An Example of Schumperterian Dynamics: Effects of Innovation and Imitation in the Long Run of A Two-Sector Disequilibirum Dynamic Model

Masanao Aoki
Department of Economics, University of California, Los Angeles
and Institute of Research Initiatives and Developments, Chuo University, Tokyo
e-mail aoki@econ.ucla.edu
Tohru Nakano, and Gen Yoshida
Department of Physics, Chuo University
e-mail nakano@phys.chuo-u.ac.jp.jp

March 8, 2004

Abstract

This paper describes a simple model of stochastic industry dynamics with two types of firms. The first type is composed of technically advanced firms with positive probability rate of generating innovations , and the other type composed of technically less advanced firms which are not capable of innovations. Firms of the latter type may succeed in joining the first type by successfully imitating firms of former type in interaction with the former. Conversely, technically advanced firms have positive probability rate of dropping out from the first sector to belong to the latter. All these changes in types are stochastic in nature, being specified as transition rates a continuous-time Markov chain by modeling industry dynamics as such.

The paper solves the backward Chapman-Kolmogorov (also known as master) equation of this Markov chain to examine disequilibrium dynamics of two types of firms evolve over time in interacting with each other. We show that long-run equilibria exist for the model in which equilibrium sizes of the clusters of both types have finite positive means and finite variances.

Introduction

Importance of innovation has received much attention in the context of Schumpeterian dynamics. See Aghion and Howitt (1992, 1997), Iwai (1997, 2001), and Aoki and Yoshikawa (2002) to cite a few recent contributions. To quote Iwai, for example,

> The industry does not approach a neoclassical equilibrium of uniform technology in the long run, but at best a statistical equilibrium of technological disequilibria which maintain a relative dispersion of efficiencies in a statistical balanced form.

In this paper we explicitly solve disequilibrium stochastic dynamics of a model with two types of firms, one type with innovations and the other with imitation but no innovations. We focus on the long-run behavior to show that the model reaches stochastic equilibria to demonstrate clearly the role of innovation in the expected sizes and associated variances of two sectors.

We solve a disequilibrium backward Chapman-Kolmogorov equation by the method of cumulant generating function to derive the long-run stochastic equilibria. As pointed out by Iwai, this type of results refute the neoclassical notion of long-run equilibria.

The Model

Our model has two sectors; one technically advanced sector and the other less so. By a suitable choice of units we denote the sizes of the two sectors by a vector (n_1, n_2). We may think of them as the number of firms in some suitably chosen standard units. Firms in sector one succeed in creating innovative firms at rate f which is, for simplicity, exogenously fixed in this model.[1]

Firms' stochastic behavior is described by a continuous time Markov chain which is uniquely determined by a set of transition rates. We write the transition rate from state a to b by $w(a, b)$. This means that the probability that the system moves from state a to b in some small time interval is given by the time interval times the transition rates up to o(time interval size). They are specified as follows: The first two describe entry (growth) rates

$$w\{(n_1, n_2), (n_1 + 1, n_2)\} = c_1 n_1 + f,$$

$$w\{(n_1, n_2), (n_1, n_2 + 1)\} = c_2 n_2.$$

Here c_i is the rate of growth of type i firm size, $i = 1, 2$.

The next two specify exit rates from the model

$$w\{(n_1, n_2), (n_1 - 1, n_2)\} = d_1 n_1,$$

$$w\{(n_1, n_2), (n_1, n_2 - 1)\} = d_2 n_2.$$

Here d_i is the exit (death) rate of type i firms from the economy, $i = 1, 2$.

The last set of two transition rates describes how firms change their types

$$w\{(n_1, n_2), (n_1 + 1, n_2 - 1)\} = \mu g_1 n_2 (n_1 + h),$$

with $g_2 = c_2/d_2$, and $h = f/c_1$, and

$$w\{(n_1, n_2), (n_1 - 1, n_2 + 1)\} = \mu g_2 n_1 n_2,$$

with $g_i = c_i/d_i$, $i = 1, 2$, and $\mu = \lambda d_1 d_2$. This parameter λ is the coefficient in the transition rates of type changes by firms in the two sectors. The first of the two shows the rate at which one of type 1 firm becomes technologically obsolete and join the cluster made up of type 2 firms. The second equation specifies how firms of type 2 successfully imitate firms of type 1 and join their cluster. for example.

The stochastic dynamic equation is easy to state. It is a backward Chapman-Kolmogorov equation, also known as the master equation. (We

[1] It will be interesting to endogenize this rate in a way that is not equivalent to increasing the birth rate c_1 in the model of this paper.

use the latter name as it is short, and implies that everything you need to know about stochastic behavior is implicit in the master equation.)

$$\frac{\partial P(n_1, n_2; t)}{\partial t} = I(n_1, n_2; t) - O(n_1, n_2; t), \tag{1}$$

where the first term collects all inflows of probability flux into state (n_1, n_2), and the second term collects all outflows of probability fluxes out of this state. There are six distinct flows. In detail we have

$$I(n_1, n_2; t) = P(n_1 + 1, n_2; t)d_1(n_1 + 1) + P(n_1, n_2 + 1; t)d_2(n_2 + 1)$$

$$+P(n_1 - 1, n_2; t)c_1(n_1 - 1 + h) + P(n_1, n_2 - 1)c_2(n_2 - 1)$$

$$+P(n_1+1, n_2-1; t)\mu g_2(n_1+1)(n_2-1)+P(n_1-1, n_2+1; t)\mu g_1(n_1-1+h)(n_2+1).$$

The second term in (1) is given by

$$O(n_1, n_2; t) = P(n_1, n_2; t)\{c_1+n_1+f+c_2n_2+d_1n_1+d_2n_2+\mu g_1n_2(n_1+h)+\mu g_2n_1n_2)\}.$$

To solve the master equation, we first convert it into the probability generating function

$$G(z_1, z_2; t) = \sum_{n_1, n_2} P(n_1, n_2; t)z_1^{n_1} z_2^{n_2}.$$

We obtain a partial differential equation for $G(z_1, z_2; t)$. It is given in Appendix. This partial differential equation is rather intractable, and for that reason we convert it into the cumulant generating function and solve for the expected values of first and second moments.[2]

Cumulant generating functions are related to the probability generating functions by

$$K(\theta_1, \theta_2; t) = \ln G(e^{-\theta_1}, e^{-\theta_2}),$$

where we change variables from z_1, and z_2 into θ_1 and θ_2.

It is known that the cumulant generating function has a Taylor series expansion of the form

$$K(\theta_1, \theta_2; t) = k_1\theta_1 + k_2\theta_2 + \frac{1}{2}(\theta_1, \theta_2)\Theta(\theta_1, \theta_2)' + \cdots,$$

where $k_1 = E(n_1)$, and $k_2 = E(n_2)$, that is, they are the expected sizes of the two types, and where Θ is a covariance matrix made up of the variances and covariances of the two sizes,

$$\Theta = \begin{pmatrix} k_{1,1} & k_{1,2} \\ k_{1,2} & k_{2,2} \end{pmatrix}.$$

See Aoki (2002, Chapt. 7) for further information on these generating functions, and some simple examples.

From the cumulant generating functions we derive a set of five ordinary differential equations for $k_1, k_2, k_{1,1}, k_{1,2}$, and $k_{2,2}$.

Appendix gives the explicit expressions.

[2]In some cases the resulting ordinary differential equations for the moments turn out to be an infinite set of coupled ordinary differential equation. Fortunately, the differential equations for the first and second cumulants are self-contained in this model.

Stationary Means and Variances

The equations for the two means are:

$$\frac{dk_1}{dt} = f - d_1(1-g_1)k_1 + \lambda f d_2 k_2 + 2\alpha\lambda A_0, \tag{2}$$

and

$$\frac{dk_2}{dt} = -d_2(1-g_2+\lambda f)k_2 - 2\alpha\lambda A_0, \tag{3}$$

where $\lambda = \mu/d_1 d_2$, where $g_i = c_i/d_i$, $A_0 = k_{1,2} + k_1 k_2$, and $2\alpha = d_1 d_2(g_1 - g_2)$. Note that $A_0 =< n_1 n_2 >\geq 0$.

Since A_0 depends on $k_{1,2}$ we need solve for it as well.

Stationary means are described by setting the left-hand sides of (2) and (3) to zero:

$$f - d_1(1-g_1)k_1 + \lambda f d_2 k_2 + 2\alpha\lambda A_0 = 0, \tag{4}$$

$$-d_2(1-g_2+\lambda f)k_2 - 2\alpha\lambda A_0 = 0. \tag{5}$$

By adding (4) and (5) to express an important relation between f, k_1 and k_2

$$f = d_1(1-g_1)k_1 + d_2(1-g_2)k_2. \tag{6}$$

This equation clearly shows that a fraction of innovation flow accounts for the new firms in sector 1, and the rest accounts for the net exit flow of firms from sector 2. We later show that the expected value of the stationary values of the size of sector 1 scales with $\gamma_1 := \lambda d_1$, and that of sector 2 scales with $\gamma_2 := \lambda d_2$,

Recalling the definition $d_i(1-g_i) = d_i - c_i$, we see that the rate of innovation f equals the sum of the expected exit rates of firms of both sectors, since $(d_i - c_i)k_i$ is the net exit rate of firms of sector i, $i = 1, 2$.

In this model it is necessary that both g_1 and g_2 are not greater than one. There are two cases: $g_1 > g_2$ and $g_2 > g_1$. In the former $\alpha > 0$ and $\alpha < 0$ in the latter. Eqs. (4) and (6) exclude the case $g_1 > g_2$. In the other cases, we could have $g_2 > 1 > g_1$ or $1 > g_2 > g_1$.

To reflect these consideration we introduce two parameters m and n[3] by

$$g_1 = 1 - n\gamma,$$

and

$$g_2 = 1 - m\gamma,$$

where $\gamma = \lambda f$. The constants are bounded by

$$\frac{1}{\gamma} > n > 0,$$

and

$$n > m > -1.$$

The inequality $m > -1$ is derived from (5).

Solving (4) and (5), we obtain the means of the sizes of the two sectors

$$k_1 = \frac{1-\theta}{n\gamma_1}, \tag{7}$$

[3]not to be confused with n_1 or n_2

with

$$\theta := \frac{m(n-m)}{1+m} d_1 d_2 \lambda^2 A_0,$$

and

$$k_2 = \frac{\theta}{m\gamma_2}. \tag{8}$$

The stationary variances $k_{1,1}$ and $k_{2,2}$ are derived in Appendix; the results are

$$\begin{aligned} k_{1,1} &= \frac{1}{2\gamma_1\left(n + \frac{n-m}{m}\theta\right)} \left\{ \frac{2}{n\gamma} + \frac{2}{\gamma\gamma_1 m n^2 (n-m)} \left[\gamma_1 n(m^2+n) \right.\right. \\ & \left.\left. -\gamma m \left(\gamma_1 n^2 (1+m) - m(1+n)\right)\right] \theta + \frac{2(2+n)}{\gamma_1 n^2}\theta^2 + \frac{2(n-m)}{\gamma_1 m n^2}\theta^3 \right\}, \end{aligned}$$

and

$$\begin{aligned} k_{2,2} &= \frac{1}{2\gamma_2\left(1 + m - \frac{n-m}{n}(1-\theta)\right)} \\ & \times \left\{ \frac{2(1-m\gamma)(1+n)}{m(n-m)\gamma}\theta + \frac{2(1+n)}{mn\gamma_2}\theta^2 + \frac{2(n-m)}{m^2 n\gamma_2}\theta^3 \right\}. \end{aligned}$$

The covariance $k_{1,2}$ is expressed in terms of θ through the definition $k_{1,2} = A_0 - k_1 k_2$ as

$$k_{1,2} = \frac{1}{\gamma_1\gamma_2} \frac{1}{nm(n-m)} \theta \left[(1+n)m + (n-m)\theta\right].$$

What remains is to determine θ. The self-consistent equation for θ is derived in Appendix. Although the equation is a fifth order equation of θ because of the five unknown quantities, the highest term vanishes so that

$$\theta F(\theta) = 0 \tag{9}$$

where

$$F(\theta) = r_0 + r_1\theta + r_2\theta^2 + r_3\theta^3. \tag{10}$$

The forms of r_i are given in Appendix.

The root $\theta = 0$ is of interest because this value of θ yields a stationary state in which sector 2 vanishes, $k_2 = 0$, $k_{2,2} = 0$, and $k_{12} = 0$.

From $F(\theta) = 0$ we obtain three values of θ The roots must be such that θ is real and the obtained values of $k_1, k_2, k_{1,1}$ and $k_{2,2}$ are positive; $k_{1,2}$ is not necessarily positive. Although the analytic solutions may be obtained for special set of parameters, such solutions are not possible in general.

Mathematica, however, enables us to numerically solve $F(\theta) = 0$. In order that those solutions exist in reality, the solutions must be the stable fixed points.

As an example we describe in detail the case where $m = .01$; $n = 2$, $\gamma = \gamma_1$, and $\gamma_2 = \gamma_1 + \epsilon$, with a small positive ϵ. In this case there is only one root for which the dynamics are locally stable. It is given by $\theta = 0.472$.

The stability of the stationary states is examined in the following way. The starting equations are (2), (3) and (11), (12) and (13) in Appendix. By setting the left hand sides of those equations we have the stationary values, which are confirmed to numerically coincide with the solutions from

$\theta F(\theta) = 0$. Then the linearized equations for deviations $\delta k_1, \delta k_2, \delta k_{11}, \delta k_{22}$ and δk_{12} from the stationary values are derived. The eigenvalues of those equations are numerically calculated with a help of Mathematica. If real parts of all five eigenvalues associated with a stationary point are negative, the stationary point is stable.

The value of $\theta = .472$ corresponds to a locally stable solution. This leads to

$$k_1 = \frac{.264}{\gamma_1}, \; k_2 = \frac{47.2}{\gamma_2}.$$

From (5) we obtain

$$A_0 = \frac{\theta}{\gamma_1\gamma_2}\frac{1+m}{m(n-m)} = \frac{23.956}{\gamma_1\gamma_2}.$$

From this we derive

$$k_{1,2} = A_0 - k_1 k_2 = \frac{11.495}{\gamma_1\gamma_2}.$$

From (9) through (11) we can obtain approximate order of magnitude values for the second moments $k_{1,1}$ and $k_{2,2}$ as follows.

$$k_{1,1} = \frac{C_{1,1}}{\gamma_1^2},$$

with

$$C_{1,1} \approx \frac{1}{n^2}(1+\theta^2) \approx .31,$$

which is close to .309 obtained in the numerical example below, and

$$k_{2,2} = \frac{C_{2,2}}{\gamma_2^2},$$

with

$$C_{2,2} \approx \frac{1}{m^2}[\theta^2 + \frac{m(1+n)}{n}\theta] = 2300.$$

We also have an approximate expression for k_{12} as $C_{12}/\gamma_1\gamma_2$ with

$$C_{12} = \frac{\theta^2}{mn} \approx 11.15,$$

which is in good agreement with the value obtained above as 11.495.

Numerical Examples

We focus on a stationary solutions. Since there are five parameters, we have many solutions.

To keep the sizes of the two sectors at reasonable values, we examine cases with the death rates close to the birth rates. Namely, we choose g_i to be close to unity. Previously we have indicated that g_2 can be either larger than one or smaller than one, while g_1 is always less than one. First, we consider the case that the death rate d_i is slightly larger than the birth rate c_i, so that $n\gamma$, $m\gamma \ll 1$. Although the death rate of sector 2 is considered to be larger than that of sector 1, we assume that both are almost the same. We focus on the following parameters; $\gamma = \gamma_1 = \gamma_2 = 0.01, n = 2.0, m = 0.01$. Then we have three types of solutions; (1) $k_1 = 50, k_{11} = 2500, k_2 = k_{22} = k_{12} = 0$, (2) $k_1 = 49.97, k_2 = 4.77, k_{11} = 2501, k_{22} = 46505, k_{12} = 3.71$ and (3) $k_1 = 26.4, k_2 = 4719, k_{11} = 3093, k_{22} = 2.37 \times 10^7, k_{12} = 114918$. The stable solution is only the first type; only sector 1 survives. The second and third types are not stable. If we increase γ_2 slightly to $\gamma_2 = 0.011$, a remarkable change occurs in the type 3 solution. The numbers for (1) are the same as the previous case. On the other hand, (2) $k_1 = 49.977, k_2 = 4.122, k_{1,1} = 2501, k_{2,2} = 40232, k_{1,2} = 3.2$ and (3) $k_1 = 23.94, k_2 = 4738, k_{1,1} = 3216.7, k_{2,2} = 2.378 \times 10^7, k_{1,2} = 127042$. The second solution is not stable, but the third solution turns out to be stable in this case.

We vary a value of γ_2 with other parameters fixed. We found that the stable fixed point exists in a narrow range such that $0.02 \gtrsim \gamma_2 \gtrsim 0.0102$.

What parameters are chosen to increase the number of companies? For that purpose we should decrease $\gamma, \gamma_1, \gamma_2$. When $n = 2.0$ and $m = 0.01$ are fixed, we employ $\gamma = \gamma_1 = 0.001, \gamma_2 = 0.0011$. Then we have the stable third solution $k_1 = 264, k_2 = 47193$ with the correlation coefficient $k_{1,2}/\sqrt{k_{1,1}k_{2,2}} = 0.42$. In other sets of parameters with n and m fixed at the above values, $\gamma = \gamma_1$ and γ_2 being slightly larger than γ, we have the following scaling relation

$$k_1 = \frac{0.264}{\gamma_1}, \quad k_2 = \frac{47.2}{\gamma_2}, \quad k_{1,1} = \frac{0.309}{\gamma^2}, \quad k_{2,2} = \frac{2374}{\gamma^2}, \quad k_{1,2} = \frac{11.5}{\gamma^2}.$$

The correlation coefficient is 0.42.

The coefficient of variations are 2.11 and 1.03 for the two sectors respectively. We also note that with γ_2 nearly the same as γ_1 only 0.6 percent of the total sizes of the capital resides in sector 1.

Next we examine negative values of m. Take $m = -0.01$ while keeping the values of the other parameters the same as before. The numerical calculation gives a negative value of k_2. Although we have not done an extensive study, a negative value of m, i.e., g_2 is larger than one, may not yield stable stationary situations.

Concluding Remarks

To the best knowledge of the authors, this is the first example of Schumpeterian dynamics with innovations and imitation for which the first two moments of a stable solution of the Chapman-Kolmogorov equation have been analytically derived, and then numerically evaluated. Our model allows one to examine parametrically the relative importance of net death rates and innovation rate and to draw important conclusions on qualitative behavior of interacting two sectors of industry. We have shown that the means of stationary locally stable equilibria scale with parameters of the innovation rate, and death rates, among other things.

References

Aghion, D., and P. Howitt (1992). "A Model of growth through creative destruction", *Econometrica* **60**

Aoki, M., and H. Yoshikawa (2002). "Demand saturation-creation and economic growth " *J. Econ. Behav. Org.*,**48**, 127-154.

Iwai, K.(1997)."A contribution to the evolutionary theory of innovation, imitation and growth,",*J. Econ. Behav. Org.*, **43**, 167-198

—— (2001), "Schumperterian dynamics: A disequilibrium theory of long run profits" in L.Punzo (ed) Cycles, Growth and Structural Change: Theories and empirical evidence, Routledge, London and New York.

Appendix

The probability generating function

With only a scalar random variable X, its probability generating function is defined by $G(z,t) = E(z^X) = \sum_k z^k P(k,t)$. Its partial differential equation is obtained by noting that that

$$\sum_k z^k P(k-1,(t) = zG(z,t),$$

$$\sum_{k=1}^{\infty}(k+1)z^k P(k+1,t) = \partial G(z,t)/\partial z,$$

$$\sum_{k=1}^{\infty} kz^k P(k,t) = z\partial G(z,t)/\partial z,$$

and

$$\sum_{k=1}^{\infty}(k-1)z^k P(k-1,t) = z^2 \partial G/\partial z.$$

With two state variables n_1 and n_2, similar relations. The result is

$$\frac{\partial G}{\partial t} = [d_1(1-z_1) + c_1 z_1(z_1-1) + \mu g_2(z_2-z_1)]\frac{\partial G}{\partial z_1}$$

$$+[d_2(1-z_2) + c_2 z_2(z_2-1) + \mu g_1 h(z_1-z_2)]\frac{\partial G}{\partial z_2}$$

$$+[\mu g_1 z_1(z_1-z_2) + \mu g_2 z_2(z_2-z_1)]\frac{\partial^2 G}{\partial z_1 \partial z_2} + f(z_1-1)G.$$

The cumulant generating function

Noting that

$$\frac{\partial G}{\partial t} = G\frac{\partial K}{\partial t},$$

$$\frac{\partial G}{\partial z_i} = -Ge^{\theta_i}\frac{\partial K}{\partial \theta_i},$$

$i = 1, 2$, and

$$\frac{\partial^2 G}{\partial z_1 \partial z_2} = Ge^{\theta_1+\theta_2}H$$

with

$$H = \frac{\partial K}{\partial \theta_1}\frac{\partial K}{\partial \theta_2} + \frac{\partial^2 K}{\partial \theta_1 \partial \theta_2},$$

we convert the partial differential equation for G into that for K

$$\frac{\partial K}{\partial t}=\frac{1}{G}\frac{\partial G}{\partial t}=-\sum_{i=1}^{2}[d_i(e^{\theta_i}-1)+c_i(e^{-\theta_i}-1)\frac{\partial K}{\partial \theta_i}+f(e^{-\theta_1}-1)$$
$$+\mu[g_1(e^{(\theta_2-\theta_1}-1)+g_2(e^{(\theta_1-\theta_2)}-1)]H.$$

We then extract coefficients of θ_i and equate them to dk_i/dt, $i=1,2$, and those of θ_1^2, θ_2^2 with the derivatives $dk_{1,1}/dt$ and $dk_{2,2}/dt$, and the coefficient of $\theta_1\theta_2$ with the derivative $dk_{1,2}/dt$.

In this way we generate a set of five differential equations for k_1, k_2, $k_{1,1}$, $k_{2,2}$, and $k_{1,2}$.

Calculations of the variances and covariance

The equations for the variance and covariance are derived as follows:

$$\begin{aligned}\dot{k}_{1,1} &= f-2d_1(1-g_1)k_{11}+d_1(1+g_1)k_1\\ &\quad +\lambda d_2 f(2k_{1,2}+k_2)+4\alpha\lambda(k_1k_{1,2}+k_2k_{1,1})+2\beta\lambda A_0, \qquad (11)\\ \dot{k}_{2,2} &= -2d_2(1-g_2+\lambda f)k_{2,2}+d_2(1+g_2+\lambda f)k_2\\ &\quad -4\alpha\lambda(k_1k_{2,2}+k_2k_{1,2})+2\beta\lambda A_0, \qquad (12)\\ \dot{k}_{1,2} &= -[d_1(1-g_1)+d_2(1-g_2+\lambda f)]k_{1,2}+\lambda d_2 f(k_{2,2}-k_2)\\ &\quad -2\alpha\lambda(k_1k_{1,2}+k_2k_{1,1}-k_1k_{2,2}-k_2k_{1,2})-2\beta\lambda A_0, \qquad (13)\end{aligned}$$

where $\beta=d_1d_2(g_1+g_2)/2$. Stationary values of variances k_{11} and k_{22} are obtained by setting the left hand sides of (11) and (12) equal to zero:

$$\begin{aligned}k_{1,1} &= \frac{1}{2\gamma_1\left(n+\frac{n-m}{m}\theta\right)}\left\{\frac{2}{n\gamma}+\frac{2}{\gamma\gamma_1 mn^2(n-m)}\left[\gamma_1 n(m^2+n)\right.\right.\\ &\left.\left.-\gamma m\left(\gamma_1 n^2(1+m)-m(1+n)\right)\right]\theta+\frac{2(2+n)}{\gamma_1 n^2}\theta^2+\frac{2(n-m)}{\gamma_1 mn^2}\theta^3\right\} \qquad (14)\end{aligned}$$

and

$$\begin{aligned}k_{2,2} &= \frac{1}{2\gamma_2\left(1+m-\frac{n-m}{n}(1-\theta)\right)}\\ &\quad\times\left\{\frac{2(1-m\gamma)(1+n)}{m(n-m)\gamma}\theta+\frac{2(1+n)}{mn\gamma_2}\theta^2+\frac{2(n-m)}{m^2n\gamma_2}\theta^3\right\} \qquad (15)\end{aligned}$$

Self consistent values of θ

Substituting (14) for $k_{1,1}$, (15) for $k_{2,2}$ and

$$k_{1,2}=A_0-k_1k_2=\frac{1}{\gamma_1\gamma_2}\frac{1+m}{m(n-m)}\theta\left[1-\frac{n-m}{n(1+m)}(1-\theta)\right]$$

into the equation which is derived by setting the left hand side of (13) equal to zero yields the fifth order equation for θ. Luckily, however, the highest term vanishes, so that the equation becomes quartic;

$$\theta F(\theta)=0 \qquad (16)$$

where

$$F(\theta)=r_0+r_1\theta+r_2\theta^2+r_3\theta^3. \qquad (17)$$

Here

$$r_0 = \frac{m(1+n)}{\gamma(n-m)}\Big\{\gamma_1\gamma_2\Big[-m+2n(1+n)\Big]+\gamma\Big[\gamma_1 n^2(1+n)$$
$$+\gamma_2\Big(-\gamma_1 n^2(1+n)+m[1+2n+(1-\gamma_1)n^2]\Big)\Big]\Big\}, \tag{18}$$

$$r_1 = -\frac{1}{\gamma m}\Big\{\gamma_1\gamma_2\Big[n^3-4mn(1+n)+m^2(2+n)\Big]+\gamma m\Big[-\gamma_1 n(1+5n+4n^2)$$
$$+\gamma_2\Big(\gamma_1 n^2(2+n)-m[4+8n+(4-\gamma_1)n^2]\Big)\Big]\Big\} \tag{19}$$

$$r_2 = -\frac{n-m}{\gamma m^2}\Big\{\gamma_1\gamma_2(n-m)^2-\gamma m\Big[5\gamma_2 m(1+n)+\gamma_1 n(3+5n)\Big]\Big\}, \tag{20}$$

$$r_3 = \frac{2(n-m)^2}{m^2}\Big\{\gamma_2 m+\gamma_1 n\Big\}. \tag{21}$$