Proposition

1) A K site organized as a single level tree has dynamics

$$\frac{d}{dt}\underline{P}(t) = W\,\underline{P}(t)$$

where $W = -K I_K + q e_K e_K'$.

This matrix has eigenvalues 0 and $-K(K-1)q$ with multiplicity $K-1$, where q is the basic transition rate between any two sites.

Let $K = 2^k$ for some positive integer k.

Ex with $k=2$, the one-level tree has 4 branches with eigenvalue $0, -4q$

pf

$$|\lambda I - W| = |(\lambda + Kq) I_K - q e_K e_K'|$$

$$= (\lambda + Kq)^K \left(1 - \frac{q}{(\lambda + Kq)} e_K' e_K\right)$$

$$= (\lambda + Kq)^{K-1} \lambda$$

~~The matrix W has eigenvalue $\lambda_0 = 0$ and $\lambda_1 = -Kq$ with multiplicity $K-1$~~

To collect 3 sites into a cluster, ii)

define $q = Sp$

with $S = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Then

$$\frac{dq}{dt} = Qq \quad \text{where } Q = SWS'(SS')^{-1}$$
$$= g\begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}$$

This matrix still has eigenvalue 0 and $-4g$.

With $S = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, we group 4 sites into 2 clusters with 2 sites each. The dynamics for $r = Sp$ is

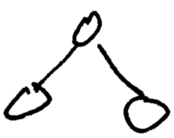
$$\frac{dr}{dt} = Rr$$

with $R = RWR'(RR')^{-1}$

$$= g\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$$

with eigenvalues 0 and $-4g$.

Prop With one level tree, the eigenvalues are invariant with the manner the sites are aggregated into 2 super sites with



$$S = \begin{array}{c} \overbrace{}^{k} \\ \begin{bmatrix} 1\,1 \cdots 1 & 0 \cdots 0 \\ 0 \cdots 0 & 1 \cdots 1 \end{bmatrix} \end{array} = \begin{bmatrix} e_k' & z_{K-k}' \\ z_k' & e_{K-k}' \end{bmatrix}$$ k and $K-k$ sites

(iii)

<u>Pf</u> ~~where~~ Let z_k be a k-dimensional vector of all zero element.

Then

$$SW = -K\cancel{q}S + q\binom{k}{K-k} e_K',$$

and

$$(SWS' + SS')^{-1} = -Kq I_2 + q\binom{k}{K-k}(k, K-k)\begin{bmatrix} \frac{1}{k} & 0 \\ 0 & \frac{1}{K-k} \end{bmatrix}$$

$$= -KqI_2 + q\begin{bmatrix} k & k \\ (K-k) & (K-k) \end{bmatrix} = q\begin{bmatrix} -(K-k) & k \\ K-k & -k \end{bmatrix}.$$

Its eigenvalues are 0 and $-Kq$.

<u>Prop</u> (Consider a tree with k levels)

~~Prop~~ With $K = 2^k$, the tree, aggregated to one level of two clusters with 2^{k-1} sites each, has eigenvalues 0 and $-(2q)^k$.

compared with a tree with one level of 2 site ~~trees~~ by grouping K leaves into L and $K-L$, $0 \le L \le K$ with K and L positive integers.