions conditional on whether the DSGE-PVM discount factor is calibrated or estimated. Three UC models calibrate the DSGE-PVM discount factor to one, which disconnects the exchange rate from the transitory(s) component of fundamentals. Transitory fundamentals restrict the

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[2]Actual data most often rejects the standard-PVM. Typical are tests Meese (1986) reported that employed the first ten years of the floating rate regime. He finds that exchange rates are infected with persistent deviations from fundamentals, which reject the standard-PVM and its cross-equation restrictions. However, Meese is unable to uncover the source of the rejections. Instead of a condemnation of the standard-PVM, we view results such as Meese's as a challenge to update and deepen its analysis.

exchange rate in three other UC models that estimate the DSGE-PVM discount factor. Within this dichotomy of the DSGE-PVM discount factor, six UC models are identified by restrictions on transitory cross-country money and consumption shocks.

We estimate six UC models on a Canadian-U.S. sample running from $1976Q1$ to $2004Q4$. The UC model yields a state space system for the DSGE-PVM, which allows us to recruit the Kalman filter to evaluate the likelihood. We compute likelihoods of the UC models using the Metropolis-Hastings (MH) simulator described by Rabanal and Rubio-Ramírez (2005) to draw Markov chain Monte Carlo (MCMC) replications from the posteriors. We conduct model comparisons using marginal posterior likelihoods of the six UC models to find which is favored by the Canadian-U.S. data. This data settles on the UC model that calibrates the DSGE-PVM discount factor to one and drives cyclical fluctuations only with the transitory shock to cross-country consumption. According to the Canadian-U.S. data, next is the UC model with the same transitory shock, but the estimated posterior mean of the DSGE-PVM discount factor is 0.9962. The posterior of this UC model reveals that permanent shocks to fundamentals dominate Canadian dollar–U.S. dollar exchange rate fluctuations. Thus, the Canadian-U.S. data prefer UC models that are consistent with the EW hypothesis. Moreover, we find that the Canadian–U.S. data fail to support UC models that tie the exchange rate to the transitory monetary shock. Rogoff (2007) also notes that exchange rates appear disconnected from 'mean reverting monetary fundamentals', but our results are still puzzling because of the key roles assigned to nominal rigidities, UIP shock persistence, and monetary disturbances in open economy monetary DSGE models.[3]

The outline of the paper follows. The next section solves the standard-PVM of the exchange rate and presents its five propositions. Section 3 constructs the DSGE-PVM and discusses its three propositions. Our Bayesian econometric strategy is discussed in section 4. Section 5 reports the posterior estimates of the six UC models. We conclude in section 6.

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[3]The open economy VAR literature provides mixed evidence on the importance of various shocks to the exchange rate. Early papers including Eichenbaum and Evans (1995), Rogers (2000) and Kim and Roubini (2000) found some significance of identified monetary shocks. Recent contributions, however, suggest that monetary policy shocks have only a minor impact on exchange rate fluctuations, consistent with Rogoff's view, for example, see Faust and Rogers, (2003) and Scholl and Uhlig (2005).

## 2. Two Present-Value Models of Exchange Rates

The standard-PVM determines the equilibrium exchange rate by combining a liquidity-money demand function, uncovered interest rate parity (UIRP) condition, purchasing power parity (PPP), and flexible prices. This is a workhorse exchange rate model used by, among others, Dornbusch (1976), Bilson (1978), Frankel (1979), Meese (1986), Mark (1995), and Engel and West (2005).

Rejection of the standard PVM is often given as a reason to discard linear rational expectations models of exchange rates. This paper does not. In this section, we also develop a PVM model of the exchange rate derived from a canonical optimizing two-country monetary DSGE model. Our aim is to generalize the standard PVM and EW hypothesis to this wider class of DSGE models. We meet this goal with the DSGE-PVM which yields an equilibrium exchange rate model whose short-, medium-, and long-run predictions are testable on actual data. We address the empirical implications of the DSGE-PVM in the next two sections.

### 2a. The Standard Present-Value Model of Exchange Rates

The standard-PVM of the exchange rate starts with the liquidity-money demand function

$$(1) \qquad m_{h,t} - p_{h,t} = \psi y_{h,t} - \phi r_{h,t}, \quad 0 < \psi, \phi,$$

where $m_{h,t}$ $p_{h,t}$, $y_{h,t}$, and $r_{h,t}$ denote the home country's natural logarithm of money stock, price level, output, and the level of the nominal interest rate. The parameter $\psi$ measures the income elasticity of money demand. Since the nominal interest rate is in its level, $\phi$ is the interest rate semi-elasticity of money demand. Define cross-country differentials $m_t = m_{h,t} - m_{f,t}$, $p_t = p_{h,t} - p_{f,t}$, $y_t = y_{h,t} - y_{f,t}$, and $r_t = r_{h,t} - r_{f,t}$, where $f$ denotes the foreign country. Assuming PPP holds, $e_t = p_t$, where $e_t$ is the log of the (nominal) exchange rate in which the U.S dollar is the home country's currency.

Under UIRP, the law of motion of the exchange rate is approximately

$$(2) \qquad \mathbf{E}_t e_{t+1} - e_t = r_t.$$

Substitute for $r_t$ in the law of motion of the exchange rate (2) with the money demand function (1) and impose PPP to produce the Euler equation $e_t - \omega \mathbf{E}_t e_{t+1} = (1 - \omega)(m_t - \psi y_t)$, where the standard-PVM

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discount factor is $\omega \equiv \dfrac{\phi}{1 + \phi}$ and $m_t - \psi y_t$ is the standard-PVM fundamental, which nets cross-country money with its income demand. Iterate on the Euler equation through date $T$ and recognize that the transversality condition $\lim_{T \to \infty} \omega^{T+1} \mathbf{E}_t e_{t+T} = 0$ to obtain the standard PVM relation

$$(3) \qquad e_t \;=\; (1 - \omega) \sum_{j=0}^{\infty} \omega^j \mathbf{E}_t \left\{ m_{t+j} - \psi y_{t+j} \right\}.$$

The standard PVM (3) sets the log exchange rate equal to the annuity value of the fundamental $m_t - \psi y_t$ at the standard-PVM discount factor $\omega \equiv \dfrac{\phi}{1 + \phi}$.[4]

## 2b.   The DSGE Model

Rejection of the PVM is often given as a reason to discard linear rational expectations models of exchange rates. This paper does not. In this section, we develop a PVM model of the exchange rate derived from a canonical optimizing two-country monetary DSGE model. Our aim is to generalize the PVM and EW hypothesis to this wider class of DSGE models. We meet this goal with the DSGE-PVM which yields an equilibrium exchange rate model whose short-, medium-, and long-run predictions are testable on actual data. We address the empirical implications of the DSGE-PVM in the next two sections.

The optimizing monetary DSGE model consists of the preferences of domestic and foreign economies and their resource constraints. For the home ($h$) and foreign ($f$) countries, the former objects take the form

$$(4) \qquad \mathcal{U}\left( C_{i,t}, \frac{M_{i,t}}{P_{i,t}} \right) \;=\; \frac{\left[ C_{i,t}^{\nu} \left( \dfrac{M_{i,t}}{P_{i,t}} \right)^{(1-\nu)} \right]^{(1-\varphi)}}{1 - \varphi}, \quad 0 < \nu < 1, \;\; 0 < \varphi,$$

where $C_{i,t}$ and $M_{i,t}$ represent the $i$th country's consumption and the $i$th country's holdings of its money stock. The resource constraint of the home country is

$$(5) \quad B_{h,t}^h + s_t B_{h,t}^f + P_{h,t} C_{h,t} + M_{h,t} \;=\; (1 + r_{h,t-1}) B_{h,t-1}^h + s_t (1 + r_{f,t-1}) B_{h,t-1}^f + M_{h,t-1} + P_{h,t} Y_{h,t},$$

where $B_{i,t}^i$, $B_{i,t}^{\ell}$, $r_{i,t-1}$, $r_{\ell,t-1}$, $Y_{i,t}$, and $s_t$ denote the $i$th country's nominal holding of its own bonds at the end of date $t$, the $i$th country's nominal holding of the $\ell$th country's bonds at the end of date $t$,

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[4]The present-value relation (3) yields the weak prediction that $e$ Granger-causes $z$. Engel and West (2005) and Rossi (2007) report that this prediction is often not rejected in G–7 data.

the return on the $i$th country's bond, the return on the $\ell$th country's bond, the output level of the $i$th country, and the level of the exchange rate. The two-country DSGE model is closed with $B_{h,t}^h + B_{h,t}^f + B_{f,t}^h + B_{f,t}^f = 0$. This condition forces the world stock of nominal debt to be in zero net supply, period-by-period, along the equilibrium path.

In section 2, analysis of the standard-PVM relies on $I(1)$ fundamentals. Likewise, we assume that the processes for labor-augmenting total factor productivity (TFP), $A_{i,t}$, and $M_{i,t}$ satisfy

**ASSUMPTION 1**: $ln[A_{i,t}]$ and $ln[M_{i,t}] \sim I(1)$, $i = h, f$.

**ASSUMPTION 2**: Cross-country TFP and money stock differentials are $I(1)$ and do not cointegrate.

Assumptions 1 and 2 impose stochastic trends on the two-country DSGE model.

## 2c. Optimizing UIRP and Money Demand

The home country maximizes its expected discounted lifetime utility over uncertainty streams of consumption and real balances,

$$\mathbf{E}_t \left\{ \sum_{j=0}^{\infty} (1 + \rho)^{-j} \mathcal{U}\left( C_{h,t+j}, \frac{M_{h,t+j}}{P_{h,t+j}} \right) \right\}, \quad 0 < \rho,$$

subject to (5). The first-order necessary conditions of economy $i$ yield optimality conditions that describe UIRP and money demand. The utility-based UIRP condition of the home country is

(6)
$$\mathbf{E}_t \left\{ \frac{\mathcal{U}_{C,h,t+1}}{P_{h,t+1}} \right\} (1 + r_{h,t}) = \mathbf{E}_t \left\{ \frac{\mathcal{U}_{C,h,t+1}}{P_{f,t+1}} \right\} \frac{(1 + r_{f,t})}{s_t},$$

where $\mathcal{U}_{C,h,t}$ is the marginal utility of consumption of the home country at date $t$. Given the utility specification (4), the exact money demand function of country $i$ is

(7)
$$\frac{M_{i,t}}{P_{i,t}} = C_{i,t} \left( \frac{1 - \nu}{\nu} \right) \frac{1 + r_{i,t}}{r_{i,t}}, \quad i = h, f.$$

The consumption elasticity of money demand is unity, while the interest elasticity of money demand is a nonlinear function of the steady state bond return.

The UIRP condition (6) and money demand equation (7) can be stochastically detrended and then linearized to produce an equilibrium DSGE-law of motion for the exchange rate. Begin by combining

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the utility function (4) and the UIRP condition (6) to obtain

$$
\mathbf{E}_t \left\{ \frac{\mathcal{U}_{h,t+1}}{P_{h,t+1} C_{h,t+1}} \right\} (1 + r_{h,t}) = \mathbf{E}_t \left\{ \frac{\mathcal{U}_{h,t+1}}{P_{f,t+1} C_{h,t+1}} \right\} \frac{(1 + r_{f,t})}{s_t},
$$

where $\mathcal{U}_{i,t}$ is the utility level of country $i$ at date $t$. Prior to stochastically detrending the previous expression, define $\widehat{\mathcal{U}}_{i,t} = \mathcal{U}_{i,t}/A_{i,t}$, $\hat{P}_{i,t} = P_{i,t} A_{i,t}/M_{i,t}$, $\hat{C}_{i,t} = C_{i,t}/A_{i,t}$, $\gamma_{A,i,t} = A_{i,t}/A_{i,t-1}$, $\gamma_{M,i,t} = M_{i,t}/M_{i,t-1}$, $\hat{s}_t = s_t A_t/M_t$, $A_t = A_{h,t}/A_{f,t}$, and $M_t = M_{h,t}/M_{f,t}$. Note that $\hat{C}_{i,t}$ is the transitory component of consumption of the $i$th economy, $\gamma_{A,i,t} (\gamma_{M,i,t})$ is the TFP (money) growth rate of country $i$, and the cross-country TFP (money stock) differential $A_t (M_t)$ are $I(1)$. Applying the definitions, the stochastically detrended UIRP condition becomes

$$
\mathbf{E}_t \left\{ \frac{\widehat{\mathcal{U}}_{h,t+1} \gamma_{A,h,t+1}^{1-\varphi}}{\gamma_{M,h,t+1} \hat{P}_{h,t+1} \hat{C}_{h,t+1}} \right\} (1 + r_{h,t}) = \mathbf{E}_t \left\{ \frac{\widehat{\mathcal{U}}_{h,t+1} \gamma_{A,f,t+1}}{\gamma_{A,h,t+1}^{\varphi} \gamma_{M,f,t+1} \hat{P}_{f,t+1} \hat{C}_{h,t+1}} \right\} \frac{(1 + r_{f,t})}{\hat{s}_t}.
$$

A log linear approximation of the stochastically detrended UIRP condition yields

(8)
$$
\mathbf{E}_t \tilde{e}_{t+1} - \tilde{e}_t = \frac{r^*}{1 + r^*} \tilde{r}_t + \mathbf{E}_t \{ \tilde{\gamma}_{A,t+1} - \tilde{\gamma}_{M,t+1} \},
$$

where, for example, $\tilde{e}_t = \ln[\hat{s}_t] - \ln[s^*]$ and $r^* (= r_h^* = r_f^*)$ denotes the steady state world real rate.

## 2d.  A DSGE-PVM of the Exchange Rate

We use the linear approximate law of motion of the exchange rate (8), and a stochastically detrended version of the money demand equation (7) to produce the DSGE-PVM. When linearized, the unit consumption elasticity-money demand equation (7) produces $-\tilde{p}_t = \tilde{c}_t - \frac{1}{1 + r^*} \tilde{r}_t$. Impose PPP on the stochastically detrended version of the money demand equation and combine it with the law of motion (8) of the transitory component of the exchange rate to find

$$
\left[ 1 - \frac{1}{1 + r^*} \mathbf{E}_t \mathbf{L}^{-1} \right] \tilde{e}_t = \frac{1}{1 + r^*} \mathbf{E}_t \{ \tilde{\gamma}_{M,t+1} - \tilde{\gamma}_{A,t+1} \} - \frac{r^*}{1 + r^*} \tilde{c}_t.
$$

Solving this stochastic difference equation forward gives a present value relation for the transitory component of the exchange rate

$$(9) \qquad \tilde{e}_t \;\; = \;\; \sum_{j=1}^{\infty} \kappa^j \mathbf{E}_t \left\{ \tilde{y}_{M,t+j} \, - \, \tilde{y}_{A,t+j} \right\} \;\; - \;\; (1 - \kappa) \sum_{j=0}^{\infty} \kappa^j \mathbf{E}_t \tilde{c}_{t+j},$$

where the relevant tranversality conditions are invoked and the DSGE-PVM discount factor $\kappa \equiv \dfrac{1}{1 + r^*}$. Note that the DSGE-PVM and permanent income hypothesis discount factors are equivalent.

The DSGE-PVM relation (9) is the equilibrium law of motion of the cyclical component of the exchange rate. Transitory movements in the exchange rate are equated with the future discounted expected path of cross-country money and TFP growth and the (negative of the) annuity-value of the transitory component of cross-country consumption. The DSGE model identifies the exchange rate's unobserved time-varying risk premium with the expected path of cross-country TFP growth and transitory consumption, which suggest additional sources of exchange rate fluctuations.

The DSGE model produces a present value relation that resembles the standard-PVM (3). The DSGE-PVM follows from unwinding the stochastic detrending of the present value (9)

$$(10) \qquad e_t \;\; = \;\; (1 - \kappa) \sum_{j=0}^{\infty} \kappa^j \mathbf{E}_t \left\{ m_{t+j} \, - \, c_{t+j} \right\}.$$

Thus, the standard-PVM (3) and DSGE-PVM (10) are identical up to differences in their discount factors and real fundamentals. The standard-PVM discount factors $\omega$ is tied to the interest rate semi-elasticity of money demand, $\phi$, while the DSGE-PVM sets $\kappa$ to the inverse of the gross steady state real world interest rate, $1 + r^*$. For the standard-PVM (DSGE-PVM), the real fundamental is cross-country output $y_t$ (consumption $c_t$). Table 1 summarizes the notable elements of the standard- and DSGE-PVMs.

## 3. GENERALIZING THE ENGEL–WEST HYPOTHESIS

This section presents five propositions that generalize the EW hypothesis. The propositions employ standard time series tools, which broadens analysis of the EW hypothesis. For example, the PVM predicts the exchange rate and fundamentals have a common feature given uncontroversial assumptions about fundamentals under one proposition. When the PVM discount factor goes to one, the common feature restriction produces a random walk in the exchange rate which satisfies the EW hypothesis.

The proposition applies to the standard-PVM *and* the DSGE-PVM because their present value relations coincide. The same is also true for the other four propositions. Thus, we generalize the EW hypothesis to the large class of two-country monetary DSGE models.

We collapse the differences in the discount factor and real fundamental of the standard-PVM (3) and DSGE-PVM (10) to stress their mutual predictions in this section. These differences are put aside by defining a PVM discount factor $\mathcal{B}$ equal to either $\omega$ or $\kappa$, while the fundamental $z_t$ is equivalent to either $m_t - \psi y_t$ or $m_t - c_t$. With these assumptions, the focus is on the PVM

$$(11) \qquad e_t \;\; = \;\; (1 - \mathcal{B}) \sum_{j=0}^{\infty} \mathcal{B}^j \mathbf{E}_t z_{t+j},$$

which subsumes the standard- and DSGE-PVMs. The PVM (11) provides several predictions given

**ASSUMPTION 3:** $z_t \sim I(1)$.

**ASSUMPTION 4:** $(1 - \mathbf{L})z_t$ has a Wold representation, $(1 - \mathbf{L})z_t \;=\; \Delta z^* + \zeta(\mathbf{L})\,\upsilon_t$, where $\mathbf{L}\,z_t = z_{t-1}$.[5] Engel and West (2005) employ Assumption 3, but they do not require restrictions as strong as Assumption 4. However, Assumption 4 is standard for linear rational expectation models; see Hansen, Roberds, and Sargent (1991). Assumption 4 is also an implication of a linear approximate solution of the open economy DSGE model, while Assumption 3 is consistent with Assumptions 1 and 2.

## 3a.  *Cointegration Restrictions*

The first prediction is that $e_t$ and $z_t$ share a common trend. This follows from subtracting the latter from both sides of the equality of the present-value relation (11) and combining terms to produce the exchange rate-fundamental cointegrating relation

$$(12) \qquad e_t \;-\; z_t \;\; = \;\; \sum_{j=1}^{\infty} \mathcal{B}^j \mathbf{E}_t \Delta z_{t+j}, \quad \Delta \equiv 1 - \mathbf{L}.$$

Equation (12) reflects the forces – expected discounted value of fundamental growth – that push the exchange rate toward long-run PPP. The explanation is

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[5]The restrictions on the moving average are $\Delta z^*$ is linearly deterministic, $\zeta_0 = 1$, $\zeta(\mathbf{L})$ is an infinite order lag polynominal with roots outside the unit circle, the $\zeta_i$s are square summable, and $\upsilon_t$ is mean zero, homoskedastic, linearly independent given history and is serially uncorrelated with itself and the past of $\Delta z_t$.

**PROPOSITION 1**: *If $z_t$ satisfies Assumptions 3 and 4, $X_t = \beta' q_t$ forms a cointegrating relation with cointegrating vector $\beta' = [1 \quad -1]$, where $q_t \equiv [e_t \quad z_t]'$.*

The proposition is a variation of results found in Campbell and Shiller (1987). We interpret the cointegration relation $X_t$ as the 'adjusted' exchange rate because movements in fundamentals are eliminated from it. According to the cointegration present value relation (12), the 'adjusted' exchange rate is stationary and forward-looking in fundamental growth. Moreover, the cointegration relation $X_t$ is an infinite-order moving average, MA($\infty$) equal to $\mathcal{B}(\mathbf{L}) \zeta_{\mathcal{B}}(\mathbf{L}) v_t$, where $\mathcal{B}(\mathbf{L}) = \sum_{j=0}^{\infty} \mathcal{B}^j \mathbf{L}^j$ and $\zeta_{\mathcal{B}}(\mathbf{L}) = \sum_{j=0}^{\infty} (\mathcal{B}\zeta)^j \mathbf{L}^{j-1}$ under Assumptions 3 and 4 (*i.e.*, $z_t$ is $I(1)$ and its growth rate has a Wold representation). Thus, the 'adjusted' exchange rate is a "*cycle generator*" – as defined by Engle and Issler (1995) – because shocks to serially correlated fundamental growth create persistent PPP deviations.

The standard- and DSGE-PVM require Assumptions 3 and 4 to satisfy Proposition 1. Rather than these assumptions, we can construct a cointegration relation from the DSGE model using Assumptions 1 and 2 because $X_t$ is implied by the balanced growth restriction, $e_t \equiv \ln[s_t] = \tilde{e}_t + m_t - a_t$, where $m_t = \ln[M_t]$ and $a_t = \ln[A_t]$. In this case, PPP deviations arise from the DSGE-PVM because of restrictions the present-value relation (9) places on the transitory component of the exchange rate, $\tilde{e}_t$.

### 3b.  *Equilibrium Currency Return Dynamics*

The second PVM prediction is that currency returns depend only on the lagged 'adjusted' exchange rate and fundamental forecast innovation. We show this by first rewriting the PVM of (11) as $e_t - (1 - \mathcal{B})z_t = (1 - \mathcal{B}) \sum_{j=1}^{\infty} \mathcal{B}^j \mathbf{E}_t z_{t+j}$. Differencing this equation produces, $\Delta e_t - (1 - \mathcal{B})\Delta z_t = (1 - \mathcal{B}) \sum_{j=1}^{\infty} \mathcal{B}^j \left[ \mathbf{E}_t z_{t+j} - \mathbf{E}_{t-1} z_{t+j-1} \right]$. Next, add and subtract $\mathbf{E}_{t-1} z_{t+j}$ inside the brackets, and substitute with the cointegration-present-value relation (12) to obtain

$$(13) \qquad \Delta e_t - \frac{1 - \mathcal{B}}{\mathcal{B}} X_{t-1} = (1 - \mathcal{B}) \sum_{j=0}^{\infty} \mathcal{B}^j [\mathbf{E}_t - \mathbf{E}_{t-1}] z_{t+j}.$$

In equilibrium, currency return dynamics are generated by the lagged cointegration relation, $X_{t-1}$, and the expected present discounted value of the forecast innovations of the fundamental. The lagged cointegration relation is the ECM (13) that reflects the only forces that restore currency returns to equilibrium and PPP in response to the shock innovation $u_{\Delta e, t}$. These ideas are summarized by

**PROPOSITION 2**: *Under Proposition 1, the PVM predicts that the equilibrium currency return is an error correction mechanism in which the lagged 'adjusted' exchange rate (or cointegration relation) is the only factor that drives the exchange rate to PPP in response to fundamental shock innovations.*

Equation (13) is an error correction mechanism (ECM), which regresses currency returns only on the lagged 'adjusted' exchange rate. The regression is $\Delta e_t = \vartheta X_{t-1} + u_{\Delta e,t}$, with factor loading $\vartheta = \frac{1-\mathcal{B}}{\mathcal{B}}$ and the error term $u_{\Delta e,t} = (1-\mathcal{B})\sum_{j=0}^{\infty}\mathcal{B}^j[\mathbf{E}_t - \mathbf{E}_{t-1}]z_{t+j}.$[6]

## 3c.  A Limiting Model of Exchange Rate Determination

Proposition 2 relies on $\mathcal{B} < 1$ to define short- to medium-run currency return dynamics. This raises the question of the impact of relaxing this bound.

**PROPOSITION 3**: *The exchange rate approaches a martingale (in the strict sense) as $\mathcal{B} \longrightarrow 1$, according to the present-value relation* (13) *assuming Proposition 1.*

Proposition 3 suggests an equilibrium path for $e_{t+1}$ in which its best forecast is $e_t$, given relevant information.[7] The hypothesis of Proposition 2 produces $\vartheta \xrightarrow{p} 0$ (or $\mathcal{B} \xrightarrow{p} 1$) and $u_{\Delta e,t} \xrightarrow{p} 0$, which implies the martingale $\mathbf{E}_t e_{t+1} = e_t$ and random walk behavior for the exchange rate.[8]

## 3d.  PVM Exchange Rate Dynamics Redux

Engel and West (2005) show that the PVM of the exchange rate yields an approximate random walk as $\mathcal{B}$ approaches one. This section affirms the EW hypothesis, but unlike Proposition 3 does not rely on Proposition 2. Rather than follow the EW proof exactly, we invoke Assumptions 3 and 4, the present-value relation (3), the Weiner-Kolmogorov prediction formula, and the *conjecture $e_t = \mathsf{a}z_t$* to find that currency returns are unpredictable.

The EW hypothesis is $plim_{\mathcal{B} \longrightarrow 1}[\Delta e_t - \mathsf{a}\zeta(\mathbf{1})\upsilon_t] = 0$. Its hypothesis test begins by noting $e_t = z_{t-1} + \sum_{j=0}^{\infty}\mathcal{B}^j\mathbf{E}_t\Delta z_{t+j}$, which is obtained from the present-value relation (3). Use this equation to construct $\Delta e_t - \mathbf{E}_{t-1}\Delta e_t = \zeta\,(\mathcal{B})\,\upsilon_t$, given Assumptions 3 and 4 and the Weiner-Kolmogorov prediction

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[6]The error $u_{\Delta e,t}$ is also justified if the econometrician's information set is strictly within that of currency traders.

[7]Hansen, Roberds, and Sargent (1991) study linear rational expectations models that anticipate Proposition 4.

[8]Maheswaran and Sims (1993) show that the martingale restriction has little empirical content for tests of asset pricing models when data is sampled at discrete moments in time.

formula. The PVM of (11) also sets currency returns equal to the annuity value of fundamental growth,
$\Delta e_t = (1 - \mathcal{B}) \sum_{j=0}^{\infty} \mathcal{B}^j \mathbf{E}_t \Delta z_{t+j}$. The last two equations yield

$$(14) \qquad \Delta e_t \;=\; \zeta\,(\mathcal{B})\,\upsilon_t \;+\; (1 - \mathcal{B}) \sum_{j=0}^{\infty} \mathcal{B}^j \mathbf{E}_{t-1} \Delta z_{t+j}.$$

By letting $\mathcal{B} \xrightarrow{p} 1$, the random walk hypothesis of EW is verified independent of the ECM of Proposition 2 (and cointegration prediction of Proposition 1).[9]

The ECM (13) and Proposition 2 maps into the EW currency return generating equation (14). First, apply the change of index $j = i - 1$ to the present value of (14) to obtain the present-value cointegration relation (12) lagged once. For the ECM (13), its present value $(1 - \mathcal{B}) \sum_{j=0}^{\infty} \mathcal{B}^j [\mathbf{E}_t - \mathbf{E}_{t-1}] z_{t+j}$ equals $\zeta\,(\mathcal{B})\,\upsilon_t$ subsequent to evoking Assumptions 3 and 4 and the Weiner-Kolmogorov prediction formula. Thus, when the PVM discount factor $\mathcal{B}$ is arbitrarily close to one, the EW hypothesis predicts $\Delta e_t = \zeta\,(\mathbf{1})\,\upsilon_t$ which is consistent with currency returns following an ECM with no own lags or lags of fundamental growth. Since the standard- and DSGE-PVMs produce the ECM, the EW hypothesis is generalized to the larger class of two-country monetary DSGE models.

### 3e.  A Common Trend-Common Cycle Model of Exchange Rates and Fundamentals

Proposition 2 predicts an ECM for currency returns that is consistent with the EW currency return generating equation (14). These results rely, at most, on assumptions 3 and 4 under which fundamentals are $I(1)$ and have a Wold representation in growth rates. However, empirical work on exchange rates often employ multivariate time series models (*i.e.*, VARs) instead of the deeper notion of a Wold representation.

This section studies the impact of endowing fundamental growth with an ECM on the bivariate exchange rate-fundamental process, $q_t = [e_t \;\; z_t]'$. In this case, $\Delta q_t$ forms a VECM(0)

$$(15) \qquad \Delta q_t \;=\; \begin{bmatrix} \vartheta \\ \eta \end{bmatrix} \mathcal{X}_{t-1} \;+\; \begin{bmatrix} u_{\Delta e,t} \\ u_{\Delta z,t} \end{bmatrix},$$

---

[9]This analysis matches equations $A.3 - A.11$ and the surrounding discussion of Engel and West (2005).

where $\eta$ is the factor loading on the lagged cointegrating relation $X_{t-1}$ and $u_{\Delta z,t}$ is the fundamental growth forecast innovation.

The VECM(0) restricts the bivariate exchange rate-fundamental process. Pre-multiplying the VECM(0) by $\overline{\beta}' = \begin{bmatrix} 1 & -\frac{\vartheta}{\eta} \end{bmatrix}$ creates the common feature

(16) $$\overline{\beta}' \Delta q_t = \overline{\beta}' \begin{bmatrix} u_{\Delta e,t} & u_{\Delta z,t} \end{bmatrix}'.$$

The vector $\overline{\beta}'$ satisfies the Engle and Kozicki (1993) notion of a common feature because it creates a linear combination of $\Delta e_t$ and $\Delta z_t$ that is unpredictable conditional on their history. Given this common feature restriction and the cointegration relation of Proposition 1, Vahid and Engle (1993) provide a method to construct a Stock and Watson (1988) multivariate Beveridge and Nelson (1981) common trend-common cycle decomposition. We summarize these results with

**PROPOSITION 4**: *Assume fundamental growth is the ECM process $\Delta z_t = \eta X_{t-1} + u_{\Delta z,t}$, where the forecast innovation $u_{\Delta z,t}$ is Gaussian. When Proposition 2 holds, $q_t$ has a common feature, $\overline{\beta}' \Delta q_t$, in the sense of Engle and Kozicki (1993), where $\overline{\beta}' = \begin{bmatrix} 1 & -\frac{\vartheta}{\eta} \end{bmatrix}$. The cointegrating and common feature vectors $\beta$ and $\overline{\beta}$ restrict the trend-cycle decomposition of $q_t$, as described by Vahid and Engle (1993).*

The common feature of Proposition 4 endows $q_t = [e_t \ z_t]'$ with a common trend and a common cycle Beveridge-Nelson-Stock-Watson (BNSW) decomposition. Vahid and Engle (1993) provide an example in which the cointegration and common feature vectors restrict the trend of $q_t$ to $\mathbf{I}_2 - \overline{\beta}(\beta' \overline{\beta})^{-1}\beta'$, which gives trend and cycle components $\frac{-\mathcal{B}\eta}{1 - \mathcal{B}(1+\eta)}\overline{\beta}' q_t$ and $\frac{1 - \mathcal{B}}{1 - \mathcal{B}(1+\eta)}\beta' q_t$, respectively.[10] The BNSW decomposition imposes a common cycle on $e_t$ and $z_t$ in the short-, medium-, and long-run, which restricts the exchange rate to be unpredictable at all forecast horizons. A prediction that is at odds with the empirical evidence of Mark (1995).

The common feature relation (16) also provides another approach to verify the EW hypothesis.

**PROPOSITION 5**: *Let the exchange rate and fundamental have the VECM(0) (15). Then, the EW hypothesis requires currency returns and fundamental growth to share a common feature defined by $\overline{\beta}' = \begin{bmatrix} 1 & -\frac{\vartheta}{\eta} \end{bmatrix}$ and that $\vartheta \xrightarrow{p} 0$ (or $\mathcal{B} \xrightarrow{p} 1$).*

---

[10]Vahid and Engle show a $n$–dimension VAR(1) with $d$ cointegrating relations has $n - d$ common feature relations.

Proposition 5 differs from other approaches to the EW hypothesis. First, the common feature relation (16) imposes cross-equation restrictions on $\Delta q_t$ because its source serial correlation, the lagged cointegrating relation $\mathcal{X}_{t-1}$ is annihilated by $\overline{\beta}'$. Second, having eliminated the cycle generator $\mathcal{X}_{t-1}$, the EW hypothesis decouples the exchange rate from the fundamental growth forecast innovation $u_{\Delta z,t}$. Observe that when $\vartheta \xrightarrow{p} 0$ (or $\mathcal{B} \xrightarrow{p} 1$), $\overline{\beta}' \xrightarrow{p} [1 \;\; 0]$. This leaves only the forecast innovation $u_{\Delta e,t}$ to generate movements in $\Delta e_t$. Thus, the EW hypothesis is affirmed by Proposition 5.[11]

A corollary of Proposition 5 is that changes in fundamentals do not Granger cause currency returns as $\mathcal{B} \longrightarrow 1$. Only if $\mathcal{B} \in (0, 1)$, do movements in fundamentals have predictive power for currency returns according to the PVM. However, currency returns Granger cause growth in the fundamental as long as it is predicted by its own lagged forecast innovations. The equilibrium currency return generating equation (13) and Proposition 2 shows that this holds even if $\mathcal{B} \longrightarrow 1$.

### 3f.   Tests of the PVM of the Exchange Rate

The propositions suggest testable restrictions on exchange rates and fundamentals. If the lag length of the levels VAR of the exchange rate and fundamental exceeds one, the VECM (15) is rejected. Cointegration tests suffice to examine Proposition 1. Vahid and Engel (1993) and Engel and Issler (1995) provide common feature tests that yield information about the EW hypothesis and Propositions 4 and 5. Table 3 summarizes the results and details the tests involved.

We estimate VARs of foreign currency-U.S. dollar exchange rates and fundamentals using Canadian, Japanese, U.K., and U.S. data on a 1976Q1 – 2004Q4 sample.[12] VAR lag lengths are chosen using likelihood ratio (LR) statistics, given a VAR(8), …, VAR(1).[13] The Canadian–, Japanese–, and U.K.–U.S.

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[11]Proposition 5 can also be cast as an implication of the BNSW representation of $\Delta q_t$. In this case, $\overline{\beta}'$ removes the vector MA($\infty$) in $u_{\Delta e,t}$ and $u_{\Delta x,t}$ from the BNSW representation of $\Delta q_t$. Only a linear combination of pure forecast innovations, $u_{\Delta e,t}$ and $u_{\Delta x,t}$, are left to drive $\Delta q_t$. Let $\vartheta \xrightarrow{p} 0$ to obtain the random walk exchange rate with innovation $u_{\Delta e,t}$.

[12]Fundamentals equal cross-country money minus cross-country output, which implies an income elasticity of money demand, $\psi$, calibrated to one. This calibration is consistent with estimates reported by Mark and Sul (2003). The money stocks (outputs) are measured in current (constant) local currency units and per capita terms.

[13]The VARs include a constant and linear time trend. The LR statistics employ the Sims (1980) correction and have standard asymptotic distribution according to results in Sims, Stock, and Watson (1990).

samples yield a VAR(8), VAR(5), and VAR(4), respectively.[14] Thus, the Canadian, Japanese, U.K., and U.S. data reject the VECM (15) because $\Delta q_t$ has more serial correlation than explained by the lagged cointegration relation $\mathcal{X}_{t-1}$.

Engel and West (2005) argue there is little evidence that exchange rates and fundamentals cointegrate. Table 3 presents Johansen (1991, 1994) trace and $\lambda-$max statistics that support this conclusion and fail to confirm the cointegration prediction of Proposition 1 for the Canadian–, Japanese–, and U.K.–U.S. samples.

Table 3 includes squared canonical correlations of currency returns and fundamental growth. The common feature null is that the smallest correlation equals zero. We use a $\chi^2$ statistic of Vahid and Engle (1993) and a $F-$statistic developed by Rao (1973) to test this null. The tests reject the null for the largest canonical correlation, but not for the smaller one in the three samples. This is evidence that currency returns and fundamental share a common feature in the Canadian–, Japanese–, and U.K.–U.S. samples. Given a common feature, the exchange rate approximates a random walk when $\mathcal{B} \xrightarrow{p} 1$. The next section explores the empirical content of this assumption in a Canadian–U.S. sample.

## 4. Econometric Models and Methods

Propositions 1–5 broaden our understanding of the EW hypothesis. The EW hypothesis is generalized to hold for the DSGE-PVM, which imposes a rich set of cross-equation restrictions on the joint behavior of the exchange rate and DSGE fundamentals. Although the previous section discusses VAR methods that yield evidence about the joint behavior of the exchange rate and standard-PVM fundamentals, this approach is not informative about the standard-PVM discount factor $\omega$.

This section presents methods to estimate $\mathcal{B}$ and to test the EW hypothesis. Instead of relying on VARs, we employ UC models to estimate the DSGE-PVM and test the EW hypothesis using Bayesian methods. A brief example motivates our approach. Consider the PVM (11) where the fundamental $z_t$ has the permanent-transitory decomposition $z_t = \tau_t + \tilde{z}_t$, $\tau_{t+1} = \tau_t + \varepsilon_{\tau,t+1}$, $(1 - \sum_{i=1}^{p_z} \mathbf{L}^i)\tilde{z}_t = \varepsilon_{\tilde{z},t}$,

---

[14]The Canadian-U.S. and Japanese-U.S. VARs are selected when the $p-$value of the LR test is five percent or less. Since the U.K.-U.S. VAR offers ambiguous results, we settle on a VAR(4).

$\mathbf{E}_t \varepsilon_{\tau,t+1} = \mathbf{E}_t \varepsilon_{\tilde{z},t+1} = 0$, $\mathbf{E}_t \varepsilon_{\tau,t+1}^2 = \sigma_\tau^2$, $\mathbf{E}_t \varepsilon_{\tilde{z},t+1}^2 = \sigma_{\tilde{z}}^2$, and $\mathbf{E}_t \varepsilon_{\tau,t+i} \varepsilon_{\tilde{z},t+j} = 0$ for all $i$ and $j$.[15] Combining the PVM (11) and the permanent-transitory decomposition of $z_t$ gives an equilibrium permanent-transitory decomposition of the exchange rate, $e_t = \tau_t + (1-\mathcal{B})\iota_{\tilde{z}}\left[\mathbf{I} - \mathcal{B}\mathcal{A}_{\tilde{z}}\right]^{-1}\tilde{z}_t$, where $\iota_{\tilde{z}}$ is a $1 \times p_z$ row vector with a first element of one and zeros elsewhere and $\mathcal{A}_{\tilde{z}}$ is the companion matrix of the AR of $\tilde{z}_t$. The exchange rate trend is identified with the random walk of $z_t$ under the permanent-transitory decomposition of $z_t$. Transitory exchange rate fluctuations are driven by the fundamental cyclical component, $\tilde{z}_t$, which creates a common dynamic factor in the exchange rate and observed fundamental $z_t$. The permanent-transitory decomposition of the exchange rate is useful for the EW hypothesis because it becomes possible to estimate $\mathcal{B}$, along with the coefficients of the permanent-transitory decomposition of $z_t$. Note also that as $\mathcal{B}$ approaches one, the permanent component $\tau_t$ comes to dominate exchange rate fluctuations as predicted by the EW hypothesis.

We exploit cross-restrictions created by permanent-transitory decompositions of fundamentals to estimate the DSGE-PVM. The DSGE-PVM has a deep underlying structure connected to the fundamentals of the cross-country money stock and cross-country consumption. Permanent-transitory decompositions of these fundamentals is the foundation of the UC models that we estimate.

This section describes the Bayesian methods we employ to estimate six UC multivariate models of the DSGE-PVM. The UC models represent different combinations of restrictions imposed by the DSGE-PVM on the exchange rate, cross-country money, and cross-country consumption. For example, $\kappa$ is estimated for three UC models, which ties the exchange rate to the transitory component(s) of fundamentals. The exchange rate is disconnected from transitory shocks in remaining three UC models because $\kappa$ is calibrated to one. We cast the UC models in state space form to evaluate numerically the likelihoods on a $1976Q1$–$2004Q4$ sample of the Canadian dollar–U.S. dollar ($CDN\$/US\$$) exchange rate and the Canadian–U.S. money and consumption differentials. The random walk MH simulator is used to generate MCMC draws from the UC model posterior distributions conditional on this sample. We compute model moments, such as parameter means, unconditional variance ratios, permanent-transitory

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[15]We thank Farshid Vahid for suggesting this example.

decompositions, and forecast error variance decompositions (FEVDs), from the posterior distributions. Model comparisons are based on marginal likelihoods, which we construct by integrating the likelihood function of each model across its parameter space where the weighting function is the model prior.

## 4a.  State Space Systems of the UC Models

The state space systems of the six UC models begin with the balanced growth restriction the DSGE model imposes on the exchange rate. This restriction is equivalent to the permanent-transitory decomposition $e_t = m_t - a_t + \widetilde{e}_t$. The DSGE-PVM (9) place cross-equation restrictions on the stationary component of the exchange rate, $\widetilde{e}_t$.

Cross-equation restrictions are conditioned on the permanent and transitory components of cross-country money and cross-country consumption. The permanent components of money and consumption are $\mu_{t+1} = \mu^* + \mu_t + \varepsilon_{\mu,t+1}$, $\varepsilon_{\mu,t+1} \sim \mathcal{N}(0, \sigma^2_{\varepsilon_\mu})$, and $a_{t+1} = a^* + a_t + \varepsilon_{a,t+1}$, $\varepsilon_{a,t+1} \sim \mathcal{N}(0, \sigma^2_{\varepsilon_a})$, respectively. Note that $\mu^*$ and $a^*$ are the deterministic trend growth rates of cross-country money and TFP. We assume $\widetilde{m}_t$ is a MA($k_{\widetilde{m}}$), $\widetilde{m}_t = \sum_{j=0}^{k_{\widetilde{m}}} \alpha_j \varepsilon_{\widetilde{m}, t-j}$, where $\alpha_0 \equiv 1$ and $\varepsilon_{\widetilde{m},t} \sim \mathcal{N}(0, \sigma^2_{\varepsilon_{\widetilde{m}}})$. For $\widetilde{c}_t$, we employ a AR($k_{\widetilde{c}}$), $\widetilde{c}_t = \sum_{j=1}^{k_{\widetilde{c}}} \theta_j \widetilde{c}_{t-j} + \varepsilon_{\widetilde{c},t}$, where $\varepsilon_{\widetilde{c},t} \sim \mathcal{N}(0, \sigma^2_{\varepsilon_{\widetilde{c}}})$. Put these elements together to form the balanced growth version of the DSGE-PVM

$$(17) \qquad e_t = \mu_t - a_t + (1 - \kappa) \sum_{j=0}^{\infty} \kappa^j \mathbf{E}_t \left\{ \widetilde{m}_{t+j} - \widetilde{c}_{t+j} \right\},$$

which satisfies the DSGE balanced growth path restrictions. The balanced growth DSGE-PVM (17) implies the cointegrating relation of Proposition 1. Thus, the exchange rate responds only to trends in cross-country money, $\mu_t$, and TFP, $a_t$, in the long-run. Serial correlation in the exchange rate is produced by the transitory components of cross-country money and consumption, $\widetilde{m}_t$ and $\widetilde{c}_t$. Also, note that if a common cycle generates these transitory components, the exchange also shares the restriction. Thus, the permanent and transitory components of cross-country money and consumption drive exchange rate fluctuations, which give rise to cross-equations in the UC models.

We classify the UC models according to whether there are two cycles or a common cycle and whether $\kappa$ is calibrated to one or estimated. Thus, the DSGE-PVM (17) is solved for the exchange rate

17

given $\widetilde{m}_t \sim \text{MA}(k_{\widetilde{m}})$ and $\widetilde{c}_t \sim \text{AR}(k_{\widetilde{c}})$ or a common cycle is imposed using either the $\text{MA}(k_{\widetilde{m}})$ or $\text{AR}(k_{\widetilde{c}})$. We double these three UC models when $\kappa$ is calibrated to one or not. However, the six UC models have in common the cross-country money and TFP trends, $\mu_t$ and $a_t$.

A rich set of cross-equation restrictions arises in the 2-trend, 2-cycle UC model with $\kappa \in (0, 1)$. In part, its state space system consists of the observation equations

$$
(18) \quad
\begin{bmatrix} e_t \\[6pt] m_t \\[6pt] c_t \end{bmatrix}
=
\begin{bmatrix}
1 & -1 & \delta_{\widetilde{m},0} & \delta_{\widetilde{m},1} & \ldots & \delta_{\widetilde{m},k_{\widetilde{m}}} & \delta_{\widetilde{c},0} & \delta_{\widetilde{c},1} & \ldots & \delta_{\widetilde{c},k_{\widetilde{c}}-1} \\[6pt]
1 & 0 & 1 & \alpha_1 & \ldots & \alpha_{k_{\widetilde{m}}} & 0 & 0 & \ldots & 0 \\[6pt]
0 & 1 & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0
\end{bmatrix}
S_{\widetilde{m},c,t},
$$

where $S_{\widetilde{m},c,t} = \left[ \mu_t \ a_t \ \varepsilon_{\widetilde{m},t} \ \varepsilon_{\widetilde{m},t-1} \ \ldots \ \varepsilon_{\widetilde{m},t-k_{\widetilde{m}}} \ \widetilde{c}_t \ \widetilde{c}_{t-1} \ \ldots \ \widetilde{c}_{t-k_{\widetilde{c}}+1} \right]'$, the factor loadings on $\varepsilon_{\widetilde{m},t}$ and its lags are

$$
(19) \quad \delta_{\widetilde{m},i} = (1 - \kappa) \sum_{j=i}^{k_{\widetilde{m}}} \kappa^{j-i} \alpha_j, \quad i = 0, \ldots, k_{\widetilde{m}},
$$

the factor loadings on $\widetilde{c}_t, \ldots, \widetilde{c}_{t-k_{\widetilde{c}}}$ are elements of the row vector

$$
(20) \quad \delta_{\widetilde{c}} = -s_{\widetilde{c}}(1 - \kappa) \left[ \mathbf{I}_{k_{\widetilde{c}}} - \kappa\Theta \right]^{-1}, \quad s_{\widetilde{c}} = [1 \ \mathbf{0}_{1 \times k_{\widetilde{c}}-1}],
$$

and $\Theta$ is the companion matrix of the $\text{AR}(k_{\widetilde{c}})$ of $\widetilde{c}_t$. The system of first-order state equations is

$$
(21) \quad S_{\widetilde{m},c,t+1} =
\begin{bmatrix} \mu^* \\ a^* \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix}
+
\begin{bmatrix}
1 & 0 & \ldots & 0 & 0 & \ldots & & 0 \\
0 & 1 & \ldots & 0 & 0 & \ldots & & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & & 0 \\
\vdots & \vdots & \mathbf{I}_{k_{\widetilde{m}}} & \vdots & \vdots & & & \vdots \\
0 & 0 & \ldots & 0 & \theta_1 & \ldots & & \theta_{k_{\widetilde{c}}} \\
\vdots & \vdots & & \vdots & \mathbf{I}_{k_{\widetilde{c}}-1} & \mathbf{0}_{(k_{\widetilde{c}}-1)\times 1} \\
\vdots & \vdots & & \vdots
\end{bmatrix}
S_{\widetilde{m},c,t}
+
\begin{bmatrix}
\varepsilon_{\mu,t+1} \\
\varepsilon_{a,t+1} \\
\varepsilon_{\widetilde{m},t+1} \\
\mathbf{0}_{k_{\widetilde{m}}\times 1} \\
\varepsilon_{\widetilde{c},t+1} \\
\mathbf{0}_{(k_{\widetilde{c}}-1)\times 1}
\end{bmatrix},
$$

with the covariance matrix $\Omega_{m,\widetilde{c}} = \varepsilon_{\widetilde{m},c,t}\varepsilon'_{\widetilde{m},c,t}$ where $\varepsilon_{\widetilde{m},c,t} = [\varepsilon_{\mu,t+1} \ \varepsilon_{a,t+1} \ \varepsilon_{\widetilde{m},t+1} \ \mathbf{0}_{k_{\widetilde{m}}\times 1} \ \varepsilon_{\widetilde{c},t+1} \ \mathbf{0}_{(k_{\widetilde{c}}-1)\times 1}]'$.

We also study the implications of imposing one common transitory factor on $m_t$ and $c_t$. When this common component is $\widetilde{m}_t$, the response of $c_t$ to $\widetilde{m}_t$ is denoted $\pi_{m,\tilde{c}}$. This implies $\tilde{c}_t = \pi_{m,\tilde{c}}\widetilde{m}_t = \pi_{m,\tilde{c}}\sum_{j=0}^{k_{\widetilde{m}}}\alpha_j\varepsilon_{\widetilde{m},t-j}$. For the 2-trend, money cycle UC model, the state vector and observer system are $S_{\widetilde{m},t} = \begin{bmatrix} \mu_t & a_t & \varepsilon_{\widetilde{m},t} & \varepsilon_{\widetilde{m},t-1} & \ldots & \varepsilon_{\widetilde{m},t-k_{\widetilde{m}}} \end{bmatrix}'$ and

$$
(22) \qquad \begin{bmatrix} e_t \\ m_t \\ c_t \end{bmatrix} = \begin{bmatrix} 1 & -1 & (1-\pi_{c,\widetilde{m}})\delta_{\widetilde{m},0} & (1-\pi_{c,\widetilde{m}})\delta_{\widetilde{m},1} & \ldots & (1-\pi_{c,\widetilde{m}})\delta_{\widetilde{m},k_{\widetilde{m}}} \\ 1 & 0 & 1 & \alpha_1 & \ldots & \alpha_{k_{\widetilde{m}}} \\ 0 & 1 & \pi_{c,\widetilde{m}} & \pi_{c,\widetilde{m}}\alpha_1 & \ldots & \pi_{c,\widetilde{m}}\alpha_{k_{\widetilde{m}}} \end{bmatrix} S_{\widetilde{m},t},
$$

respectively. The state equation of this system is

$$
(23) \qquad S_{\widetilde{m},t+1} = \begin{bmatrix} \mu^* \\ a^* \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{bmatrix} S_{\widetilde{m},t} + \begin{bmatrix} \varepsilon_{\mu,t+1} \\ \varepsilon_{a,t+1} \\ \varepsilon_{\widetilde{m},t+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix},
$$

with covariance matrix $\Omega_{\widetilde{m}} = \varepsilon_{\widetilde{m},t}\varepsilon'_{\widetilde{m},t}$, where $\varepsilon_{\widetilde{m},t} = \begin{bmatrix} \varepsilon_{\mu,t+1} & \varepsilon_{a,t+1} & \varepsilon_{\widetilde{m},t+1} & 0 & \ldots & 0 \end{bmatrix}'$.

Identifying the common transitory component with $\tilde{c}_t$ restricts $\widetilde{m}_t = \pi_{m,\tilde{c}}\tilde{c}_t$. This yields the system of observer equations of the 2-trend, consumption cycle UC model

$$
(24) \qquad \begin{bmatrix} e_t \\ m_t \\ c_t \end{bmatrix} = \begin{bmatrix} 1 & -1 & (1-\pi_{m,\tilde{c}})\delta_{\tilde{c},0} & (1-\pi_{m,\tilde{c}})\delta_{\tilde{c},1} & \ldots & (1-\pi_{m,\tilde{c}})\delta_{\tilde{c},k_{\tilde{c}}-1} \\ 1 & 0 & \pi_{m,\tilde{c}} & 0 & \ldots & 0 \\ 0 & 1 & 1 & 0 & \ldots & 0 \end{bmatrix} S_{\tilde{c},t},
$$

and the system of state equations

$$
(25) \qquad S_{\tilde{c},t+1} =
\begin{bmatrix}
\mu^* \\
a^* \\
0 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
+
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \theta_1 & \theta_2 & \cdots & \theta_{k_{\tilde{c}}-1} & \theta_{k_{\tilde{c}}} \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
S_{\tilde{c},t}
+
\begin{bmatrix}
\varepsilon_{\mu,t+1} \\
\varepsilon_{a,t+1} \\
\varepsilon_{\tilde{c},t+1} \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix},
$$

where $S_{\tilde{c},t} = \begin{bmatrix} \mu_t & a_t & \tilde{c}_t & \tilde{c}_{t-1} & \cdots & \tilde{c}_{t-k_{\tilde{c}}+1} \end{bmatrix}'$, $\Omega_{\tilde{c}} = \varepsilon_{\tilde{c},t}\varepsilon'_{\tilde{c},t}$, and $\varepsilon_{\tilde{c},t} = \begin{bmatrix} \varepsilon_{\mu,t+1} & \varepsilon_{a,t+1} & \varepsilon_{\tilde{c},t+1} & 0 & \cdots & 0 \end{bmatrix}'$.

The three remaining UC models set $\kappa = 1$ in the state space systems of the 2-trend, 2-cycle UC model, the 2-trend, money cycle UC model, and the 2-trend, consumption cycle UC model. The restriction on the state space of these UC models is that beyond the second column only zeros occupy the first row of the observation equations (18), (22), and (24). Thus, we are able to compare DSGE-PVMs with $\kappa$ estimated on $(0, 1)$ to limiting DSGE models in which $\kappa \longrightarrow 1$ on our Canadian-U.S. sample. This provides an empirical appraisal of the EW hypothesis.

## 4b. The UC Model and Its Likelihood Function

We label the 2-trend, 2-cycle UC model with $\kappa \in (0, 1)$ $UC_{2,2,\kappa}$. Likewise, $UC_{2,\widetilde{m},\kappa}$ and $UC_{2,\tilde{c},\kappa}$ denote the 2-trend, money cycle and 2-trend, consumption cycle, $\kappa \in (0, 1)$ UC models. The state space systems of $UC_{2,2,\kappa}$, $UC_{2,\widetilde{m},\kappa}$, and $UC_{2,\tilde{c},\kappa}$ are (18) and (21), (22)–(23), and (24)–(25), respectively. These state space systems represent the dynamics of $y_t = \begin{bmatrix} e_t & m_t & c_t \end{bmatrix}'$ restricted by the DSGE-PVM and permanent-transitory specifications of $m_t$ and $c_t$. These state space systems are mapped into the Kalman filter to evaluate the likelihood function as proposed by Harvey (1989) and Hamilton (1994).[16] Denote the likelihood $\mathcal{L}(y_t \mid \Gamma_{2,i,\kappa}, \; UC_{2,i,\kappa})$, where $i = 2, \widetilde{m}, \tilde{c}$ and $\Gamma_{2,i,\kappa}$ is the parameter vector of $UC_{2,i,\kappa}$. The parameter vector of $UC_{2,2,\kappa}$ contains $11 + k_{\widetilde{m}} + k_{\tilde{c}}$ elements, $\Gamma_{2,2,\kappa} = \begin{bmatrix} \kappa & \alpha_1 & \cdots & \alpha_{k_{\widetilde{m}}} & \theta_1 & \cdots & \theta_{k_{\tilde{c}}} \end{bmatrix}$
$\mu^* \; a^* \; \sigma_\mu \; \sigma_a \; \sigma_{\widetilde{m}} \; \sigma_{\tilde{c}} \; \varrho_{a,\tilde{c}} \; \pi_{e,0} \; \pi_{e,t} \; \pi_{e,a} \end{bmatrix}'$.

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[16] A related example is Harvey, Trimbur, and van Dijk (2007) who use Bayesian methods to estimate permanent-transitory decompositions of aggregate time series, but without rational expectations cross-equation restrictions.

We add the parameters $\varrho_{a,\tilde{c}}$, $\pi_{e,0}$, $\pi_{e,t}$, and $\pi_{e,a}$ to $\Gamma_{2,2,\kappa}$ to better fit the UC models to the data. For example, the Canadian-U.S. TFP differential exhibits more variation than $c_t$ if the correlation coefficient of innovations to $a_t$ and $\tilde{c}_t$, $\mathbf{E}\{\varepsilon_{a,t}\,\varepsilon_{\tilde{c},t}\} = \varrho_{a,\tilde{c}}$, is negative.[17] The remaining three parameters allow for an unrestricted exchange rate intercept, $\pi_{e,0}$, a linear exchange rate time trend, $\pi_{e,t}$, and a factor loading on the Canadian-U.S. TFP differential, $\pi_{e,a}$, that differs from negative one for the (1, 2) element in the matrix of the observation systems (18), (22), and (24).[18] We estimate $\pi_{e,a}$ to ask if the data supports the cointegration- balanced growth path restriction imposed on the DSGE-PVM (17).

The parameter vectors of the other five UC models are smaller. The $UC_{2,\widetilde{m},\kappa}$ model drops two plus $k_{\tilde{c}}$ parameters from $\Gamma_{2,\widetilde{m},\kappa} = \begin{bmatrix} \kappa & \alpha_1 & \ldots & \alpha_{k_{\widetilde{m}}} & \mu^* & a^* & \sigma_\mu & \sigma_a & \sigma_{\widetilde{m}} & \pi_{e,0} & \pi_{e,t} & \pi_{e,a} & \pi_{c,\widetilde{m}} \end{bmatrix}'$, while adding the factor loading on $\widetilde{m}_t$ for $c_t$, $\pi_{c,\widetilde{m}}$. The factor loading $\pi_{m,\tilde{c}}$ enters the parameter vector of $UC_{2,\widetilde{m},\kappa}$, while $\alpha_1 \ldots \alpha_{k_{\widetilde{m}}}$ and $\sigma_{\widetilde{m}}$ are dropped from $\Gamma_{2,\tilde{c},\kappa} = \begin{bmatrix} \kappa & \theta_1 & \ldots & \theta_{k_{\tilde{c}}} & \mu^* & a^* & \sigma_\mu & \sigma_a \end{bmatrix}$ $\begin{matrix} \sigma_{\tilde{c}} & \varrho_{a,\tilde{c}} & \pi_{e,0} & \pi_{e,t} & \pi_{e,a} & \pi_{m,\tilde{c}} \end{matrix}\Big]'$. The parameter vectors of the UC models $UC_{2,2,\kappa=1}$, $UC_{2,\widetilde{m},\kappa=1}$, and $UC_{2,\tilde{c},\kappa=1}$ are identical to $\Gamma_{2,2,\kappa}$, $\Gamma_{2,\widetilde{m},\kappa}$, and $\Gamma_{2,\tilde{c},\kappa}$ except that $\kappa = 1$.

## 4c. The Data

The sample runs from 1976Q1 to 2004Q4, $T = 116$. We have observations on the Canadian dollar–U.S. dollar exchange rate (average of period). The Canadian monetary aggregate is M1 in current Canadian dollars, while for the U.S. it is the Board of Governors Monetary Base (adjusted for changes in reserve requirements) in current U.S. dollars. Consumption is the sum of non-durable and services expenditures in constant local currency units.[19] The aggregate quantity data is seasonally adjusted and converted to per capita units. The data is logged and multiplied by 100, but is neither demeaned nor detrended.

## 4d. Estimation Methods

The likelihood function of the UC models do not have analytic solutions. We approximate the

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[17]Morley, Nelson, and Zivot (2003) show that this restriction applied to an univariate UC model resolves its differences with the Beverage and Nelson (1981) decomposition.

[18]The factor loading on the permanent component of $m_t$ remains (normalized to) one.

[19]Canadian consumption includes semi-durable expenditures.

likelihoods $\mathcal{L}(y_t| \Gamma_{2,i,\kappa=1}, UC_{2,i,\kappa=1})$ and $\mathcal{L}(y_t| \Gamma_{2,i,\kappa}, UC_{2,i,\kappa})$ with posterior distributions of $\Gamma_{2,i,\kappa=1}$ and $\Gamma_{2,i,\kappa}$, generated by the MCMC replications of the random walk MH simulator. Our estimates of $\Gamma_{2,i,\kappa=1}$ and $\Gamma_{2,i,\kappa}$ and marginal likelihoods build on the Bayesian estimation tools of Fernández-Villaverde and Rubio-Ramírez (2004), Rabanal and Rubio-Ramírez (2005), Geweke (1999, 2005), An and Schorfheide (2007), and Gelman, Carlin, Stern, and Rubin (2004). The MH simulator is asked to create 1.5 million MCMC draws from the posterior. The initial 750,000 draws are treated as a burn-in sample and therefore discarded. We base our estimates on the remaining 750,000 draws from the posteriors of the $UC_{2,2,\kappa=1}$, $UC_{2,\widetilde{m},\kappa=1}$, $UC_{2,\widetilde{c},\kappa=1}$, $UC_{2,2,\kappa}$, $UC_{2,\widetilde{m},\kappa}$, and $UC_{2,\widetilde{c},\kappa}$ models.[20]

## 4e.  Priors

The second column of table 4 (5) list the priors of $\Gamma_{2,i,\kappa=1}$ ($\Gamma_{2,i,\kappa}$), $i = 2, \widetilde{m}, \widetilde{c}$. Under a normal prior, the first element is the degenerate mean and second its standard deviation. The inverse-gamma priors are parameterized by its degrees of freedom, the first element, and its mean, the second element. The left and right end points of a uniform prior is denoted by its first and second elements.

We choose degenerate priors for the lag lengths of the MA($k_{\widetilde{m}}$) of $\widetilde{m}_t$ and AR($k_{\widetilde{c}}$) of $\widetilde{c}_t$ that set $k_{\widetilde{m}} = k_{\widetilde{c}} = 2$. Normal priors for the MA ($\alpha_1$ and $\alpha_2$) and AR ($\theta_1$ and $\theta_2$) coefficients allow for disparate transitory behavior in $\widetilde{m}_t$ and $\widetilde{c}_t$. The prior means of $\alpha_1$, $\alpha_2$, $\theta_1$, and $\theta_2$ guarantee that the relevant eigenvalues are strictly less than one. The eigenvalues of the MA(2) (AR(2)) of $\widetilde{m}_t$ ($\widetilde{c}_t$) are $0.60 \pm 0.20i$ (0.95 and -0.10). The standard deviation of the normal priors of the MA and AR coefficients provide for a wide set of realizations for $\alpha_1$, $\alpha_2$, $\theta_1$, and $\theta_2$. However, when a draw generates an eigenvalue greater than one (in absolute value) for either the MA or AR coefficients, the draw is discarded. Nonetheless, the MA and AR priors admit transitory cycles in cross-country money and consumption that allow for

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[20]The posterior distributions are based on acceptance rates of between 25 and 36 percent. Besides the 750,000 MCMC draws used to compute the moments reported below, four more sequences of 750,000 MCMCs are generated from disparate starting values to assess across chain and with chain convergence. We compute the $\hat{R}$ statistic of Gelman, Carlin, Stern, and Rubin (2004) to evaluate across chain across and the separated partial means test of Geweke (2005) convergence, which is distributed asymptotically $\chi^2$. Across the 77 parameters of the six UC models, the two largest $\hat{R}$s are 1.20 and 1.04, while Gelman, et al suggest a $\hat{R}$ of about 1.10. On five subsamples, The Geweke separated partial means test has no $p$-value smaller than 0.21 across the six UC models and five MCMC simulation sequences.

power at the business cycle frequencies, if the data wants.

We opt for priors of $\mu^*$ and $a^*$ that rely on the Canadian–U.S. money stock and consumption differentials samples. Since $\mu^*$ and $a^*$ represent deterministic trend growth, we ground the priors on normal distributions. The prior standard deviations of $\mu^*$ and $a^*$ match sample moments.

Priors on the standard deviations of the shock innovations reflect standard practice for estimating DSGE models with Bayesian methods. For example, Adolfson, Laséen, Lindé, and Villani (2007) employ inverse-gamma priors for the standard deviations of the shock innovations of their sticky price open economy DSGE model. However, there is a lack of good information about $\sigma_\mu$, $\sigma_a$, $\sigma_{\widetilde{m}}$, and $\sigma_{\widetilde{c}}$. This explains why we impose a prior with two degrees of freedom, which forces these standard deviations to be positive. On the other hand, we attach a normally distributed prior to the correlation of innovations to $a_t$ and $\widetilde{c}_t$, $\varrho_{a,\widetilde{c}}$. Its mean is negative to capture our prior that $a_t$ is smoother than $c_t$. Since we have no information about the extent of the smoothness, the mean is $-0.5$ with a standard deviation of $0.2$ that places draws close to negative one or zero in the 95 percent coverage interval of the prior. Draws greater than one or less than negative one are ignored. The correlation of innovations to $\mu_t$ and $\widetilde{m}_t$ is fixed at zero because our belief is that the sources and causes of permanent and transitory monetary shocks are orthogonal.

The exchange rate intercept and linear time trend priors are set according to a linear regression of the sample Canadian dollar–U.S. dollar exchange rate on these objects. This motivates our choice of normally distributed priors for $\pi_{e,0}$ and $\pi_{e,t}$ and of their degenerate means and standard deviations.

The remaining factor loadings have priors that reflect a dearth of information on our part. The uniform priors of $\pi_{e,a}$, $\pi_{c,\widetilde{m}}$, and $\pi_{c,\widetilde{m}}$ are wide and include zero. If, for example, $\pi_{e,a}$ is small it indicates the inadequacy of the balanced growth restriction and the impact of permanent fluctuations in Canadian–U.S. TFP differentials on the exchange rate. The same holds for the response of $c_t$ ($m_t$) to transitory movements in the Canadian–U.S. money stock (consumption) differential.

The $UC_{2,i,\kappa}$ models have only one 'economic' parameter, the DSGE-PVM discount factor $\kappa \equiv \frac{1}{1+r^*}$, in common. We adopt the Engel and West (2005) prior for $\kappa$. They argue that it is necessary for

$\kappa \in [0.9,\ 0.999]$ to generate an approximate random walk exchange rate from the standard-PVM. Hence, our prior on $\kappa$ is constructed to provide information about the EW hypothesis from the posteriors of the $UC_{2,i,\kappa}$ models. We impose an inverse-gamma prior on the DSGE-PVM discount factor $\kappa$, which follows Del Negro and Schorfheide (2006). The degenerate prior means of $\kappa = 0.988$ and $\exp([a^* = 0.158]/400)$ imply an annual average real world interest rate of about five percent. Although a five percent real world interest rate is large for the floating rate period, the standard deviation of 0.038 guarantees draws for $\kappa$ that cover a wide interval. However, MCMC draws from the random walk MH simulator of the $UC_{2,i,\kappa}$ models obey the EW prior because we ignore draws for which $\kappa \notin [0.9,\ 0.999]$.

## 5. RESULTS

This section presents the results of implementing our empirical strategy. Tables 4 and 5 provide the posterior means and standard deviations of the $\Gamma_{2,i,\kappa=1}$ and $\Gamma_{2,i,\kappa}$ vectors, $i = 2,\ \widetilde{m},\ \widetilde{c}$, for the $UC_{2,2,\kappa=1}$, $UC_{2,\widetilde{m},\kappa=1}$, $UC_{2,\widetilde{c},\kappa=1}$, $UC_{2,2,\kappa}$, $UC_{2,\widetilde{m},\kappa}$, and $UC_{2,\widetilde{c},\kappa}$ models. We present densities of the prior and posteriors of $\kappa$ for the latter three UC models in figure 1. The posterior distributions of the six UC models are used to construct the marginal likelihoods of the six UC model, as described by Fernández-Villaverde and Rubio-Ramírez (?), Rabanal and Rubio-Ramírez (2005), and Geweke (1999), to conduct inference across these models. We also report the factor loadings on the $CDN\$/US\$$ exchange rate of the transitory components of the Canadian-U.S. money stock and consumption differentials, unconditional variance ratios of the present discounted value of the shock innovations to the $CDN\$/US\$$ exchange rate, FEVDs of the trend-cycle decomposition of the exchange rate with respect to these shocks, and summary statistics of trend-cycle decompositions in tables 6, 7, 8, and 9, respectively. Figures 2, 3, and 4 plot the trend-cycle decomposition of the $CDN\$/US\$$ exchange rate.

### 5a. Parameter Estimates

Tables 4 and 5 list the posterior means and standard deviations of the parameters of the six UC models. Estimates of the limiting DSGE models appear in table 4. These are $UC_{2,2,\kappa=1}$, $UC_{2,\widetilde{m},\kappa=1}$, $UC_{2,\widetilde{c},\kappa=1}$. These three models exhibit persistence in the transitory components of the Canadian-U.S. money and consumption differentials, $\widetilde{m}_t$ and $\widetilde{c}_t$. For example, the $UC_{2,\widetilde{m},\kappa=1}$ $(UC_{2,\widetilde{c},\kappa=1})$ model yield

AR (MA) estimates that imply the half life of shock to $\tilde{c}_t$ ($\widetilde{m}_t$) is 17 (7) years.[21] However, only $\tilde{c}_t$ is persistent in the $UC_{2,2,\kappa=1}$ model. The half life of a shock to $\widetilde{m}_t$ is less than two quarters, while for $\tilde{c}_t$ it is between nine and ten years. Note also that the priors and posterior means of the MA coefficients, $\alpha_1$ and $\alpha_2$, only differ for the $UC_{2,\widetilde{m},\kappa=1}$ model. Although the posterior means of the AR coefficients have moved away from the prior means, a one standard deviation of the posterior of $\theta_2$ covers zero for the $UC_{2,2,\kappa=1}$ and $UC_{2,\tilde{c},\kappa=1}$ models.

The posterior means of $\mu^*$ and $a^*$ show that Canada experiences slower (faster) trend money (TFP) growth than for the U.S. from 1976Q1 to 2004Q4. Trend U.S. money growth is on average about 0.05 percent higher annually according to the posteriors of the $UC_{2,2,\kappa=1}$, $UC_{2,\widetilde{m},\kappa=1}$, and $UC_{2,\tilde{c},\kappa=1}$ models. Across these models, $a^* \approx 0.16$ indicates Canadian deterministic trend TFP growth dominates its U.S. counterpart by about 0.06 percent at an annual rate.

The $UC_{2,2,\kappa=1}$, $UC_{2,\widetilde{m},\kappa=1}$, and $UC_{2,\tilde{c},\kappa=1}$ models show differences across estimates of the posterior means of the shock innovation standard deviations the Canadian-U.S. sample. Only the estimated impulse structure of the $UC_{2,\widetilde{m},\kappa=1}$ model is dominated by movements in the permanent innovations of the Canadian–U.S. money differential shock, $\sigma_\mu$. The converse is that this model yields the smallest posterior means of the standard deviation of the Canadian–U.S. TFP differential shock and $\widetilde{m}_t$ shock innovations, $\sigma_a$ and $\sigma_{\widetilde{m}}$. The $UC_{2,\tilde{c},\kappa=1}$ model yields the largest estimates of $\sigma_a$ and $\sigma_{\tilde{c}}$, but these posterior means are about the same magnitude. Note also that the correlation of the innovations to the Canadian–U.S. TFP differential shock and $\tilde{c}_t$ shock is estimated to be $\varrho_{a,\tilde{c}} = -0.88$ and $-0.95$ by the $UC_{2,2,\kappa=1}$ and $UC_{2,\tilde{c},\kappa=1}$ models, respectively. Thus, these models are consistent with the Canadian–U.S. TFP trend differential being more volatile than observed Canadian–U.S. consumption.

Estimates of the exchange rate intercept and linear time trend indicate that the $UC_{2,\widetilde{m},\kappa=1}$ model provides the largest value for the $US\$$ in steady state and the largest deterministic growth rate for the $CDN\$/US\$$ exchange rate. The posterior means of $\pi_{e,0}$ and $\pi_{e,t}$ imply that the steady state $CDN\$/US\$$ exchange rate is 1.23 with a deterministic annual growth rate of about 0.8 percent. For the

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[21]The half life equals $\log[0.5]/\log[q]$, where $q$ is the largest modulus of the companion matrix of the AR or MA coefficients.

$UC_{2,2,\kappa=1}$ ($UC_{2,\tilde{c},\kappa=1}$) model, the analogous values are 1.10 (1.03) and 0.3 (0.2) percent per annum. Thus, the $UC_{2,\widetilde{m},\kappa=1}$ model places more emphasis on deterministic elements to fit the data compared to the other two UC models of the limiting DSGE-PVM.

The remaining coefficients are the factor loadings $\pi_{e,a}$, $\pi_{e,\widetilde{m}}$, and $\pi_{e,\tilde{c}}$. The posterior mean estimates of $\pi_{e,a}$ $-2.68$ and $-9.02$ reveal that there are statistically and economically large deviations from the balanced growth path by the $UC_{2,2,\kappa=1}$ and $UC_{2,\widetilde{m},\kappa=1}$ models. The $UC_{2,\tilde{c},\kappa=1}$ model is closer to satisfying the balanced growth hypothesis that $\pi_{e,a} = -1$. This UC model has a posterior mean of $-0.72$ for $\pi_{e,\tilde{c}}$ whose two standard deviation interval contains the balanced growth restriction. The response of the Canadian–U.S. money stock differential to $\tilde{c}_t$ is also close to negative one for the $UC_{2,\tilde{c},\kappa=1}$ because the posterior mean of $\pi_{m,\tilde{c}} = -0.90$ with a standard deviation of 0.21. The $UC_{2,\widetilde{m},\kappa=1}$ model reveals that a one percent rise in $\tilde{c}_t$ results in a 4.4 percent rise in the Canadian–U.S. consumption differential.

The key economic parameter of the DSGE-PVM is its discount factor $\kappa$. Table 5 lists the posterior means and standard deviations of the $\Gamma_{2,2,\kappa}$ and $\Gamma_{2,\widetilde{m},\kappa}$, and $\Gamma_{2,\tilde{c},\kappa}$ vectors that include estimates of $\kappa$. Aside from the inclusion of the prior, posterior mean, and standard deviation $\kappa$ at the top of table 5, the posterior means and standard deviations of the remaining coefficients resemble those reported on table 4. The only notable exceptions are that the posterior means of $\sigma_a$, $\sigma_{\tilde{c}}$, $\pi_{e,a}$, and $\pi_{e,\tilde{c}}$ are smaller for the $UC_{2,\tilde{c},\kappa}$ model compared to its cousin with the calibration $\kappa = 1$. The result is that $\sigma_\mu$ is the largest innovation shock standard deviation of the $UC_{2,\tilde{c},\kappa}$ model. Also, this UC model and the data produce an estimate of $\pi_{e,a}$ whose one standard deviation coverage interval contains negative one. Thus, the $UC_{2,\tilde{c},\kappa}$ model is closer to the balanced growth hypothesis and relies to a greater extent on permanent shocks to the Canadian–U.S. money stock differential.

The posterior means of $\kappa$ range from 0.966 for the $UC_{2,2,\kappa}$ model, 0.974 for the $UC_{2,\widetilde{m},\kappa}$ model, to the largest estimate of 0.9962 for the $UC_{2,\tilde{c},\kappa}$ model. These estimates are consistent with annual world real interest rates of 15.1, 11.4, and 1.7 percent given the posteriors of the $UC_{2,2,\kappa}$, $UC_{2,\widetilde{m},\kappa}$ and $UC_{2,\tilde{c},\kappa}$ models, respectively. Although the $UC_{2,2,\kappa}$, $UC_{2,\widetilde{m},\kappa}$ models have posteriors that suggest unreasonably large world real interest rates, these UC models yield 95 percent coverage intervals whose upper end

is 0.999. The $UC_{2,\tilde{c},\kappa}$ model produces a posterior of $\kappa$ with a 95 percent coverage interval whose lower end equals 0.987. This value of $\kappa$ is greater than the posterior means of $\kappa$ of the $UC_{2,2,\kappa}$ for the $UC_{2,\widetilde{m},\kappa}$ models. Thus, the $UC_{2,\tilde{c},\kappa}$ model generates a posterior distribution of $\kappa$ that is to the right of those produced by the $UC_{2,2,\kappa}$ and $UC_{2,\widetilde{m},\kappa}$ models.

Figure 1 reinforces the idea that the posteriors of the $UC_{2,2,\kappa}$ and $UC_{2,\widetilde{m},\kappa}$ models yield estimates of $\kappa$ that are shifted to the right of those of the $UC_{2,\tilde{c},\kappa}$ model. Posterior densities of $\kappa$ are plotted in figure 1 for these UC models, along with the density of the inverse-gamma prior of $\kappa$ on the EW prior of $\kappa \in [0.9, 0.999]$. The solid (black) line is the density of the $\kappa$ prior and is close to the posterior density of $\kappa$ derived from the $UC_{2,2,\kappa}$ model, which is the dashed (blue) line. The $UC_{2,\widetilde{m},\kappa}$ model generates a posterior density of kappa, the dot-dash (green) plot, which moves off the prior by placing less weight on $\kappa$s less than 0.97 and more weight above this value. The dot-dot (red) plot is the density of $\kappa$ from the posterior of the $UC_{2,\tilde{c},\kappa}$ model. This density is deflated by ten percent for ease of comparison to the other densities. A striking feature of figure 1 is that the posterior of the $UC_{2,\tilde{c},\kappa}$ model pushes $\kappa$ off of its prior because its mass lays between 0.98 and 0.999.

Table 6 contains the posterior means of the exchange rate factor loadings with respect to $\widetilde{m}_t$ and $\tilde{c}_t$, the $\delta_{\widetilde{m},i}$s and $\delta_{\tilde{c},i}$s.[22] A striking aspect of the estimates of $\delta_{\widetilde{m},0}$, $\delta_{\widetilde{m},1}$, and $\delta_{\widetilde{m},2}$ is that the response of the $CDN\$/US\$$ exchange rate to innovations in $\widetilde{m}_t$ is economically small for either the $UC_{2,2,\kappa}$ or $UC_{2,\widetilde{m},\kappa}$ models. The large posterior standard errors on these factor loading also indicate the imprecision the Canadian–U.S. data give to these estimates. The data have less problems yielding a precise estimate of $\delta_{\tilde{c},0}$ for the $UC_{2,2,\kappa}$ model. The posterior mean of this factor loading shows that the exchange rate falls by 0.6 percent given an one percent increase in $\tilde{c}_t$. These estimates drops to $-0.33$ for the $UC_{2,\tilde{c},\kappa}$ model. Also, the associated 95 percent coverage interval contains zero. In summary, the $UC_{2,2,\kappa}$, $UC_{2,\widetilde{m},\kappa}$, and $UC_{2,\tilde{c},\kappa}$ models have posteriors in which there is either a negligible exchange rate response to $\widetilde{m}_t$ shocks or an economically large negative reaction by the $CDN\$/US\$$ exchange rate to $\tilde{c}_t$ fluctuations. However, the latter exchange rate response can be estimated imprecisely.

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[22]For the $UC_{2,\widetilde{m},\kappa}$ and $UC_{2,\tilde{c},\kappa}$ models, the relevant factor loadings are multiplied by $1 - \pi_{e,\widetilde{m}}$ or $1 - \pi_{e,\tilde{c}}$.

## 5b. Unconditional Variance Ratios and FEVDs of the Exchange Rate

Tables 7 and 8 present unconditional variance ratios and FEVDs computed using the posteriors of the $UC_{2,2,\kappa}$, $UC_{2,\widetilde{m},\kappa}$, and $UC_{2,\widetilde{c},\kappa}$ models. We calculate the variances of the present discounted values (PDVs) of the Canadian–U.S. money, TFP, and consumption differential shock innovations using the DSGE-PVM version of the equilibrium currency return generating equation (14) and UC model restrictions when $\kappa$ is estimated. The variance ratios are these values divided by the sample variance of the $CDN\$/US\$$ exchange rate (= 2.04). According to the unconditional variance ratios, only permanent shocks to the Canadian–US money differential, $\varepsilon_{\mu,t}$, and the TFP differential, $\varepsilon_{a,t}$, explain variation in the $CDN\$/US\$$ exchange rate. The variances of the PDVs of shock innovations to $\widetilde{m}_t$ and $\widetilde{c}_t$ are small and lack precision. Note that except for the $UC_{2,\widetilde{m},\kappa}$ model, the variance of the PDV of $\varepsilon_{a,t}$ is larger than that of $\varepsilon_{\mu,t}$.

We report FEVDs in table 8 with implications similar to the unconditional variance ratios.[23] The top panel of figure 8 shows that the posterior of the $UC_{2,2,\kappa}$ yields a FEVD in which the Canadian–US TFP differential shock $\varepsilon_{a,t}$ makes a large and increasing contribution to $CDN\$/US\$$ exchange rate fluctuations at longer forecast horizons. The Canadian–US money differential shock $\varepsilon_{\mu,t}$ remains economically important for $CDN\$/US\$$ exchange rate movements out to a three to five year forecast horizon, but shocks to $\widetilde{m}_t$ and $\widetilde{c}_t$ are unimportant at any forecast horizon. Much the same is true for the FEVDs found using the posterior of $UC_{2,\widetilde{m},\kappa}$. However, the relative shares of the $\varepsilon_{\mu,t}$ and $\varepsilon_{a,t}$ shocks are unchanged at a two-thirds/one-third split from the one quarter to ten year forecast horizons.

The posterior of the $UC_{2,\widetilde{c},\kappa}$ model imbues a slowly changing dynamic to the $CDN\$/US\$$ exchange rate FEVDs as found in the bottom panel of table 8. The $\varepsilon_{\mu,t}$ and $\varepsilon_{a,t}$ shocks are responsible for about 60 and 40 percent, respectively, of fluctuations in the $CDN\$/US\$$ exchange rate at the short horizons. At a 10 year horizon, the contribution of $\varepsilon_{\mu,t}$ ($\varepsilon_{a,t}$) only falls (rises) to 55 (45) percent. Thus, only the posterior of the $UC_{2,\widetilde{c},\kappa}$ model predicts that permanent shocks to the Canadian–US money differential dominate $CDN\$/US\$$ exchange rate movements at the longer forecast horizons.

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[23]The FEVDs are computed using the vector ECM implied by equation (13) and the permanent-transitory specifications of Canadian-U.S. money and consumption differentials. The vector ECM is placed in state space form as outlined by Heqc, Palm, and Urbain (2000) and iterated upon to create the FEVDs of table 8.

## 5c. Trend-Cycle Decompositions

Trend-cycle decompositions of the $CDN\$/US\$$ exchange rate and Canadian–U.S. money and consumption differentials are plotted in figures 2 and 3 with summary statistics given in table 9. The posteriors of the $UC_{2,2,\kappa}$, $UC_{2,\widetilde{m},\kappa}$, and $UC_{2,\widetilde{c},\kappa}$ models are run through the Kalman smoother to create the trend-cycle decompositions and summary statistics. Figures 2 and 3 and table 9 contain moments that are averages over the 750,000 draws from the UC model posteriors. We label trend exchange rate growth $\Delta e^{\tau}$ in table 9.

The top window of figure 2 contains plots of the $CDN\$/US\$$ exchange rate and smoothed trends taken from the posteriors of the $UC_{2,2,\kappa}$ and $UC_{2,\widetilde{c},\kappa}$ models.[24] The solid (black) line is $e_t$, the log of the actual $CDN\$/US\$$ exchange rate. The smoothed trends of the $UC_{2,2,\kappa}$ and $UC_{2,\widetilde{c},\kappa}$ models are the dashed (blue) and dotted (red) plots, respectively. Note that these UC models generate smoothed $CDN\$/US\$$ exchange rate trends that are more volatile than the actual exchange rate. The top row of table 9 indicate that the posteriors of the $UC_{2,2,\kappa}$ and $UC_{2,\widetilde{c},\kappa}$ models generate standard deviations of $\Delta e^{\tau}$ are 2.66 and 2.44, respectively. The standard deviation of $\Delta e^{\tau}$ equals 2.04.

The smoothed $CDN\$/US\$$ exchange rate cycles appear in the bottom window of figure 2. The dotted (blue) line is the smoothed exchange rate cycle, $\widetilde{e}_t$, based on the posterior of the $UC_{2,2,\kappa}$ model, while the dotted (red) line is associated with the $UC_{2,\widetilde{c},\kappa}$ model. Although the former $\widetilde{e}_t$ exhibits more variability than the latter (the standard deviations are 3.68 and 2.44), these $\widetilde{e}_t$s are persistent with AR1 correlation statistics of 0.97 and 0.98. Note that only the posterior of the $UC_{2,\widetilde{m},\kappa}$ model yields (close to) a non-zero correlation for $\Delta e^{\tau}$ and $\widetilde{e}$, according to table 9.

Figure 3 shows the smoothed permanent-transitory decompositions of the Canadian-U.S. money and consumption differentials. The actual differentials and smoothed trends appear in the top row of windows, while the smoothed cycles are found in the bottom row of windows. The Canadian-U.S. money

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[24]We do not present the trend-cycle decompositions based on the posterior of the $UC_{2,\widetilde{m},\kappa}$ model because its log marginal likelihood is far below those of the other UC models. Table 9 includes standard deviations of $\Delta e^{\tau}$ and $\widetilde{e}$ from the posterior of the $UC_{2,\widetilde{m},\kappa}$ model that are larger by a factor of 30 or compared to these statistics from the $UC_{2,2,\kappa}$ and $UC_{2,\widetilde{c},\kappa}$ models, which is a signal of its lack of acceptance by our Canadian–U.S. sample.

(consumption) differentials are the right (left) side windows. The posterior of the $UC_{2,2,\kappa}$ model produces a money trend, $\mu_t$ that almost perfectly mimics the actual Canadian-U.S. money differentials, as found in the top left window of figure 3. The result is that the smoothed $\widetilde{m}_t$ is much less volatile, a standard deviation of 0.68, compared to a standard deviation of 1.62 for $\mu_t$. The bottom left window of figure 3 shows a saw-toothed pattern in $\widetilde{m}_t$, conditional on the posterior of the $UC_{2,2,\kappa}$ model. This explains the AR1 correlation statistic of $-0.68$ for $\widetilde{m}_t$ that appears in the middle of second column of table 9.

Table 9 reveals that the posterior of the $UC_{2,\widetilde{c},\kappa}$ model produces a smoothed money trend, $\mu_t$, that is about as volatile as does the $UC_{2,2,\kappa}$ model. The relevant standard deviations are 1.62 and 1.71 in the second and fourth columns of table 9 for the smoothed money growth trend of these UC models. However, smoothed $\Delta\mu$ and $\Delta e^\tau$ share a positive correlation of 0.62 only in the posterior of the $UC_{2,\widetilde{c},\kappa}$ model as shown in the bottom half of the fourth column of table 9.

The posteriors of the $UC_{2,2,\kappa}$ and $UC_{2,\widetilde{c},\kappa}$ models yield qualitatively similar plots for the smoothed TFP trend, $a_t$, and consumption cycle $\widetilde{c}_t$ in the top right window of figure 3. These plots appear in the top right window of figure 3 as dashed (blue) for the and dotted (red) for the $UC_{2,2,\kappa}$ and $UC_{2,\widetilde{c},\kappa}$ models, respectively, where observed cross country consumption is the solid (black) line. The smoothed $\Delta a$ is 50 percent more volatile for the $UC_{2,\widetilde{c},\kappa}$ model than it is for the $UC_{2,2,\kappa}$ model. The posteriors of these UC models also produce correlations of $-0.71$ and $-0.85$ for $\Delta e^\tau$ and $\Delta a$, which suggest that rising U.S. TFP is associated with a depreciation in the Canadian dollar.

The bottom right window of figure 3 presents the smoothed $\widetilde{c}_t$ of the $UC_{2,2,\kappa}$ and $UC_{2,\widetilde{c},\kappa}$ models. The former cycle is the dashed (blue) line and the latter is the dotted (red) plot. These cycles are persistent because their AR1 correlation statistics 0.97 and 0.98, but the $UC_{2,2,\kappa}$ model generates a third less volatility in smoothed $\widetilde{c}_t$ than found for the $UC_{2,\widetilde{c},\kappa}$ model.

The posteriors of the $UC_{2,2,\kappa}$ and $UC_{2,\widetilde{c},\kappa}$ models generate economically meaningful trends and cycles in the Canadian–U.S. $a_t$ and $\widetilde{c}$. The smoothed TFP differential is falling in the latter 1970s, which reflects a greater productivity slowdown in the Canada. By the 1980s, Canadian TFP is growing more rapidly than in the U.S., which continues into the early 1990s. Subsequently, U.S. TFP recovers relative

to Canadian TFP. At the end of the sample, the Canadian–U.S. TFP differential is expanding once more.

The smoothed $\tilde{c}_t$ has peaks and troughs that coincide with several U.S.-Canadian business cycle dates. For example, troughs in the posterior mean of $\tilde{c}_t$ appear in 1981 and 1990 which also represent recessions dates in the U.S. and Canada. Since the end of the 1990 – 1991 recession, the rise in $\tilde{c}_t$ points to persistent, but transitory, rise in U.S. consumption relative to Canada. Nonetheless, $\tilde{c}_t$ has been falling rapidly since a peak in late 2001, which corresponds to the end of the last U.S. recession.

The bottom row of table 9 shows that $\tilde{c}_t$ and $\tilde{e}_t$ are perfectly negatively correlated in the posteriors of the $UC_{2,2,\kappa}$ and $UC_{2,\tilde{c},\kappa}$ models. The negative correlation of the transitory component of the exchange rate with $\tilde{c}_t$ help to interpret the $CDN\$/US\$$ exchange rate fluctuations. Peaks in the transitory component of the $CDN\$/US\$$ exchange rate occur either at or shortly after the end of recessions. For example, the smoothed $CDN\$/US\$$ exchange rate cycles have a tendency to peak and trough around dates usually associated with U.S. and Canadian business cycle dates (*i.e.*, the late 1970s, early 1990s, and 2001). A specific case is the peak in $\tilde{e}_t$ during the 1990 – 1991 recession in the U.S., which is a moment at which the Canadian dollar approached par against the U.S. dollar. An exception is the end of the 2001 recession when the Canadian dollar reached a low of nearly 0.62 to the U.S. dollar.

## 5d.  Comparing the UC models

The bottom row of table 4 reports the log marginal likelihoods, $\ln \hat{\mathcal{L}}$ of the $UC_{2,2,\kappa=1}$, $UC_{2,\tilde{m},\kappa=1}$, and $UC_{2,\tilde{c},\kappa=1}$ models. These marginal likelihoods show that our Canadian-U.S. sample gives most support $UC_{2,\tilde{c},\kappa=1}$ model. The difference between this model and the $UC_{2,2,\kappa=1}$ model is about 29, or the Bayes factor prefers the UC model with only transitory consumption. For the data to give more credence to the latter model, its prior probability must be raised by the prior probability of the $UC_{2,\tilde{c},\kappa=1}$ model multiplied by $4.7 \times 10^{12}$ [$= \exp(29.18)$]. Since the magnitude of this factor is large, it seems unreasonable to include the transitory money shock in the UC model when $\kappa$ is calibrated to one.

The last row of table 5 contains the log marginal likelihoods of the $UC_{2,2,\kappa}$, $UC_{2,\tilde{m},\kappa}$, and $UC_{2,\tilde{c},\kappa}$ models. The ranking of these models matches that of the UC models with the $\kappa = 1$ calibration. The $UC_{2,\tilde{m},\kappa}$ model dominates the $UC_{2,2,\kappa}$ and $UC_{2,\tilde{m},\kappa}$ models. A key reason is that the posteriors of these

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models yield economically implausible estimates of the DGSE-PVM discount factor $\kappa$.

This raises the question of whether our Canadian–U.S. sample will find it difficult to choose between the $UC_{2,\widetilde{c},\kappa=1}$ and $UC_{2,\widetilde{c},\kappa}$ models. Our Canadian–U.S. sample favors the $UC_{2,\widetilde{c},\kappa=1}$ and $UC_{2,\widetilde{c},\kappa}$ models compared to the other four. The former UC model has a larger marginal likelihood, which suggest the data support it over the latter UC model with $\kappa \in [0.9, 0.999]$. This choice relies on the belief that scaling up the prior probability of the $UC_{2,\widetilde{c},\kappa}$ model by about $167.3 = \exp(5.12)$ is too large to be justified. If, on the other hand, this factor is regarded as inconclusive in rejecting the $UC_{2,\widetilde{c},\kappa}$ model, it could be argued that our Canadian–U.S. sample cannot pick between the $UC_{2,\widetilde{c},\kappa=1}$ and $UC_{2,\widetilde{c},\kappa}$ models.[25] In either case, we argue that this is support for the EW hypothesis.

## 5e.  Exchange Rate Dynamics as $\kappa \longrightarrow 1$

Engel and West (2005) argue that the exchange rate will approximate a random walk when the discount factor is close to one and fundamentals have a unit root. Propositions 7 and 8 also predict that $\widetilde{e}_t$ will collapse to random walk, as $\kappa \longrightarrow 1$.

We extract evidence about the EW hypothesis from the posteriors of the $UC_{2,2,\kappa}$ and $UC_{2,\widetilde{c},\kappa}$ models. The focus is on these UC models because the $UC_{2,\widetilde{c},\kappa=1}$ model attributes all $CDN\$/US\$$ exchange rate movements to permanent shocks. The $UC_{2,2,\kappa}$ model is included for comparison. We conduct this comparison with $\kappa$s at the 16th and 84th percentiles, along with the largest $\kappa$s, from the $\Gamma_{2,2,\kappa}$ and $\Gamma_{2,\widetilde{c},\kappa}$ vectors. For the $UC_{2,2,\kappa}$ ($UC_{2,\widetilde{c},\kappa}$) model, the 16th percentile, 84th percentile, and largest $\kappa$s are 0.9425, 0.9883, and 0.9990 (0.9943, 0.9987, and 0.9990), respectively. Fixing $\kappa$ at these values, we simulate the $UC_{2,2,\kappa}$ and $UC_{2,\widetilde{c},\kappa}$ models drawing 2000 sequences from the posteriors, discard the first 1000, run the Kalman smoother on the remaining 1000 sequences, and average the ensemble to generate $CDN\$/US\$$ exchange rate cycles to respect the rational expectations hypothesis.

Figure 4 plots the smoothed $CDN\$/US\$$ exchange rate cycles. The top (bottom) window contains the $\widetilde{e}_t$ created from the posterior of the $UC_{2,2,\kappa}$ ($UC_{2,\widetilde{c},\kappa}$) model. The dot-dash (blue), dotted (green), and dotted (red) lines are conditional on the 16th percentile, 84th percentile, and largest $\kappa$s, respectively.

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[25] Jeffreys (1998, p. 432) contends that Bayes factors differing by 3.16 is evidence about the two models just between 'not worth more than a bare mention' and substantially in favor of the model with the larger marginal likelihood.

Across the top and bottom windows, the volatility of $\tilde{e}_t$ is compressed as $\kappa$ approaches 0.999. This is reflected in the standard deviations of $\tilde{e}_t$ that equal 4.44, 2.84, and 0.57 moving from the smallest to largest $\kappa$ for the $UC_{2,2,\kappa}$ model. The equivalent standard deviations are 3.74, 1.64, and 1.30 for the $UC_{2,\tilde{c},\kappa}$ model. Although the $UC_{2,2,\kappa}$ generates $CDN\$/US\$$ exchange rate cycles that are smoother than at its posterior mean only for the largest $\kappa$s, the $UC_{2,2,\kappa}$ model is able to produce smoother exchange rate cycles at the 84th percentile and largest $\kappa$s. Thus, pushing $\kappa$ increases the smoothness of the $CDN\$/US\$$ exchange rate cycle which we argue is evidence in support of the EW hypothesis.

## 6. CONCLUSION

Economists have little to say about the impact of policy on currency markets without an equilibrium theory of exchange rate determination that is empirically relevant. According to Engel and West (2005), the near random walk behavior of exchange rates explains the failure of equilibrium models to fit the data or to find any model that systematically beats it at out-of-sample forecasting. They conjecture that the standard-present value model (PVM) of exchange rates yields the random walk prediction when fundamentals are persistent and the discount factor is close to one.

This paper generalizes the Engel and West (EW) hypothesis by constructing a PVM from a two-country monetary dynamic stochastic general equilibrium (DSGE) model. The standard and DSGE-PVMs place identical restrictions on the exchange rate and fundamentals up to the PVM discount factors. Thus, we generalize the EW hypothesis to the larger class of open economy DSGE models.

We present five propositions that also generalize the EW hypothesis. Besides predicting the exchange rate and fundamental cointegrate, the PVMs predict currency returns respond to the lagged cointegration relation and the fundamental forecast innovation. When the PVM discount factor goes to one, only the fundamental forecast innovation drives currency returns which verifies the EW hypothesis. We also show that the EW hypothesis is equivalent to eliminating the common dynamic factor(s) or serial correlation in the exchange rate and fundamentals. This affirms the EW hypothesis within a wider empirical environment.

Our empirical results support the view that it is difficult for the data to choose between a random

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walk exchange rate model and a DSGE-PVM with a discount factor estimated to be near one. At the same time we obtain evidence on the nature of the shocks driving exchange rates. Bayesian estimates of the DSGE-PVM suggest that the Canadian dollar–U.S. dollar exchange rate is dominated by permanent shocks whether the discount factor is estimated or calibrated to one. Thus, the DSGE-PVM and the Canadian-U.S. sample yield estimates that support the Engel-West hypothesis. Our evidence is also consistent with the recent VAR literature suggesting that monetary policy shocks have only a minor impact on exchange rate fluctuations. Monetary policy shocks are also found to be unimportant for exchange rate movements by Lubik and Schorfheide (2006) within the context of an estimated open economy DSGE model. Whether this result holds across a wider set of open economy DSGE is a worthy goal of future research.

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# Table 1:  Summary of Standard PVM and DSGE-PVM

## *Standard-PVM*

**ECM(0):** $\quad$ (13) $\Delta e_t - \frac{1-\omega}{\omega}X_{t-1} = (1-\omega)\sum_{j=0}^{\infty}\omega^j[\mathbf{E}_t - \mathbf{E}_{t-1}]z_{t+j}.$

**EW Equation:** $\quad$ (14) $\Delta e_t = \zeta(\omega)\upsilon_t + (1-\omega)\sum_{j=0}^{\infty}\omega^j\mathbf{E}_{t-1}\Delta z_{t+j}.$

**Parameters:** $\quad \omega \equiv \dfrac{\phi}{1+\phi} =$ Discount Factor,

$\phi =$ Money Demand Interest Rate Semi-Elasticity,

$\psi =$ Money Demand Income Elasticity.

**Fundamentals:** $X_t = e_t - z_t, \quad z_t = m_t - \psi y_t,$
$m_t =$ Cross Country Money,
$y_t =$ Cross-Country Output.

## *DSGE-PVM*

**ECM(0):** $\quad$ (13) $\Delta e_t - \dfrac{(1-\kappa)}{\kappa}X_{DSGE,t-1}$

$\qquad\qquad = (1-\kappa)\sum_{j=0}^{\infty}\kappa^j[\mathbf{E}_t - \mathbf{E}_{t-1}]\left\{m_{t+j} - c_{t+j}\right\}.$

**EW Equation:** $\quad$ (14) $\Delta e_t = \sum_{j=0}^{\infty}\kappa^j[\mathbf{E}_t - \mathbf{E}_{t-1}]\left\{\Delta m_{t+j} - \Delta c_{t+j}\right\}$

$\qquad\qquad + (1-\kappa)\sum_{j=0}^{\infty}\kappa^j\mathbf{E}_t\left\{\Delta m_{t+j} - \Delta c_{t+j}\right\}.$

**Parameters:** $\quad \kappa \equiv \dfrac{1}{1+r^*} =$ Discount Factor,

$r^* =$ Steady State Real World Interest Rate.

**Fundamentals:** $X_{DSGE,t} = e_t - m_t + c_t,$
$m_t =$ Cross-Country Money,
$c_t =$ Cross-Country Consumption.

## Table 2:  Summary of Propositions
## for Standard- and DSGE-PVMs

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**Proposition 1**:  PVM Predicts Exchange Rate and Fundamentals Cointegrate; Campbell and Shiller (1987).


**Proposition 2**:  Currency Returns Are an ECM(0).


**Proposition 3**:  Exchange Rate Approximates a Martingale as $\mathcal{B} \longrightarrow 1$.


**Proposition 4**:  VECM(0) Imply Common Trend and Common Cycle for Exchange Rate and Fundamental.


**Proposition 5**:  EW's (2005) Hypothesis Needs Currency Returns and Fundamental Growth Share a Co-Feature and $\mathcal{B} \longrightarrow 1$.

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# Table 3:  Tests of Propositions 1, 3, and 5

## Sample:   1976Q1 – 2004Q4

|  | Canada & U.S. | Japan & U.S. | U.K. & U.S. |
|---|---|---|---|
| **Proposition 3**: VECM(0) | | | |
| Levels VAR Lag Length | 8 | 5 | 4 |
| LR statistic $p$−value | (0.02) | (0.01) | (0.09) |
| | | | |
| **Proposition 1**: Common Trend Cointegration Tests | | | |
| Model | Case 2∗ | Case 1 | Case 1 |
| $\lambda$−Max statistic | 4.86 | 0.20 | 2.27 |
| | 17.28 | 4.64 | 12.32 |
| Trace statistic | 4.86 | 0.20 | 2.27 |
| | 12.42 | 4.43 | 10.04 |
| | | | |
| **Proposition 5**: Common Cycle | | | |
| Sq. Canonical Correlations | 0.30 | 0.44 | 0.19 |
| | 0.09 | 0.08 | 0.07 |
| $\chi^2$−statistic $p$−value | (0.01) | (0.00) | (0.00) |
| | (0.69) | (0.21) | (0.12) |
| $F$−statistic $p$−value | (0.00) | (0.00) | (0.00) |
| | (0.61) | (0.19) | (0.11) |

The level of fundamentals equals cross-country money netted with cross-country output calibrated to a unitary income elasticity of money demand. The money stocks (outputs) are measured in current (constant) local currency units and per capita terms. A constant and linear time trend are included in the level VARs. The LR statistics employ the Sims (1980) correction and have standard asymptotic distribution according to results in Sims, Stock, and Watson (1990). The case 2∗ and case 1 model definitions are based on Osterwald-Lenum (1992). MacKinnon, Haug, and Michelis (1999) provide five percent critical values of 8.19 (8.19) and 18.11 (15.02) for the case 2∗ model $\lambda$−max (trace) tests and 3.84 (3.84) and 15.49 (14.26) for the case 1 model. The common feature tests compute the canonical correlations of $\Delta e_t$ and $\Delta m_t - \Delta y_t$. The common feature null is all or a subset of the canonical correlations are zero. See Engle and Issler (1995) and Vahid and Engle (1993) for details.

# Table 4: UC Model Posterior Means, $\omega = 1$

| Parameter | Priors | $UC_{2,2,\kappa=1}$ | $UC_{2,\widetilde{m},\kappa=1}$ | $UC_{2,\tilde{c},\kappa=1}$ |
|---|---|---|---|---|
| $\alpha_1$ | Normal [−1.2, 0.10] | −1.1906 (0.0613) | −0.8691 (0.0470) | − |
| $\alpha_2$ | Normal [0.40, 0.17] | 0.4133 (0.1092) | 0.9501 (0.0528) | − |
| $\theta_1$ | Normal [0.85, 0.10] | 0.9407 (0.0491) | − | 0.9830 (0.0296) |
| $\theta_2$ | Normal [0.10, 0.15] | 0.0403 (0.0488) | − | 0.0069 (0.0291) |
| $\mu^*$ | Normal [−0.126, 0.015] | −0.1260 (0.0150) | −0.1258 (0.0150) | −0.1258 (0.0120) |
| $a^*$ | Normal [0.158, 0.025] | 0.1615 (0.0229) | 0.1645 (0.0213) | 0.1571 (0.0199) |
| $\sigma_\mu$ | Inv-Gamma [2.0, 1.5] | 1.8838 (0.1436) | 2.4629 (0.1507) | 1.6784 (0.1264) |
| $\sigma_a$ | Inv-Gamma [2.0, 0.4] | 1.0471 (0.2206) | 0.3971 (0.0345) | 1.9461 (0.3831) |
| $\sigma_{\widetilde{m}}$ | Inv-Gamma [2.0, 0.6] | 0.6002 (0.0899) | 0.4728 (0.1168) | − |
| $\sigma_{\tilde{c}}$ | Inv-Gamma [2.0, 0.7] | 1.2874 (0.2354) | − | 2.0135 (0.3823) |
| $\varrho_{a,\tilde{c}}$ | Normal [−0.5, 0.2] | −0.8758 (0.0462) | − | −0.9475 (0.0234) |
| $\pi_{e,0}$ | Normal [100.0, 15.0] | 80.1528 (6.8283) | 138.8984 (5.7281) | 62.9442 (2.9991) |
| $\pi_{e,t}$ | Normal [1.0, 0.5] | 0.7038 (0.1710) | 1.9366 (0.1106) | 0.3831 (0.1306) |
| $\pi_{e,a}$ | Uniform [−10.0, 0.0] | −2.6822 (0.6316) | −9.0223 (0.5377) | −0.7208 (0.1886) |
| $\pi_{c,\widetilde{m}}$ | Uniform [−2.0, 7.5] | − | 4.3973 (1.1057) | − |
| $\pi_{m,\tilde{c}}$ | Uniform [−7.5, 2.0] | − | − | −0.8985 (0.2099) |
| $\ln \widehat{\mathcal{L}}$ | | −53.95 | −226.76 | −24.76 |

## Table 5: UC Model Posterior Means, $\omega \in [0.9, 0.999]$

| Parameter | Priors | $UC_{2,2,\kappa}$ | $UC_{2,\widetilde{m},\kappa}$ | $UC_{2,\widetilde{c},\kappa}$ |
|---|---|---|---|---|
| $\kappa$ | Inv-Gamma [0.988, 0.038] | 0.9658 (0.0219) | 0.9738 (0.0196) | 0.9962 (0.0046) |
| $\alpha_1$ | Normal [−1.20, 0.10] | −1.1892 (0.0673) | −0.8828 (0.0369) | − |
| $\alpha_2$ | Normal [0.40, 0.17] | 0.4131 (0.1179) | 0.8465 (0.0223) | − |
| $\theta_1$ | Normal [0.85, 0.10] | 0.9396 (0.0431) | − | 0.9799 (0.0315) |
| $\theta_2$ | Normal [0.10, 0.15] | 0.0421 (0.0422) | − | 0.0018 (0.0314) |
| $\mu^*$ | Normal [−0.126, 0.015] | −0.1256 (0.0148) | −0.1260 (0.0149) | −0.1260 (0.0147) |
| $a^*$ | Normal [0.158, 0.025] | 0.1621 (0.0227) | 0.1630 (0.0205) | 0.1578 (0.0243) |
| $\sigma_\mu$ | Inv-Gamma [2.0, 1.5] | 1.8914 (0.1484) | 2.4241 (0.1663) | 1.7188 (0.1112) |
| $\sigma_a$ | Inv-Gamma [2.0, 0.4] | 1.0842 (0.2477) | 0.4043 (0.0378) | 1.5738 (0.2727) |
| $\sigma_{\widetilde{m}}$ | Inv-Gamma [2.0, 0.6] | 0.6068 (0.0865) | 0.5752 (0.0975) | − |
| $\sigma_{\widetilde{c}}$ | Inv-Gamma [2.0, 0.7] | 1.3828 (0.2346) | − | 1.6742 (0.2772) |
| $\varrho_{a,\widetilde{c}}$ | Normal [−0.5, 0.2] | −0.8990 (0.0409) | − | −0.9256 (0.0291) |
| $\pi_{e,0}$ | Normal [100.0, 15.0] | 85.2271 (6.7620) | 135.7241 (6.3363) | 67.8714 (3.7506) |
| $\pi_{e,t}$ | Normal [1.0, 0.5] | 0.7955 (0.1774) | 1.9076 (0.1211) | 0.4526 (0.1112) |
| $\pi_{e,a}$ | Uniform [−10.0, 0.0] | −3.2825 (0.6426) | −8.8023 (0.5900) | −1.2605 (0.3087) |
| $\pi_{c,\widetilde{m}}$ | Uniform [−2.0, 7.5] | − | 4.4186 (0.7952) | − |
| $\pi_{m,\widetilde{c}}$ | Uniform [−7.5, 2.0] | − | − | −1.0380 (0.2179) |
| $\ln\widehat{\mathcal{L}}$ | | −53.94 | −253.03 | −29.88 |

## Table 6: UC Model Posterior Means, $\omega \in [0.9, 0.999]$, Factor Loadings on Money and Consumption Cycles

| Parameter | $UC_{2,2,\kappa}$ | $UC_{2,\widetilde{m},\kappa}$ | $UC_{2,\widetilde{c},\kappa}$ |
|---|---|---|---|
| $(1 - \pi_{c,\widetilde{m}})\delta_{\widetilde{m},0}$ | 0.0086 (0.0096) | $-0.0841$ (0.0676) | – |
| $(1 - \pi_{c,\widetilde{m}})\delta_{\widetilde{m},1}$ | $-0.0269$ (0.0188) | 0.0064 (0.0082) | – |
| $(1 - \pi_{c,\widetilde{m}})\delta_{\widetilde{m},2}$ | 0.0143 (0.0108) | $-0.0762$ (0.0624) | – |
| $(1 - \pi_{c,\widetilde{m}})\Sigma_i\delta_{\widetilde{m},i}$ | $-0.0040$ (0.0202) | $-0.1542$ (0.1237) | – |
| $(1 - \pi_{m,\widetilde{c}})\delta_{\widetilde{c},0}$ | $-0.6044$ (0.2042) | – | $-0.3252$ (0.2052) |
| $(1 - \pi_{m,\widetilde{c}})\delta_{\widetilde{c},1}$ | $-0.0238$ (0.0261) | – | $-0.0004$ (0.0111) |
| $(1 - \pi_{m,\widetilde{c}})\Sigma_i\delta_{\widetilde{c},i}$ | $-0.6283$ (0.2117) | – | $-0.3256$ (0.2053) |

[†]The factor loadings $\pi_{c,\widetilde{m}}$ and $\pi_{m,\widetilde{c}}$ are zero for the 2-trend, 2-cycle UC model.

## Table 7: UC Model Posterior Means, $\omega \in [0.9, 0.999]$, Variance(PDV–$\varepsilon$) / Variance($\Delta e$)

| Parameter | $UC_{2,2,\kappa}$ | $UC_{2,\widetilde{m},\kappa}$ | $UC_{2,\widetilde{c},\kappa}$ |
|---|---|---|---|
| $Var(PDV - \varepsilon_\mu)/Var(\Delta e)$ | 0.92 (0.15) | 1.48 (0.22) | 0.71 (0.09) |
| $Var(PDV - \varepsilon_a)/Var(\Delta e)$ | 3.04 (0.73) | 0.36 (0.06) | 0.96 (0.53) |
| $Var(PDV - \varepsilon_{\widetilde{m}})/Var(\Delta e)$ | 0.00 (0.00) | 0.00 (0.00) | – |
| $Var(PDV - \varepsilon_{\widetilde{c}})/Var(\Delta e)$ | 0.22 (0.16) | – | 0.12 (0.22) |

# Table 8:  UC-Models, $\kappa \in [0.9, 0.999]$,
# Exchange Rate FEVDs[†]

### $UC_{2,2,\kappa}$ Model

| Forecast Horizon | $\varepsilon_\mu$ | $\varepsilon_A$ | $\varepsilon_{\tilde{c}}$ |
|:---:|:---:|:---:|:---:|
| 1 | 0.23 | 0.77 | 0.00 |
| 4 | 0.23 | 0.77 | 0.00 |
| 12 | 0.21 | 0.79 | 0.00 |
| 20 | 0.19 | 0.80 | 0.01 |
| 40 | 0.15 | 0.82 | 0.03 |

### $UC_{2,\widetilde{m},\kappa}$ Model

| Forecast Horizon | $\varepsilon_\mu$ | $\varepsilon_A$ | $\varepsilon_{\tilde{c}}$ |
|:---:|:---:|:---:|:---:|
| 1 | 0.32 | 0.68 | – |
| 4 | 0.32 | 0.68 | – |
| 12 | 0.32 | 0.68 | – |
| 20 | 0.32 | 0.68 | – |
| 40 | 0.32 | 0.68 | – |

### $UC_{2,\tilde{c},\kappa}$ Model

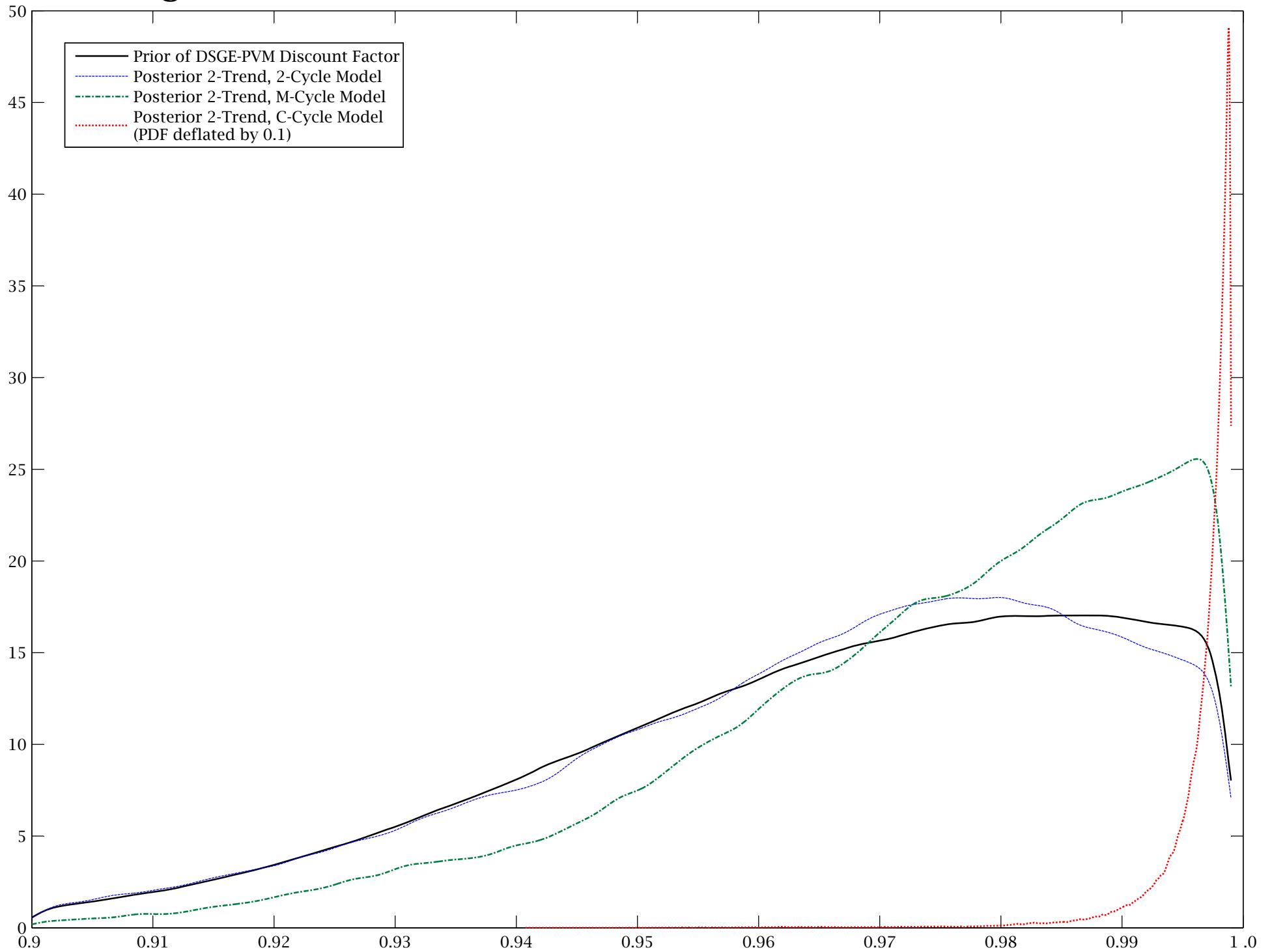| Forecast Horizon | $\varepsilon_\mu$ | $\varepsilon_A$ | $\varepsilon_{\tilde{c}}$ |
|:---:|:---:|:---:|:---:|
| 1 | 0.59 | 0.40 | 0.01 |
| 4 | 0.58 | 0.41 | 0.01 |
| 12 | 0.57 | 0.42 | 0.01 |
| 20 | 0.57 | 0.43 | 0.01 |
| 40 | 0.55 | 0.45 | 0.01 |

[†]The summary statistics are the mean of the posterior distributions of the exchange rate FEVDs with respect to permanent and transitory cross-country money and cross-country consumption shocks.

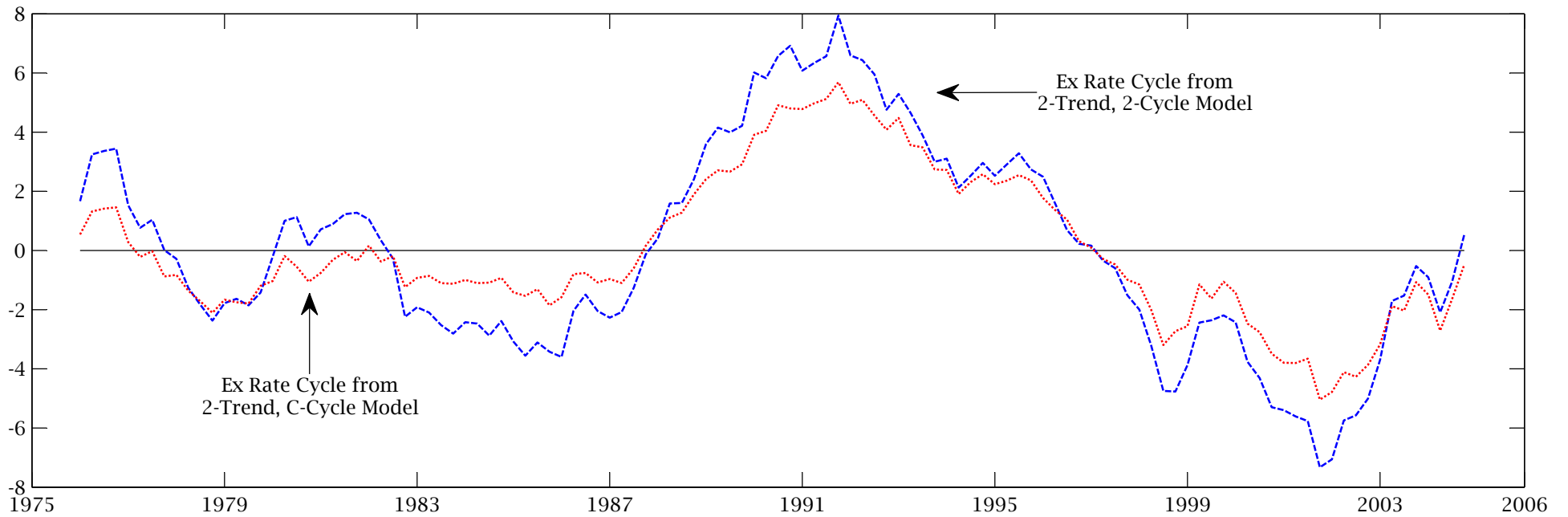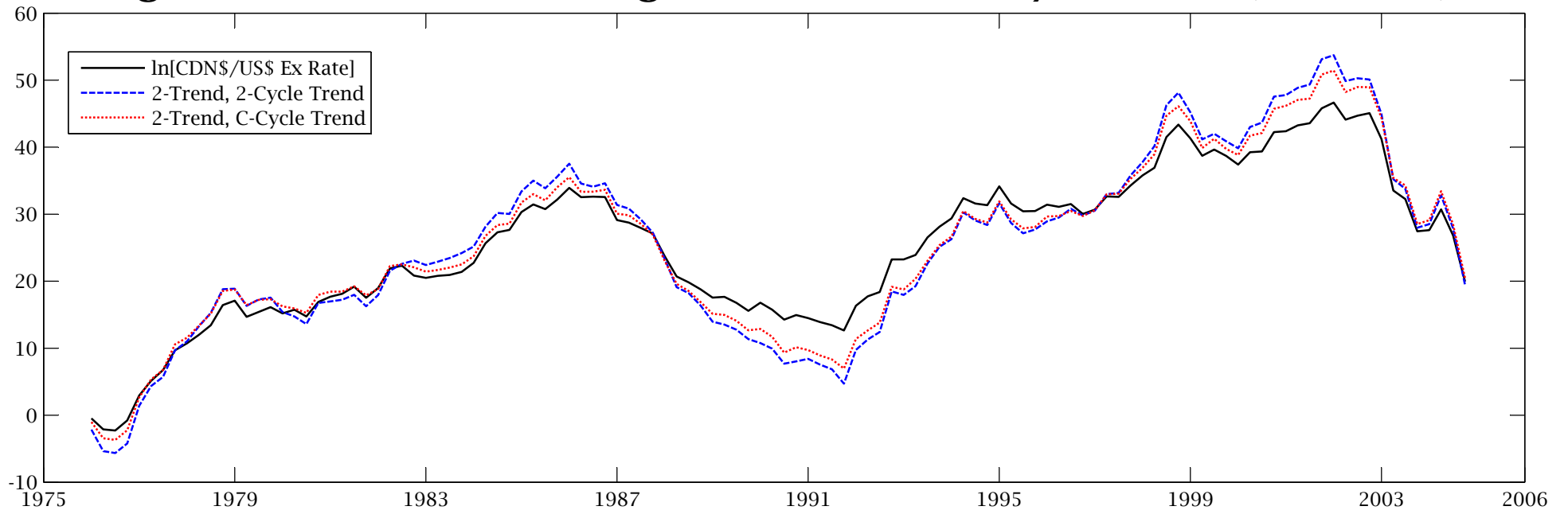## Table 9:  UC-Models, $\kappa \in [0.9, 0.999]$, Summary of the Trend-Cycle Decomposition[†]

| Parameter | $UC_{2,2,\kappa}$ | $UC_{2,\widetilde{m},\kappa}$ | $UC_{2,\widetilde{c},\kappa}$ |
|---|---|---|---|
| $STD(\Delta e^{\tau})$ | 2.66 | 104.28 | 2.44 |
| $STD(\widetilde{e})$ | 3.68 | 106.84 | 2.55 |
| $AR1(\widetilde{e})$ | 0.97 | 0.55 | 0.98 |
| $Corr(\Delta e^{\tau}, \widetilde{e})$ | −0.06 | −0.37 | −0.02 |
| | | | |
| $STD(\Delta\mu)$ | 1.62 | 4.54 | 1.71 |
| $STD(\widetilde{m})$ | 0.68 | 3.03 | – |
| $AR1(\widetilde{m})$ | −0.68 | 0.13 | – |
| $Corr(\Delta\mu, \widetilde{m})$ | 0.26 | −0.24 | – |
| | | | |
| $STD(\Delta a)$ | 0.99 | 12.18 | 1.56 |
| $STD(\widetilde{c})$ | 5.73 | – | 7.73 |
| $AR1(\widetilde{c})$ | 0.97 | – | 0.98 |
| $Corr(\Delta a, \widetilde{c})$ | −0.16 | – | −0.14 |
| | | | |
| $Corr(\Delta e^{\tau}, \Delta\mu)$ | 0.01 | −0.40 | 0.62 |
| $Corr(\Delta e^{\tau}, \Delta a)$ | −0.85 | −0.93 | −0.71 |
| $Corr(\Delta\mu, \Delta a)$ | 0.52 | 0.55 | 0.10 |
| | | | |
| $Corr(\widetilde{e}, \widetilde{m})$ | 0.04 | −0.73 | – |
| $Corr(\widetilde{e}, \widetilde{c})$ | −1.00 | – | −1.00 |

[†]The summary statistics are taken the mean of the summary statistics of the posterior distributions of the trends and cycle of exchange rate, cross-country money, and cross-country consumption. The trend growth rate of the exchange rate is denoted $\Delta e^{\tau}$.
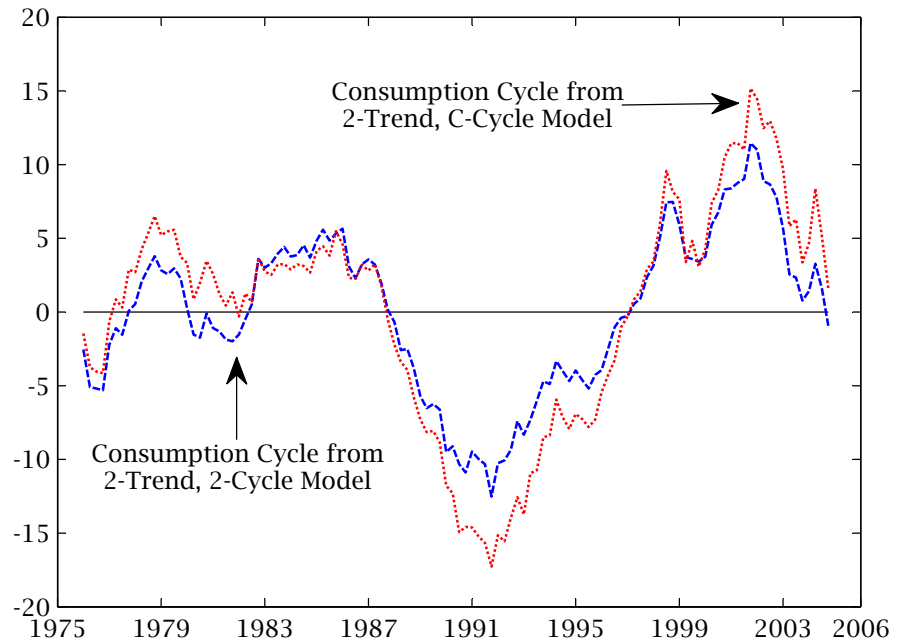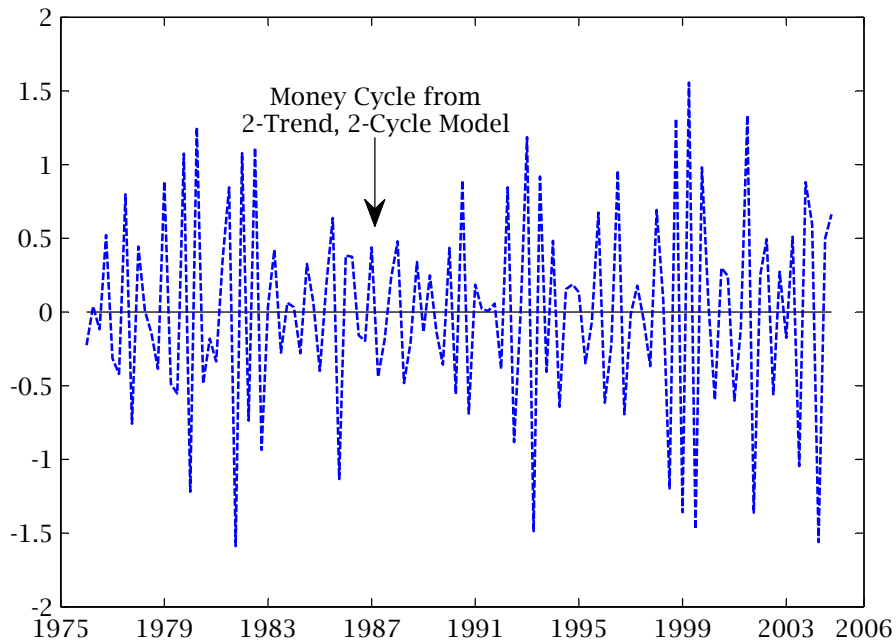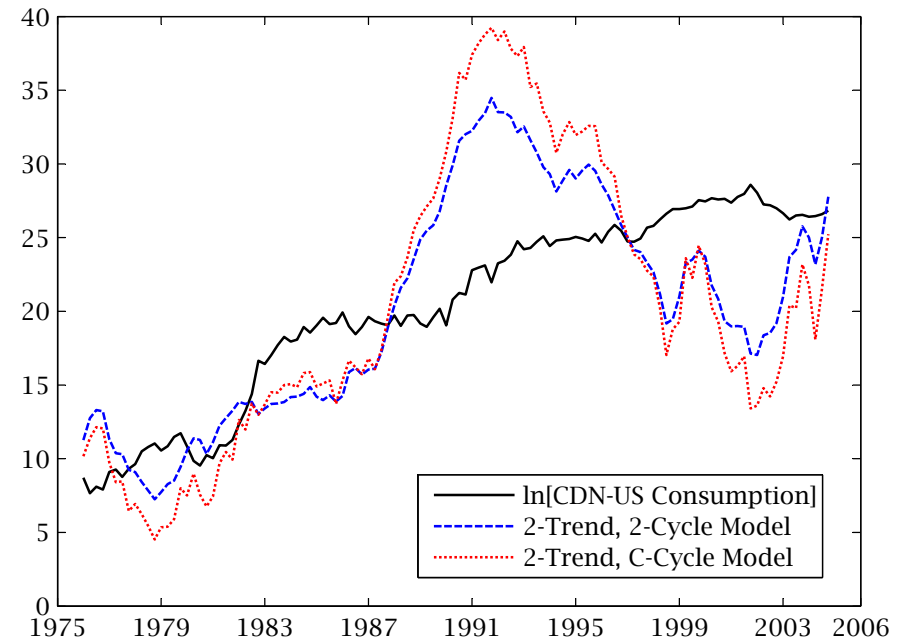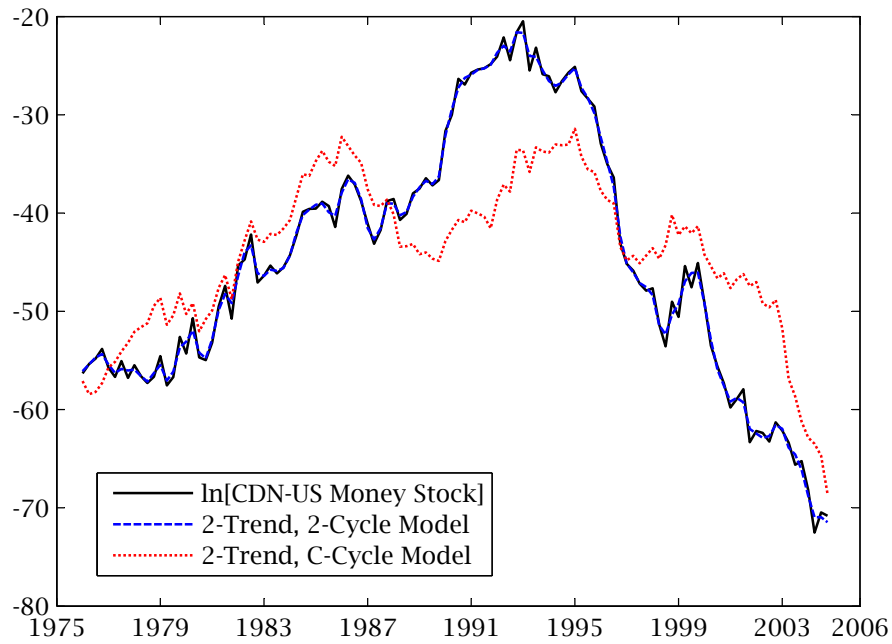
Figure 1: Prior and Posterior PDFs of DSGE-PVM Discount Factor

**Legend:**
- Prior of DSGE-PVM Discount Factor
- Posterior 2-Trend, 2-Cycle Model
- Posterior 2-Trend, M-Cycle Model
- Posterior 2-Trend, C-Cycle Model (PDF deflated by 0.1)

Figure 2: CDN$/US$ Exchange Rate Trend and Cycle, 1976Q1 - 2004Q4

Figure 3: CDN-US Money, Consumption Trends and Cycles, 1976Q1 - 2004Q4

# Figure 4: CDN$/US$ Ex Rate Cycles at Different DSGE-PVM Discount Factors



2-Trend, 2-Cycle UC Model

Discount Factor, 16th Percentile = 0.943

Discount Factor, 84th Percentile = 0.988

Discount Factor = 0.999

2-Trend, C-Cycle UC Model

Discount Factor, 16th Percentile = 0.994

Discount Factor, 84th Percentile = 0.9987

Discount Factor = 0.999