Liquidity trap and optimal monetary policy in open economies

Tomoyuki Nakajima*
Kyoto University

February, 2005

Abstract

We consider an open-economy model with the Calvo-type sticky prices. We mainly analyze the situation in which the monetary authority in each country cooperates so as to maximize the world welfare. In the case where the zero lower bound (ZLB) on nominal interest rates never binds, the optimal inflation targeting rule in our open-economy model has exactly the same form as in the closed-economy model. This is not the case, however, when the ZLB may bind. The optimal paths are characterized in such a situation. In contrast with what Svensson (2001, 2003, 2004) suggests, the optimal paths of the nominal exchange rate in our model typically exhibit appreciation of the currency of the country where the ZLB binds.

JEL Classification Numbers: E52, F31, F41

Key words: optimal monetary policy, zero lower bound, open economies, nominal rigidities.

*Address: Institute of Economic Research, Kyoto University, Kyoto 606-8501, Japan. Tel: 81-75-753-7132. Email: nakajima@kier.kyoto-u.ac.jp. I have benefitted from discussions with Makoto Saito, and Akihisa Shibata. I also thank seminar participants at Hitotsubashi University, Osaka University and the University of Tokyo for helpful comments and criticisms.

1 Introduction

How should monetary policy be conducted when the zero lower bound (ZLB) for nominal interest rates may bind? The recent experience of Japan is a well-known example of such a "liquidity trap." There, the call rate[1] has been below 0.5 percent per annum since October 1995 and below 0.1 percent since March 1999 (except for the period August 2000-March 2001). Krugman (1998) argues that, even when the nominal interest rate hits the zero bound, the central bank could still stimulate the current level of output by raising expectations of future inflation. This point is further elaborated in a more fully dynamic framework with staggered pricing by Eggertsson and Woodford (2003).[2] Based on an optimization-based, quadratic approximate welfare measure, they analyze the state-contingent paths of inflation, output gap, and nominal interest rates under optimal policy commitment. Furthermore, they derive a price-level targeting rule that could implement the optimal paths.

In this paper, we extend the analysis of Eggertsson and Woodford (2003) to a two-country open-economy model.[3] A continuum of differentiated products are produced in each country, and each good price is adjusted at random intervals as in Calvo (1983). We assume perfect exchange-rate pass-through, so that the law of one price holds. We analyze the optimal state-contingent paths of various variables, and compare our results to the proposal of Svensson (2001, 2003, 2004) that the currency of a country in a liquidity trap should depreciate.

We start with a result on equilibrium shares of consumption across countries. In the literature on open-economy monetary models, it is often assumed that the equilibrium shares of consumption across countries are determined independently of the monetary policy rule adopted by each country. This assumption is a bit problematic. It is true that with complete asset markets (and isoelastic preferences), given initial financial asset holdings and policy rules, the equilibrium shares of consumption are constant at all dates and all contingencies. In general, however, even under these assumptions, given initial asset holdings, different policy rules would result in different equilibrium shares of consumption across countries. We provide sufficient conditions on preferences and initial asset holdings for the equilibrium shares of consumption across the two countries to be independent of the policy rule adopted by each country. They include unit elasticity between goods produced

[1]The call rate is an overnight interest rate, which is analogous to the federal funds rate in the US.

[2]A related problem is studied by Jung, Teranishi and Watanabe (2001).

[3]There is growing literature on open-economy models with nominal rigidities. Examples include, among others, Corsetti and Pesenti (2001), Obstfeld and Rogoff (2002), Clarida, Galí and Gertler (2001), Galí and Monacelli (2002), Benigno and Benigno (2003), Svensson (2004).

in different countries. For simplicity, we assume that those conditions hold in this paper.

As for the policy objective, we mostly focus on the case in which the two monetary authorities cooperate each other to maximize the world welfare, which is defined as the world average of expected lifetime utility of households. Based on a second-order approximation, the objective function of the monetary authorities is given by a quadratic loss function in output gaps and inflation rates. Here, the welfare-relevant inflation rates are not the CPI inflation rates, but the producer-price inflation rate in each country, as in Benigno and Benigno (2003) and Clarida, Galí and Gertler (2001). Thus, fluctuations in the nominal exchange rate per se does not affect the welfare.

We first examine the optimal policy rule when the ZLB is assumed never to bind. Surprisingly, in this case, the optimal inflation targeting rule in our open economy model takes exactly the same form with the same parameter value as the one obtained for the closed economy by Woodford (2003, Section 7.5). That is, the (producer-price) inflation rate in each country must be targeted at the level given by a constant times the rate of change of its output gap. This is true even though the aggregate supply relation shows international dependence, that is, the inflation-output-gap tradeoff in each country is affected by the output gap in the other country. Thus, as long as the ZLB never binds, the monetary authority in each country may forget about international dependence and set the target rate of inflation independently, in order to maximize the world welfare.

This is not the case when the ZLB may bind. In such a case, as our optimal price-level target rule shows, the price-level target in each country is not determined independently. To examine quantitative properties of the optimal state-contingent paths of key variables, we conduct a numerical experiment similar to the one in Eggertsson and Woodford (2003). It shows that the optimal state-contingent paths of inflation, output gaps, and nominal interest rates of the country in a liquidity trap look very similar to those obtained for the closed economy by Eggertsson and Woodford (2003). Such paths in our open economy model are, however, made possible by active policy coordination of the other country.

Regarding exchange rates, our numerical experiment shows that the currency of a country in a liquidity trap must appreciate, rather than depreciate. This makes a sharp contrast to what Svensson (2001, 2003, 2004) proposes. A theoretical justification for his proposal is made in Svensson (2004). While our model and Svensson's (2004) differ in several ways,[4] the difference in the evolution of nominal exchange rates under optimal policy arises from his assumption that (i) the shock that generates a liquidity trap lasts

[4]For instance, Svensson (2004) considers prices that are set one period in advance, rather than staggered pricing. Also, in his model, a country in a liquidity trap may or may not be small; there may or may not be international coordination of monetary policy.

only for one period and (ii) the country in a liquidity trap has a productivity level which is higher than the steady-state level. Note that a country falls into a liquidity trap, say, at date 0, if the expected growth rate of productivity in that country from date 0 to date 1 is sufficiently negative. Svensson (2004) creates such a situation by assuming that the productivity of a country at date 0 is unusually high (and the expected level of productivity at date 1 is normal). This is why he observes depreciation at date 0 under optimal policy: Since the country produces more than normally the relative price of goods produced in that country to those produced abroad must fall. Because the price of a good is set one period in advance, this fall in the relative price is achieved by a depreciation. However, even in this scenario, if the shock lasts for more than one period so that the expected growth rate of productivity is negative, say, for dates $t = 0, \ldots, \tau$, then the currency would depreciate at all $t = 1, \ldots, \tau$ under optimal policy commitment. In this sense, our claim that the currency of a country in a liquidity trap depreciates under optimal policy commitment holds more robustly.

The rest of the paper is organized as follows. In Section 2, we describe households and give a proposition on the equilibrium share of consumption. In Section 3, the aggregate-supply relations are derived. In Section 4, a quadratic approximate world welfare measure is computed, and an optimal inflation targeting rule is derived for the case where the ZLB is assumed never to bind. In Section 5, the natural rates of interest are defined and the "intertemporal IS equations" are obtained. In Section 6, optimal policy in a liquidity trap is analyzed. In Section 7, the state-contingent paths of nominal exchange rates under optimal policy is discussed. Section 8 is concluding remarks.

2 Households

The model economy is an open-economy version of the sticky-price model developed by Woodford (2003). In particular, goods prices are adjusted at random intervals as in Calvo (1983); households supply differentiated labor; and monetary frictions are abstracted so that money is not modelled explicitly (a "cashless economy" in the terminology of Woodford, 2003).

The world economy consists of two countries, Home (H) and Foreign (F). The size of population in country $j \in \{H, F\}$ is n_j. We normalize the world population to unity: $n_H + n_F = 1$. A set of differentiated products are produced in each country and they are traded between the two countries. Let N_j denote the set of those products. We assume that

$$N_H = [0, n_H], \quad \text{and} \quad N_F = (n_H, 1].$$

Each country is resided by identical households, who consume differentiated commodities, supply labor, and own firms in their country. The monetary and fiscal policy is set by the government in each country. Governments do not consume, and set fiscal policy in the "Ricardian" way in the sense used by Benhabib, Schmitt-Grohé and Uribe (2001), among others. The detail of monetary policy is discussed later.

2.1 Preferences

A representative household in H has preferences given by

$$U_0 = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \tilde{u}(C_t) - \frac{1}{n_H} \int_{N_H} \tilde{v}\left[\ell_t(i)\right] di \right\}, \tag{1}$$

where $0 < \beta < 1$, $\sigma > 0$, and $\ell_t(i)$, $i \in N_H$, is the supply of type-i labor, which is used to produce differentiated product i. We assume that $\tilde{u}$ and $\tilde{v}$ have constant elasticity:

$$\tilde{u}(C) \equiv \frac{C^{1-\sigma}}{1-\sigma}, \qquad \tilde{v}(\ell) \equiv \frac{1}{1+\omega}\ell^{1+\omega}.$$

The consumption index for the home household, C_t, is given by

$$C_t = \left[n_H^{\frac{1}{\rho}} C_{H,t}^{\frac{\rho-1}{\rho}} + n_F^{\frac{1}{\rho}} C_{F,t}^{\frac{\rho-1}{\rho}} \right]^{\frac{\rho}{\rho-1}}, \tag{2}$$

where $C_{H,t}$ and $C_{F,t}$ are the consumption indexes of home and foreign goods consumed by the home household, respectively, which are defined by:

$$C_{j,t} = \left[n_j^{-\frac{1}{\theta}} \int_{N_j} c_t(i)^{\frac{\theta-1}{\theta}} di \right]^{\frac{\theta}{\theta-1}}, \qquad j = H, F. \tag{3}$$

Here, $\theta > 1$ and $c_t(i) \in N_j$ is the home household's consumption of good i produced in country $j \in \{H, F\}$. It is convenient to define the function $u(C_H, C_F)$ by

$$u(C_H, C_F) \equiv \tilde{u}\left(\left[n_H^{\frac{1}{\rho}} C_H^{\frac{\rho-1}{\rho}} + n_F^{\frac{1}{\rho}} C_F^{\frac{\rho-1}{\rho}} \right]^{\frac{\rho}{\rho-1}} \right)$$

The lifetime utility of a representative household in F takes the same form as that of the home household:

$$U_0^* = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \tilde{u}(C_t^*) - \frac{1}{n_F} \int_{N_F} \tilde{v}\left[\ell_t^*(i)\right] di \right\}. \tag{4}$$

The consumption indexes for the foreign household, $\{C_t^*, C_{H,t}^*, C_{F,t}^*\}$, are defined as in (2) and (3):

$$C_t^* = \left[n_H^{\frac{1}{\rho}} C_{H,t}^{*\,\frac{\rho-1}{\rho}} + n_F^{\frac{1}{\rho}} C_{F,t}^{*\,\frac{\rho-1}{\rho}} \right]^{\frac{\rho}{\rho-1}}, \tag{5}$$

$$C_{j,t}^* = \left[n_j^{-\frac{1}{\theta}} \int_{N_j} c_t^*(i)^{\frac{\theta-1}{\theta}}\, di \right]^{\frac{\theta}{\theta-1}}, \qquad j = H, F. \tag{6}$$

Corresponding to the consumption indexes in country H, C_t, $C_{j,t}$, $j = H, F$, the prices indexes, P_t, $P_{j,t}$, $j = H, F$, are defined as

$$P_t = \left[n_H P_{H,t}^{1-\rho} + n_F P_{F,t}^{1-\rho} \right]^{\frac{1}{1-\rho}},$$

$$P_{j,t} = \left[\frac{1}{n_j} \int_{N_j} p_t(i)^{1-\theta}\, di \right]^{\frac{1}{1-\theta}}, \qquad j = H, F, \tag{7}$$

where $p_t(i)$, $i \in N_j$, $j \in \{H, F\}$, is the price of good i produced in country j quoted in the home currency. The price indexes in country F, P_t^*, $P_{j,t}^*$, $j = H, F$, are defined similarly by individual good prices, $p_t^*(i)$, $i \in N_j$, $j \in \{H, F\}$, quoted in the foreign currency. We assume that the law of one price holds:

$$p_t(i) = \mathcal{E}_t p_t^*(i),$$

for all $i \in N_j$, $j \in \{H, F\}$, where $\mathcal{E}_t$ is the nominal exchange rate, defined as the price of foreign currency in terms of home currency. It follows that $P_{j,t} = \mathcal{E}_t P_{j,t}^*$, $j = H, F$ and $P_t = \mathcal{E}_t P_t^*$.

Cost minimization leads to the derived demands of the home household:

$$C_{j,t} = n_H C_t \left(\frac{P_{j,t}}{P_t} \right)^{-\rho}, \qquad j = H, F,$$

$$c_t(i) = \frac{1}{n_j} C_{j,t} \left(\frac{p_t(i)}{P_{j,t}} \right)^{-\theta}, \qquad j = H, F. \tag{8}$$

The derived demands of the foreign household are written similarly.

2.2 Utility maximization

We assume worldwide complete markets. The flow budget constraint for the home household is

$$P_t C_t + E_t[Q_{t,t+1} W_{t+1}] = W_t + \int_{N_H} \left[w_t(i) \ell_t(i) + \Pi_t(i) \right] di + T_t, \tag{9}$$

where E_t is the conditional expectation operator, $Q_{t,t+1}$ is the stochastic discount factor between dates t and $t+1$ for nominal payoffs in the home country, W_{t+1} is the portfolio of one-period state-contingent bonds, $w_t(i)$ is the date-t nominal wage rate for type $i \in N_H$ labor, $\Pi_t(i)$ is the date-t nominal profits from sales of good $i \in N_H$, and T_t is the nominal lump-sum transfers from the home government.

To prevent "Ponzi schemes," the home household also faces the "natural debt limit" (Ljungqvist and Sargent, 2000):

$$W_{t+1} \geq -\sum_{s=t+1}^{\infty} E_t \left[Q_{t+1,s} \left\{ \int_{N_H} \left[w_s(i)\ell_s(i) + \Pi_s(i) \right] di + T_s \right\} \right], \tag{10}$$

where $Q_{t,s}$ is the stochastic discount factor between dates t and s, defined by $Q_{t,s} = \prod_{\tau=t+1}^{s} Q_{\tau-1,\tau}$. Given the initial asset holding, W_0, the home household maximizes the lifetime utility (1) subject to (9) and (10).

The flow budget constraint for the foreign household is analogously expressed:

$$\mathcal{E}_t P_t^* C_t + E_t[Q_{t,t+1}\mathcal{E}_{t+1} W_{t+1}^*] = \mathcal{E}_t W_t^* + \int_{N_F} \mathcal{E}_t \left[w_t^*(i)\ell_t^*(i) + \Pi_t^*(i) \right] di + T_t^*,$$

where W_{t+1}^* is the portfolio of the state-contingent bonds in the foreign currency, $w_t^*(i)$, is the nominal wage rate of type $i \in N_F$ labor, $\Pi_t^*(i)$ is the nominal profit from sales of good $i \in N_F$, and T_t^* is the nominal transfer from the foreign government. Given the initial asset holding W_0^*, the utility maximization problem for the foreign household is defined similarly to that for the home household.

The first-order conditions that $\{C_t, C_t^*, \ell_t(i), \ell_t^*(i)\}$ must satisfy are given by

$$\frac{\beta \tilde{u}_c(C_{t+1})}{\tilde{u}_c(C_t)} = \frac{\beta \tilde{u}_c(C_{t+1}^*)}{\tilde{u}_c(C_t^*)} = Q_{t,t+1}\frac{P_{t+1}}{P_t}, \tag{11}$$

and

$$\frac{1}{n_H}\frac{\tilde{v}_\ell[\ell_t(i)]}{\tilde{u}_c(C_t)} = \frac{w_t(i)}{P_t}, \quad i \in N_H, \qquad \frac{1}{n_F}\frac{\tilde{v}_\ell[\ell_t^*(i)]}{\tilde{u}_c(C_t^*)} = \frac{w_t^*(i)}{P_t^*}, \quad i \in N_F.$$

As we shall see, what the policy makers should stabilize is not P_t or P_t^*, but $P_{H,t}$ and $P_{F,t}^*$. For this reason, the first-order conditions in terms of $C_{j,t}$ and $C_{j,t}^*$, $j = N, H$, turn out to be more relevant. They are given by

$$\frac{\beta u_H(C_{H,t+1}, C_{F,t+1})}{u_H(C_{H,t}, C_{F,t})} = \frac{\beta u_H(C_{H,t+1}^*, C_{F,t+1})^*}{u_H(C_{H,t}^*, C_{F,t}^*)} = Q_{t,t+1}\frac{P_{H,t+1}}{P_{H,t}} \tag{12}$$

$$\frac{\beta u_F(C_{H,t+1}, C_{F,t+1})}{u_F(C_{H,t}, C_{F,t})} = \frac{\beta u_F(C_{H,t+1}^*, C_{F,t+1})^*}{u_F(C_{H,t}^*, C_{F,t}^*)} = Q_{t,t+1}\frac{P_{F,t+1}}{P_{F,t}} \tag{13}$$

$$\frac{1}{n_H}\frac{\tilde{v}_\ell[\ell_t(i)]}{u_H(C_{H,t}, C_{F,t})} = \frac{w_t(i)}{P_{H,t}} \tag{14}$$

$$\frac{1}{n_F}\frac{\tilde{v}_\ell[\ell_t^*(i)]}{u_F(C_{H,t}^*, C_{F,t}^*)} = \frac{w_t^*(i)}{P_{F,t}^*} \tag{15}$$

$$\frac{u_F(C_{H,t}, C_{F,t})}{u_H(C_{H,t}, C_{F,t})} = \frac{u_F(C_{H,t}^*, C_{F,t}^*)}{u_H(C_{H,t}^*, C_{F,t}^*)} = \frac{\mathcal{E}_t P_{F,t}^*}{P_{H,t}} \tag{16}$$

Here, $u_j(C_H, C_F)$ denotes the partial derivative of $u(C_H, C_F)$ with respect to C_j, $j = H, F$.

2.3 Equilibrium shares of consumption

Since asset markets are complete, our formulation of the lifetime utility, (1) and (4), implies that, given a policy rule of each country and the initial asset holdings, W_0 and W_0^*, the relative amount of consumption, C_t/C_t^*, becomes constant at all dates and all contingencies in equilibrium. But this amount, C_t/C_t^*, depends, in general, upon the policy rule adopted in each country. In particular, the equilibrium amount of C_t/C_t^* would depend on the process of prices, $\{P_{H,t}, P_{F,t}^*\}$, which, in turn, depends on the monetary policy conducted in the two countries. For our welfare analysis, it is convenient if the share of consumption is determined independently of policy. The next proposition shows that this is the case when the elasticity of substitution between C_H and C_F is unity and when the initial financial asset of the household in country j equals the initial liabilities of the country-j government. The latter assumption implies that

$$n_H W_0 = E_0 \sum_{t=0}^{\infty} Q_{0,t} \left[P_{H,t} Y_{H,t} - n_H \left\{ T_t + \int_{N_H} \left[w_t(i)\ell_t(i) + \Pi_t(i) \right] di \right\} \right]$$

$$n_F \mathcal{E}_0 W_0^* = E_0 \sum_{t=0}^{\infty} Q_{0,t} \mathcal{E}_t \left[P_{F,t}^* Y_{F,t} - n_F \left\{ T_t^* + \int_{N_F} \left[w_t^*(i)\ell_t^*(i) + \Pi_t^*(i) \right] di \right\} \right],$$

where $Y_{j,t} \equiv n_H C_{j,t} + n_F C_{j,t}^*$ is the aggregate amount of country-j goods, $j \in \{H, F\}$.

Proposition. *Assume that*

(a) the elasticity of substitution between home-goods and foreign-goods is unity: $\rho = 1$; and

(b) the initial financial asset of the household in country $j \in \{H, F\}$ equals the (per-capita amount of the) initial liabilities of the country-j government.

Then the relative amount of consumption between the two countries is determined independently from the policy rule adopted in each country. Indeed, the equilibrium relative consumption equals unity:

$$\frac{C_t}{C_t^*} = \frac{C_{H,t}}{C_{H,t}^*} = \frac{C_{F,t}}{C_{F,t}^*} = \frac{c_t(i)}{c_t^*(i)} = 1, \qquad i \in [0, 1].$$

The proof is in Appendix. In what follows we assume (a)-(b) in the proposition. Thus, the consumption indexes (2) and (5) become

$$C = \left(\frac{C_H}{n_H} \right)^{n_H} \left(\frac{C_F}{n_F} \right)^{n_F}, \qquad C^* = \left(\frac{C_H^*}{n_H} \right)^{n_H} \left(\frac{C_F^*}{n_F} \right)^{n_F}.$$

It follows that $u(C_H, C_F)$ is written as

$$u(C_H, C_F) = \frac{1}{1-\sigma}\left(\frac{C_H}{n_H}\right)^{n_H(1-\sigma)}\left(\frac{C_F}{n_F}\right)^{n_F(1-\sigma)},$$

and the price index, P_t, become

$$P_t = P_{H,t}^{n_H} P_{F,t}^{n_F} = \mathcal{E}_t P_t^* = \mathcal{E}_t P_{H,t}^{*\,n_H} P_{F,t}^{*\,n_F}.$$

Let $y_t(i)$, $i \in N_H$, and $y_t^*(i)$, $i \in N_F$, be the aggregate supply of home and foreign goods, respectively:

$$y_t(i) = n_H c_t(i) + n_F c_t^*(i), \quad i \in N_H, \qquad y_t^*(i) = n_H c_t(i) + n_F c_t^*(i), \quad i \in N_F.$$

The corresponding production indexes of home and foreign goods are

$$\begin{aligned}
Y_{H,t} &\equiv \left[n_H^{-\frac{1}{\theta}} \int_{N_H} y_t(i)^{\frac{\theta-1}{\theta}}\, di\right]^{\frac{\theta}{\theta-1}} = n_H C_{H,t} + n_F C_{H,t}^*, \\
Y_{F,t} &\equiv \left[n_F^{-\frac{1}{\theta}} \int_{N_F} y_t^*(i)^{\frac{\theta-1}{\theta}}\, di\right]^{\frac{\theta}{\theta-1}} = n_H C_{F,t} + n_F C_{F,t}^* \\
Y_t &\equiv \left(\frac{Y_{H,t}}{n_H}\right)^{n_H}\left(\frac{Y_{F,t}}{n_F}\right)^{n_F} = n_H C_t + n_F C_t^*
\end{aligned}$$

It follows from the proposition that

$$C_{H,t} = C_{H,t}^* = Y_{H,t}, \qquad C_{F,t} = C_{F,t}^* = Y_{F,t}, \qquad C_t = C_t^* = Y_t. \tag{17}$$

3 Aggregate supply

3.1 Technology

For simplicity, we assume that the technology to produce each good is linear in labor:

$$\begin{aligned}
y_t(i) &= A_t n_H \ell_t(i), \qquad i \in N_H, \\
y_t^*(i) &= A_t^* n_F \ell_t^*(i), \qquad i \in N_F,
\end{aligned}$$

where A_t and A_t^* represent country-specific technology shocks.

For later use, it is convenient to define random variables ξ_t and ξ_t^* by

$$\xi_t \equiv -(1+\omega)\ln A_t, \qquad \xi_t^* \equiv -(1+\omega)\ln A_t^*,$$

and also functions $v(y;\xi)$ and $v^*(y^*;\xi^*)$ by

$$\begin{aligned}
v(y;\xi) &\equiv \frac{e^{\xi}}{1+\omega}\left(\frac{y}{n_H}\right)^{1+\omega} = \tilde{v}\left(\frac{y}{n_H A}\right), \\
v^*(y^*;\xi^*) &\equiv \frac{e^{\xi^*}}{1+\omega}\left(\frac{y^*}{n_F}\right)^{1+\omega} = \tilde{v}\left(\frac{y^*}{n_F A^*}\right).
\end{aligned}$$

Thus, $v(y;\xi)$ and $v^*(y^*;\xi^*)$ measure the disutility of producing y and y^* in the home and foreign countries, respectively, when their technology shocks are ξ and ξ^*. Note that

$$v_y(y;\xi) = \frac{\tilde{v}_\ell(\ell)}{n_H A}, \qquad v_y^*(y^*;\xi^*) = \frac{\tilde{v}_\ell(\ell^*)}{n_F A^*}.$$

Thus, first-order conditions (14) and (15) are rewritten as

$$\frac{v_y[y_t(i);\xi_t]}{u_H(Y_{H,t},Y_{F,t})} = \frac{1}{A_t}\frac{w_t(i)}{P_{H,t}}, \qquad \frac{v_y^*[y_t^*(i);\xi_t^*]}{u_F(Y_{H,t},Y_{F,t})} = \frac{1}{A_t^*}\frac{w_t^*(i)}{P_{F,t}^*}, \tag{18}$$

where we have used equilibrium condition (17).

3.2 Natural rates of output

Each producer is assumed to be a wagetaker.[5] Using (8) and (17), nominal profits of a home supplier of good $i \in N_H$ at date t are given by

$$\left[(1-\tau)p_t(i) - \frac{w_t(i)}{A_t}\right] y_t(i) = \left[(1-\tau)p_t(i) - \frac{w_t(i)}{A_t}\right] \frac{Y_{H,t}}{n_H}\left[\frac{p_t(i)}{P_{H,t}}\right]^{-\theta} = n_H \Pi_t(i),$$

where τ is the constant tax rate on firms' revenue. Monopoly profits of a foreign firm is defined similarly with τ^* as the tax rate on its revenue.

Let us define the "natural rates of output" (Woodford, 2003) at date t, $Y_{H,t}^n$ and $Y_{F,t}$, as the levels of home and foreign output which would prevail in the flexible-price equilibrium. Suppose, momentarily, that all prices are fully flexible. Profit maximization leads to:

$$\begin{aligned} (1-\Phi)\frac{p_t(i)}{P_{H,t}} &= \frac{w_t(i)}{P_{H,t}A_t} = \frac{v_y[y_t(i);\xi_t]}{u_H(Y_{H,t},Y_{F,t})}, \qquad i \in N_H, \\ (1-\Phi^*)\frac{p_t^*(i)}{P_{F,t}^*} &= \frac{w_t^*(i)}{P_{F,t}^*A_t^*} = \frac{v_y^*[y_t^*(i);\xi_t^*]}{u_F(Y_{H,t},Y_{F,t})}, \qquad i \in N_F, \end{aligned}$$

where we have used (18) and Φ and Φ^* are the measures of distortions due to market power defined by

$$1-\Phi = \frac{\theta-1}{\theta}(1-\tau), \qquad 1-\Phi^* = \frac{\theta-1}{\theta}(1-\tau^*).$$

In the flexible-price equilibrium, $p_t(i) = P_{H,t}$ and $y_t(i) = Y_{H,t}/n_H$ for all $i \in N_H$, and $p_t^*(i) = P_{F,t}^*$ and $y_t^*(i) = Y_{F,t}/n_F$ for all $i \in N_F$. Thus, the natural rates of output, $Y_{H,t}^n$ and $Y_{F,t}^n$, are determined by

$$\frac{v_y[Y_{H,t}^n/n_H;\xi_t]}{u_H(Y_{H,t}^n,Y_{F,t}^n)} = 1-\Phi, \qquad \frac{v_y^*[Y_{F,t}^n/n_F;\xi_t^*]}{u_F(Y_{H,t}^n,Y_{F,t}^n)} = 1-\Phi^*. \tag{19}$$

[5]See Woodford (2003, Section 3.1) for how to make this assumption consistent with the supposition that each producer uses a different type of labor.

3.3 New Keynesian aggregate supply relation

Now suppose that goods prices are adjusted at random intervals as in Calvo (1983). Let α be the probability that each good price remains unchanged in each period. We assume that this probability is identical in the two countries.

Consider the price adjustment in the home country. Suppose that the price of good $i \in N_H$ can be adjusted at date t. The supplier of that good chooses $p_t(i)$ to maximize the expected discounted profits:

$$E_t \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left\{ \left[(1-\tau) p_t(i) - \frac{w_T(i)}{A_T} \right] \frac{Y_{H,T}}{n_H} \left[\frac{p_t(i)}{P_{H,T}} \right]^{-\theta} \right\}$$

The first-order condition for profit maximization is written as

$$E_t \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} \left\{ \frac{Y_{H,T}}{n_H} \left(\frac{p_t(i)}{P_{H,T}} \right)^{-\theta-1} \left[\frac{v_y[y_T(i);\xi_T]}{u_H(Y_{H,T}, Y_{F,T})} - (1-\Phi) \frac{p_t(i)}{P_{H,T}} \right] \right\} = 0. \tag{20}$$

It follows that all producers that change their prices at date t choose the same price. Let z_t denote this common new price.

As is shown in Appendix, log-linearization of (20) and the corresponding equation for the foreign country leads to the "New Keynesian" aggregate-supply relations:

$$\pi_{H,t} = \gamma_H x_{H,t} + \gamma_{HF} n_F x_{F,t} + \beta E_t \pi_{H,t+1}, \tag{21}$$

$$\pi^*_{F,t} = \gamma_{HF} n_H x_{H,t} + \gamma_F x_{F,t} + \beta E_t \pi^*_{F,t+1}. \tag{22}$$

Here $\pi_{H,t}$ and $\pi^*_{F,t}$ are the inflation rates of goods produced in the home and foreign countries, respectively:

$$\pi_{H,t} \equiv \ln P_{H,t} - \ln P_{H,t-1}, \qquad \pi^*_{F,t} \equiv \ln P^*_{F,t} - \ln P^*_{F,t-1},$$

$x_{j,t}$ is the "output gap" in country $j = H, F$:

$$x_{j,t} \equiv \ln Y_{j,t} - \ln Y^n_{j,t},$$

and the coefficients are given by

$$\gamma_H \equiv \zeta \big[1 + \omega + (\sigma - 1) n_H \big] > 0, \tag{23}$$

$$\gamma_{HF} \equiv \zeta(\sigma - 1), \tag{24}$$

$$\gamma_F \equiv \zeta \big[1 + \omega + (\sigma - 1) n_F \big] > 0, \tag{25}$$

$$\zeta \equiv \frac{1-\alpha}{\alpha} \frac{1-\alpha\beta}{1+\omega\theta}. \tag{26}$$

4 Welfare approximation

In this section we follow Woodford (2003) and Benigno and Woodford (2004) to derive an approximate world welfare criterion and discuss optimal policy rules for the case where the zero lower bound (ZLB) on the nominal interest rates never binds. We assume that the monetary authorities in the two countries cooperate to maximize aggregate utility. The case where monetary policy is set in the noncooperative fashion is discussed in Appendix.

Given that preferences of the home and foreign household are given by (1) and (4), respectively, the level of average expected utility between the two countries is given by

$$n_H U_0 + n_F U_0^* = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u(Y_{H,t}, Y_{F,t}) - \int_{N_H} v\left[y_t(i); \xi_t\right] di - \int_{N_F} v^*\left[y_t^*(i); \xi_t^*\right] di \right\}$$

Using the conditions that

$$y_t(i) = \frac{1}{n_H} Y_{H,t} \left[\frac{p_t(i)}{P_{H,t}}\right]^{-\theta}, \qquad y_t^*(i) = \frac{1}{n_F} Y_{F,t} \left[\frac{p_t^*(i)}{P_{F,t}^*}\right]^{-\theta},$$

we obtain

$$\int_{N_H} v\left[y_t(i); \xi_t\right] di = n_H v\left(\frac{Y_{H,t}}{n_H}; \xi_t\right) \Delta_{H,t},$$
$$\int_{N_F} v^*\left[y_t^*(i); \xi_t^*\right] di = n_F v^*\left(\frac{Y_{F,t}}{n_F}; \xi_t^*\right) \Delta_{F,t},$$

where Δ_H and Δ_F are the measures of price dispersion defined by

$$\Delta_{H,t} \equiv \frac{1}{n_H} \int_{N_H} \left(\frac{p_t(i)}{P_{H,t}}\right)^{-\theta(1+\omega)} di, \tag{27}$$
$$\Delta_{F,t} \equiv \frac{1}{n_F} \int_{N_F} \left(\frac{p_t^*(i)}{P_{F,t}^*}\right)^{-\theta(1+\omega)} di.$$

Then the world welfare measure is given by

$$n_H U_0 + n_F U_0^* = E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ u(Y_{H,t}, Y_{F,t}) \tag{28}$$
$$- n_H v\left(\frac{Y_{H,t}}{n_H}; \xi_t\right) \Delta_{H,t} - n_F v^*\left(\frac{Y_{F,t}}{n_F}; \xi_t^*\right) \Delta_{F,t} \Bigg\}$$

Now, fix a non-stochastic steady state with zero inflation. In what follows, a bar over a variable denotes its steady state value, and a hat indicates the log-deviation from the steady-state value. Following Benigno and Woodford (2004), we take a second-order approximation of (28) around that steady state in terms of $\Xi \equiv (\xi, \xi^*, \hat{\Delta}_{H,-1}^{1/2}, \Delta_{F,-1}^{1/2}, \varphi)$, where φ are parameters of policy rules normalized in such a way that $\varphi = 0$ implies long-run output levels $Y_{j,\infty} = \bar{Y}_j$, $j = H, F$, and for any small enough φ, $\ln Y_{j,\infty} - \ln \bar{Y}_j = \mathcal{O}(\|\varphi\|)$, $j = H, F$. Then, as shown in Appendix,

$$
\begin{aligned}
&u(Y_{H,t}, Y_{F,t}) - n_H v\left(\frac{Y_{H,t}}{n_H}; \xi_t\right) \Delta_{H,t} - n_F v\left(\frac{Y_{F,t}}{n_F}; \xi_t^*\right) \Delta_{F,t} \qquad (29)\\
&\quad = \frac{\bar{u}_H \bar{Y}_H}{n_H} \Big\{ n_H \Phi \hat{Y}_{H,t} + n_F \Phi^* \hat{Y}_{F,t} + \frac{n_H}{2}\Big[(1-\sigma)n_H - (1+\omega)(1-\Phi)\Big]\hat{Y}_{H,t}^2 \\
&\qquad + (1-\sigma)n_H n_F \hat{Y}_{H,t}\hat{Y}_{F,t} + \frac{n_F}{2}\Big[(1-\sigma)n_F - (1+\omega)(1-\Phi^*)\Big]\hat{Y}_{F,t}^2 \\
&\qquad - n_H(1-\Phi)\hat{Y}_{H,t}\xi_t - n_F(1-\Phi^*)\hat{Y}_{F,t}\xi_t^* - \frac{n_H(1-\Phi)}{1+\omega}\hat{\Delta}_{H,t} - \frac{n_F(1-\Phi^*)}{1+\omega}\hat{\Delta}_{F,t} \Big\} \\
&\qquad + \text{t.i.p.} + \mathcal{O}(\|\Xi\|^3),
\end{aligned}
$$

where t.i.p. denotes the terms independent of policy. Also, it is shown that

$$\sum_{t=0}^{\infty} \beta^t \frac{n_H}{1+\omega}\hat{\Delta}_{H,t} = \frac{\alpha\theta(1+\omega\theta)}{(1-\alpha)(1-\alpha\beta)} \sum_{t=0}^{\infty} \beta^t \frac{n_H}{2}\pi_{H,t}^2 + \text{t.i.p.} + O(\|\Xi\|^3), \qquad (30)$$

$$\sum_{t=0}^{\infty} \beta^t \frac{n_F}{1+\omega}\hat{\Delta}_{F,t} = \frac{\alpha\theta(1+\omega\theta)}{(1-\alpha)(1-\alpha\beta)} \sum_{t=0}^{\infty} \beta^t \frac{n_F}{2}\pi_{F,t}^{*2} + \text{t.i.p.} + O(\|\Xi\|^3). \qquad (31)$$

4.1 Small distortions at the steady state

The existence of linear terms in (29), $n_H \Phi \hat{Y}_{H,t} + n_F \Phi^* \hat{Y}_{F,t}$, might complicate the analysis. Here, for simplicity, we follow Woodford (2003) and assume that Φ and Φ^* are small and treat them as expansion parameters in the Taylor series approximation.[6] Let Y_j^e, $j = H, F$, denote the efficient levels of output in the absence of shocks, that is,

$$\frac{v_y(Y_H^e/n_H; 0)}{u_H(Y_H^e, Y_F^e)} = 1, \qquad \frac{v_y^*(Y_F^e/n_F; 0)}{u_F(Y_H^e, Y_F^e)} = 1.$$

Then, let x_j^e, $j = H, F$, denote the efficient levels of the output gaps:

$$x_j^e \equiv \ln Y_j^e - \ln \bar{Y}_j, \qquad j = H, F,$$

where $\bar{Y}_j$, $j = H, F$, are the steady-state levels of output. When Φ and Φ^* are small,

$$
\begin{aligned}
\big[1 + \omega + (\sigma-1)n_H\big]x_H^e + (\sigma-1)n_F x_F^e &= \Phi + O(\|\Phi\|^2), \\
(\sigma-1)n_H x_H^e + \big[1 + \omega + (\sigma-1)n_F\big]x_F^e &= \Phi^* + O(\|\Phi^*\|^2)
\end{aligned}
$$

[6] Benigno and Woodford (2004) study the case where Φ is not small.

Then, using (29)-(31), a second-order approximation of the world welfare measure (28) is given by

$$n_H U_0 + n_F U_0^* = -\frac{\bar{u}_H \bar{Y}_H}{n_H}\frac{\theta}{\zeta}\sum_{t=0}^{\infty}\beta^t L_t + \text{t.i.p.} + O(\|\Phi, \Phi^*, \Xi\|^3),$$

where ζ is as defined in (26), and the loss function L_t is given by

$$L_t \equiv \frac{1}{2}(x_t - x^e)'\Lambda(x_t - x^e) + \frac{n_H}{2}\pi_{H,t}^2 + \frac{n_F}{2}\pi_{F,t}^{*2}, \tag{32}$$

Here, $x_t \equiv (x_{H,t}, x_{F,t})'$, $x_t^e \equiv (x_H^e, x_F^e)'$, and Λ is defined by

$$\Lambda \equiv \frac{1}{\theta}\begin{bmatrix} \gamma_H n_H & \gamma_{HF} n_H n_F \\ \gamma_{HF} n_H n_F & \gamma_F n_F \end{bmatrix},$$

where γ_H, γ_{HF}, and γ_F are defined in (23)-(25).

Our welfare measure clearly shows that what must be stabilized are "domestic inflation," $\pi_{H,t}$ and $\pi_{F,t}^*$, rather than CPI inflation rates, π_t and π_t^*, as shown by Clarida, Galí and Gertler (2001) and Benigno and Benigno (2003) in different contexts. In particular, fluctuations in nominal exchange rate per se do not affect the world welfare.

4.2 Optimal targeting rule in the case where the ZLB never binds

In this subsection we analyze optimal policy when the zero bound on the nominal interest rates never binds. In particular, we derive "robustly optimal target criteria" (Giannoni and Woodford, 2002; Woodford 2003). Interestingly, those targeting rules have exactly the same form with the same parameter values as those obtained in the closed-economy model.[7]

In order to introduce a tradeoff between inflation and output stabilization, we modify the aggregate-supply relation (21)-(22) so that

$$\pi_{H,t} = \gamma_H x_{H,t} + \gamma_{HF} n_F x_{F,t} + \beta E_t \pi_{H,t+1} + u_{H,t}, \tag{33}$$

$$\pi_{F,t}^* = \gamma_{HF} n_H x_{H,t} + \gamma_F x_{F,t} + \beta E_t \pi_{F,t+1}^* + u_{F,t}, \tag{34}$$

where $u_{j,t}$, $j = H, F$, are interpreted as "cost-push shocks" (Woodford, 2003).

Suppose that the zero bounds on the nominal interest rates never binds. Then the optimal policy commitment is derived by solving $\min(1/2)E_0\sum_t L_t$ subject to (33)-(34).

[7] Clarida, Galí and Gertler (2001) derive an optimal inflation targeting rule for a small open economy.

The Lagrangian is formed as

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ \frac{1}{2}(x_t - x_t^e)' \Lambda (x_t - x_t^e) + \frac{n_H}{2}\pi_{H,t}^2 + \frac{n_F}{2}\pi_{F,t}^{*2}$$
$$+ \psi_{H,t}\Big(n_H \pi_{H,t} - \gamma_H n_H x_{H,t} - \gamma_{HF} n_H n_F x_{F,t} - \beta n_H \pi_{H,t+1} \Big)$$
$$+ \psi_{F,t}\Big(n_F \pi_{F,t}^* - \gamma_{HF} n_H n_F x_{H,t} - \gamma_F n_F x_{F,t} - \beta n_F \pi_{F,t+1}^* \Big) \Big\}$$

The first-order conditions are

$$\Lambda(x_t - x_t^e) = \theta \Lambda \psi_t,$$
$$\pi_{j,t} = -\psi_{j,t} + \psi_{j,t-1}, \qquad j = H, F,$$

where $\psi_t \equiv (\psi_{H,t}, \psi_{F,t})'$. Eliminating ψ_t, we obtain a robustly optimal target rule:[8]

$$\pi_{H,t} + \frac{1}{\theta}(x_{H,t} - x_{H,t-1}) = 0, \qquad \pi_{F,t}^* + \frac{1}{\theta}(x_{F,t} - x_{F,t-1}) = 0. \tag{35}$$

Note that these targeting rules have exactly the same form with the same parameter values as those obtained for the closed economy (for example, by Woodford, 2003). It follows that, under our parametric assumptions and as long as the zero bound does not bind, the world welfare is maximized when each monetary authority acts independently and adopts the policy rule which is optimal for the closed economy. Later, we shall see that this is not the case when the zero bound binds.

5 Nominal interest rates and the ZLB

Let $i_{H,t}$ and $i_{F,t}$ be the (one-period) nominal interest rates at date t in the home and foreign countries, respectively. By definition,

$$\frac{1}{1+i_{H,t}} = E_t[Q_{t,t+1}], \qquad \frac{1}{1+i_{F,t}} = E_t\left[Q_{t,t+1}\frac{\mathcal{E}_{t+1}}{\mathcal{E}_t}\right].$$

Using first-order conditions (11)-(13) and equilibrium condition (17), we could relate the nominal interest rates and the growth rates of output in the following way:

$$\frac{1}{1+i_{H,t}} = E_t\left[\frac{\beta \tilde{u}_{c,t+1}}{\tilde{u}_{c,t}}\frac{P_t}{P_{t+1}}\right] = E_t\left[\frac{\beta u_{H,t+1}}{u_{H,t}}\frac{P_{H,t}}{P_{H,t+1}}\right] = E_t\left[\frac{\beta u_{F,t+1}}{u_{F,t}}\frac{P_{F,t}}{P_{F,t+1}}\right],$$
$$\frac{1}{1+i_{F,t}} = E_t\left[\frac{\beta \tilde{u}_{c,t+1}}{\tilde{u}_{c,t}}\frac{P_t^*}{P_{t+1}^*}\right] = E_t\left[\frac{\beta u_{H,t+1}}{u_{H,t}}\frac{P_{H,t}^*}{P_{H,t+1}^*}\right] = E_t\left[\frac{\beta u_{F,t+1}}{u_{F,t}}\frac{P_{F,t}^*}{P_{F,t+1}^*}\right],$$

[8]The fully date-0 optimal solution only satisfy (35) for $t \geq 1$. As is well known, such a solution is not time consistent. As is discussed in Woodford (2003), however, requiring (35) for $t \geq 0$ makes the solution "optimal from a timeless perspective," and the policy problem time consistent.

where $\tilde{u}_{c,s} = \tilde{u}_c(Y_s)$ and $u_{j,s} = u_j(Y_{H,s}, Y_{F,s})$, for $s = t, t+1$ and $j = H, F$.

Whether or not the zero bound binds depends on the policy rules that the monetary authorities adopt. A given shock may or may not result in $i_{H,t} = 0$ depending on, for example, whether the home monetary authority tries to stabilize CPI, P_t, or the domestic price level, $P_{H,t}$. In the previous section, we have seen that the prices which have to be stabilized are not CPI's, P_t or P_t^*, but the producer price levels, $P_{H,t}$ and $P_{F,t}^*$. In this sense, it is natural to use

$$\frac{1}{1+i_{H,t}} = E_t\left[\frac{\beta u_{H,t+1}}{u_{H,t}}\frac{P_{H,t}}{P_{H,t+1}}\right], \quad \text{and} \quad \frac{1}{1+i_{F,t}} = E_t\left[\frac{\beta u_{F,t+1}}{u_{F,t}}\frac{P_{F,t}^*}{P_{F,t+1}^*}\right] \tag{36}$$

for our policy analysis.

As in Woodford (2003), it is convenient to introduce new variables, $r_{H,t}^n$ and $r_{F,t}^n$, called the "natural rates of interest." They are defined as the real interest rates which would realize if the domestic price levels, $P_{H,t}$ and $P_{F,t}^*$, were completely stabilized for all t. It follows that $r_{H,t}^n$ and $r_{F,t}^n$ satisfy

$$\frac{1}{1+r_{H,t}^n} = E_t\left[\frac{\beta u_H(Y_{H,t+1}^n, Y_{F,t+1}^n)}{u_H(Y_{H,t}^n, Y_{F,t}^n)}\right], \quad \text{and} \quad \frac{1}{1+r_{F,t}^n} = E_t\left[\frac{\beta u_F(Y_{H,t+1}^n, Y_{F,t+1}^n)}{u_F(Y_{H,t}^n, Y_{F,t}^n)}\right] \tag{37}$$

At the non-stochastic steady state, $r_{j,t}^n = \bar{r} \equiv \beta^{-1} - 1$, $j = H, F$. Note that even if $P_{H,t}$ and $P_{F,t}^*$ were completely stabilized, the nominal exchange rate, $\mathcal{E}_t$, may fluctuate. Thus, $r_{H,t}^n \neq r_{F,t}^n$ in general.

Using these, log-linearizing (36) yields

$$\begin{aligned} 0 \leq i_{t,H} = E_t\Big\{&\big[1 + (\sigma-1)n_H\big](x_{H,t+1} - x_{H,t}) \\ &+ (\sigma-1)n_F(x_{F,t+1} - x_{F,t}) + \pi_{H,t+1} + r_{H,t}^n\Big\} \end{aligned} \tag{38}$$

$$\begin{aligned} 0 \leq i_{t,F} = E_t\Big\{&\big[1 + (\sigma-1)n_F\big](x_{F,t+1} - x_{F,t}) \\ &+ (\sigma-1)n_H(x_{H,t+1} - x_{H,t}) + \pi_{F,t+1}^* + r_{F,t}^n\Big\} \end{aligned} \tag{39}$$

6 Optimal policy commitment in the case where the ZLB may bind

We say that country j is in a liquidity trap if $i_{j,t}$ hits the zero lower bound. In this section, we consider optimal policy coordination in the case where the home country may fall into a liquidity trap, that is, the case in which the inequality in (38) may bind but (39) does not. Our numerical exercise is an open-economy extension of Eggertsson and Woodford (2003).

We restrict attention to the case where there is no distortion at the steady state, $\Phi = \Phi^* = 0$, and there are no cost push shocks, $u_t = u_t^* = 0$, all t. Thus, the aggregate-supply relations are given by (21)-(22), and the loss function L_t in (32) becomes

$$L_t \equiv \frac{1}{2} x_t' \Lambda x_t + \frac{n_H}{2} \pi_{H,t}^2 + \frac{n_F}{2} \pi_{F,t}^{*2}, \tag{40}$$

It follows that, without the zero bound, the optimal policy would simply be the one that attains $\pi_{H,t} = \pi_{F,t}^* = x_{H,t} = x_{F,t} = 0$, for all t.

Assume that the world economy is in the steady state with zero inflation at date $t = -1$, namely, the one around which we have approximated the world welfare, the aggregate-supply relations, and the intertemporal Euler equations. Regarding exogenous shocks, we suppose that the productivity processes, ξ_t and ξ_t^*, are such that the natural rates of interest, $r_{H,t}^n$ and $r_{F,t}^n$, follow the following stochastic processes: (i) The natural rate of interest in the foreign country equals the steady-state level at all times, $r_{F,t}^n = \bar{r}$, all t. (ii) The natural rate of interest in the home country gets negative at date 0, $r_{H,0}^n = \underline{r}$, for some $\underline{r} < 0$. In the following periods, it returns to the steady-state level at a constant probability, $1 - \mu \in (0, 1]$. That is, given that $r_{H,t-1}^n = \underline{r}$, the conditional distribution of $r_{H,t}^n$ is given by

$$r_{H,t} = \begin{cases} \underline{r}, & \text{with probability } \mu, \\ \bar{r}, & \text{with probability } 1 - \mu. \end{cases}$$

Once it gets back to $\bar{r}$, it stays there from then on: if $r_{H,t-1}^n = \bar{r}$, then $r_{H,t}^n = \bar{r}$ with probability one.

The parameters are calibrated as follows. One period in the model corresponds to a quarter. We set $\alpha = 0.66$, $\beta = 0.99$, $\theta = 7.88$, $\omega = 0.47$, which are taken from Woodford (2003, Table 5.1). We follow Eggertsson and Woodford (2003) to set $\underline{r} = -0.02/4$. For σ, three values are considered: $\sigma = 0.5, 1, 5$. As we shall see below, the value of σ affects how the foreign monetary authority should react to the liquidity trap that occurs in the home country. The two countries have an equal size: $n_H = n_F = 1/2$. Finally, $\mu = 0.25$, that is, the home natural rate of interest remains negative for a year on average. As discussed below, the value of μ is chosen so that zero-inflation targeting generates deflation during a liquidity trap.

6.1 Zero inflation targeting

We first examine the consequences of strict zero inflation targeting, that is, the policy that achieves zero inflation whenever possible. It follows from the aggregate-supply relations (21)-(22) and the Euler equations (38)-(39) that it is indeed possible to attain $\pi_{H,t} = \pi_{F,t}^* =$

$x_{H,t} = x_{F,t} = 0$ after r_H^n returns to the steady-state level, $\bar{r}$. When $r_{H,t}^n = \underline{r}$, however, setting both of the two inflation rates to zero is not possible. Under our parametric assumptions, during the period when $r_{H,t}^n = \underline{r}$,

$$\pi_{H,t} < 0, \qquad x_{H,t} < 0, \qquad \pi_{F,t}^* = 0,$$

and $x_{F,t} \gtreqless 0$ as $\sigma \gtreqless 1$. Here, $x_{F,t} = 0$ when $\sigma = 1$, because, as the aggregate-supply relations (21)-(22) show, when $\sigma = 1$, the inflation rate and the output gap in each country are completely separately determined. The state-contingent paths of $\pi_{H,t}$, $x_{H,t}$, and $x_{F,t}$ are drawn in Figure 1.[9] In each panel, the j-th (solid, dashed, dashdot) line corresponds to the sample path where $r_{H,t}^n = \underline{r}$ for $t = 0, \ldots, j-1$, and $r_{H,t}^n = \bar{r}$ for $t \geq j$.

This example fits well with the casual statement that a "bad policy" during a liquidity trap results in deflation ($\pi_{H,t} < 0$) and a negative output gap ($x_{H,t} < 0$). It is not generally true, however. For example, Figure 2 depicts the state-contingent paths of the same variables under strict zero inflation targeting when $\mu = 0.1$. In this case, there is inflation, $\pi_{H,t} > 0$, and a positive output gap, $x_{H,t} > 0$, during a liquidity trap. Indeed, we can show that there is a threshold value, $\underline{\mu}$, such that the home country experiences deflation for $\mu > \underline{\mu}$ during a liquidity trap, and inflation otherwise. To gain an intuition, consider the closed-economy case, and suppose that $\mu = 0$, that is, the natural rate of interest remains negative for all t. It follows from the Fisher equation that the inflation rate equals the difference between the nominal and the real interest rates. Since the latter is negative and the former is zero, the inflation rate is positive.

In what follows, we assume $\mu = 0.25$ so that strict zero inflation targeting implies deflation in the home country during the liquidity trap. We have checked numerically that the qualitative properties of the optimal state-contingent paths of inflation and output gaps are not affected much by the choice of the value of μ.

6.2 Optimal policy

The optimal policy problem is formulated as the minimization problem of the discounted expected value of the loss function (40) subject to the aggregate-supply relations (21)-(22) and the inequality constraints (38)-(39). Letting $\psi_{j,t}$ and $\phi_{j,t}$, $j = H, F$, be the Lagrange

[9] In all figures, inflation rates and interest rates show annual rates.

multipliers associated with those constraints, the Lagrangian is formed as

$$
\begin{aligned}
\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \bigg\{ & \frac{1}{2} x_t' \Lambda x_t + \frac{n_H}{2} \pi_{H,t}^2 + \frac{n_F}{2} \pi_{F,t}^{*2} \\
& + \psi_{H,t} \Big(n_H \pi_{H,t} - \gamma_H n_H x_{H,t} - \gamma_{HF} n_H n_F x_{F,t} - \beta n_H \pi_{H,t+1} \Big) \\
& + \psi_{F,t} \Big(n_F \pi_{F,t}^* - \gamma_{HF} n_H n_F x_{H,t} - \gamma_F n_F x_{F,t} - \beta n_F \pi_{F,t+1}^* \Big) \\
& + \phi_{H,t} \Big(n_H [1 + (\sigma - 1) n_H] (x_{H,t} - x_{H,t+1}) \\
& \qquad + (\sigma - 1) n_H n_F (x_{F,t} - x_{F,t+1}) - n_H \pi_{H,t+1} - r_{H,t}^n \Big) \\
& + \phi_{F,t} \Big((\sigma - 1) n_H n_F (x_{H,t} - x_{H,t+1}) \\
& \qquad + n_F [1 + (\sigma - 1) n_F] (x_{F,t} - x_{F,t+1}) - n_F \pi_{F,t+1}^* - r_{F,t}^n \Big) \bigg\}
\end{aligned}
$$

Letting $\pi_t \equiv (\pi_{H,t}, \pi_{F,t}^*)'$, $x_t \equiv (x_{H,t}, x_{F,t})'$, $\psi_t \equiv (\psi_{H,t}, \psi_{F,t})'$, and $\phi_t \equiv (\phi_{H,t}, \phi_{F,t})'$, the first-order conditions are written as

$$\Lambda x_t - \theta \Lambda \psi_t + A(\phi_t - \beta^{-1} \phi_{t-1}) = 0, \tag{41}$$

$$\pi_t + \psi_t - \psi_{t-1} - \beta^{-1} \phi_{t-1} = 0, \tag{42}$$

where the matrix A is defined by

$$A \equiv \begin{bmatrix} n_H[1 + (\sigma - 1) n_H] & (\sigma - 1) n_H n_F \\ (\sigma - 1) n_H n_F & n_F[1 + (\sigma - 1) n_F] \end{bmatrix}$$

Eliminating ψ and ϕ in (41)-(42), we obtain the following optimal targeting rule for the logged price levels, $p_t \equiv (\ln P_{H,t}, \ln P_{F,t}^*)'$, with possibly binding ZLB. It extends the result of Eggertsson and Woodford (2003) to the open-economy model.[10]

(i) In each period, there is a predetermined price-level target $\check{p}_t = (\check{p}_{H,t}, \check{p}_{F,t}^*)$. The central banks choose the interest rates $i_{H,t}$ and $i_{F,t}$ to achieve the target relation:

$$\tilde{p}_t = \check{p}_t$$

if this is possible; if it is not possible because of the zero bound, then

$$i_{H,t} = 0 \quad \text{and/or} \quad i_{F,t} = 0.$$

Here, $\tilde{p}_t = (\tilde{p}_{H,t}, \tilde{p}_{F,t}^*)'$ is output-gap adjusted price indexes defined by

$$\tilde{p}_t \equiv p_t + \frac{1}{\theta} x_t.$$

[10] In the terminology of Woodford (2003), this rule is optimal from a timeless perspective, and hence time-consistent.

(ii) The target for the next period is determined as

$$\check{p}_{t+1} = \check{p}_t + \beta^{-1}(I + \theta A^{-1}\Lambda)\delta_t - \beta^{-1}\delta_{t-1} \tag{43}$$

where δ_t is the target shortfall in period t:

$$\delta_t \equiv \check{p}_t - \tilde{p}_t.$$

In the price-target equation (43),

$$\theta A^{-1}\Lambda = \frac{\zeta}{\sigma}\begin{bmatrix} \sigma + \omega + (\sigma - 1)\omega n_F & -(\sigma - 1)\omega n_F \\ -(\sigma - 1)\omega n_H & \sigma + \omega + (\sigma - 1)\omega n_H \end{bmatrix},$$

which is not a diagonal matrix. It follows that the target price levels, $\check{p}_{H,t}$ and $\check{p}^*_{F,t}$, are not separately determined. This is in contrast to the case where the zero bound never binds, as we have seen in the previous section.

Figure 3 shows the optimal state-contingent paths of inflation, output gaps, and nominal interest rates in the two countries. Figure 4 compares the paths of those variables under optimal policy and under strict zero inflation targeting in the case where $r^n_{H,t} = \underline{r}$ for $t = 0, 1, \ldots, 9$ and $\sigma = 5$. The optimal state-contingent paths of $\pi_{H,t}$, $x_{H,t}$ and $i_{H,t}$ are remarkably similar to those obtained for the closed-economy by Eggertsson and Woodford (2003). When the shock hits the home country at date 0, the home inflation rate π_H and the home output gap x_H become negative, but by committing to generate inflation after the natural rate r^n_H turns back to normal, its impact is curbed to minimal. As Eggertsson and Woodford (2003) emphasize, under optimal policy commitment, the home nominal interest rate, $i_{H,t}$, typically is set to zero for a few periods after the home natural rate, $r^n_{H,t}$, returns to the steady-state level, $\bar{r}$. As Figure 4 shows, the deviations of $\pi_{H,t}$ and $x_{H,t}$ are much smaller under optimal policy commitment than under strict zero inflation targeting. Note that such stabilization is made possible by the coordinating monetary policy in the foreign country unless $\sigma = 1$. During periods where $r^n_{H,t} = \underline{r}$, the foreign nominal interest rate, $i_{F,t}$, must be set lower (higher) than the steady state level when $\sigma < 1$ ($\sigma > 1$). It is true that such policy coordination generates (small) inflation/deflation in the foreign country, the output gaps in the foreign country are made much smaller than under zero inflation targeting.

7 Liquidity trap and the exchange rate

In this section, we discuss the optimal state-contingent paths of the nominal exchange rate. We are particularly interested in examining the extent to which Svensson's (2001, 2003, 2004) proposal that emphasizes currency depreciation to "escape optimally from a liquidity trap" holds true in our setting.

First-order condition (16) for utility maximization implies that the nominal exchange rate, $\mathcal{E}_t$, must satisfy

$$\mathcal{E}_t = \frac{n_H}{n_F} \frac{P_{H,t}}{P^*_{F,t}} \frac{Y_{H,t}}{Y_{F,t}}.$$

Note that $\mathcal{E}_t$ is the price of the foreign currency quoted in the home currency.

To determine dynamic paths of the nominal exchange rate, it is not enough to specify the stochastic process of the natural rates of interest, $r^n_{H,t}$ and $r^n_{F,t}$. We need to identify the process of the (growth rates of the) natural rates of output, $Y^n_{H,t}$ and $Y^n_{F,t}$. By definition of the natural rates of interest, (37), we obtain

$$\frac{1}{\beta(1+r^n_{H,t})} = E_t\left[\left(\frac{Y^n_{H,t+1}}{Y^n_{H,t}}\right)^{(1-\sigma)n_H-1}\left(\frac{Y^n_{F,t+1}}{Y^n_{F,t}}\right)^{(1-\sigma)n_F}\right]$$

$$\frac{1}{\beta(1+r^n_{F,t})} = E_t\left[\left(\frac{Y^n_{H,t+1}}{Y^n_{H,t}}\right)^{(1-\sigma)n_H}\left(\frac{Y^n_{F,t+1}}{Y^n_{F,t}}\right)^{(1-\sigma)n_F-1}\right]$$

Log-linearization of these equations yields

$$-\ln[\beta(1+r^n_{H,t})] = [(1-\sigma)n_H - 1]E_t g^n_{H,t+1} + (1-\sigma)n_F E_t g^n_{F,t+1}$$
$$-\ln[\beta(1+r^n_{F,t})] = (1-\sigma)n_H E_t g^n_{H,t+1} + [(1-\sigma)n_F - 1]E_t g^n_{F,t+1}$$

where $g^n_{j,t+1}$, $j = H, F$, are the growth rates of the natural rates of output in country j:

$$g^n_{j,t+1} \equiv \ln(Y^n_{j,t+1}/Y^n_{j,t}), \qquad j = H, F.$$

Since $\beta(1+r^n_{F,t}) = 1$, all t, we obtain

$$\begin{bmatrix} E_t g^n_{H,t+1} \\ E_t g^n_{F,t+1} \end{bmatrix} = \frac{1}{\sigma}\begin{bmatrix} (1-n_F+\sigma n_F) \\ (1-\sigma)n_H \end{bmatrix} \ln[\beta(1+r^n_{H,t})]$$

This equation relates the natural rate of interest $r^n_{H,t}$ at date t and the expected growth rates of the natural rates of output in the next period, $E_t g^n_{j,t+1}$, $j = H, F$. Thus, given a process of $r^n_{H,t}$, there are many processes of $g^n_{j,t+1}$, $j = H, F$, which are consistent of it. To be specific, we assume that, given $r^n_{H,t}$ at date t, the growth rate of the natural rates of output in the next period, $g^n_{j,t+1}$, $j = H, F$, are perfectly forecastable. That is, we assume that $g^n_{j,t+1}$, $j = H, F$, $t \geq 0$, follow the process given by

$$\begin{bmatrix} g^n_{H,t+1} \\ g^n_{F,t+1} \end{bmatrix} = \frac{1}{\sigma}\begin{bmatrix} (1-n_F+\sigma n_F) \\ (1-\sigma)n_H \end{bmatrix} \ln[\beta(1+r^n_{H,t})]$$

For normalization, we assume that $\mathcal{E}_{-1} = 1$ and $g^n_{j,0} = 0$, for $j = H, F$. Note that the value of $g_{j,0}$ is irrelevant for $r^n_{H,t}$, $t \geq 0$. We discuss the normalization regarding $g^n_{j,0}$ later, when we compare our result to Svensson's proposal.

Let $\epsilon_t \equiv \ln \mathcal{E}_t$. Then our parametric assumptions imply that

$$\epsilon_t = \epsilon_{t-1} + \ln[\beta(1 + r^n_{H,t-1})] + \pi_{H,t} - \pi^*_{F,t} + x_{H,t} - x_{H,t-1} - (x_{F,t} - x_{F,t-1}).$$

Figure 5 draws the optimal state-contingent paths of the log nominal exchange rate, ϵ_t. The figure shows that, regardless of the value of σ, the optimal nominal exchange rate of the home currency appreciates during periods when $r^n_{H,t} < 0$. The optimal evolution of the nominal exchange rate does not exhibit a depreciation as opposed to the proposal of Svensson.

Where does the difference between our results and Svensson's come from? His theoretical analysis is given in Svensson (2004). His model is different from ours in several ways. For instance, in his model, all prices are set one period in advance; the home country may be small or large; in the large-country case, there may or may not be international policy coordination. However, the difference in the results is not due to those differences in the models considered. Instead, it owes to the difference in the normalization of $g^n_{H,0}$.

To illustrate it, following Svensson (2004), consider the case in which the home natural rate of interest becomes negative only at date 0:

$$r^n_{H,0} = \underline{r}, \qquad r^n_{H,t} = \bar{r}, \quad \text{for } t \geq 1.$$

This implies that the natural rate of home output, $Y^n_{H,t}$, evolves as $Y^n_{H,0} > Y^n_{H,1} = Y^n_{H,2} = \cdots$, or equivalently, $g^n_{H,1} < 0$ and $g^n_{H,t} = 0$, for $t \geq 2$. The process of $r^n_{H,t}$ does not pin down $g^n_{H,0}$. What we have assumed above is that $g^n_{H,0} = 0$, i.e., $Y^n_{H,-1} = Y^n_{H,0}$. In contrast, what Svensson (2004) assumes is that $Y^n_{H,0} > Y^n_{H,-1} = Y^n_{H,t}$, $t \geq 1$, and so,

$$g^n_{H,0} > 0, \qquad g^n_{H,1} < 0, \qquad g^n_{H,t} = 0, \quad \text{for } t \geq 1.$$

Thus, he considers the case in which $r^n_{H,0} < 0$ at date 0 because the home country experiences an unusually high level of productivity. Figure 6 shows the processes of $g^n_{H,t}$ assumed here (panel a) and in Svensson (2004) (panel b).

Figure 7 plots the optimal paths of the nominal exchange rate, ϵ_t, (a) when $g^n_{H,0} = 0$ and (b) when $g^n_{H,0} > 0$, for the three values of σ. Case (a), corresponding to our results in Figure 5, shows that the home currency appreciates during a liquidity trap. In contrast, corresponding to Svensson (2004), case (b) shows depreciation of the home currency at date 0. The home currency depreciates in case (b), because the home country produces goods more than usually so that the relative price of home goods must fall.

Note, however, that our difference lies only in the choice of $g^n_{H,0}$, and hence, in ϵ_0. It follows that in the case where the shock lasts for more than one period, the evolution of nominal exchange rates for $t \geq 1$ would be independent of the assumption on $g^n_{H,0}$, and thus, the home currency would appreciate for $t \geq 1$ even in the model of Svensson (2004). Hence, our previous remark that the home currency appreciates as long as $r^n_{H,t} = \underline{r}$ along the optimal path holds true for $t \geq 1$, regardless of the choice of $g^n_{H,0}$.

Finally, Figure 8 compares the optimal path of the nominal exchange rate and its path under strict zero inflation targeting in the case where $r^n_{H,t} = \underline{r}$ for $t = 0, 1, \ldots, 9$. The nominal exchange rate under strict zero inflation targeting shows much larger appreciation of the home currency than under optimal policy commitment. This is because, under our parametric assumptions, strict zero inflation targeting leads to very large deflation as shown in Figure 1.

8 Conclusions

We have analyzed optimal policy in an open economy. First, when the ZLB for nominal interest rate is assumed never to bind, the optimal inflation targeting rule for each country is exactly the same with the same parameter value as the one for the closed-economy model (Woodford, 2003). Indeed, in such a case, each monetary authority can forget about international dependence in setting its inflation target, in order to maximize the world welfare. Second, this is not the case when the ZLB may bind. The optimal price-level target rule shows significant international dependence. The optimal state-contingent paths of inflation, output gaps, and nominal interest rates for the country in a liquidity trap look very similar to those obtained for the closed-economy model by Eggertsson and Woodford (2003). However, such paths are made possible by policy coordination by the other country. Third, in spite of the proposal of Svensson (2001, 2003, 2004), the nominal exchange rate of the country in a liquidity trap appreciates under optimal policy (except possibly at the initial date).

We have restricted our attention to the case where the steady-state distortions due to market power is (close to) zero. Benigno and Woodford (2004) have developed the approach to deal with the case of a distorted steady state. Applying their approach to open economies is left for future research.

References

[1] Benhabib, J., S. Schmitt-Grohé and M. Uribe (2001), "Monetary policy and multiple equilibria," *American Economic Review*, 91, 167-186.

[2] Benigno, G. and P. Benigno (2003), "Price stability in open economies," *Review of Economic Studies*, 70, 743-764.

[3] Benigno, P. and M. Woodford (2004), "Inflation stabilization and welfare: The case of a distorted steady state," mimeo.

[4] Calvo, G. (1983), "Staggered prices in a utility-maximizzing framework," *Journal of Monetary Economics*, 12, 383-398.

[5] Clarida, R., J. Galí and M. Gertler (1999), "The science of monetary policy: A new Keynesian perspective," *Journal of Economic Literature*, 37, 1661-1707.

[6] Clarida, R., J. Gali and M. Gertler (2001), "Optimal monetary policy in open versus closed economies: An integrated approach," *American Economic Review*, 91, 248-252.

[7] Corsetti, G. and P. Pesenti (2001), "Welfare and macroeconomic interdependence," *Quarterly Journal of Economics*, 116, 421-446.

[8] Eggertsson, G. and M. Woodford (2003), "The zero bound on interest rates and optimal monetary policy," *Brookings Papers on Economic Activity*, 2003:1, 139-211.

[9] Galí, J. and T. Monacelli (2002), "Monetary policy and exchange rate volatility in a small open economy," NBER Working Paper 8905.

[10] Giannoni, M. P. and M. Woodford (2002), "Optimal interest-rate rules: I. General theory," NBER Working Paper No. 9419.

[11] Jung, T., Y. Teranishi and T. Watanabe (2001), "Zero bound on nominal interest rates and optimal monetary policy," Discussion Paper no. 525, Kyoto Institute of Economic Research.

[12] Krugman, P. (1998), "It's baaack! Japan's slump and the return of the liquidity trap," *Brookings Papers on Economic Activity*, 1998:2, 137-187.

[13] Ljungqvist, L. and T. J. Sargent (2000), *Recursive macroeconomic theory*, Cambridge, MA: MIT Press.

[14] Obstfeld, M. and K. Rogoff (2002), "Global implications of self-oriented national monetary rules," *Quarterly Journal of Economics*, 117, 503-536.

[15] Svenssion, L. E. O. (2001), "The zero bound in an open-economy: A foolproof way of escaping from a liquidity trap," *Monetary and Economic Studies*, 19, pp. 277-312.

[16] Svensson, L. E. O. (2003), "Escaping from a liquidity trap and deflation: The foolproof way and others," *Journal of Economic Perspective*, 17:4, 145-166.

[17] Svensson, L. E. O. (2004), "The maginc of the exchange rate: Optimal escape from a liquidity trap in small and large open economies," mimeo (Version 1.2, July 2004).

[18] Woodford, M. (2003), *Interest and prices*, Princeton, NJ: Princeton University Press.

A Appendix

A.1 Proof of the proposition

Here, we prove the proposition in Section 2.3. Then the consumption indexes become

$$C = \left(\frac{C_H}{n_H}\right)^{n_H}\left(\frac{C_F}{n_F}\right)^{n_F}, \qquad C^* = \left(\frac{C_H^*}{n_H}\right)^{n_H}\left(\frac{C_F^*}{n_F}\right)^{n_F}.$$

Under the natural debt limit, the household's utility maximization problem is reexpressed by using the lifetime budget constraints:

$$\begin{aligned} & E_0 \sum_{t=0}^{\infty} Q_{0,t}\big(P_{H,t}C_{H,t} + P_{F,t}C_{F,t}\big) \\ & \qquad \leq W_0 + E_0 \sum_{t=0}^{\infty} Q_{0,t}\left\{\int_{N_H}\big[w_t(i)\ell_t(i) + \Pi_t(i)\big]\,di + T_t\right\}, \end{aligned} \tag{44}$$

and

$$\begin{aligned} & E_0 \sum_{t=0}^{\infty} Q_{0,t}\mathcal{E}_t\big(P_{H,t}^*C_{H,t}^* + P_{F,t}^*C_{F,t}^*\big) \\ & \qquad \leq \mathcal{E}_0 W_0^* + E_0 \sum_{t=0}^{\infty} Q_{0,t}\mathcal{E}_t\left\{\int_{N_F}\big[w_t^*(i)\ell_t^*(i) + \Pi_t^*(i)\big]\,di + T_t^*\right\}. \end{aligned} \tag{45}$$

Let λ and λ^* be the Lagrange multipliers associated with the lifetime budget constraint for the home household and for the foreign household, respectively. The first-order conditions for utility maximization of the home household with respect to $C_{j,t}$ are given by

$$\begin{aligned} \beta^t\left(\frac{C_{H,t}}{n_H}\right)^{n_H(1-\sigma)-1}\left(\frac{C_{F,t}}{n_F}\right)^{n_F(1-\sigma)} &= \lambda Q_{0,t}P_{H,t}, \\ \beta^t\left(\frac{C_{H,t}}{n_H}\right)^{n_H(1-\sigma)}\left(\frac{C_{F,t}}{n_F}\right)^{n_F(1-\sigma)-1} &= \lambda Q_{0,t}P_{F,t}. \end{aligned}$$

Analogous conditions hold for the foreign household. Those conditions imply that

$$\frac{P_{H,t}C_{H,t}}{P_{F,t}C_{F,t}} = \frac{P_{H,t}C_{H,t}^*}{P_{F,t}C_{F,t}^*} = \frac{n_H}{n_F} \tag{46}$$

$$\frac{C_{H,t}}{C_{H,t}^*} = \frac{C_{F,t}}{C_{F,t}^*} = \left(\frac{\lambda}{\lambda^*}\right)^{-\frac{1}{\sigma}} \tag{47}$$

The net revenue of the home government at date t is

$$P_{H,t}Y_{H,t} - n_H\left\{T_t + \int_{N_H}\big[w_t(i)\ell_t(i) + \Pi_t(i)\big]\,di\right\},$$

where $Y_{H,t} \equiv n_H C_{H,t} + n_F C_{F,t}$. Since W_0 equals the initial liabilities of the home government, the lifetime budget constraint for the home government implies that

$$n_H W_0 = E_0 \sum_{t=0}^{\infty} Q_{0,t} \left[P_{H,t} Y_{H,t} - n_H \left\{ T_t + \int_{N_H} \left[w_t(i) \ell_t(i) + \Pi_t(i) \right] di \right\} \right]. \tag{48}$$

Similarly, the lifetime budget constraint for the foreign government is given by

$$n_F \mathcal{E}_0 W_0^* = E_0 \sum_{t=0}^{\infty} Q_{0,t} \mathcal{E}_t \left[P_{F,t}^* Y_{F,t} - n_F \left\{ T_t^* + \int_{N_F} \left[w_t^*(i) \ell_t^*(i) + \Pi_t^*(i) \right] di \right\} \right]. \tag{49}$$

Note that the lifetime budget constraint for each household holds with equality in equilibrium. Then, combining the governments' and households' lifetime budget constraints, we obtain the equilibrium condition:

$$E_0 \sum_{t=0}^{\infty} Q_{0,t} (n_H P_{F,t} C_{F,t} - n_F P_{H,t} C_{H,t}^*) = 0. \tag{50}$$

Using (46)-(47), this equation is rewritten as

$$\left\{ 1 - \left(\frac{\lambda}{\lambda^*} \right)^{\frac{1}{\sigma}} \right\} E_0 \sum_{t=0}^{\infty} Q_{0,t} P_{H,t} C_{H,t} = 0.$$

Thus, $\lambda = \lambda^*$ in equilibrium, and

$$\frac{C_t}{C_t^*} = \frac{C_{H,t}}{C_{H,t}^*} = \frac{C_{F,t}}{C_{F,t}^*} = 1.$$

A.2 Steady-state equilibrium

It is convenient to define the "real marginal cost" of producing good $i \in N_j$ in country $j \in \{H, F\}$ as

$$s\big[y_t(i), Y_{H,t}, Y_{F,t}; \xi_t\big] \equiv \frac{v_y[y_t(i); \xi_t]}{u_H(Y_{H,t}, Y_{F,t})} = \frac{1}{A_t} \frac{w_t(i)}{P_{H,t}} \tag{51}$$

$$s^*\left[y_t^*(i), Y_{H,t}, Y_{F,t}; \xi_t^*\right] \equiv \frac{v_y^*[y_t^*(i); \xi_t^*]}{u_F(Y_{H,t}, Y_{F,t})} = \frac{1}{A_t^*} \frac{w_t^*(i)}{P_{F,t}^*} \tag{52}$$

Consider a non-stochastic steady state with zero inflation: $\xi_t = \xi_t^* = 0$, and

$$p_t(i) = P_{H,t} = P_H, \quad i \in N_H, \qquad p_t^*(i) = P_{F,t}^* = P_F^*, \quad i \in N_F, \qquad \mathcal{E}_t = \mathcal{E}.$$

In such a steady state, aggregate output levels, $\bar{Y}_H$ and $\bar{Y}_F$, are determined by

$$s\left(\frac{\bar{Y}_H}{n_H}, \bar{Y}_H; 0 \right) = 1 - \Phi, \quad \text{and} \quad s^*\left(\frac{\bar{Y}_F}{n_F}, \bar{Y}_F; 0 \right) = 1 - \Phi^*.$$

Output of individual products are

$$y_t(i) = \frac{\bar{Y}_H}{n_H}, \quad i \in N_H, \quad \text{and} \quad y_t^*(i) = \frac{\bar{Y}_F}{n_F}, \quad i \in N_F.$$

A.3 Derivation of the aggregate supply relations (21)-(22)

The natural rates of output are then implicitly defined by the conditions:

$$s\big[Y^n_{H,t}/n_H, Y^n_{H,t}, Y^n_{F,t}; \xi_t\big] = 1 - \Phi, \qquad s^*\big[Y^n_{F,t}/n_F, Y^n_{H,t}, Y^n_{F,t}; \xi^*_t\big] = 1 - \Phi^*.$$

Let a hat over a variable denote the log-deviation from its steady-state value. Then, log-linearization of those equations yields

$$\begin{aligned}\xi_t &= \big[(1-\sigma)n_H - (1+\omega)\big]\hat{Y}^n_{H,t} + (1-\sigma)n_F\hat{Y}^n_{F,t},\\ \xi^*_t &= (1-\sigma)n_H\hat{Y}^n_{H,t} + \big[(1-\sigma)n_F - (1+\omega)\big]\hat{Y}^n_{F,t}.\end{aligned}$$

Using these, log-linearization of the real marginal cost function (51) leads to

$$\begin{aligned}\hat{s}_t(i) = \omega\hat{y}_t(i) &+ \big[1-(1-\sigma)n_H\big]\hat{Y}_{H,t} - (1-\sigma)n_F\hat{Y}_{F,t}\\ &+ \big[(1-\sigma)n_H - (1+\omega)\big]\hat{Y}^n_{H,t} + (1-\sigma)n_F\hat{Y}^n_{F,t}\end{aligned} \tag{53}$$

Let s_t denote the date-t average of the real marginal cost in the home country. Then

$$\hat{s}_t = \big[1+\omega-(1-\sigma)n_H\big](\hat{Y}_{H,t} - \hat{Y}^n_{H,t}) - (1-\sigma)n_F(\hat{Y}_{F,t} - \hat{Y}^n_{F,t}) \tag{54}$$

Remember that z_t is the new price chosen by all home producers that revise their prices at date t. Let $y_{t,T}$ be the date-T supply of goods whose prices equal z_t:

$$y_{t,T} \equiv \frac{Y_{H,T}}{n_H}\left(\frac{z_t}{P_{H,T}}\right)^{-\theta},$$

and define $s_{t,T} \equiv s(y_{t,T}, Y_{H,T}, Y_{F,T}; \xi_T)$.

Then the log-linearization of the profit-maximizing condition (20) becomes

$$E_t\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}\big[(\ln z_t - \ln P_{H,T}) - \hat{s}_{t,T}\big] = 0. \tag{55}$$

Note that

$$\ln z_t - \ln P_{H,T} = (\ln z_t - \ln P_{H,t}) - \sum_{\tau=t+1}^{T}\pi_{H,\tau}.$$

The home-good price index, $P_{H,t}$, evolves as

$$P^{1-\theta}_{H,t} = \alpha P^{1-\theta}_{H,t-1} + (1-\alpha)z^{1-\theta}_t,$$

so that, to a first-order approximation,

$$\ln z_t - \ln P_{H,t} = \frac{\alpha}{1-\alpha}\pi_{H,t}.$$

It follows that equation (55) reduces to

$$E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left\{ \frac{\alpha}{1-\alpha} \pi_{H,t} - \sum_{\tau=t+1}^{T} \pi_{H,\tau} - \hat{s}_{t,T} \right\} = 0. \tag{56}$$

From (53)-(54), it follows that

$$\begin{aligned}
\hat{s}_{t,T} &= \hat{s}_T + \omega(\hat{y}_{t,T} - \hat{Y}_{H,T}) \\
&= \hat{s}_T - \omega\theta(\ln z_t - \ln P_{H,T}) \\
&= \hat{s}_T - \omega\theta \left(\frac{\alpha}{1-\alpha} \pi_{H,t} - \sum_{\tau=t+1}^{T} \pi_{H,\tau} \right),
\end{aligned}$$

where we have used the demand function (8) in the second line. Then (56) is rewritten as

$$E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left\{ \frac{\alpha(1+\omega\theta)}{1-\alpha} \pi_{H,t} - \hat{s}_T - (1+\omega\theta) \sum_{\tau=t+1}^{T} \pi_{H,\tau} \right\} = 0$$

Solving this equation for $\pi_{H,t}$, we obtain

$$\pi_{H,t} = \frac{1-\alpha}{\alpha} \frac{1-\alpha\beta}{1+\omega\theta} \hat{s}_t + \beta E_t \pi_{H,t+1}.$$

Finally, using (54), we obtain

$$\begin{aligned}
\pi_{H,t} = \frac{1-\alpha}{\alpha} \frac{1-\alpha\beta}{1+\omega\theta} \Big\{ \big[(1+\omega) - (1-\sigma)n_H\big](\hat{Y}_{H,t} - \hat{Y}^n_{H,t}) - (1-\sigma)n_F(\hat{Y}_{F,t} - \hat{Y}^n_{F,t}) \Big\} \\
+ \beta E_t \pi_{H,t+1},
\end{aligned}$$

which is equation (21) in the text. The aggregate-supply relation (22) for the foreign country is derived in the similar fashion.

A.4 Derivation of second-order approximations (29)-(31)

We follow the procedure described in Benigno and Woodford (2004). The home-goods price level, $P_{H,t}$, evolves as

$$P_{H,t}^{1-\theta} = (1-\alpha) z_t^{1-\theta} + \alpha P_{H,t-1}^{1-\theta},$$

where z_t is the newly chosen price at date t. We can rewrite this as

$$\left(\frac{z_t}{P_{H,t}} \right)^{-\theta(1+\omega)} = \left(\frac{1-\alpha \Pi_{H,t}^{\theta-1}}{1-\alpha} \right)^{\frac{\theta(1+\omega)}{\theta-1}},$$

where $\Pi_{H,t} \equiv P_{H,t}/P_{H,t-1}$. It follows that the evolution of $\Delta_{H,t}$, defined in (27), is written as

$$\Delta_{H,t} = (1-\alpha)\left(\frac{1-\alpha\Pi_{H,t}^{\theta-1}}{1-\alpha}\right)^{\frac{\theta(1+\omega)}{\theta-1}} + \alpha\Pi_{H,t}^{\theta(1+\omega)}\Delta_{H,t-1}$$

A second-order approximation of this equation is given as

$$\hat{\Delta}_{H,t} = \alpha\hat{\Delta}_{H,t-1} + \frac{1}{2}\frac{\alpha}{1-\alpha}\theta(1+\omega)(1+\omega\theta)\pi_{H,t}^2 + O(\|\Xi\|^3),$$

and a similar expression holds for $\Delta_{F,t}$. These expressions are used to obtain (30)-(31).

A quadratic approximation of $u(Y_{H,t}, Y_{F,t})$ in terms of the log-deviations $\hat{Y}_{j,t}$, $j = H, F$, is given by

$$\begin{aligned} u(Y_{H,t}, Y_{F,t}) = {} & u_H\bar{Y}_H\hat{Y}_{H,t} + u_F\bar{Y}_F\hat{Y}_{F,t} + \frac{1}{2}(\bar{Y}_H u_H + \bar{Y}_H^2 u_{HH})\hat{Y}_{H,t}^2 \\ & + u_{HF}\bar{Y}_H\bar{Y}_F\hat{Y}_{H,t}\hat{Y}_{F,t} + \frac{1}{2}(\bar{Y}_F u_F + \bar{Y}_F^2 u_{FF})\hat{Y}_{F,t}^2 + \text{t.i.p.} + O(\|\Xi\|^3) \end{aligned} \tag{57}$$

A quadratic approximation of $n_H v(Y_{H,t}/n_H;\xi_t)\Delta_{H,t}$ is

$$\begin{aligned} n_H v\left(\frac{Y_{H,t}}{n_H};\xi_t\right)\Delta_{H,t} = {} & \bar{v}_y\bar{Y}_H\left\{\frac{\hat{\Delta}_{H,t}}{1+\omega} + \hat{Y}_{H,t} + \frac{1+\omega}{2}\hat{Y}_{H,t}^2 + \hat{Y}_{H,t}\xi_t\right\} \\ & + \text{t.i.p.} + O(\|\Xi\|^3) \end{aligned} \tag{58}$$

Also,

$$\begin{aligned} n_F v\left(\frac{Y_{F,t}}{n_F};\xi_t^*\right)\Delta_{F,t} = {} & \bar{v}_y^*\bar{Y}_F\left\{\frac{\hat{\Delta}_{F,t}}{1+\omega} + \hat{Y}_{F,t} + \frac{1+\omega}{2}\hat{Y}_{F,t}^2 + \hat{Y}_{F,t}\xi_t^*\right\} \\ & + \text{t.i.p.} + O(\|\Xi\|^3) \end{aligned} \tag{59}$$

Then, using the fact that

$$\begin{aligned} \bar{v}_y = (1-\Phi)\bar{u}_H, \qquad & \bar{v}_y^* = (1-\Phi^*)\bar{u}_F, \\ \frac{u_F Y_F}{u_H Y_H} = \frac{n_F}{n_H}, \qquad & \frac{u_{HH}Y_H^2}{u_H Y_H} = (1-\sigma)n_H - 1, \\ \frac{u_{HF}Y_H Y_F}{u_H Y_H} = (1-\sigma)n_F, \qquad & \frac{u_{FF}Y_F^2}{u_F Y_F} = (1-\sigma)n_F - 1, \end{aligned}$$

we can combine (57)-(59) to obtain (29).

A.5 Noncooperation

In the main text we have assumed that there is no (or small) distortion at the non-stochastic steady state, $\Phi = \Phi^* = 0$, and analyzed the case in which the two countries coordinate so as to maximize the world welfare. Remember that our linear-quadratic approach is based on this assumption of zero (or small) distortions at the steady state, which is introduced to deal with the linear terms in (29). As is pointed out by Benigno and Benigno (2003), when we consider the noncooperative case, however, the linear terms do not vanish under our assumptions on Φ and Φ^*.

To see this, suppose that the monetary authority in each country seeks to maximize the lifetime utility of its representative household. The home household's lifetime utility is given by

$$U_0 = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u(Y_{H,t}, Y_{F,t}) - v\left(\frac{Y_{H,t}}{n_H}; \xi_t\right) \Delta_{H,t} \right\}$$

A second-order approximation of the flow utility is

$$\begin{aligned} & u(Y_{H,t}, Y_{F,t}) - v\left(\frac{Y_{H,t}}{n_H}; \xi_t\right) \Delta_{H,t} \\ &= \frac{\bar{u}_H \bar{Y}_H}{n_H} \left\{ n_H \hat{Y}_{H,t} + \frac{(1-\sigma) n_H^2}{2} \hat{Y}_{H,t}^2 + n_H n_F (1-\sigma) \hat{Y}_{F,t} \hat{Y}_{H,t} + n_F \hat{Y}_{F,t} + \frac{(1-\sigma) n_F^2}{2} \hat{Y}_{F,t}^2 \right. \\ & \qquad \left. - (1-\Phi) \hat{Y}_{H,t} - (1-\Phi) \frac{1+\omega}{2} \hat{Y}_{H,t}^2 - (1-\Phi) \xi_t \hat{Y}_{H,t} - (1-\Phi) \frac{\hat{\Delta}_{H,t}}{1+\omega} \right\} \end{aligned}$$

Thus, the linear term, $(n_H - (1-\Phi))\hat{Y}_{H,t}$, does not vanish unless $\Phi = 1 - n_H$. Similarly, the linear term in the corresponding quadratic measure of the welfare of the foreign household does not go away unless $\Phi^* = 1 - n_F$.

The existence of such linear terms would make our simple approach invalid. Such linear terms would give an incentive to each domestic monetary authority to generate unexpected deflation when $\Phi = \Phi^* = 0$. There is a simple intuition for this result. Suppose that $\Phi = \Phi^* = 0$. Then at the non-stochastic steady state the average utility flow across the two countries is maximized:

$$(\bar{Y}_H, \bar{Y}_F) = \arg\max_{Y_H, Y_F} u(Y_H, Y_F) - n_H v\left(\frac{Y_H}{n_H}; 0\right) - n_F v\left(\frac{Y_F}{n_F}; 0\right)$$

Thus, $\bar{v}_y = \bar{u}_H$ and $\bar{v}_y^* = \bar{u}_F$. The home household's flow utility at the steady state, however, is given by

$$u(\bar{Y}_H, \bar{Y}_F) - v\left(\frac{\bar{Y}_H}{n_H}; 0\right),$$

which decreases with $\bar{Y}_H$, since $\bar{u}_H - \bar{v}_y/n_H < 0$. Thus, from the noncooperative perspective, the efficient level of output is too large. This is because one unit of decrease in the world supply of Y_H reduces only $n_H < 1$ unit of its consumption in the home country. A similar argument holds for Y_F, and this is where an incentive to generate deflation arises.

Benigno and Woodford (2004) have shown how to extend the linear-quadratic approach to the case in which such linear terms remain in the quadratic approximate welfare measure. Applying their approach to the open-economy model is left for future research.

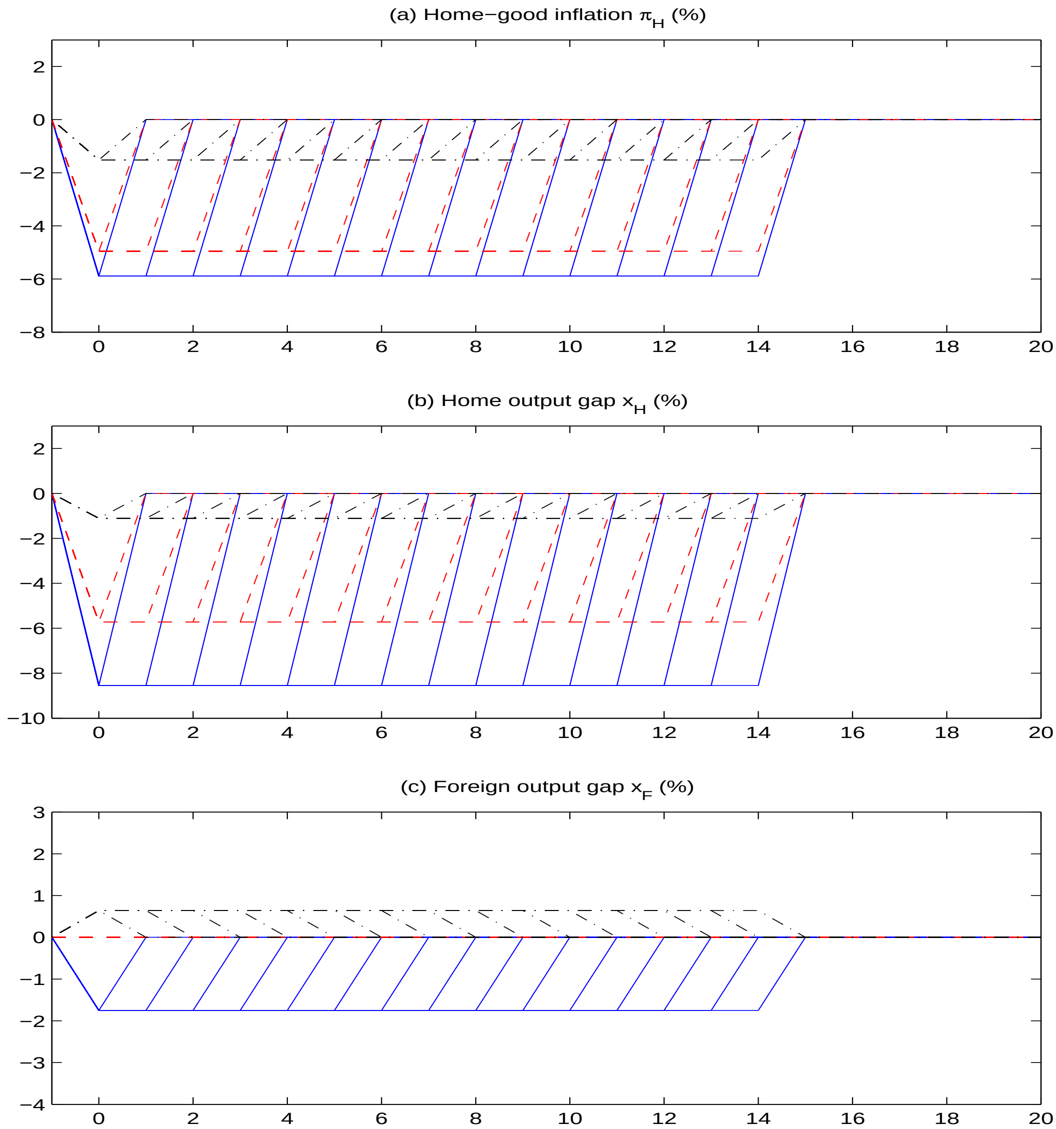

Figure 1: State-contingent paths of $\pi_{H,t}$, $x_{H,t}$, and $x_{F,t}$ under strict zero inflation targeting when $\mu = 0.25$. In each panel, the solid line corresponds to $\sigma = 0.5$, the dashed line to $\sigma = 1$, and the dashdot line to $\sigma = 5$.

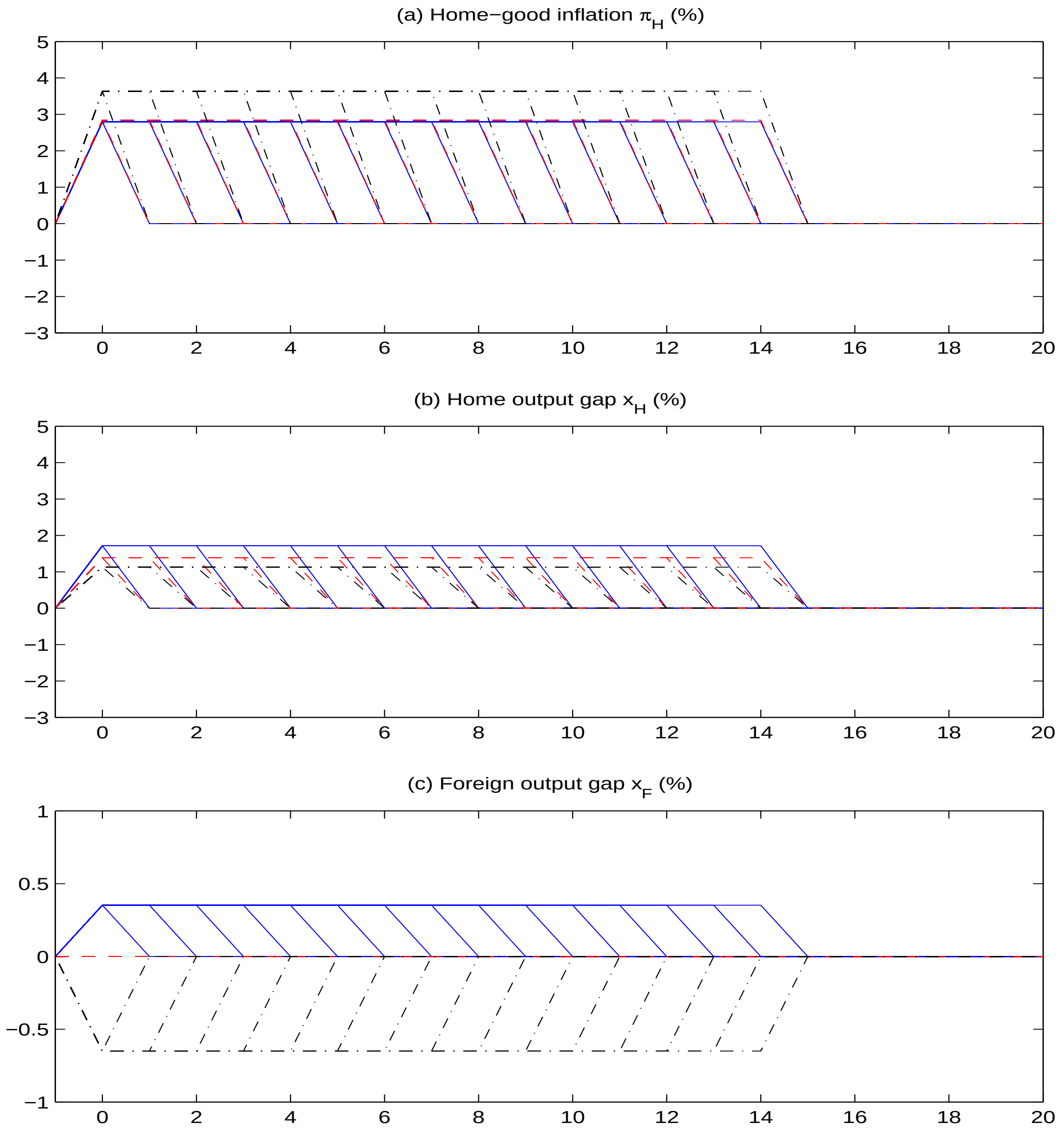

Figure 2: State-contingent paths of $\pi_{H,t}$, $x_{H,t}$, and $x_{F,t}$ under strict zero inflation targeting when $\mu = 0.1$. In each panel, the solid line corresponds to $\sigma = 0.5$, the dashed line to $\sigma = 1$, and the dashdot line to $\sigma = 5$.

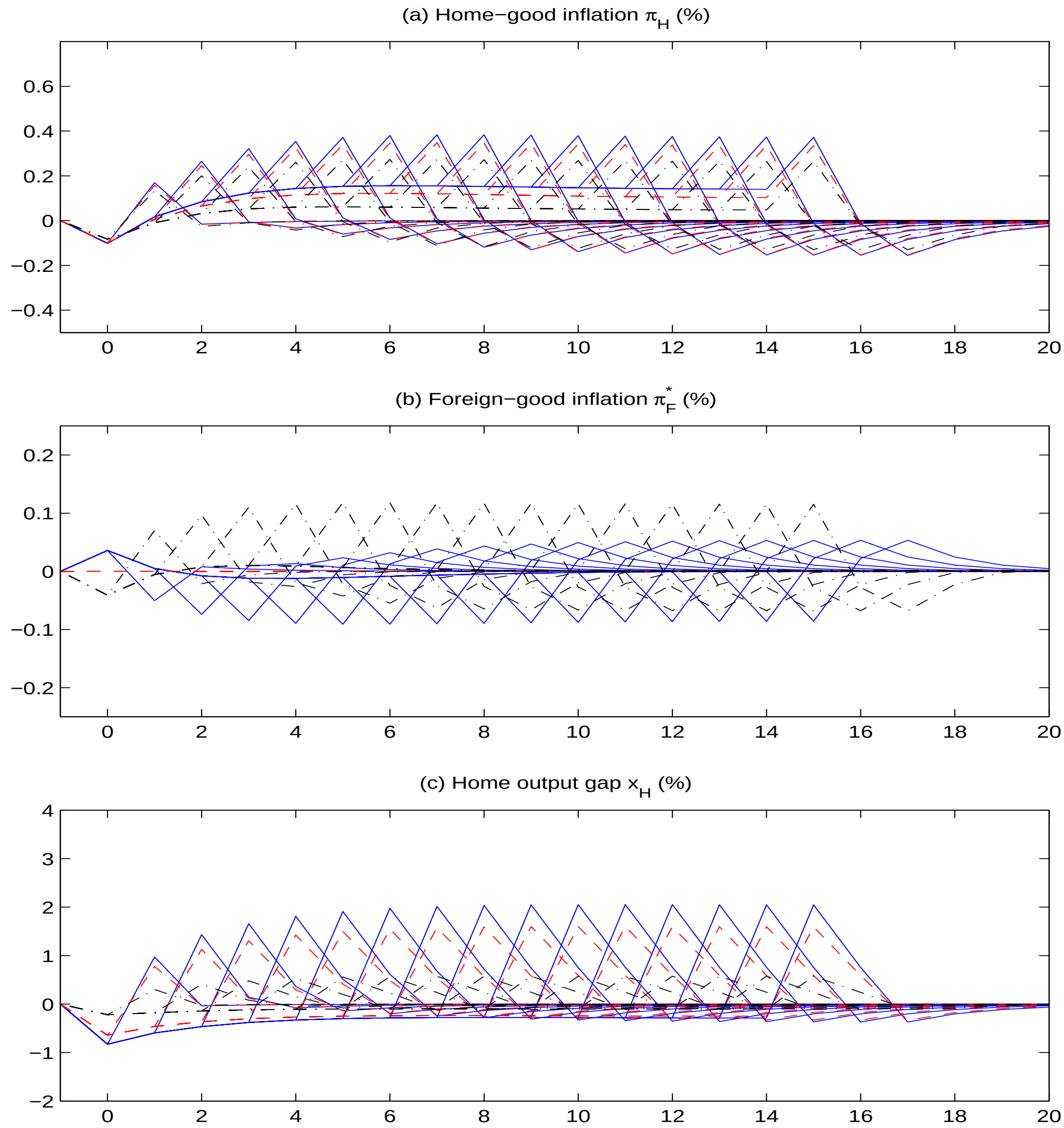

Figure 3: Optimal state-contingent paths of $\pi_{H,t}$, $\pi^*_{F,t}$, $x_{H,t}$, $x_{F,t}$, $i_{H,t}$, and $i_{F,t}$. In each panel, the solid line corresponds to $\sigma = 0.5$, the dashed line to $\sigma = 1$, and the dashdot line to $\sigma = 5$.

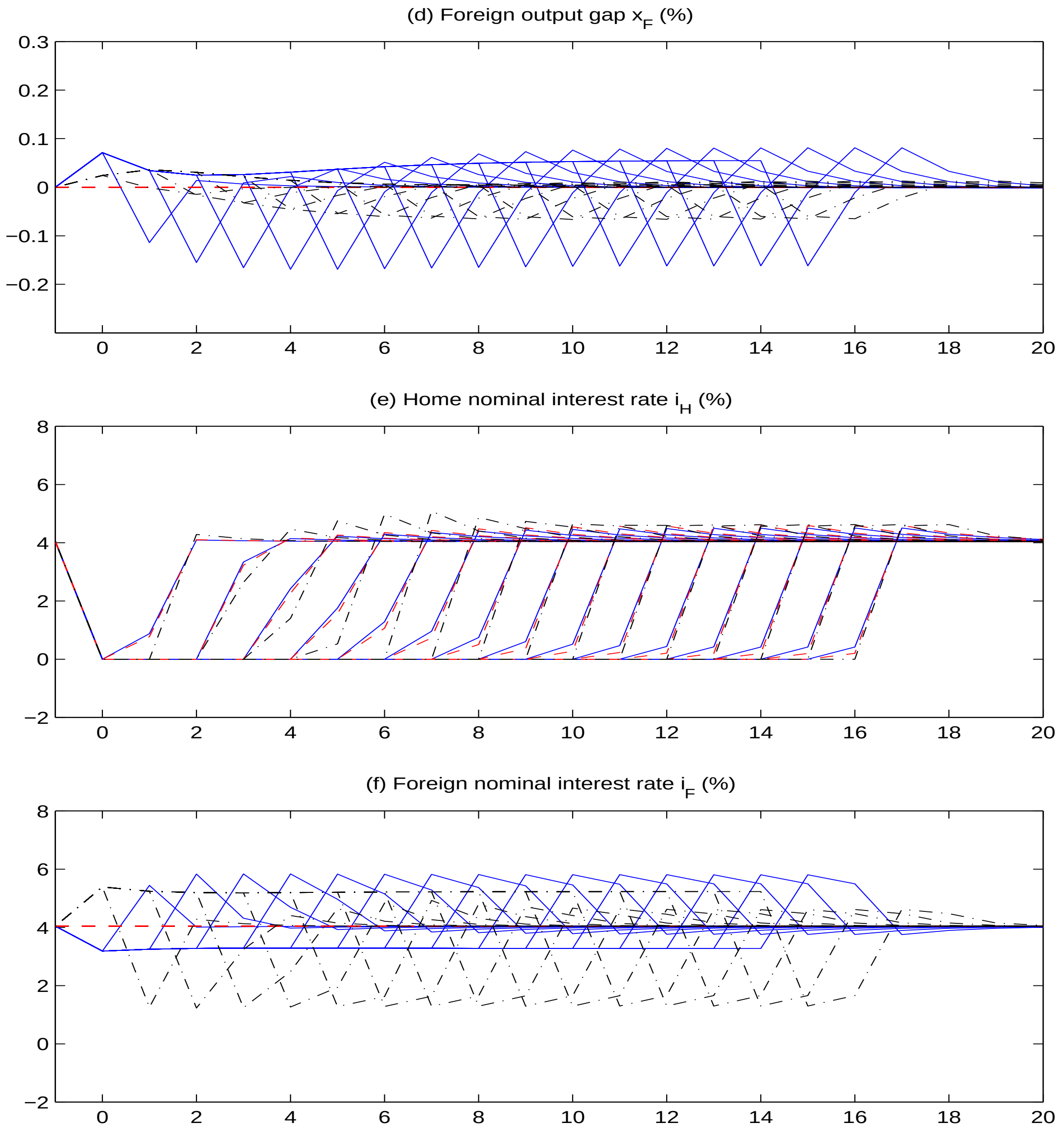

Figure 3 (continued): Optimal state-contingent paths of $\pi_{H,t}$, $\pi^*_{F,t}$, $x_{H,t}$, $x_{F,t}$, $i_{H,t}$, and $i_{F,t}$. In each panel, the solid line corresponds to $\sigma = 0.5$, the dashed line to $\sigma = 1$, and the dashdot line to $\sigma = 5$.

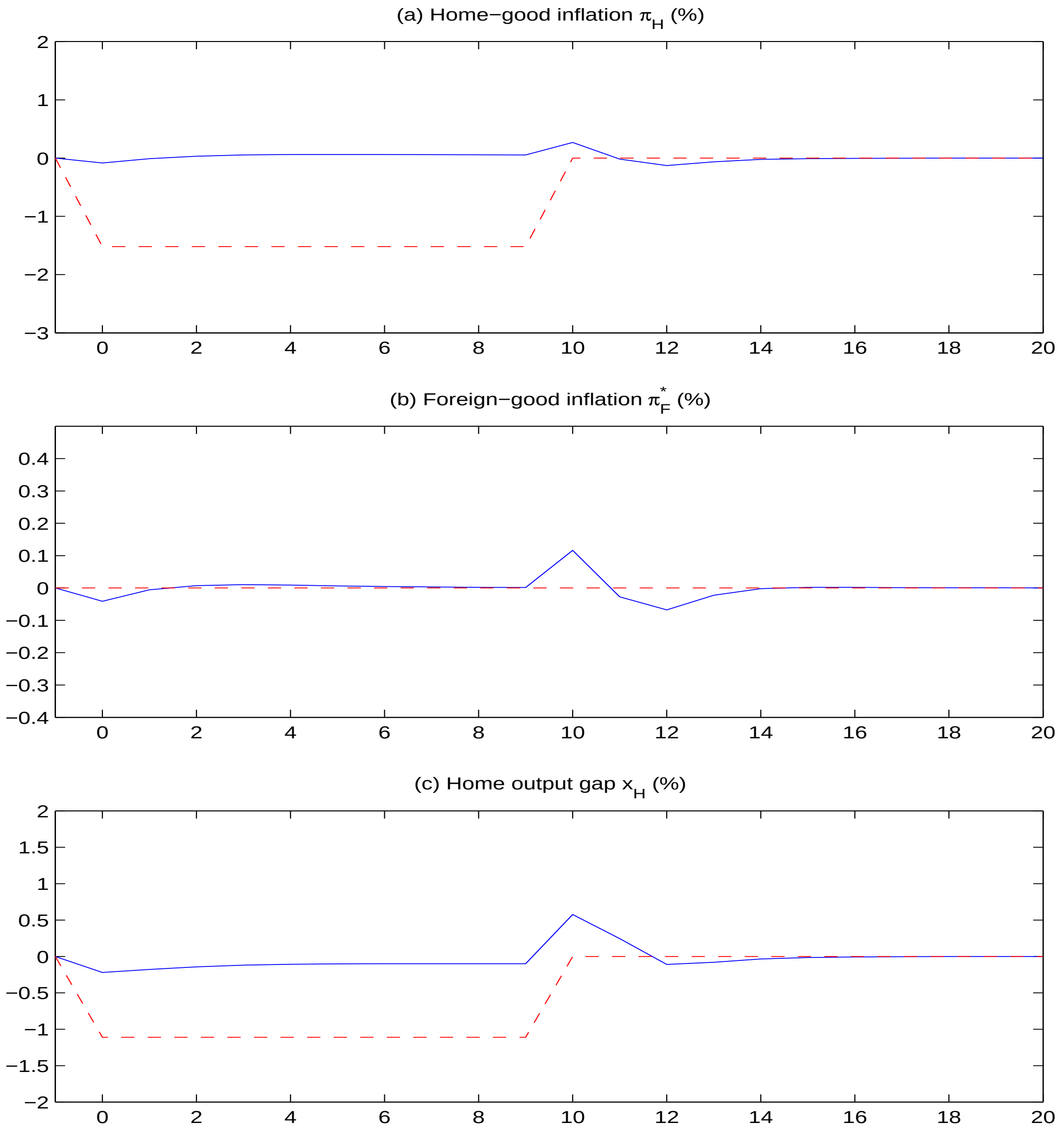

Figure 4: The paths of $\pi_{H,t}$, $\pi^*_{F,t}$, $x_{H,t}$, $x_{F,t}$, $i_{H,t}$, and $i_{F,t}$ under optimal policy commitment (solid lines) and under strict zero inflation targeting (dashed line) in the case where $r^n_{H,t}$ is negative for $t = 0, 1, \ldots, 9$ and $\sigma = 5$.

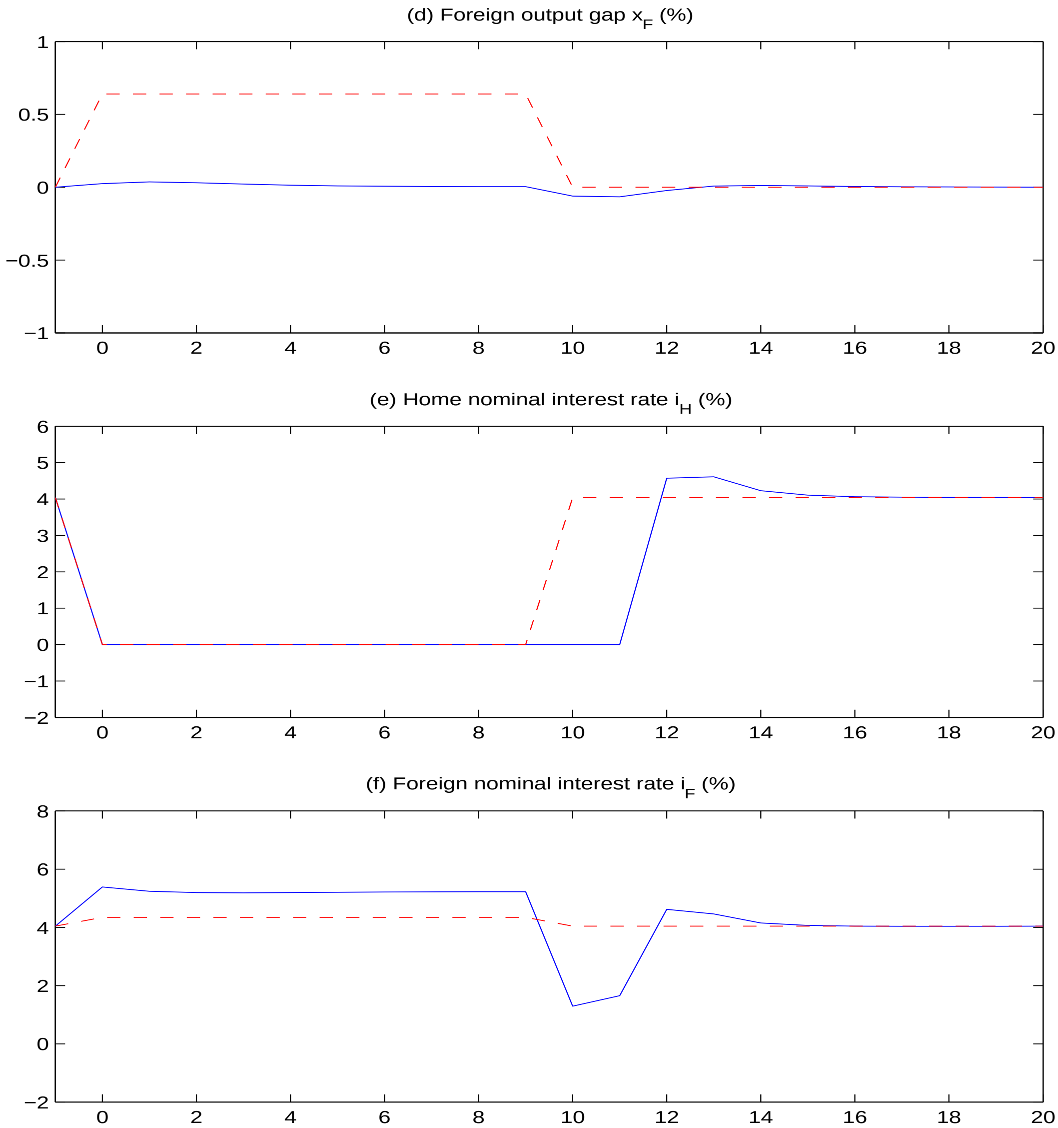

Figure 4 (continued): The paths of $\pi_{H,t}$, $\pi^*_{F,t}$, $x_{H,t}$, $x_{F,t}$, $i_{H,t}$, and $i_{F,t}$ under optimal policy commitment (solid lines) and under strict zero inflation targeting (dashed line) in the case where $r^n_{H,t}$ is negative for $t = 0, 1, \ldots, 9$ and $\sigma = 5$.

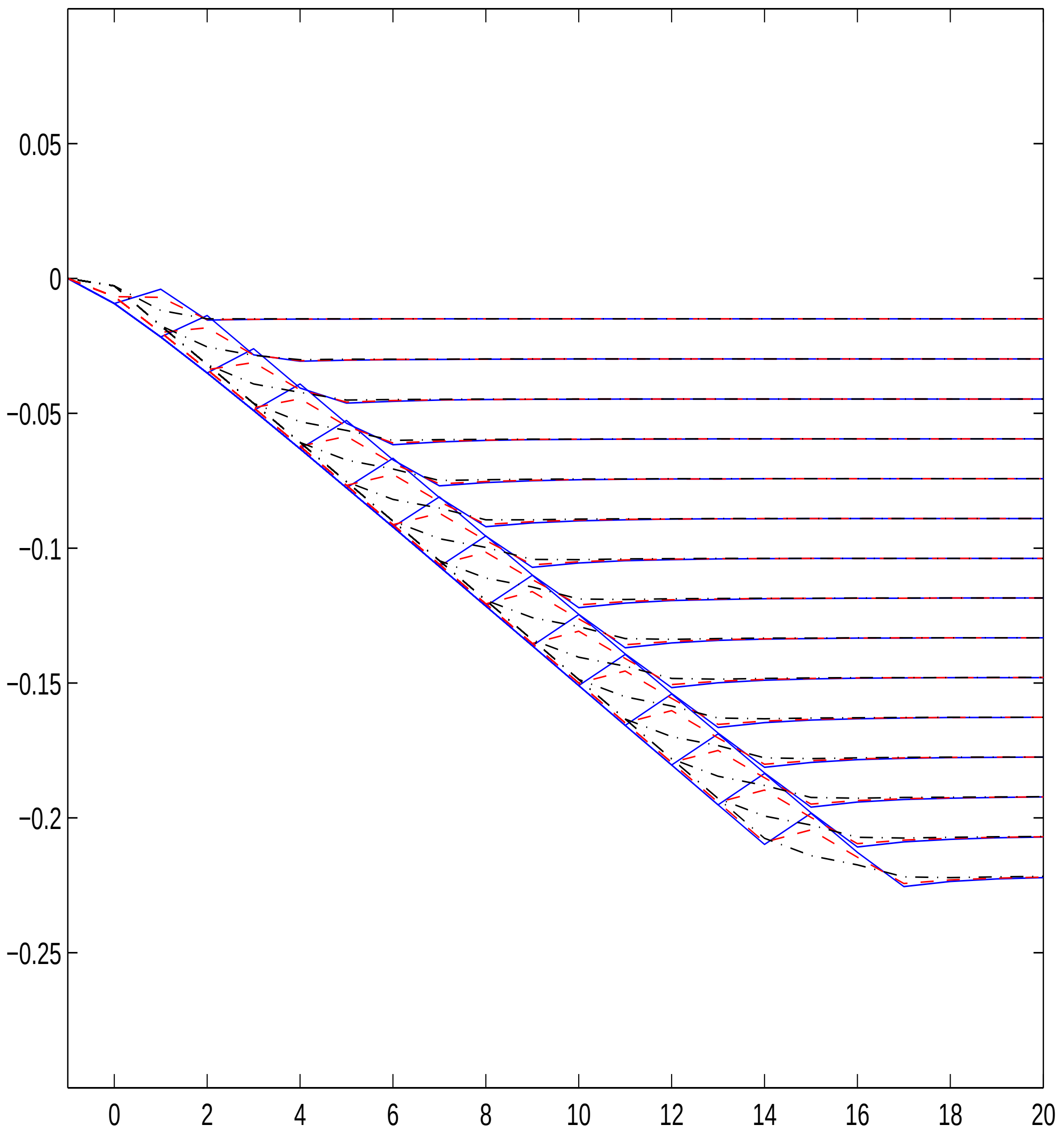

Figure 5: Optimal state-contingent paths of the (log) nominal exchange rate ϵ_t for $\sigma = 0.5$ (solid line), $\sigma = 1$ (dashed line), and $\sigma = 5$ (dashdot line).

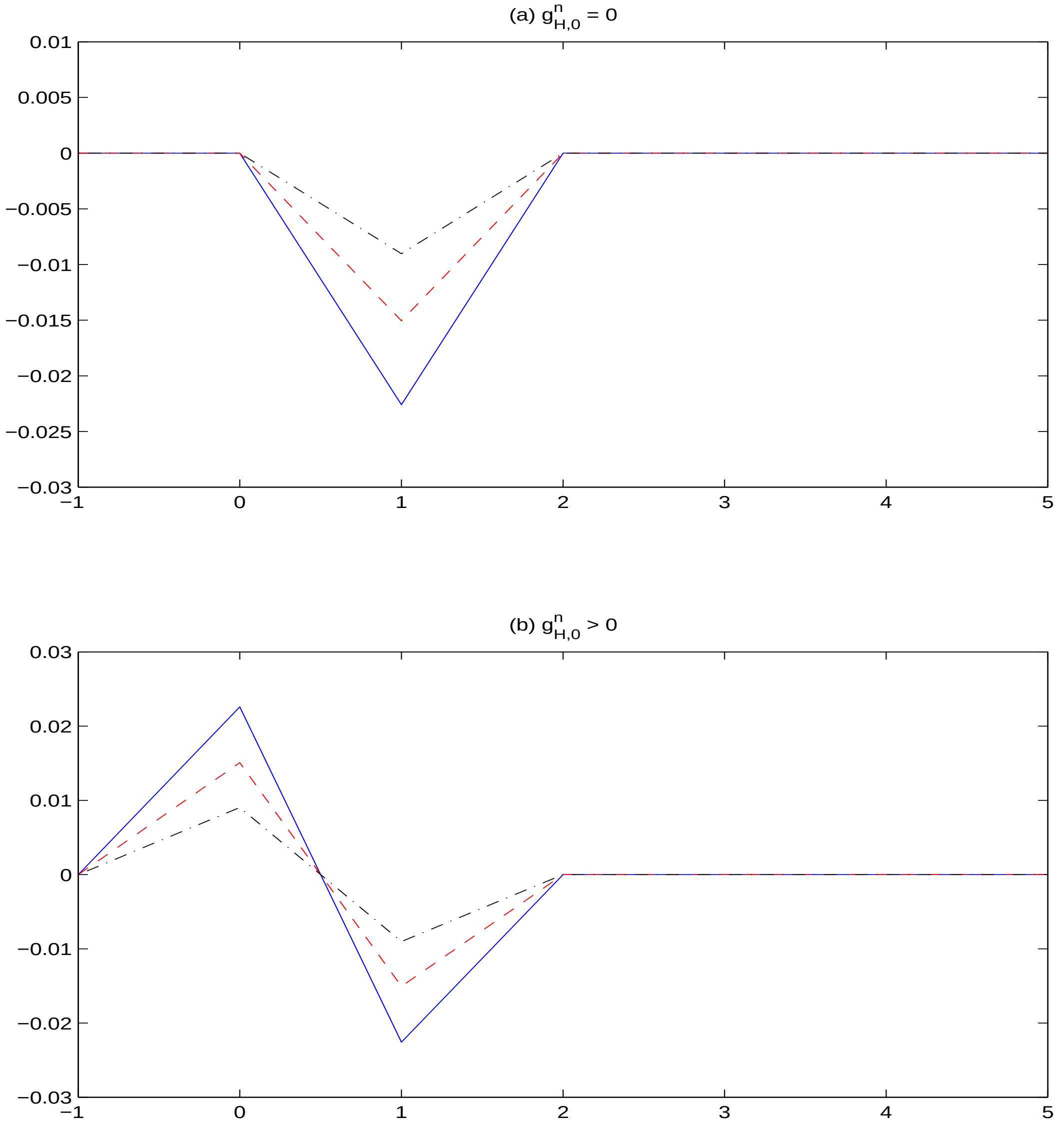

Figure 6: The paths of $g^n_{H,t}$ in the case where the natural rate of interest evolves deterministically as $r^n_{H,0} = \underline{r}$ and $r^n_{H,t} = \bar{r}$ for $t \geq 1$. Panel (a) is the case in which $g^n_{H,0} = 0$ and panel (b) is the case in which $g^n_{H,0} > 0$. The solid lines, dashed lines, and dashdot lines correspond to $\sigma = 0.5$, $\sigma = 1$, and $\sigma = 5$, respectively.

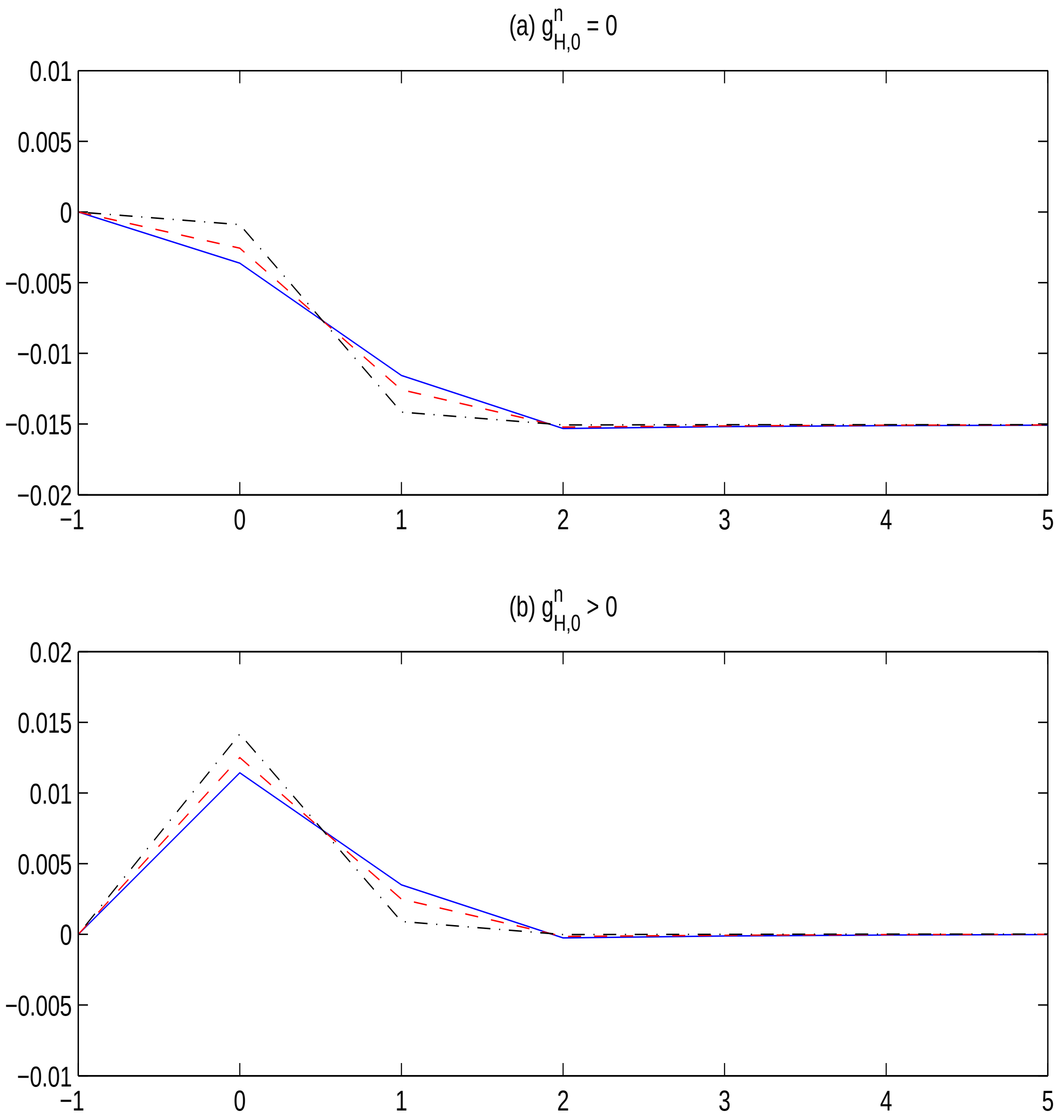

Figure 7: The optimal evolution of the (log) nominal exchange rate, ϵ_t, when $r^n_{H,0} = \underline{r}$ and $r^n_{H,t} = \bar{r}$ for $t \geq 1$. Panel (a) is the case in which $g^n_{H,0} = 0$ and panel (b) is the case in which $g^n_{H,0} > 0$. The solid lines, dashed lines, and dashdot lines correspond to $\sigma = 0.5$, $\sigma = 1$, and $\sigma = 5$, respectively.

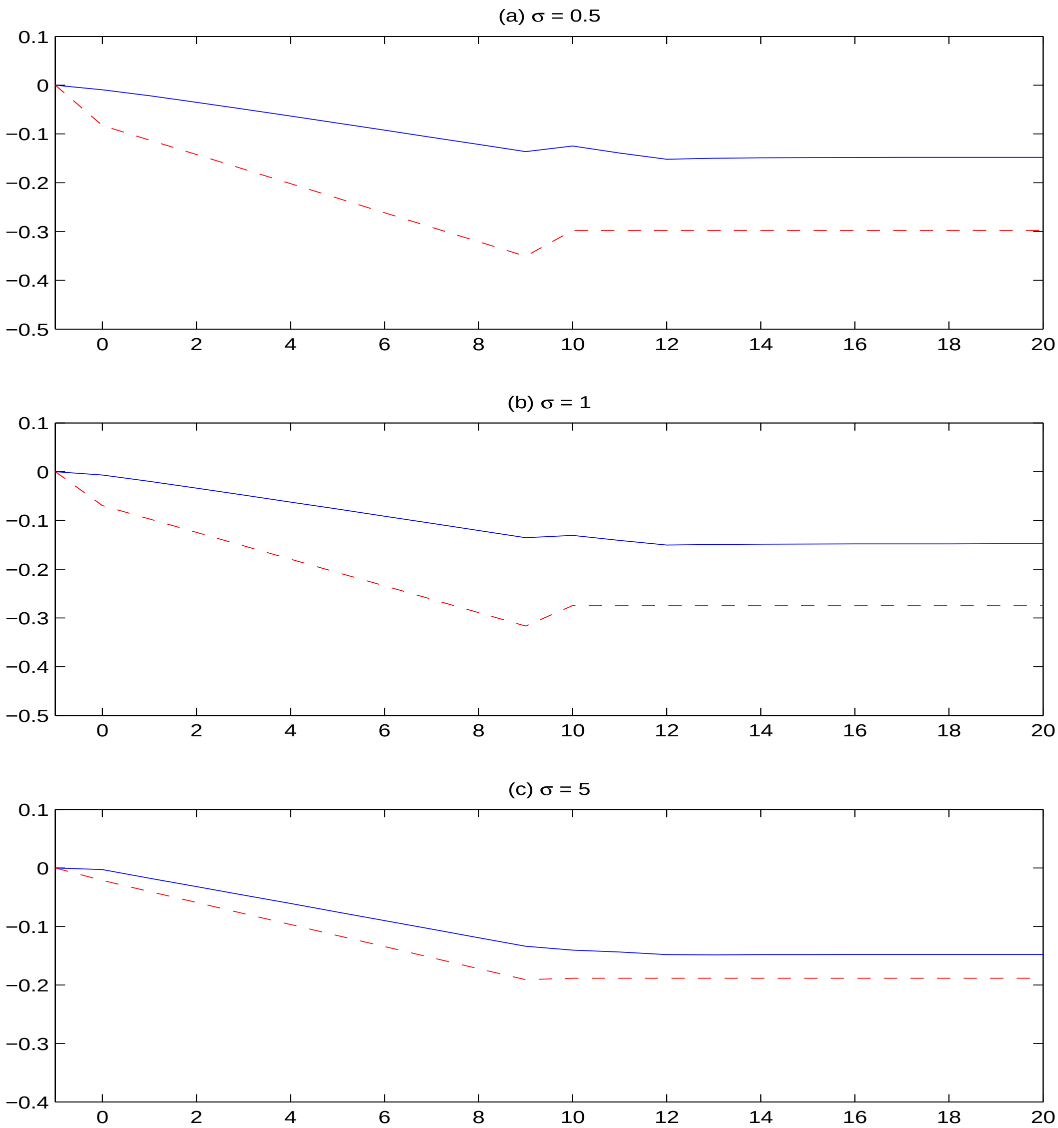

Figure 8: The paths of ϵ_t under optimal policy commitment (solid lines) and under strict zero inflation targeting (dashed line) in the case where $r^n_{H,t} = \underline{r}$ for $t = 0, 1, \ldots, 9$.