Inflation under a monetary policy game

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Abstract

This paper presents a full characterization of the equilibrium value set of a Calvo (1978) monetary environment. We will consider this environment as a monetary policy game, and study its properties with a particular interest in the way inflation occurs within the model. The description of this paper is largely based on Chang (1998), with two new contributions. The first contribution is to introduce the public randomization device and solve the problem using Judd et al. (2000), which is a more recently developed procedure. Secondly, inflation is studied under a monetary policy game environment.

Keywords: Computation, optimum quantity of money, policy game, recursive method.

1 Introduction

This paper presents a full characterization of the equilibrium value set of a Calvo (1978) monetary environment. We will consider this environment as a monetary policy game, and study its properties with a particular interest in the way inflation occurs within the model.

What is a monetary policy game? In Calvo's case, a government decides the growth rate of money in each period, and then households move competitively and decide their action.

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It is generally known that there is a time inconsistency problem in Calvo's case, which is due to: (a) The nature of the demand for money, and (b) The fact that this inconsistency implies in general the use of distortionary taxation (Calvo (1978)). The intuitive understanding is as follows: In Calvo's environment, the government earns an inflationary tax by printing money and transferring it to households. To reduce the distortion, the government satiates the economy with real balance money and by generating a deflation that drives the net interest rate to zero. This is the so-called "Friedman rule" (Friedman (1969)). At first households forecast the government's money growth rate and act competitively based on the forecast. After that, however, the government is tempted to levy an unannounced inflation tax (raising the growth rate of money) by deviating from the Friedman rule to earn seigniorage. However, such an action by the government will change the forecast of households and this is not optimal from a long-term viewpoint. In other words, once such a deviation occurs, the reputation of the government vis-a-vis the households is damaged.

To achieve the Friedman rule in a long-run environment, the government will be punished by households if it deviates. Such a punishment takes the form of inflation in this case. We will formulate this fact under study of a monetary policy game.

The description of this paper is largely based on Chang (1998), and our contribution are twofold. First of all, we introduce a public randomization device and solve the problem using Judd et al. (2000), which is a more recently developed procedure. Secondly, we study inflation under a monetary policy game environment.

1.1 Related literature

Regarding the policy problem between the government and households, the seminal papers are Kydland & Prescott (1977) and Calvo (1978). They firstly pointed out the time inconsistency problem when the government cannot commit to date-state contingent policies. The usual optimal control techniques are inappropriate to apply to such problems directly, because this class of problems are not recursive in the natural state variables (Ljungqvist & Sargent (2004)(hereafter LS, p.615)) However, if we assume that the government can commit at the beginning of time to a policy specifying its actions for all future dates and states of nature, we can study the nature of this problem. This is called as “the Ramsey problem,” and studies into this question have progressed in two different areas: the Ramsey tax problem and the optimum quantity of money.

In terms of the Ramsey tax problem, Chamley (1986) and Judd (1985) studied a perfect foresight model with commitment and they found the optimal tax rate on capital is zero. Chari et al. (1994) also studied the stochastic version of this model by performing numerical simulations and found the ex ante capital tax rate is approximately zero.

In the optimum quantity of money literature, which stemmed from Friedman (1969), the Ramsey problem with commitment is discussed too.[1] The nature of the optimality of the Friedman rule is such that a policy satiates the economy with real balances by generating deflation that drives the net interest rate to zero. In a stationary economy, there can be deflation only if the government takes up currency from circulation with a government surplus. When all government revenues must be raised through distortionary tax, the optimality of the Friedman rule may be controversial.

Such an optimal government policy with commitment is called “the Ramsey plan,” and in past studies of this area, mainly investigated the Ramsey plan under the assumption of commitment to a future policy. If we do not assume commitment, the agent’s decision depends on all past histories and the problem becomes too difficult to solve.

Until now, we discussed the case with commitment. Recently, however, the optimal government policy *without* commitment has been studied in recent papers in both two areas as I noted, the Ramsey tax problem and the optimum quantity of money.

The basic idea of optimal policy without commitment is based on Chari & Kehoe (1990), which developed the concept of time inconsistency presented in Kydland & Prescott (1977), who argued that a policy problem is better viewed as a dynamic game between the government and a continuum of households. The strategic dynamic programming approach of Abreu et al. (1986, 1990) is a natural starting point to study these policy games.

[1] Readers might refer the Chapter 24 of LS for the related literature.

In addition, to consider the infinite action of households after any history, we introduce a much simpler accounting system, wherein we only keep track of "the marginal utility of state" (which will be explained later) for households. Marcet & Marimon (1998), basing their work on Kydland & Prescott (1980), have demonstrated that by keeping track of the marginal utility of state, we can show the Ramsey plan has a recursive structure.

We can use this "marginal utility of state" concept in a strategic setting without commitment. The crucial point is that since each "small" agent (except the government, a "large" agent) is anonymous and cannot affect the path of prices, the households' problem can be summarized into Euler equation conditions and co-state variable (=the marginal utility of state). By incorporating the marginal utility of state into the strategic dynamic programming framework, we wed the two different techniques that are usually considered competing alternatives (Phelan & Stacchetti (2001)(hereafter PS, p.1492)).

There are a few papers analysing cases without commitment. In the Ramsey tax literature, PS analyzes a dynamic game and present a full characterization (not only optimal one) of the Ramsey tax problem by also incorporating the marginal utility of state into the strategic dynamic programming framework. They then check the validity of the celebrated result of Chamley (1986) and Judd (1985) under the case with commitment. On the other hand, with regard to the optimum quantity of money, Chang (1998) studies full characterization of the sustainable equilibrium values in the context of the optimum quantity of money using a similar method as PS, and shows that the Friedman rule is sustainable even without commitment.

Up to now, we have discussed about the policy game with public information. It should be noted that we treat only the public information in this paper. In terms of private information, however, three other papers should be mentioned, since they offer a key to the actual case with punishment, meaning the penalty of inflation.

First, Green & Porter (1984) studies an oligopoly game with private information, and Abreu et al. (1986) shows the optimal strategy for games which have the "bang-bang" property. With this property, the optimal recursive equilibria in private information games necessarily jump between the extreme points of an equilibrium value set.

Second, Zarazaga (1995) studies the policy game between the (more than two) governments and households, and shows that hyperinflation will occur as punishment of a subgame perfect equilibrium in the game with private information. His simulation results capture well the moves in price in Latin American countries after the 1980s. The method used there, however, seems incomplete in the treatment of households' behavior.

Sleet (2001) also studies the set of sustainable equilibria in the policy game with private information, using a new Keynesian model. He shows that the inflation will occur along an optimal equilibrium path.

The paper proceeds as follows. Section 2 introduces Chang's(1998) paper. We first explain the dynamic general equilibrium model and define the competitive equilibrium. Then, we define a solution concept Chari & Kehoe (1990)'s Sustainable Plans within the framework of a monetary policy game. After that, we show the set of outcomes of the solution and its values.

Section 3 incorporates public randomization devices into Chang (1998)'s environment. This changes Chang (1998)'s environment in some aspects, but the fundamental characters are preserved. Then we introduce a operator $E(Z)$ inspired by APS and Kydland & Prescott (1980) and modify it with public randomization; An algorithm is also shown to compute the set of equilibria in the computer. Section 4 displays the numerical results for a parametric version of the model. Section 5 offers a conclusion. Some proofs and details in computation are left in the Appendix.

2 Chang's (1998) paper

In this section, we will explain Chang's original paper. His paper is an expansion of Calvo (1978)'s monetary model with a policy game approach, and using a solution concept which is advocated as Sustainable Plans in Chari & Kehoe (1990). Such a solution is called the sustainable outcome.

2.1 The model[2]

We will analyze a discrete time version of a dynamic general equilibrium model first proposed by Calvo (1978). Time is discrete and indexed by $t = 0, 1, 2, ...$ In each period, there is only one consumption good and currency is the only asset. The economy is populated by a continuum of identical households on $[0, 1]$ and a benevolent government. The representative household lives forever and has preferences over consumption and real money holding given by:

$$w = \sum_{t=0}^{\infty} \beta^t [u(c_t) + v(m_t)] \tag{1}$$

The functions u and v satisfy:

(A1) $u : \mathbf{R}_+ \to \mathbf{R}$ is C^2, strictly concave and strictly increasing.

(A2) $v : \mathbf{R}_+ \to \mathbf{R}$ is C^2, and strictly concave.

(A3) $\lim_{c \to 0} u'(c) = \lim_{m \to 0} v'(m) = \infty$.

(A4) There is a finite $m = m^f > 0$ such that $v'(m^f) = 0$.

The household will maximize (1) subject to $c_t, m_t = \frac{M_t}{P_t} \geq 0$ and

$$c_t + m_t + x_t \leq y_t + m_{t-1} \frac{P_{t-1}}{P_t} \tag{2}$$

$$m_t \leq \bar{m} \tag{3}$$

for all $t \geq 0$, where y_t denote a period t endowment of consumption good, x_t is a lump sum tax(or transfer, if negative), and $\bar{m} \geq m^f$ is an exogenously given constraint. The household takes the sequences $\{P_t\}, \{x_t\}, \{y_t\}$ and its initial currency holding M_{-1} as given. The constraint (3) is needed to make the set of real money balance bounded. It could be arbitrarily large.

The government chooses how much money to create or to withdraw from circulation. Given M^s_{-1}, the path of the money supply is completely described. We denote the growth rate of money by θ_t, and we can express money growth as

[2] We use the word "model" just for the description of a dynamic general equilibrium model, whereas "policy game" as a broader concept which incorporates a strategic behavior of agents after any histories, as in Chari & Kehoe (1990).

$M_t^s = (1+\theta_t)M_{t-1}^s, \forall t \geq 0$. We can also express it in terms of the inverse growth rate, $h_t = \frac{M_{t-1}^s}{M_t^s}$, and assume that

(A5) For some $\underline{\pi}, \bar{\pi}$ such that $0 < \underline{\pi} < 1 < \frac{1}{\beta} \leq \bar{\pi}, h_t \in [\underline{\pi}, \bar{\pi}] \equiv \Pi$.

(A5) bounds admissible rate of money creation and, like (3), is needed for technical reasons.

The government's budget constraint is:

$$M_t^s - M_{t-1}^s = -P_t x_t \tag{4}$$

Using market clearing condition, $\frac{M_t^s}{P_t} = m_t$, and $h_t = \frac{M_{t-1}^s}{M_t^s}$, this can be written as:

$$m_t(1-h_t) = -x_t \tag{5}$$

$$\text{or,} \quad \frac{\theta_t}{1+\theta_t}\frac{M_t^s}{P_t} = -x_t$$

So the negative value of x_t is the seigniorage of government and it transfers all to the household.

Since $m_t \in [0, \bar{m}]$ and $h_t \in \Pi$, x_t must belong to the interval $[(\underline{\pi}-1)\bar{m}, (\bar{\pi}-1)\bar{m}] \equiv X$. (5) emphasizes that h_t, the inverse of the money growth rate in period t, can be thought as the gross rate of the inflation tax on holding money.

Finally, taxes or transfers are assumed to be distortionary. The simplest way to introduce this, is to assume the household's endowment y_t is a function $f(x_t)$ of the tax collected in period t. Now, $f : \mathbf{R} \to \mathbf{R}$ is at least C^2 and is assumed to satisfy:

(A6) $f(0) > 0, f'(0) = 0, f''(x) < 0$.

(A7) f is symmetric about zero: $f(x) = f(-x)$, all $x \in \mathbf{R}$.

(A8) f is strictly positive on X.

$u'[f(x_t)]$ is a uniformly bounded sequence.

In this model, it is clearly desirable to bring the quantity of money to the satiation level m^f. However, this can only be achieved by steadily reducing the supply of money because of the distortionary tax, which is assumed in (A6) and (A7). They have negative effects on output. Hence it is clear that an optimal policy will imply some positive deflation. In this point, the optimal policy is similar to the Friedman rule, but it is not clear that the deflation rate may or may not be the rate of time preference.

Then we can define competitive equilibria as follows:

DEFINITION. Given a policy $\{h_t\}_{t=0}^{\infty}$, *a competitive equilibrium* is characterized by the allocation $\{c_t, m_t, x_t, y_t\}_{t=0}^{\infty}$, the price level $\{P_t\}_{t=0}^{\infty}$, and given M_{-1} such that

(i) The pair $\{c_t, m_t\}_{t=0}^{\infty}$ solves the household's problem, Max(1) s.t.(2), given $\{P_t\}_{t=0}^{\infty}, \{x_t, y_t\}_{t=0}^{\infty}$

(ii) The allocation satisfies the aggregate feasibility constraint: $c_t \leq y_t = f(x_t)$.

(iii) The government budget constraint is satisfied and $\frac{M_{t-1}}{M_t} = h_t$.

Let $E = [0, \bar{m}] \times \Pi \times X$, and $E^{\infty} = E \times E \times ....$ Under the assumptions (A1)-(A8), we can prove that:

Proposition 1. A competitive equilibrium is completely characterized by *an outcome path*[3] $\{m_t, x_t, h_t\}_{t=0}^{\infty}$ such that, for all t, $m_t \in [0, \bar{m}], h_t \in \Pi, x_t \in X$, and:

$$m_t(1 - h_t) = -x_t \tag{6}$$

$$m_t\{u'(c_t) - v'(m_t)\} \leq \beta[u'(c_{t+1})(m_{t+1} + x_{t+1})] \quad \text{with equality if } m < \bar{m} \tag{7}$$

Proof. See appendix.

Proposition 1 says that *an outcome path* $\{m_t, x_t, h_t\}_{t=0}^{\infty}$ is consistent with a competitive equilibrium if it belongs to E^{∞} and if it satisfies the government budget constraint and the household's Euler condition in all periods.[4] Hence the conditions

[3] We use the word "path" for the deterministic case, whereas "process" for the stochastic case later.

[4] Notice that a competitive equilibrium path $\{m_t, x_t, h_t\}_{t=0}^{\infty}$ does not depend on M_{-1}. Also see appendix.

connects at most two periods, it has a recursive structure. We heavily use this idea in our approach later.

The argument about household's transversality condition is given in the Appendix. Here, (A8) assures that consumption is positive. Without it, given (A3), the marginal utility of consumption would be unbounded, invalidating the argument for ignoring the transversality condition.

An element in E^∞ satisfying (6)-(7) will be called *a competitive equilibrium path* and the set of all such sequences is denoted by CE.

2.2 Recursive structure of the Ramsey problem

From now on we consider the benevolent government. However, we assume that the government can commit the entire path of money growth rate, $\{h_t\}_{t=0}^\infty$ once and for all in time 0 and never revises them.

Let denote the value and the marginal value (utility) of state[5] formed by an outcome path

$$
\begin{gathered}
w = \sum_{t=0}^\infty \beta^t [u(c_t) + v(m_t)] \\
\theta = u'[f(x_0)](m_0 + x_0) \\
\text{for some } \{c_t, m_t, h_t\}_{t=0}^\infty \in E^\infty
\end{gathered}
$$

The government's problem under commitment is to choose the policy $\{h_t\}_{t=0}^\infty$ and associated competitive equilibrium which gives the highest consumer's welfare under the constraints.

Also, given Proposition 1, we know that the problem must solve: Max (1) subject to (6)-(7) and $c_t = f(x_t)$, where the maximization is over the sequences in E^∞. This is called as the Ramsey problem, and there are some variants of the way to solve it. Here we explain briefly the approach with the marginal value of money.

The key to the procedure is to use a recursive description of competitive equilibria. From the perspective of any period t, a competitive equilibrium can be seen as the collection of a current policy and allocation, together with a "promise" of poli-

[5] In this model, the natural state variable is the real money balance, so hereafter we call it the marginal value of *money*.

cies and allocations from period $(t+1)$ on that satisfies some conditions. Roughly speaking, the RHS of Euler equation, $\theta_{t+1} = u'[f(x_{t+1})](m_{t+1} + x_{t+1})$ can be seen as the period $t+1$ marginal value of money "promised" by the equilibrium in period t. Define the set

$$\Omega = \{\theta \in \mathbf{R} : \theta = u'[f(x_0)](m_0 + x_0) \text{ for some } \{x_t, m_t, h_t\}_{t=0}^{\infty} \in CE\}.$$

Proposition 2. Ω is nonempty and compact subset of $\mathbf{R}_+$.

Proof. Since CE is not empty (there is a competitive equilibrium with a constant supply of money), Ω is not empty. In any competitive equilibria, $m_t + x_t = h_t m_t \in [0, \bar{\pi}\bar{m}]$ from (3), (5), and (A5). $u'[f(x_t)]$ is a positive and continuous function on X, its range is a bounded subset of $\mathbf{R}_+$.

To see that Ω is compact, it is enough to show that Ω is closed. Let $\{\theta^n\}$ be a sequence in Ω converging to $\theta \in [0, \bar{\theta}]$. By definition, there is a corresponding sequence in CE such that $\theta^n = u'[f(x_0^n)](m_0^n + x_0^n)$ for each n. Since CE is compact, such a sequence converges to some $\{m_t, x_t, h_t\}_{t=0}^{\infty}$ in CE. Continuity of u' and f implies that $\theta = u'[f(x_0)](m_0 + x_0)$. Hence, Ω is closed and compact. ■

Chang (1998) also shows that the Ramsey problem can be represented as the recursive problem in which θ_t is the co-state variable. This means that any "optimal" action (m_t, x_t, h_t) and the next period state θ_{t+1} are the time invariant functions of θ_t. Namely, the Ramsey problem can be transformed into the usual optimal control problem.[6]

We can also state the following corollary about the competitive equilibrium.

Corollary 1. The continuation of a competitive equilibrium is itself a competitive equilibrium. In other words, if $\{m_t, x_t, h_t\}_{t=0}^{\infty} \in CE$, then $\{m_s, x_s, h_s\}_{s=t}^{\infty} \in CE$ for all t.

The proof is immediately from Proposition 1. This is a crucial aspect of the model: It makes precise a sense in which the set of competitive equilibrium path has a recursive structure with the co-state variable θ.

[6]It is also shown in Marcet & Marimon (1998), based on Kydland & Prescott (1980).

2.3 Sustainable plans

Now, we drop the assumption that the government has the ability to commit. It is known from Calvo (1978) and Kydland & Prescott (1977), that the government faces a "credibility" or "time consistency" problem. In this section, following Chari & Kehoe (1990), we define such a concept that is required to characterize the "credible" outcomes of the model. Chari & Kehoe (1990) calls such a equilibrium concept as *sustainable plans*.

The sustainable plans consist of a strategy and an allocation rule, (σ, α) which map the observed history of the economy into the outcome path $\{m_t, x_t, h_t\}_{t=0}^{\infty}$. Such a strategy profile is defined for *any* histories, so there exists continuation competitive equilibrium path induced by a sustainable plan after any histories.[7]

A history in period t is denoted by $\{h_s\}_{s=0}^{t} = (h_0, h_1, ..., h_t), h_s \in \Pi, \forall s$ describes the actual sequence of money growth rates in every periods up to t. Notice that the public history depends on only the government policy.[8]

Now we define a strategy for the government, an allocation rule for households, and then show the definition of sustainable plans.

A strategy for the government is a sequences of functions $\sigma = \{\sigma_t\}_{t=0}^{\infty}$ such that $\sigma_0 \in \Pi$ and $\sigma_t : \Pi^{t-1} \to \Pi$. The strategy space for the government is more restricted by the following condition:

$$CE_\pi = \{\{h_t\}_{t=0}^{\infty} \in \Pi^{\infty}\text{: there is some } \{m_t, x_t\}_{t=0}^{\infty} \text{ such that } \{m_t, x_t, h_t\}_{t=0}^{\infty} \in CE\}$$

. CE_π is the set of infinite horizon money growth sequences that are consistent with competitive equilibria. σ is *admissible* if after any histories $\{h_s\}_{s=0}^{t-1}$, the continuation history $\{h_s\}_{s=t}^{\infty}$ defined by the continuation of σ in the natural way such that

$$CE_\pi^0 = \{h \in \Pi\text{: there is } \{h_t\}_{t=0}^{\infty} \in CE_\pi \text{ with } h = h_0\}$$

[7] A strategy profile here is also called a subgame perfect equilibrium(SPE) strategy profile in LS (p.792), borrowing the language from game theory. However, the object under study is not a game, because we do not specify all of the objects that formally define a game (LS, p.786)

[8] There is a "too many infinities" problem(Chang, p.432). Namely, given any history, we must solve for infinite horizon competitive equilibrium problem, and this has to be done for every one of an infinite number of histories. In N-person repeated game, the equilibrium value set V is a subset of $\mathbf{R}^N$. A game between a government and households distributed on $[0, 1]$ requires infinite dimensional equilibrium value(PS p.1500). Introducing an adequate state variable(the RHS of Euler equation), we can represent any competitive equilibrium behavior, and their behavior doesn't affect the equilibrium path because the households are infinitely distributed on $[0, 1]$ and cannot affect the path of prices.

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An allocation rules which describe the market behavior is a sequence of functions $\alpha = \{\alpha_t\}_{t=0}^{\infty}$ such that, for each t, $\alpha_t : \Pi^t \to [0, \bar{m}] \times X$. α is *competitive* if given after any $\{h_s\}_{s=0}^{t-1}$ and $h_t \in CE_\pi^0$, σ and α induce a competitive equilibrium path.

DEFINITION. A government policy and an allocation rules $\{\alpha, \sigma\}$ constitute *a Sustainable Plan* if

(i) σ is admissible

(ii) α is competitive given σ.

(iii) After any $\{h_s\}_{s=0}^{t-1}$, the continuation of σ is optimal for government, that is, $\{h_s\}_{s=t}^{\infty}$ induced by σ after $\{h_s\}_{s=0}^{t-1}$ maximizes (1) over CE_π, given α.

Especially, the condition (iii) in the definition of a sustainable plan can be represented the following incentive constraint:

$$
\begin{gathered}
w = u(f(x)) + v(m) + \beta w' \\
\geq \max_h \min_{m,x}[u(f(x)) + v(m)] + \beta \underline{w}' \\
\text{subject to } h \in CE_\pi^0
\end{gathered}
$$

where $x = m(h-1)$(the government budget constraint), and $\underline{w}'$ is the worst sustainable value. The government will take the harshest available punishment in response to its deviation. We need to consider only the best deviation of the government. The intuition for the optimality of such a harshest punishment is that the households must provide incentives for the government to follow the equilibrium recommendations which is prescribed by sustainable plans.

There is also the property of a sustainable plan which enable us to apply recursive methods.

Proposition 3. Given any history $h^{t-1} = \{h_s\}_{s=0}^{t-1}$, the continuation of a sustainable plan is itself a sustainable plan.

Proof. This is just a matter of accounting. Namely, given a history $h^{t-1} = \{h_s\}_{s=0}^{t-1}$, a sustainable plan attains the outcomes $h_t = \sigma(h^{t-1}), (m_t, x_t) =$

$\alpha(h^t)$. Given these outcomes, we can define the set $CE_\pi^t = \{h \in \Pi$: there is $\{h_t\}_{t=0}^\infty \in CE_\pi$ with $h = h_t\}$, so the continuation of $\sigma, \sigma\ |_{h^{t-1}}$ is also admissible. Also, given h^{t-1} and $h_t \in CE_\pi^t$, $\sigma\ |_{h^{t-1}}$ and the continuation of allocation rule $\alpha\ |_{h^t}$ induces the continuation outcome path $\{m_s, x_s, h_s\}_{s=t}^\infty$ which is a part of competitive equilibrium path, so $\alpha\ |_{h^t}$ is competitive given $\sigma\ |_{h^{t-1}}$. Finally, the continuation outcome path $\{m_s, x_s, h_s\}_{s=t}^\infty$ must be restricted such that it satisfies the government incentive constraint after period t, and there is at least such a outcome path which is a part of the original competitive equilibrium path. ■

Notice that any sustainable plan induces a competitive equilibrium path $\{c_t, m_t, h_t\}_{t=0}^\infty$. Such a outcome is called as *a sustainable outcome*, and forms the total discounted value w. The recursive character in Proposition 3, as Corollary 1 in the last subsection, can be represented by the value w as the state variable.

From Proposition 3 and Corollary 1, we can guess that (w, θ) characterize the set of sustainable equilibrium values in a recursive manner. To summarize previous arguments, two important aspects of the model need to be taken into account. (Chang(1998), p.445)

First, because the government has a time consistency problem, any SP must provide incentives for government not to deviate from equilibrium behavior. These incentive constraints can be handled by introducing state the continuation value w.

Second, after any history, the continuation of a SP is consistent with a competitive equilibrium for the infinite future. This constraint can be handled with the promised marginal utility of money θ. Hence one would guess that a recursive approach to the set of sustainable plans should include at least two state variables, w and θ. In next section, we will introduce the public randomization, after that define a sustainable outcome in this case and show the way to compute the set of sustainable values by using the recursive structure with these state variables.

3 The public randomization

Here we introduce the public randomization into Chang's environment.

From (A1), (A2), and Proposition 1, the value w must belong to some compact interval, say $W = [\underline{w}, \bar{w}]$. Let Z compact subset of $W \times \Omega$ whose elements are (w, θ). From Proposition 2, this set is also compact, and we assume that there is at least one SP so it is nonempty.[9]

We introduce the public randomization device to make the set $Z \subset W \times \Omega$ convex. At the beginning of period t, the outcome r_t of uniform $[0,1]$ random variable R_t is publicly observed. $\{R_t\}$ are serially uncorrelated and independent of any choices made by the government or the households. $\{R_t\}$ can be used as coordination devices to synchronize the government and the household's moves and beliefs in a similar fashion as sunspot equilibria.

3.1 Sustainable outcomes and values

By introducing the public randomization device, the public history needs to be expanded to record the outcomes of random devices: $s_t = (r_t, h_t)$. The public history consists of the government policy and the outcome of random variable. Now, the public history is modified as $s_k = (r_k, h_k), k = 0, ...$

$$s^t = \{s_k\}_{k=0}^t = (s_0, s_1, ..., s_t) = (r_0, h_0, r_1, h_1, ..., r_t, h_t)$$
$$r_k \in [0,1], h_k \in \Pi, \forall k$$

Then, a government strategy and an allocation rule become

$$h_t = \sigma_t(s^{t-1}, r_t)$$
$$(m_t, x_t) = \alpha_t(s^t)$$

Notice that both are measurable function of r_t. Sustainable plans are need to be defined by these strategy and allocation rules.

Now, the stochastic version of household's problem is:

[9] For the proof, see Chang (1998).

$$w = \max_{\{c_t, m_t\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t [u(c_t) + v(m_t)]\right]$$

$$\text{s.t.} \qquad c_t + m_t + x_t \le y_t + m_{t-1}\frac{P_{t-1}}{P_t}, \qquad y_t = f(x_t)$$

Competitive equilibria is also well defined as before, but now the household's problem is stochastic, and sequences of competitive equilibria are all random variables. *An outcome process*[10] $\{h_t, m_t, x_t\}_{t=0}^{\infty}$ is now generated by a sequence of measurable functions: $h_t, m_t, x_t : [0,1]^{t+1} \to \mathbf{R}_+$. $h_t(r^t), m_t(r^t), x_t(r^t)$ are random and depend on the sequence of random outcomes $r^t = (r_0, ..., r_t)$.

We can rewrite Proposition 1 using stochastic Euler equation. Let $\tilde{E} = [0, \bar{m}] \times \Pi \times X \times [0,1]$, and $\tilde{E}^{\infty} = \tilde{E} \times \tilde{E} \times ....$ Under the assumptions (A1)-(A8), we can prove that:

Proposition 1'. A competitive equilibrium is completely characterized by *an outcome process* $\{m_t(r^t), x_t(r^t), h_t(r^t)\}_{t=0}^{\infty}$ such that, for all t, $m_t \in [0, \bar{m}], h_t \in \Pi, x_t \in X, r_t \in [0,1], r^t = (r_0, ..., r_t)$, and:

$$m_t(r^t)(1 - h_t(r^t)) = -x_t(r^t) \tag{8}$$

$$m_t(r^t)\{u'(c_t(r^t)) - v'(m_t(r^t))\} \tag{9}$$

$$\le \beta E_{r_{t+1}}[u'(c_{t+1}(r^{t+1}))(m_{t+1}(r^{t+1}) + x_{t+1}(r^{t+1})) \mid r^t](= \beta E_{r_{t+1}}\theta^R(r_t)) \quad \text{with equality if } m < \bar{m}$$

Proof. Similar to the one of Proposition 1. However, we must consider stochastic Euler equation (9) here.

An element in $\tilde{E}^{\infty}$ satisfying (8)-(9) will be called *a competitive equilibrium process* and the set of all such sequences is denoted by $\tilde{CE}$. $\tilde{CE}_{\pi}, \tilde{CE}^{0}_{\pi}$ can be defined in a similar way as before, such that $\sigma\mid_{r_0}$(the strategy after the realization of r_0) is admissible and $\alpha\mid_{s_0}$ is competitive, so sustainable plans for the stochastic case is well defined.

We define the value and the marginal value of money, given the outcome r_0 of R_0:

[10] We use the word "path" for a deterministic sequence (also see Proposition 1), whereas "process" for a stochastic sequence.

$$w^R(r_0) = E\left[\sum_{t=0}^{\infty} \beta^t [u(c_t(r^t)) + v(m_t(r^t))] \mid r_0\right]$$
$$\theta^R(r_0) = u'[f(x_0(r_0))](m_0(r_0) + x_0(r_0))$$

The corresponding *expected* marginal value of money and value(before the realization of r_0):

$$\tilde{w} = E_{r_0} w^R(r_0) = \int_0^1 w^R(r_0) dr_0$$
$$\tilde{\theta} = E_{r_0} \theta^R(r_0) = \int_0^1 \theta^R(r_0) dr_0$$

These values are now formed by a stochastic outcome process. Now we define ex ante and ex post sustainable equilibrium value set S and S^R:

$$\begin{aligned} S \;\; = \;\; & \{(\tilde{w}, \tilde{\theta}) \mid \text{there is a SP } \{\sigma, \alpha\} \text{ whose outcome process is } \{m_t, x_t, h_t\}_{t=0}^{\infty} \in \tilde{CE} \\ & \text{with expected value } w, \text{ and expected marginal utility of money } \theta\}. \end{aligned}$$

$$\begin{aligned} S^R \;\; = \;\; & \{(w^R(r_0), \theta^R(r_0)) \mid \text{given } r_0, \text{ there is a SP } \{\sigma \mid_{r_0}, \alpha \mid_{s_0}\} \\ & \text{whose outcome process } \{m_t, x_t, h_t\}_{t=0}^{\infty} \in \tilde{CE} \\ & \text{with value } w^R(r_0), \text{ and the marginal utility of money } \theta^R(r_0)\}. \end{aligned}$$

The set S is the set of all pairs of continuation values and promised marginal values of money that emerge in the first period of a sustainable plan. Our objective here is compute S in a recursive manner using Judd et al. (2000)'s method. For the implementation, next lemma is very important.

LEMMA S is the convex hull of S^R.

Proof. The outcome r_0 doesn't affect any decisions of agents, therefore it doesn't matter which r_0 is selected to define S^R. That is, the *set* of possible payoffs after r_0 realized doesn't depend on the actual realization r_0. However, under a particular SP, $w(r_0)$, the expected value after realization r_0 depends on r_0. For instance, $r_0 \in [0, \lambda]$ and $r_0' \in (\lambda, 1]$, then $w = \lambda w(r_0) + (1-\lambda) w(r_0')$. Hence, S is the convex hull of S^R.

3.2 Computing the set of sustainable values

Here we define the operator to compute the sustainable value set and show the algorithm. Now we have incorporated the public randomization, the expected value $(\tilde{w}, \tilde{\theta})$ should be considered. Let $Z \subset W \times \Omega$ from which tomorrow's pairs $(\tilde{w}', \tilde{\theta}')$ can be chosen. Define a new set $E(Z) \subset W \times \Omega$ as follows

$$
\begin{aligned}
E(Z) = \quad & \{(\tilde{w}, \tilde{\theta}) \mid \text{there is } (m, x, h, \tilde{w}', \tilde{\theta}') \in \tilde{E} \times Z \text{ and} \\
& \tilde{w} = u(f(x)) + v(m) + \beta \tilde{w}' && (10) \\
& \tilde{\theta} = u'(f(x))(m + x) && (11) \\
& \tilde{w} \geq \max_h \min_{m,x} [u(f(x)) + v(m)] + \beta \underline{\tilde{w}}' && (12) \\
& -x = m(1 - h) && (13) \\
& m\{u'[f(x)] - v'(m)\} \leq \beta \tilde{\theta}' \text{ with equality if } m < \bar{m}\}. && (14)
\end{aligned}
$$

The constraints (10)-(11) are usually called "regeneration constraints," while (12) is the government's incentive constraint. (13) and (14) are necessary to ensure that the continuation of a sustainable plan after any deviation is consistent with a competitive equilibrium.

E satisfies the following properties. They are needed for the proof of the algorithm to compute the set, and temporarily we conjecture them.

Proposition 4.

(i) Self generation: If $Z \subset E(Z)$, then $E(Z) \subset S$.

(ii) Factorization: $S = E(S)$.

(iii) Monotonicity: $Z \subset Z'$ implies $E(Z) \subset E(Z')$.

In other words, S is the largest fixed point of the operator E.

Proposition 5.

(i) If Z is compact, then $E(Z)$ is compact.

(ii) S is compact.

An immediate implication of prop.5(ii) is that there exists each best and worst SP. It is different from Chari & Kehoe (1990) in that these SPs are derived as the result of computation.

Proposition 6. (Algorithm): Let $Z_0 = W \times \Omega$ and $Z_n = E(Z_{n-1}), n = 1, 2, \ldots$. Then $Z_\infty = S$.

Proof. Notice that Z_0 is the largest possible sustainable equilibrium value set. Monotonicity implies that the sequence $\{Z_n\}_{n=0}^{\infty}$ is decreasing. Namely, suppose $Z_1 \subset Z_0$, then $Z_1 = E(Z_0) \supset E(Z_1) = Z_2$, and so on. Also, by Proposition 5.(i), Z preserves compactness, hence each Z_n is a compact set. Now, define $Z_\infty = \cap_{n=0}^{\infty} Z_n$, i.e. Z_∞ is obtained in the limit by repetition of the operator E, starting with Z_0.

So, $Z_\infty = E(Z_\infty)$ and Z_∞ is the largest fixed point of E, then by factorization, $Z_\infty = S$.[11] ■

To implement this algorithm, we use the way of Judd et al. (2000)'s outer approximation. Next we will explain the way of implementation briefly and show the numerical results.

4 Numerical examples

We implement operator E on computer using the method of Judd et al. (2000). Details are in appendix. Our objective here is just to illustrate the implementation of the theory and not to be necessarily realistic. Our specific functional form is as follows:

[11] I basically follow the statement of PS which does not refer particular aspects of the operator.

$$u(c) = 10000 \log c, \qquad f(x) = 64 - (0.2x)^2, \qquad v(m) = m^f m - \frac{m^2}{2},$$

$$u'(c) = 10000c^{-1}, \qquad v'(m) = m^f - m,$$

$$m \in [0, \bar{m}], \quad h \in \Pi = [\underline{\pi}, \bar{\pi}] = [0.25, 1.25]$$

The parameter is varied as $\beta = \{0.9, 0.95\}$. The satiation level $\bar{m} = m^f = 30$. We use the same functional form as Chang (1998), without the satiation level.

This specification of the model met the assumptions (A1)-(A7).[12] With these values, the set of possible values of seigniorage revenue is given by $X = [-22.5, 7.5]$, from the government budget constraint, $x = m(h-1)$. Within this range, (A8) is satisfied. Using these values, we can compute the boundary of w, θ:

$$\underline{\theta} = 0$$

$$\bar{\theta} = u'[f(\underline{x})]\bar{\pi}m^f$$

$$w^{min} = (1-\beta)^{-1}(u[\min\{f(x)\}] + v(0))$$

$$w^{max} = (1-\beta)^{-1}(u[f(0)] + v(m^f))$$

Here, w is total discounted payoff, not the average one w^*. $w = \frac{w^*}{1-\beta}$.

For the reference, non monetary equilibrium is $w^{worst} = (1-\beta)^{-1}(u[f(0)] + v(0))$,the value of constant money supply equilibrium[13] is $w^C = (1-\beta)^{-1}(u[f(0)] + v(m^{ss}))$, where m^{ss} is from eular equation, and $v'(m^{ss}) = (1-\beta)u'[f(0)]$, so $m^{ss} = m^f - (1-\beta)u'[f(0)]$, and the competitive equilibrium with Friedman rule is $w^R = (1-\beta)^{-1}(u[f(x)] + v(m^f))$, where $x = m^f(\frac{1}{\beta} - 1)$. For each parameter, the corresponding values are shown in table 1.

To implement the method by Judd et al. (2000), the set S is approximated by a number of hyperplanes and the actions of the government and households are discretized, so it becomes a linear programing of w, θ. The details of computations are in appendix. We wrote MATLAB program and ran it in my PC (AMD Sempron

[12] Actually, the assumption (A3) is not met. Chang (1998) suggests that $v(m)$ to be a square root function for m very small.

[13] There is a competitive equilibrium with a constant supply of money which is supported by a sustainable plan.

2600+).[14] This task still requires some computational burdens, and the amount of computation depends on the number of hyperplanes and the grids of each player's action. The computations starts with the initial rectangle set $Z_0 = W \times \Omega$. First we compute the sustainable value set with commitment, using the operator E without the government incentive constraint. After that, we compute the set without commitment, using the operator E.

Figure 1 displays the results of the approximation of set S. Compared with the value in table 1, the upper and lower bound almost coincide the Friedman rule and worst(non monetary) equilibrium.

The value of worst equilibrium is slightly higher than non monetary equilibrium. It may be problematic because the non-monetary equilibrium itself is not a sustainable outcome with this set.[15] Notice also that the set is outer approximation: within the set, there might be not equilibrium value.

The worst value means inflation, but such a value is never realized in the optimal path. To check it, we should compute the corresponding path with values by using Judd et al. (2000)'s outer approximation method.

The set S is convex, due to the introduction of the public randomization. However, the shape of the set is not a convex combination of values in the deterministic economy in Chang (1998). In figure 1, the lower bound of w is vertical with the case of commitment. One of the possible interpretations is that the government incentive constraint is only required to truncate the set when the constraint is neglected.

We showed the same results as Chang (1998) in that the best and worst sustainable values themselves are obtained as the result of computation. Computing path remains to be done with Judd et al. (2000)'s inner ray approximation method.

[14] The MATLAB codes I used are available from `http://www.grad.e.u-tokyo.ac.jp/~takeki/mthesis/`.

[15] The footnote 17 in Chang (p.432) suggests that the non-monetary equilibrium is a sustainable outcome.

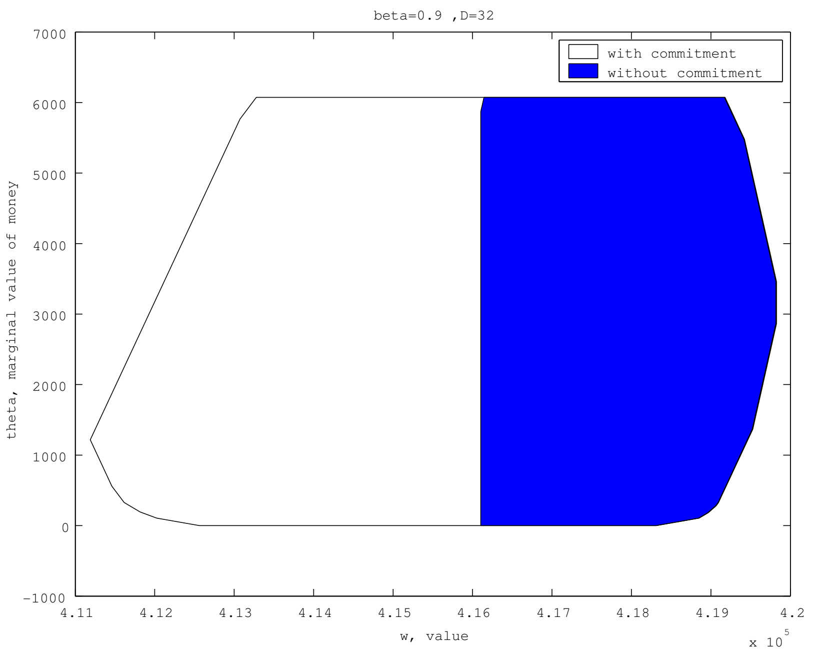
	w^{min}	w^{worst}	w^C	w^R	w^{max}
w	377849	415888	419168	419691	420388
x	-22.50	0	0	3.3333	0
h	—	0	1	1.1111	1
m	0	0	14.375	30	30

	w^{min}	w^{worst}	w^C	w^R	w^{max}
w	755698	831777	840166	840464	840777
x	-22.50	0	0	1.5789	0
h	—	0	1	1.0526	1
m	0	0	22.187	30	30

Table 1: The values and outcomes

From above, the value of $\beta = \{0.9, 0.95\}$.

Notice that $x = m(h-1)$. In non monetary equilibrium, the inflation rate is infinite, so $h = 0$. (It is out of the range of Π)



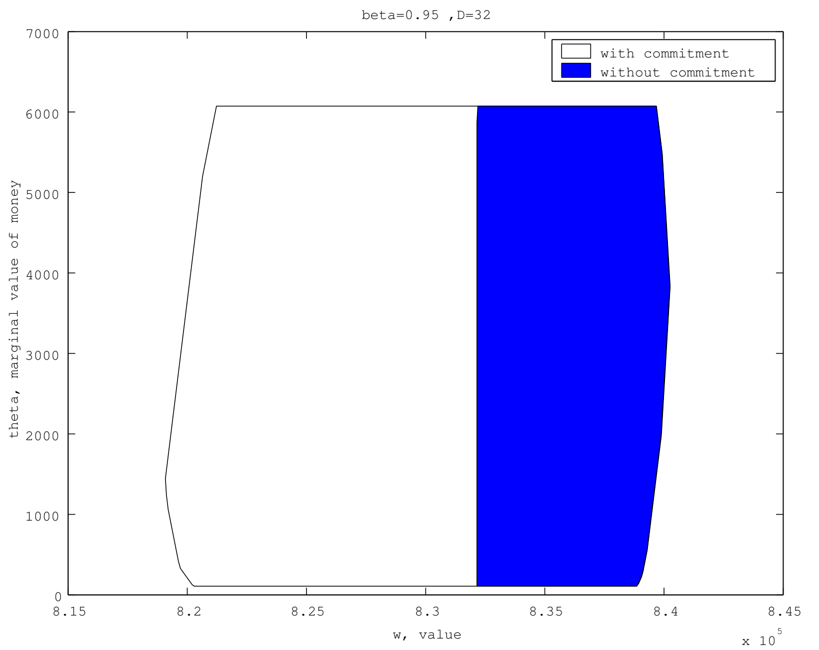


Figure 1: The result of iteration E. $D = 32, \#m = 12, \#h = 5, h \in [0.25, 1.25], m \in [0, m^f]$.
The value of $\{\underline{w}, \bar{w}\}$ (without commitment) is: $\{416105, 419825\}, \{832161, 840255\}$.

5 Concluding Remarks

This paper has studied Calvo's monetary economy based on Chang (1998) and expanded it to the stochastic economy by introducing public randomization. In the analysis, we applied Judd et al. (2000)'s outer approximation method to Chang (1998)'s environment and obtained almost the same result. This result is due to the introduction of public randomization, and it seems that the fundamental characters of the economy are preserved, although some proofs remain to be done. As pointed out in Chang (1998), by using Judd et al. (2000)'s more recently developed procedure, the computational burdens are somehow moderated.

As in Chang (1998), we can see the government can achieve the Friedman rule by threat of punishment, even in the case of no commitment. The worst value of punishment means inflation, and to derive the path corresponding such a worst value remains to be done using Judd et al. (2000)'s inner ray approximation method.

In this paper, we have treated the case with only public history, so the actual punishment is never exercised; namely, inflation never happens in this model. Such a punishment is only a threat in the case of public information, and it is actually exercised if agents have private information. (Green & Porter (1984); Zarazaga (1995); Sleet (2001)) The study of the case under the private information is also an interesting topic for future research.

Appendix

- *Proof of Proposition 1.*

From assumptions, in any competitive equilibrium, $m_t \in [0, \bar{m}], h_t \in \Pi, x_t \in X, \forall t \geq 0$. (6) follows from the government budget constraint. The representative household's problem is as follows:

$$\begin{array}{ll} & \max_{c_t, M_t} \sum_{t=0}^{\infty} \beta^t [u(c_t) + v(m_t)] \\ \text{s.t.} & c_t + m_t + x_t \leq y_t + m_{t-1} \frac{P_{t-1}}{P_t}, \quad y_t = f(x_t) \end{array}$$

FONCs are

$$
\begin{aligned}
\frac{\partial}{\partial \lambda}: &\quad f(x_t) + m_{t-1}\tfrac{P_{t-1}}{P_t} - c_t - m_t - x_t = 0 \\
\frac{\partial}{\partial c}: &\quad u'(c_t) - \lambda_t = 0 \\
\frac{\partial}{\partial m}: &\quad v'(m_t) - \lambda_t + \beta[\lambda_{t+1}\tfrac{P_t}{P_{t+1}}] = 0
\end{aligned}
$$

Eliminating λ from last two,

$$
\begin{aligned}
v'(m_t) - u'(c_t) + \beta[u'(c_{t+1})\tfrac{P_t}{P_{t+1}}] = 0 \\
v'(m_t) - u'(c_t) + \beta[u'(c_{t+1})\tfrac{m_{t+1}h_{t+1}}{m_t}] = 0 \quad (\because \frac{P_t}{P_{t+1}} = \frac{P_t}{P_{t+1}}\frac{M_{t+1}}{M_t}h_{t+1})
\end{aligned}
$$

Using (6), $m_{t+1}h_{t+1} = m_{t+1} + x_{t+1}$. Then we obtain:

$$
v'(m_t) - u'(c_t) + \beta[u'(c_{t+1})\frac{m_{t+1} + x_{t+1}}{m_t}] = 0
$$

This is the equation (7).

Conversely, suppose $\{m_t, x_t, h_t\}_{t=0}^{\infty}$ satisfy (6), (7), and $(m_t, x_t, h_t) \in E, \forall t \geq 0$. Define $\frac{M_{t-1}}{M_t} = h_t$, $\frac{M_t}{P_t} = m_t$, and the market clearing, $c_t = f(x_t)$, $\frac{M_t^s}{P_t} = m_t, \forall t \geq 0$. Then we can check that a policy $\{h_t\}_{t=0}^{\infty}$, the allocation $\{c_t, m_t, x_t, y_t\}_{t=0}^{\infty}$, and the price level $\{P_t\}_{t=0}^{\infty}$ form a competitive equilibrium.

Initial price level is $P_0 = \frac{M_0}{m_0} = \frac{1}{m_0 h_0}M_{-1}$, so it depends on the initial nominal money holdings. However, the competitive equilibrium path $\{m_t, x_t, h_t\}_{t=0}^{\infty}$ does not depend on M_{-1}, because the change of M_{-1} affects only the price level.

(6) and (7) ensures that the government budget constraint and the representative household's Euler condition are satisfied. It is then sufficient to prove that the transversality condition for the representative agent holds, that is, $\beta^t u'[f(x_t)]m_t h_t \to 0$ as $t \to \infty$, where $m_t h_t = \frac{M_{t-1}}{P_t}$ is the natural state variable in period t. Now, since E is compact, the continuity of u' and f ensures that $u'[f(x)]mh$ must belong to a compact interval for any (x, m, h). Hence $u'[f(x_t)]m_t h_t$ is a uniformly bounded sequence, and $\beta^t u'[f(x_t)]m_t h_t$ converges to zero. ■

- *Proof of Proposition 4,5*

[We temporarily conjecture them.]

- COMPUTING MAPPING E: We use the method of Judd et al. (2000) to obtain the outer approximation of the set of sustainable equilibrium.

Original operator E is as follows:

$$
\begin{aligned}
E(Z) = \quad & \{(\tilde{w}, \tilde{\theta}) \mid \text{there is } (m, x, h, \tilde{w}', \tilde{\theta}') \in \tilde{E} \times Z \text{ and} \\
& \tilde{w} = u(f(x)) + v(m) + \beta \tilde{w}' \\
& \tilde{\theta} = u'(f(x))(m + x) \\
& \tilde{w} \geq \max_h \min_{m,x} [u(f(x)) + v(m)] + \beta \underline{\tilde{w}}' \\
& -x = m(1 - h) \\
& m\{u'[f(x)] - v'(m)\} \leq \beta \tilde{\theta}' \text{ with equality if } m < \bar{m}\}
\end{aligned}
$$

We map this on the following JYC's algorithm of the outer monotone approximation(the case of $N = 2$).

1. For $l = 1, ..., D$,

(a) For each $a \in A$,

$$
\begin{aligned}
b_l(c)(a) = \quad & \max h_{l1} u_1 + h_{l2} u_2 \\
\text{s.t.} \quad & u_i = (1 - \delta)\pi_i(a_1, a_2) + \delta w_i = u_i^* + \delta w_i, \quad i = 1, 2 \\
& u_i \geq \max_{a_i} \pi_i(a_i, a_j) + \delta \underline{w}_i = \bar{u}_i^* + \delta \underline{w}_i, \quad i, j = 1, 2, i \neq j \\
& \begin{bmatrix} h_{11} & h_{12} \\ \vdots & \vdots \\ h_{D1} & h_{D2} \end{bmatrix} \begin{bmatrix} w_1 & w_2 \end{bmatrix} \leq \begin{bmatrix} c_1 \\ \vdots \\ c_D \end{bmatrix}
\end{aligned}
$$

where (h_{l1}, h_{l2}) is the gradient and c_l is the intercept for the lth direction.

This is a linear programing. Using matrices, this can be represented as

$$\min_x f'x$$

$$\text{s.t. } Ax \leq b, \qquad A_{eq}x = b_{eq}$$

$$f = -\begin{bmatrix} h_{l1} & h_{l2} & 0 & 0 \end{bmatrix}', \quad x = \begin{bmatrix} u_1 & u_2 & w_1 & w_2 \end{bmatrix}',$$

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & h_{11} & h_{12} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & h_{D1} & h_{D2} \end{bmatrix}, \qquad b = \begin{bmatrix} -(\bar{u}_1^* + \delta \underline{w}_1) \\ -(\bar{u}_2^* + \delta \underline{w}_2) \\ c_1 \\ \vdots \\ c_D \end{bmatrix}$$

$$A_{eq} = \begin{bmatrix} 1 & 0 & -\delta & 0 \\ 0 & 1 & 0 & -\delta \end{bmatrix} \qquad b_{eq} = \begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix}$$

Notice the negative sign of f so as to transform maximization into minimization. This is due to the use of MATLAB routine `linprog`.

(b) Choose best action $a \in A$, by computing $b_l(c) = \max\{b_l(c)(a) \mid a \in A\}$.

2. Then update $c = [b_1(c), ..., b_D(c)]'$ and iterate 1-2 until convergence.

In Chang's case, step 1(a) can be modified as follows.

1(a)' For each $a = \{m, h\} \in A$, (in the case of $m < \bar{m}$; the Euler equation binds.)

$$b_l(c)(a) = \max h_{l1}w + h_{l2}\theta$$

$$\text{s.t.} \quad w = (1-\beta)[u[f(x)] + v(m)] + \beta w' = w^* + \beta w'$$

$$w \geq \max_h \min_m (1-\beta)[u[f(m(h-1))] + v(m)] + \beta \underline{w}' = \bar{w}^* + \beta \underline{w}'$$

$$\theta = u'[f(x)](m + x)$$

$$\beta\theta' = m\{u'[f(x)] - v'(m)\}$$

$$x = m(h-1)$$

$$\begin{bmatrix} h_{11} & h_{12} \\ \vdots & \vdots \\ h_{D1} & h_{D2} \end{bmatrix} \begin{bmatrix} w' & \theta' \end{bmatrix} \leq \begin{bmatrix} c_1 \\ \vdots \\ c_D \end{bmatrix}$$

Using matrices,

$$
\begin{aligned}
&\min_x f'x \\
&\text{s.t. } Ax \le b, \qquad A_{eq}x = b_{eq}
\end{aligned}
$$

$$
f = -\begin{bmatrix} h_{l1} & h_{l2} & 0 & 0 \end{bmatrix}', \qquad x = \begin{bmatrix} w & \theta & w' & \theta' \end{bmatrix}',
$$

$$
A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & h_{11} & h_{12} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & h_{D1} & h_{D2} \end{bmatrix}, \qquad b = \begin{bmatrix} -(\bar{w}^* + \delta \underline{w}') \\ c_1 \\ \vdots \\ c_D \end{bmatrix}
$$

$$
A_{eq} = \begin{bmatrix} 1 & 0 & -\beta & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix} \quad b_{eq} = \begin{bmatrix} w^* \\ u'[f(m(h-1))]mh \\ m\{u'[f(m(h-1))] - v'(m)\} \end{bmatrix}
$$

If $m = m^f$, $-\beta\theta' \le -m\{u'[f(m(h-1))] - v'(m)\}$ and it doesn't necessarily bind.

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