How Beauty Contests Have Been Organized

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Abstract To the model of the rational expectations of Muth in origin, economic contexts have introduced asymmetric information and after that, noise, which means agents have to expect taking expectations of others into account. This paper makes their factor of the beauty contest clear in a simple model, applying the form of conditional expectations from Morris and Shin and perceives the implication of them. Comparing some cases, this paper examines the meaning of rationality, the law of iterated expectations, homo/heterogeneity of information and expectation. For example, we divide the homogeneity and the law of iterated expectations from the rational expectations based on the correct information and also divide the rationality of a single agent and that of agents on average. Thus we can see more clearly the relationship between these conditions of information and expectations and prices; the deviation from the 'fundamentals.' In the sense that this deviation does not always converge on zero, we can say that the beauty contest has its own meanings. Similarly, higher order expectations also have their meanings. In addition, this paper applies the beauty contest through the tatonnement process to the general equilibrium model along with the liquidity premium curve and finds equilibrium, where opinions are so diverged that the liquidity premium curve exists, and then it examines the stability of the price; in the equilibrium with heterogeneous opinions, the price is shown to be stable, while that with a homogeneous opinion is not stable.

1. Introduction

Contests over beauty have a long history since Eris tossed the Apple of Discord, which said "For the most beautiful one" and caused the Trojan War. When we want to answer this kind of problem—who, or which, the most beautiful is, it is natural to ask how the beauty is to be decided, that is one of traditional problems of philosophy and we can find some examples of answers in literature. Contexts of economics also imply some answers on this point. Here we can use a model of a financial market as a metaphor for the contest. It is a valuable aspect to obtain that implication when studying models of financial markets relating them to beauty contests. Needless to say, to understand financial markets itself can be a strong motivation for studying models of them, as we can read everyday in economic literature.

In models of financial markets, we substitute the expected profit from an asset for the beauty. This enables us, contrary to the above, to use beauty contests as a metaphor for financial markets. This rhetoric has its origin in the famous insight of Keynes.

"Or, to change the metaphor slightly, professional investment may be likened to those newspaper competitions in which competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole; so that each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, all of whom are looking at the problem from the same point of view. It is not a case of choosing those which, to the best of one's judgment, are really the prettiest, nor even those which average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practice the fourth, fifth, and higher degrees."—John Maynard Keynes(1936), p.156

It is not clear whether this view is appropriate or not. That at least depends on what financial market is described—what kind of asset, what kind of market; general or partial. Those lead to the next problem—how the profits are decided, what agents know and how expectations are constructed. Keynes himself seems to describe about the general case without enough details, as is usual in early literature.

In decades there had not been so much studies on the modeling of this beauty contest with details in decades. One of the notable reasons is that economic contexts had focused on rational expectations from 70s and after and got a lot of fruit that bore. Rational expectations first suggested in a model of partial equilibrium of an commodity market in Muth(1961) as a conclusion of discussion about adaptive expectations and state that if agents are rational and informed enough, their expectations correspond to the mathematical expectation. The most notable feature is that it is a single common value among all agents; they become homogeneous on expectation. This is what the thesis of Muth(1961) says—"The profit opportunities would no longer exist if the aggregate expectation of the firms is the same as the prediction of the theory", that means expectations are in equilibrium when they become an identical value. This feature makes arguments that contain expectations by agents quite simple and tractable. As to the newspaper competition, it is exogenously given which faces are prettiest and thus competitors become homogeneous. They may 'anticipate what average opinion expects the average opinion to be' and 'practice higher degrees' in vain. There will be no more beauty contests because all the people know who the most beautiful one is, which even gods in the ancient myth could not judge.

On the other hand, this favorable feature itself has a difficult matter and an unfavorable aspect, this logic supposing the following. i) There is correct information about the distribution of the variable in question. ii) If there is any incorrect information agents can distinguish it from true one. iii) The effect from costs of information, if any, can be negligible. The problem is not so much that these suppositions are unrealistic as that this ignores divergence of expectations and how this divergence affects the equilibrium and does not[1]. There are many papers which doubt about the rational expectations. From this standpoint, the statement of rational expectations must be rewritten in the subjunctive mood. And we need something to write ahead.

Asymmetric information over rational expectations is significant invention in this context. In Grossman and Stiglitz(1980) there are two types of agents; one knows the correct information at some cost and another has information which stochastically contains errors. Supposed that the utility function is common, relative utility (the ratio of the utility of the informed to that of the uninformed) is a function of λ, the ratio of the population of the informed to that of the uninformed, i.e. their expected revenue whom their utility function depends on depends on the distribution of heterogeneous expectations. Agents decide whether to become informed or not, according to the relative utility. It is shown that the share of either type is not necessarily zero when the value of relative utility is one, i.e. in equilibrium, where $\lambda = \lambda^e$. While their paper provides a kind of equilibrium, we can find a single Nash equilibrium where all agents take a mixed strategy to be informed at the probability of λ^e. Thus asymmetric information gives models under rational expectations an aspect of the beauty contest. Grossman(1981) says "Agents are faced with the problem of forecasting future states of nature and more importantly of forecasting the impact of these states on the actions of other agents." This development can be a reply to the criticism mentioned above. However, economic contexts take a turn that can give other

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[1] There is other criticism on rational expectations. See Yoshikawa(2000) p.14 - p.23 e.g.

more definite replies when the concept of 'noise' is introduced.

In the world with noise, there is incorrect information in addition to the correct information that corresponds to rational expectations. Black(1986) often calls the latter information 'noise' and the follower 'information'. Some agents "trade on noise as if it were information." This departure from the homogeneity of information under rational expectations has critical meanings because the other rational agents have to take into account that there are noise traders, immediately they have to expect the distribution of expectations among agents. Black(1986) contains a fragment of discussions on that. "Taking the other side's information into account, is it still worth trading? From the point of view of someone who knows what both the traders know, one side or the other must be making a mistake."

This argument is expressed more definitely in De Long, Shleifer, Summers and Waldmann(1987), which says, "In the presence of noise traders the optimal behavior of sophisticated investors would involve paying attention to pseudo-signals and acting to exploit noise traders' irrational misperceptions. Sophisticated traders would then optimally exploit noise traders, buying when noise traders depress prices and selling when noise traders push prices up." When a sophisticated trader wants to trade, he knows that every sophisticated trader wants to trade. Thus they have to take not only noise traders but also sophisticated traders themselves into account. This structure enables them to practice higher order reasoning just as in the beauty contest. The model in their paper implies, and actually shows halfway, the consequence of this reasoning; whether the expectations generally can converge or not. More interestingly, it can be shown that expectations, or those of higher degree, vanish away from the pricing on the limit when traders consider to the limit, whether the expectations are so-called 'rational' or not, though there remains some room where that is not the case. The latter of this paper sees the details of these.

No model above calls itself that of the beauty contest, however. It would be natural that there is a model of the beauty contest with heterogeneous information; the rational expectations and some noise.[2] Here, having this property, Allen, Morris, and Shin(2003) ought to be put on the frontier of this context. In their model under higher order expectations in the beauty contest, the price converges on the value that corresponds to the rational expectations but that is not necessarily what the model basically requires. In other words, prices under the beauty contest can converge but it

[2] There is Seabright(1989), whose model is equivalent to Bray(1981), which contains 'diverse private information' as a general case of Grossman(1976), one of the papers dealing with the asymmetric information.

depends on agents' belief where to converge, unless they do not converge. This is notable because the possibility of convergence is not clear in the above argument by Keynes. However, their paper shows that incompletely because it sticks to the rational expectations in the sense that agents know the distribution and use the information correctly. This paper examines the model of the beauty contest when such information is not available. Although agents in this model might be thought less rational, that is an important factor in the aim of this paper; how the equilibrium depends on the rationality, and on what rationality, when the rational expectation brings equilibrium. What is more important, taking off from the rational expectations opens out the prospect of equilibrium where expectations that agent's action depends on are diversified. The equilibrium of this type provides a foundation of the liquidity preference curve, which used to be a more popular tool for understanding financial markets and is still the counterpart of neo-classical understanding.[3]

Tobin(1958) introduces a criticism on the liquidity preference curve by Leontief; "Divergence between the current and expected interest rate is bound to vanish as investors learn from experience", that means the curve must be a horizontal line on the level where no capital gain is expected i.e. there exists no such curve called the liquidity preference curve. This argument is very similar to that of the rational expectations; if he said "expect rationally" instead of "learn from experience", that would be the same to the thesis of Muth(1961). This paper actually tries to examine this point. If there exists the liquidity preference curve with equilibrium price that does not diverge, there must be equilibrium where expectations are distributed. Agents know their expectations to be distributed and make expectations of their own, which end in a distribution. That cannot but be the beauty contest.

2. Models and Arguments

We can see various models in the literature above that have essentially the same structure. It is a model of simple portfolio between two assets, one of which is riskless and the other is risky. The price of riskless asset is fixed to one and pays no dividend, or it is, if any, certain.[4] Another price is decided in market so it is variable. There is some expected capital gain or loss which brings about the risk. The dividend of risky asset might be fixed but it is stochastic in some models[5]. That is not essential here because the risk comes enough from the price.

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[3] See Horiuchi(1990) e.g.
[4] C.f. De Long, Shleifer, Summers, and Waldmann(1987) e.g.
[5] C.f. Grossman and Stiglitz(1980) e.g.

In addition, it has frequently been supposed that agents have a short horizon, say, two or three periods. This supposition is critical[6] but what we should do here is just to follow such models here. Thus the problem for each agent is to maximize his utility function $U(W_1)$, where W_t, the value of his wealth at period t, is expected to be

$$W_{t+1} = M_t + (r + P_{t+1})B_t , \tag{1}$$

where r represents the dividend, $r \geq 0$. His budget constraint is $M_t + P_t B_t = W_t$ where W_t is given.

The utility function is often supposed to be exponential,

$$U(W_1) = -e^{-aW_1} , \quad a>0 \tag{2}$$

where a is the coefficient of absolute risk aversion. Then his demand for the risky asset is

$$\frac{1}{aVar(P_1)}(r + E_i(P_1) - P_0) , \tag{3}$$

if he, the agent i, believes that the yield from the risky asset is normally distributed. Here, the dividend is fixed and the price at the next period is thought to be distributed normally with mean $E_i(P)$ and variance Var(P). $E_i(\bullet)$ denotes i's subjective expectation. The anticipated variance can naturally be subjective but we suppose that this value is common among agents. Standardizing the population to be one, the price of the risky asset in current period is to be

$$P_0 = r + \overline{E}(P_1) - aVar(P_1)S_0 \tag{4}$$

where the supply of the risky asset at period t is S_t; $S \geq 0$[7]. $\overline{E}(\bullet)$ denotes the average of the agents' expectations.

We suppose that agents assume the same structure on the market at the next period,

$$P_1 = r + \overline{E}(P_2) - aVar(P_2)S_1 , \tag{5}$$

and that $\overline{E}(P_t)$ is stochastic and its distribution is normal, which is known to agents. Agents at period 0 know the variance, which is common among agents at period 1. Then the subjective expectation would be

$$E_i(P_1) = r + E_i\overline{E}(P_2) - aVar(P_2)E_i(S_1) , \tag{6}$$

i.e. agents must expect the average opinion about P_2. That makes the price be revised,

$$P_0 = 2r + \overline{E}\overline{E}(P_2) - aVar(P_2)\overline{E}(S_1) - aVar(P_1)S_0 , \tag{7}$$

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[6] There is some discussion about a long and short horizon in De Long, Shleifer, Summers, and Waldmann(1987). And there has been criticism on the short horizon since early times; Leontief(1947) e.g.

[7] The supplied quantity of the risky asset matters here because this model is of partial equilibrium. The supply of riskless asset is supposed to be 'perfectly elastic', c.f. De Long et al.(1987) e.g.

and thus agents would next practice the third degree of the expectations in the beauty contest, and the higher, then

$$P_0 = kr + \overline{E}^k(P_k) - a\sum_{j=1}^{k} Var(P_{j+1})\overline{E}(S_j) - aVar(P_1)S_0 . \quad (8)$$

The superscript k denotes the time of the iteration of $\overline{E}$, for example, $\overline{E}^2(x) = \overline{E}\overline{E}(x)$. This modeling is mathematically the same to that of Allen, Morris and Shin(2003), in which r=0 and $E_i(S) = 0$, accompanied with a suitable description of the market. $\overline{E}^k(P_k)$, which is brought by $E_i\overline{E}^{k-1}(P_k)$, is considered to be a possible expression of the beauty contest in the paper.

However, as mentioned in the introduction, these expressions are nonsense when agents have identical opinion. That is because, putting the opinion y, $E_i(P_k) = \overline{E}(P_k) = \overline{E}^k(P_k) = y_k$ and so on. The opinion cannot diverge by any order expectations. Instead, the paper applies the expectation conditional on two signals; a public signal and a private one. Suppose that P in question is distributed normally with mean y and variance $\frac{1}{\alpha}$ and that private signals is $z_i = \overline{P} + \varepsilon_i$, where $\overline{P}$ is a 'price' tentatively given and ε is distributed normally with mean zero and variance $\frac{1}{\beta}$. Looking on y, which corresponds to the rational expectations, as the public signal, agent i expects

$$E_i(P) = \frac{\alpha y + \beta z_i}{\alpha + \beta} . \quad (9)$$[8]

Agents expect the average of their expectations, using $E_i(P)$, and $E_i(\overline{E}^h(P))$ in the higher degree, as the private signal. Thus iterating this operation,

$$\overline{E}^k(P_k) = \left\{1 - \left(\frac{\beta}{\alpha+\beta}\right)^k\right\} y + \left(\frac{\alpha}{\alpha+\beta}\right)^k \overline{P}_k , \quad (10)$$

which converges on y as k becomes infinitely larger. In the case where k is less than a fixed T, as in Allen, Morris and Shin(2003), this fact is not so critical. In the model here, agents can make the degree of their expectations infinitely higher, which result in

[8] Expectations of this form comes from Morris and Shin(2002).

the rational expectations on the whole.

That objects to an objection against one of the re-criticism for the skepticism of the rational expectations. Introduced in Shleifer(2000), “to the extent that some investors are not rational, their trades are random and therefore cancel each other out without affecting prices” but “It is this argument that the Kahneman and Tversky theories dispose of entirely. The psychological evidence shows precisely that people do not deviate from rationality randomly, but rather most deviate in the same way.” The cancel of the noise in the private signals is observed by the equation (9) and averaging it. Here, the price is some weighted average of a value which corresponds to the rational expectations and that which represents the common deviation among agents. However, the equation (10) shows that even if agents ‘deviate in the same way’, i.e. $\overline{P}$ is some unreasonable value, they reach the rational expectations, under the structure of information and the belief above. In this case, the deviation of agents and the so-called rational expectations cannot coexist in the limit value. However, this is not all this structure of expectations implies. We will examine another case later.

On the other hand, from the equation (8), the price necessarily diverges as k becomes infinitely larger if the dividend r is positive. That is because the first term, the aggregation of the dividend in the future, diverges. This means that the convergence of the expectations, the second term of (8), is not significant. We would better think the model above is not proper when we examine infinitely higher order expectations, although that is useful when we see how the beauty contest is definitely introduced to markets. The model needs some improvement.

As mentioned in the beginning of this chapter, the dividend of the riskless asset can be positive according to the description of the market. We now consider about the riskless asset with positive dividend as the alternative to the risky asset, or else regard the riskless asset as money and suppose that agents receive liquidity premium in proportion to the holdings of ‘money.’ These two means can be mathematically equivalent but verbal expressions should be different. This paper applies the follower. Putting this dividend or the liquidity premium l,

$$V_{t+1} \equiv W_{t+1} + lM_t \,, \tag{11}$$

and each agent maximize

$$U(V_1) = -e^{-aV_1}, \quad a>0, \tag{12}$$

subject to $M_0 + P_0 B_0 = W_0$, where W_0 is given. The demand for the risky asset becomes

$$\frac{1}{aVar(P_1)}\left\{r + E_i(P_1) - (1+l)P_0\right\} \tag{13}$$

and the price is to be

$$P_0 = \frac{r}{1+l} + \frac{\overline{E}(P_1)}{1+l} - \frac{aVar(P_1)S_0}{1+l}. \tag{14}$$

By an analogous argument to the above, expectations in the k-th degree make the price be

$$P_0 = \sum_{j=1}^{k} \frac{r}{(1+l)^j} + \frac{\overline{E}^k(P_k)}{(1+l)^k} - a\sum_{h=1}^{k-1} \frac{Var(P_{h+1})\overline{E}(S_h)}{(1+l)^{h+1}} - \frac{aVar(P_1)S_0}{1+l}. \tag{15}$$

Notice that the second term on the right hand converges on zero as long as $\overline{E}^k(P_k)$ converges as k becomes infinitely larger. This means that the accumulation of expectations in the beauty contest vanishes away through the beauty contest itself, if the average of the higher order expectations does not diverge. Moreover, the same holds for the cases where agents have an identical expectation and where opinions always diverge randomly.

To see more simply these cases, which being argued above, suppose that the variance of the next price and the supply of the risky asset are constant over periods and known by agents. In addition, suppose that there are noisy agents, in constant measure μ, whose expectations are misled according to normal distribution with mean ρ_t^* and variance σ_ρ^2. ρ_t^* is distributed with mean ρ^* [9]. Subjective expectations by noisy traders j is

$$E_j(P_t) = y_t + \rho_t^j, \tag{16}$$

where y is the value which comes from expecting rationally. If sophisticated traders know the share, thus the average of subjective expectations becomes

$$\overline{E}(P_t) = y_t + \mu\rho_t^*. \tag{17}$$

[9] This model is almost equivalent to De Long, Shleifer, Summers, and Waldmann(1987), which applies overlapping generations model though. This model requires the supply of the risky asset S to be positive. More importantly, this model gives an implication about the uncertainty of the risky asset at the next price, which is determined by intentions of the 'young' at that period. It becomes more appropriate to think the asset priced by something similar entity to the agents themselves than in such case that the asset is automatically exchanged to money with errors or to commodities, say, crops. The model of Muth(1961) is by nature to be applied to the follower case, while the market of securities is considered to be more suitable for the latter case, not because of traders' myopia but of securities which are usually sold to other traders, who are more similar to the trader herself than something exogenous like weather.

This is mathematically equivalent to suppose that agents observe public signal y and private signal $z_{i_t} = y_t + 2\mu\rho_t^* + \varepsilon_{i_t}$ where ε is distributed with mean zero and z is distributed normally with a variance which is equal to that of y. Agents give weight equally to these two signals. y corresponds to the rational expectations and ρ_t^* represents the bias of noise against y. μ is translated into the volume of the noise. Then the numerator of the second term of the equation (15) is calculated and we obtain

$$\overline{E}^k(P_k) = y_k + \mu(\rho_0^* - \rho^*)(1+l)^{k-1} + \mu\rho^* \sum_{h=1}^{k}(1+l)^h , \tag{18}$$

which converges as long as $\mu = 0$ or $\rho_0^* = \rho^* = 0$ [10]. Therefore, from propositions above, the second term do not converge on zero if there is some noise. Then the price converges on

$$P_0 = \frac{r}{l} + \frac{\mu(\rho_0^* - \rho^*)}{1+l} + \frac{\mu\rho^*}{l} - \frac{aVar(P)S}{l} \tag{19}$$

as expectations become infinitely higher degree. The meaning of this equation is summarized in the following propositions;

i If there is no noise and agents practice the rational expectations i.e. μ =0, the price reflects only the 'fundamentals' and the risk premium.

ii If there is some noise and the noise is unbiased at each period i.e. $\rho_t^* = 0$ for any t, namely $\rho^* = 0$ and $\sigma_\rho^2 = 0$, the price reflects only the 'fundamentals.'[11]

iii If there is some noise and the noise is biased[12] at each period but unbiased over periods i.e. $\rho^* = 0$ and $\sigma_\rho^2 > 0$, the price reflects not only the 'fundamentals' and the risk premium but also expectations through the beauty contest. In this case if the current price is expected beforehand, the price is equal to the price in the cases above.

iv If there is some noise and the noise is biased over periods i.e. $\rho^* \neq 0$, the price reflects not only the 'fundamentals' and the risk premium but also expectations through the beauty contest.

v When the expectations through the beauty contest are reflected, the expectations

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10 ρ_t^* becomes zero in measure zero when $\sigma_\rho^2 > 0$.

11 There is no risk premium if $\sigma_\rho^2 = 0$, because $Var(P) = \frac{\mu^2\sigma_\rho^2}{(1+l)^2}$.

12 Cf. footnote 8.

form a function of the volume of the noise, the yield from the riskless asset, the bias of noise at the primary period and, if any, the bias anticipated over periods.

From the result above, we can examine what is strictly caused when agents 'deviate from rationality randomly' and 'deviate in the same way' as in the quotations from Shleifer(2000). If the deviation is as random as the case iii supposes, the price is affected only by the accidental bias, which the stochastic variance namely the average of deviation, necessarily has. This means that the price is expected to be unaffected by the deviation from rationality beforehand. If the deviation is as the case ii, the price is not affected. As implied in the equation (14), the actions of agents cancel each other in the sense that the total demand is the average of the demand by each agent. Moreover, the bias of the expectations, which the actions depend on, is cancelled in the sense that the bias is expected to be the mean of its distribution; some purchase too much, while others purchase too little. These facts are so trivial that can be definitely confirmed without the rational expectations represented by y. Suppose that the expected value y is misperceived and agents have not yet been aware of the error. This supposition of wrong expectations does not affect the result above except replacing y with the wrong value x. The price converges on the value, 'fundamentals,' which does not depend on x. Thus the assumed 'rational expectations' are washed away even though they are misperceived. In this situation no agent recognizes that they are misled, therefore the deviation of their expectations from the rational expectations cannot be observed. Despite that, the price reflects only the 'fundamentals' in the pseudo case ii. The reason is considered to be the fact that agents think that they know only what all the agents know, which they think they know, and so on. Thus they think the expectations of the price at a future period do not affect the price at the previous period, from this backward induction, expectations do not affect the current price whether they are correct or not.

This fact responds to the thesis of Muth(1961) quoted above. Although it seems to apply the case iii, the result is essentially similar. All the price reflects is only the 'fundamentals' and the risk premium except an accidental deviation happened at the current period. The average expectation is equal to the value which is induced rationally from the average of another mass of expectations which the variance in question depends on i.e.

$$\overline{E}^{k}(x) = E\left[\overline{E}^{k-1}(x)\right], \tag{20}$$

therefore $\overline{E}^{k}(x)=\overline{E}^{k-1}(x)$, which can be reduced to

$$\overline{E}\,\overline{E}(x)=\overline{E}(x)\ .^{13} \tag{21}$$

Then 'the profit opportunities would no longer exist,' according to the thesis. We can see here that the price is not affected, or not essentially affected, by expectations. The price reflects only what agents can actually receive. This implies that there is no more profit opportunities. This is not because the expectations are based on the correct information but because the average expectation does not violate the law of iterated expectations. This does not mean that the name of the 'rational' expectations is not so much after their correctness as after their reasoning. It matters how the average of expectations is to be. Each single agent can be rational in the sense that his subjective expectation satisfies the law of iterated expectations, which does not imply that they on average satisfy the law. Even if each agent is rational in this sense, the law can be violated when expectations are heterogeneous, based on different information or different belief. Agents on the whole can be 'irrational' in this sense therefore we cannot anthropomorphize the aggregation of agents.[14]

To capture the relationship between the difference among agents and the average expectation, it is useful to suppose that there are two signals and both of them are different among agents. One signal for agent i is

$$z_i=\overline{z}+\varepsilon_i \tag{22}$$

and another is

$$x_i=P+\eta_i\ , \tag{23}$$

where ε and η are independently distributed with mean zero. In analogy with the argument deriving the equation (9), we obtain

$$E_i(P)=\frac{\alpha}{\alpha+\beta}z_i+\frac{\beta}{\alpha+\beta}x_i\ . \tag{24}$$

Taking the average of subjective expectations among agents,

$$\overline{E}(P)=\frac{\alpha}{\alpha+\beta}\overline{z}+\frac{\beta}{\alpha+\beta}P\ . \tag{25}$$

Suppose that each agent knows that the signal z is not identical among agents but believes in the signal of his own. Then his expectation about the average expectation is

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[13] Allen, Morris, and Shin(2003) and Morris and Shin(2002) typically have $\overline{E}\,\overline{E}(x)\neq\overline{E}(x)$ as the counterpart of the rational expectations.

[14] This is the same problem as those which Kirman(1992) points out, whether the aggregation of agents behaves as if it were a single agent, so-called 'representative agent.' .

$E_i \overline{E}(P) = \frac{\alpha}{\alpha+\beta} z_i + \frac{\beta}{\alpha+\beta} E_i(P)$ and calculated

$$E_i \overline{E}(P) = \left\{ 1 - \left(\frac{\alpha}{\alpha+\beta} \right)^2 \right\} z_i + \left(\frac{\beta}{\alpha+\beta} \right)^2 x_i \qquad (26)$$

Thus the average opinion in the second degree becomes

$$\overline{E}\overline{E}(P) = \left\{ 1 - \left(\frac{\alpha}{\alpha+\beta} \right)^2 \right\} \overline{z} + \left(\frac{\beta}{\alpha+\beta} \right)^2 P, \qquad (27)$$

and in the k-th degree is to be

$$\overline{E}^k(P) = \left\{ 1 - \left(\frac{\alpha}{\alpha+\beta} \right)^k \right\} z_i + \left(\frac{\beta}{\alpha+\beta} \right)^k P, \qquad (28)$$

which converges on $\overline{z}$ as k infinitely becomes larger. The average expectation converges on the average opinion while private expectations converge on his belief. From the equations (25) and (27), we can observe that this average expectation violate the law of iterated expectations while private expectations can satisfy it. However, the average opinion become close to being rational, leaving agents diverged. This structure of expectations expressed eloquently in the equation (26) provides a form of equilibrium where agents are rational in private and heterogeneous but irrational on the whole.

Returning to the former argument, around the equations (18) and (19), what is less trivial in the previous chapter is that the divergence of the higher order expectations is cancelled by the discount rate, $\frac{1}{1+l}$. This discount is the critical factor not only in the convergence of the price in the equation (15) but also in the convergence, or the divergence, of the expectation under the beauty contest. In such models the degree of expectations in the beauty contest is substituted for the distance to the future period. This substitution makes the models mathematically equivalent to those before in the case above. This is the reason why the recursive structure is translated into the beauty contest in our model. However, this is not necessarily what Keynes says in the quotation above. In our model, the average expectation of the price in the first degree $\overline{E}(P_1)$ has been substituted with the second order average expectation of the price at the period two, not one, and with the k-th order expectation of the price at period k. Reading Keynes frankly, we should consider about the iteration of expectations with a fixed object[15] i.e. $\overline{E}^k(P_1)$. It remains to be examined how this expectation under the beauty contest affects the price at the primary period when we consider in a model where the substitution mentioned above make some difference. In such case that we cannot equivalently do that substitution, the style of tatonnement is just usable for these problems[16].

Tatonnement has been applied when examining the stability of prices in the general equilibrium, which has not been discussed in the context surveyed above. There are two expected profits in introducing a general equilibrium model in addition to examining the stability. i) It is very meaningful to examine precisely what Keynes says because he seems to use the metaphor of the beauty contest for the general equilibrium. ii) It can be an alternative approach of Tobin(1958) to show how the liquidity preference curve exists and to object the denial.

Introducing a general equilibrium model, the equation (1) and the budget constraint still hold. We regard the riskless asset as money and call the risky asset bonds. To simplify the problem, the utility function (12) is changed into the sum of the amount of the wealth and the liquidity,

$$U(V_1) = V_1.$$

Maximizing the utility, each agent wants the ratio of money and bonds to be adjusted as follows. Putting the proportion to hold in bonds for agent i A_{2i}, he wants A_{2i} to be zero, i) if the price of bonds is higher than the critical value he presumes. ii) If the price is lower than the critical value, he wants A_{2i} to be one. iii) If the price is just equal to the critical value, it makes no difference for him whether A_{2i} is to be. The critical value against the current price P_{0i}^C depends on the expectation of the price at the next period,

$$P_{0i}^C = \frac{r + E_i(P_1)}{1+l}. \tag{29}$$

Aggregating A_{2i} over i we obtain the proportion on bonds desired by agents in the market, A_2^d. Thus the demand of bonds depends on the distribution of the critical values. Suppose that each single sheet of bonds is accompanied by a critical value i.e. by a personality, or that agents have the same ex ante amount of bonds in each period. Then the cumulative distribution function $F(P_0^C)$ of the critical values[17] represents

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[15] In Allen et al.(2003), the object is fixed but the primary period goes backward as the degree of iteration k increases. The primary period should be fixed in such case.

[16] C.f. Mas-Collel, Whinston, and Green(1995) Chapter 17.H. and Negishi(1962).

[17] The constancy of this distribution is supported by the tatonnement process.

the proportion of bonds which is wanted to be sold at the price. Thus we obtain

$$A_2^d = 1 - F(P), \tag{30}$$

which is a kind of demand curve, called the liquidity preference curve, 'the inverse relationship of demand for cash to the rate of interest,'[18] though we use the price instead of the interest. The higher the price becomes, the smaller the proportion is desired. In the equilibrium, A_2^d is equal to the proportion based on the supplied A_2, so as to have demands for and supplies of each assets be in equilibrium i.e.

$$A_2 = \frac{PB}{M + PB} = A_2^d, \tag{31}$$

where M and B are fixed and denote the supplied quantity of money and bonds respectively.

At first we examine the case where expectations of agents are homogeneous. Then the critical values become identical and the cumulative distribution function, and therefore the demand curve also, becomes stair-shaped. In this case, the current price is equal to the common critical value in the equilibrium,

$$P_0 = \frac{r + \overline{E}_I(P_1)}{1 + l}, \tag{32}$$

where the subscript I denotes the identical expectation. Suppose that agents assume the same structure on the next period and after. They expect the next price by the analogy of the current price and substitute it into the expectation at the current period. Then they revise the analogy and the expectation[19]. Thus the current price is to be

$$P_0 = \sum_{j=1}^{k} \frac{r}{(1+l)^j} + \frac{\overline{E}_I^k(P_1)}{(1+l)^k}. \tag{33}$$

If the expectation is rational in the sense that it satisfies the law of iterated expectations, the expectation has to be $\frac{r}{l}$ and the current price becomes $\frac{r}{l}$. In this case the price converges and is expected to be steady forever. On the other hand, if the expectation is not rational in the sense, the price does not converge. Solving the equation (33), we obtain the general solution,

$$\overline{E}_I^k(P_1) = \frac{r}{l} - (1+l)^k \left(P_0 - \frac{r}{l}\right). \tag{34}$$

"Recontract is always possible and no actual trade of commodities among individual participants is permitted," Negishi(1962).

[18] Tobin(1958)

[19] See appendix I.

If agents are rational on average, it must be $P_0 = \frac{r}{l}$ because the expectations must satisfy $\overline{E}_I^k(P_1) = \overline{E}_I^{k-1}(P_1)$ on the equation (34). If the average expectation do not behave rationally, $P_0 \neq \frac{r}{l}$, $E_I^k(P_1)$ is not on the steady state. Notice that the current price P_0 is not decided by the equation (33), but by

$$P_0 = \frac{r + \overline{E}_I \overline{E}_I^{k-1}(P_1)}{1+l}, \tag{35}$$

analogous to the equation (32). Thus the current price is not steady; which converges on zero or diverge. This path of the price is equivalent to the 'bubble' in Okina(1985), which call this bubble 'rational' though. The paper says that the 'fundamentals' price requires a supposition that the price is strictly equal to the 'fundamentals' in each period in the future. We saw above that equality between the price and the 'fundamentals' is a result of the rationality in the original sense; not the tautology of the supposition.

Next, to examine the case of heterogeneous expectations, suppose that the critical values have such a distribution that

$$F(P) = 1 - \frac{1}{P/\overline{z} + 1}, \tag{36}$$

where $\overline{z}$ denotes the average of the critical values at the period and this cumulative distribution function satisfies the property $F(\overline{z}) = \frac{1}{2}$. Thus the current price in equilibrium becomes

$$P_0 = \left(\frac{M}{B}\right)^{\frac{1}{2}} \left(\overline{z}\right)^{\frac{1}{2}}. \tag{37}$$

Applying the supposition above, each agent has a belief z_i, which is normally distributed with mean $\overline{z_0}$ and variance $\frac{1}{\alpha}$, and receive a signal $x_i = \overline{z_0} + \eta_i$, where η is distributed with mean zero and the variance of this signal is fixed to be $\frac{1}{\beta}$. He constructs expectations taking weighted average of these two signals. The critical value of agent i is then

$$P_{0i}^C = \frac{r}{1+l} + \frac{1}{1+l}\left(\frac{\alpha}{\alpha+\beta} z_i + \frac{\beta}{\alpha+\beta} x_i\right). \tag{38}$$

Thus the presumed current price at this level of reasoning is calculated

$$P_0 = \left(\frac{M}{B}\right)^{\frac{1}{2}} \left(\frac{r + \overline{z_0}}{1+l}\right)^{\frac{1}{2}} \tag{39}$$

Agents assume that the market has the same structure as the current period at the next period and after. Then the calculated price at the current period is used as the signal on the next period. To give the same weight as the previous level for the simplification, we need assumptions about the variance of the signal, (39), to be same to that of the primary signal x. Each agent believe the variance of z_i to be $\frac{1}{\alpha}$ and another variance of $\overline{z_0}$. The variance of η can be appropriately defined to make the variance of signal x be as defined. However, it confront a difficulty on the next step; if the assumption about variance of $\overline{z_0}$ is constant, the weight have to be changed. What should be done is to allow this model to become more complex, or to abandon the assumption of the constancy of the variance. The follower solution is not unreasonable in the tatonnement process. We could assume a higher degree of distribution; the distribution of $\overline{z_0}$, and more higher. At any rate, we keep the simple structure of this model. Thus the critical value of each agent and the presumed current price are revised,

$$P_{0i}^C = \frac{r}{1+l} + \frac{1}{1+l}\left[\frac{\alpha}{\alpha+\beta} z_i + \frac{\beta}{\alpha+\beta}\left\{\left(\frac{M}{B}\right)^{\frac{1}{2}}\left(\frac{r+\overline{z_0}}{1+l}\right)^{\frac{1}{2}}\right\}\right] \tag{40}$$

$$P_0 = \left(\frac{M}{B}\right)^{\frac{1}{2}}\left[\frac{r}{1+l} + \frac{1}{1+l}\left[\frac{\alpha}{\alpha+\beta}\overline{z_0} + \frac{\beta}{\alpha+\beta}\left\{\left(\frac{M}{B}\right)^{\frac{1}{2}}\left(\frac{r+\overline{z_0}}{1+l}\right)^{\frac{1}{2}}\right\}\right]\right]^{\frac{1}{2}}. \tag{41}$$

Using this value instead of the signal x, each agent revises his critical value. This make the ex ante price, presumed to be (39), revised. Iterating this operation, the current price converges on

$$P_0^* = \frac{(1-\delta)\, M/B + \left\{ \left(M/B \right)^2 (1-\delta)^2 + 4(1+l)\, M/B\, (r + \delta \overline{z_0}) \right\}^{\frac{1}{2}}}{2(1+l)}, \text{[20]} \tag{42}$$

where $\delta = \dfrac{\alpha}{\alpha+\beta}$ and the superscript * denote the price in equilibrium. Thus the beauty contest brings an equilibrium price which does neither diverge nor converge on zero and is not the same to that of the rational expectations. In equilibrium, opinions of agents are heterogeneous according to the equation (40), where remains each z. Each agent believes the signal z of his own so each opinion does not converge on the average opinion though he takes it into account. This heterogeneity in equilibrium means that there exists the liquidity preference curve. Thus this provides another foundation of the liquidity premium curve, whose significant model of Tobin(1958) does not focus on the divergence of opinion among agents.

Comparing equilibria in these two cases, subjective expectations of each agent is rational in both case; homogeneous expectations and heterogeneous ones while the average expectation is rational in the latter case but not in the follower. The stability of prices are grasped by the relation between A_2 and P_0. This is exactly what governs the tatonnement process; namely the differential equation system

$$\frac{dP_0}{d\tau} = A_2(q_{t+1}) - A_2(q_t), \text{[21]} \tag{43}$$

where q_k is the current price presumed at the k-th level and τ denotes notional time. This is based on the fact that A_2 is always equal to the supplied amount taking the price into account, $\dfrac{PB}{M+PB}$, which is a monotonically increasing function of P. Therefore, the more A_2 is desired, the higher bonds are priced. In the case of the homogeneous expectations,

$$\frac{dP_0}{d\tau} = \frac{(1+l)P_0 - r}{M/B + (1+l)P_0 - r} - \frac{P_0}{M/B + P_0}, \tag{44}$$

from the equation (32). Putting the right side of the equation (43) $T(P_0)$, calculations give us

$T\left(\dfrac{r}{l}\right) = 0$, $\dfrac{dT(P_0)}{dP_0} > 0$, and, if desired, $\dfrac{d^2T(P_0)}{dP_0^{\,2}} < 0$. Then as drawn in the figure 1,

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[20] This is shown in appendix II.

[21] As seen in Negishi(1962), as the equation (T).

we can see that the current price, and the steady state equilibrium, is unstable and the ratio between the market value of bonds and the balance of money converges on zero or one if the average of expectations is not rational.

In the case of hererogeneous expectations, analogously,

$T(q^*)=0$, $\frac{dT(P_0)}{dP_0}<0$, and, if desired, $\frac{d^2T(P_0)}{dP_0^{\ 2}}>0$. Contrary to the previous case, the steady state equilibrium is stable, as drawn in figure 2. While the price converges, the private opinions, the equation (40) are diverged. The heterogeneity of agents has a critical meaning on the stability of the price.

3. **Concluding Remarks**

When an agent tries to make a better answer to the problem "who the most beautiful is," and when the correct information, if any, is not available, the decisive things are what agents subjectively believe on average and whether they are rational or not. The rationality which has critical meaning on this competition is not whether it is based on the correct information but whether the average expectation satisfies the law of iterated expectations or not. It is considered to be natural that each agent is rational in the sense that his or her expectation satisfies the law. The aggregation of agents i.e. the average expectation is rational if agents are homogeneous though it is not generally the case.

On the other hand, the first significant result in the context of the beauty contest, or of the rational expectations with noise, is that if there are some structures which keep the average expectation rational, say, randomness, the price reflects the 'fundamentals', if available, well and the iterated expectation under the beauty contest vanishes away through the beauty contest itself. However, it is not fatal for those who expect the average of the expectation, and practice higher degree, because the average expectation is not generally rational. Even though each agent is rational, if there is not the correct information which can be distinguished from the mass of those which are similar or approximated poorly, the belief of agents can be distributed stochastically and the most convenient sufficient condition, the homogeneity or the randomness around a fixed point, is not necessarily valid. It is not each human being but the mass of them that rationality matters. Our second significant result is that expectations converge through the beauty contest with opinions of each agent remaining to be heterogeneous in the same structure to that derived the convergence of expectations of each agent.

Along with the argument above, we could examine various statements and answer definitely to them in our model. Especially, we introduced the beauty contest to a simplified general equilibrium model on financial market with the tatonnement process and examine the stability; prices are more stable when agents are not rational i.e. diverged enough, than when they are homogeneous and thus rational in aggregation. This is the third significant result. Here, the existence of the liquidity preference curve is also indicated. This is the fourth.

Surveying roughly the context from the rational expectations to the beauty contest via the noises, it is presumed that the perspective is directed to the theory of Keynes, which is abundant in insights without definite model. Modeling them is useful for arguments of economics today. This paper is an example of that as well.

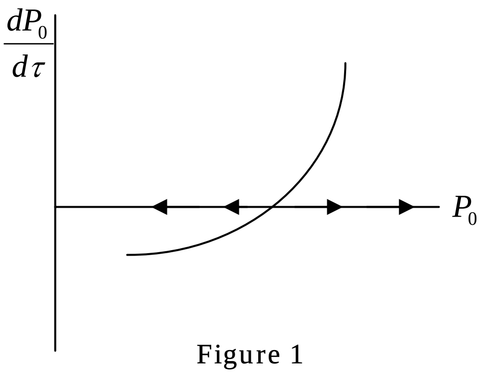

Figure 1

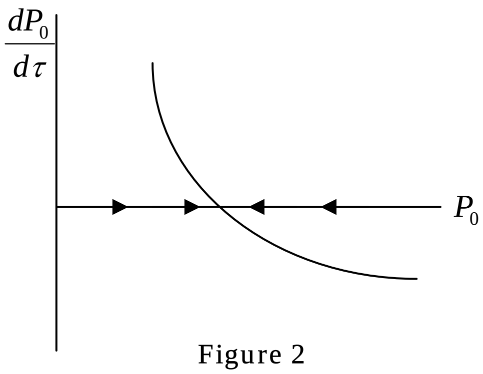

Figure 2

Appendix

I. Beginning from the equation (32), P_1 is replaced with the anticipated $E_I(P_1)$. Then $P_0 = \frac{r}{1+l} + \frac{E_I E_I(P_1)}{1+l}$. Replacing again P_1 with $E_I(P_1)$ and again and so on, we obtain the equation (33). When iterating the recursive substitution $E_I P_k = \frac{r + \overline{E}^{k+1}(P_{k+1})}{1+l}$, we obtain $P_0 = \sum_{j=1}^{k} \frac{r}{(1+l)^j} + \frac{\overline{E}_I^k(P_k)}{(1+l)^k}$,

which we can consider to be equivalent to the equation (33) despite the difference on the subscript of P in the right side.

II. Put P_0 at the k-th level q_k. From the equation (41),

$$q_{k+1}(q_k)=\left(\frac{M}{B}\right)^{\frac{1}{2}}\left[\frac{r}{1+l}+\frac{1}{1+l}\left[\delta\overline{z_0}+(1-\delta)q_k\right]\right]^{\frac{1}{2}}$$

. From the fact that $q_{t+1}(0)>0$ and $\frac{dq_{k+1}}{dq_k}>0$ (and $\frac{d^2q_{k+1}}{dq_k{}^2}<0$), we can visually grasp that the sequence $\{q_k\}$ converges in figure 3. When $q_{k+1}=q_k$, the value of them are the right side of the equation (42).

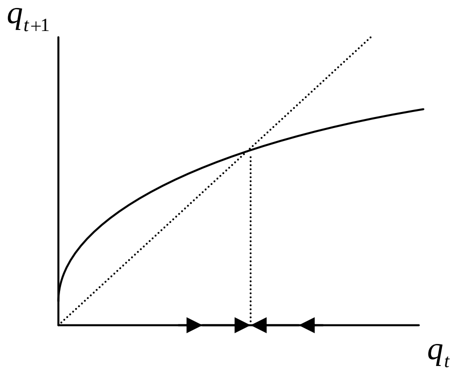

Figure 3

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