Dynamics of Integration in East Asian Equity Markets

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Motivations

1. Correlations in international financial markets have profound implications on
   1. Asset allocation
   2. Risk management
   3. Policy making
2. Understanding the correlations in international equity markets is not be easy due to the time variation
3. Analyzing the correlation dynamics in international equity markets is an important issue

Motivations

4. Number of prevous studies investigate the correlation dynamics in international equity markets
   1. Longin and Solnik (1995, JIMF)
   2. Berben and Jansen (2005, JIMF)
   3. Bekaert, Hodrick, and Zhang (2009, JF)
   4. Christoffersen, Errunza, Jacobs and Langlois (2012, RFS)
   5. Okimoto (2014, JBF)
5. Few studies focus on the East Asian equity markets

Contributions

1. Examines the dynamics of correlation (integration) in East Asian Equity Markets
2. East Asian equity markets open at almost the same time
3. Decompose equity returns into two returns
$$RCC_t \ = \ RCO_t + ROC_t$$
4. Investigate the extent to which returns contribute more to the changes in integration

Main results

1. No international integration in the East Asian equity market around 1995
2. China-related pairs' international integration increased significantly after 2007
3. Integration for the pairs excluding China increased significantly between 1998 and 2001
4. Increase in integration is largely attributable to after-trading-hours returns
5. Effects of international asset allocation in East Asian markets are vastly reduced

Marginal model for each country

1. r_{it}: stock return for country i
2. Model conditional expectation of r_{it} with AR(2) model
3. Also try VAR model, but past returns of other countries have little effects
4. Model conditional variance of r_{it} with GARCH(1,1) model
5. $$\begin{cases} r_{it} = c_i + \phi_{i1} r_{i,t-1} + \phi_{i2} r_{i,t-2} + u_{it} \\ u_{it} = \sqrt{h_{ii,t}}\varepsilon_{it}, \\ h_{ii,t} = \omega_i + \alpha_i u_{i,t-1}^2 + \beta_i h_{ii,t-1}, \end{cases}$$

Conditional Correlation Model

1. $\mathbf{u}_t = (u_{1t}, u_{2t}, \ldots, u_{nt})'$: vector of disturbance of each country
2. Assume $\mathbf{u}_t = \mathbf{H}_t^{1/2}\mathbf{v}_t, \quad \mathbf{v}_t \sim \text{iid } N(\mathbf{0}, \mathbf{I}_M)$
3. $\mathbf{H}_t$: conditional variance-covariance matrix of $\mathbf{r}_t = (r_{1t}, r_{2t}, \ldots, r_{nt})'$
4. $\mathbf{H}_t$ can be decomposed as $\mathbf{H}_t = \mathbf{D}_t\mathbf{R}_t\mathbf{D}_t$
5. $\mathbf{D}_t = \text{diag}(h_{11,t}, \ldots, h_{nn,t})^{1/2}$
6. $\mathbf{R}_t$: conditional correlation of $\mathbf{r}_t$
7. Use smooth transition correlation (STC) model to analyze the long-run trends in $\mathbf{R}_t$

STC model

1. Developed by Teräsvirta (1994, JASA) in the AR framework
2. Applied to the time-varying correlation model
   1. Berben and Jansen (2005, JIMF): Equity
   2. Kumar and Okimoto (2011, JBF): Bond
   3. Ohashi and Okimoto (2016, JCOM): Commodity
3. $\mathbf{R}_t = (1 - G(s_t; \gamma, c))\mathbf{R}^{(1)} + G(s_t; \gamma, c)\mathbf{R}^{(2)}$
4. One of the regime switching models
   1. Regime 1: $G = 0 \Longrightarrow \mathbf{R}_t = \mathbf{R}^{(1)}$
   2. Regime 2: $G = 1 \Longrightarrow \mathbf{R}_t = \mathbf{R}^{(2)}$
5. $\mathbf{R}_t$ takes the value between $\mathbf{R}^{(1)}$ and $\mathbf{R}^{(2)}$ depending on $G(s_t; \gamma, c)$

STC model

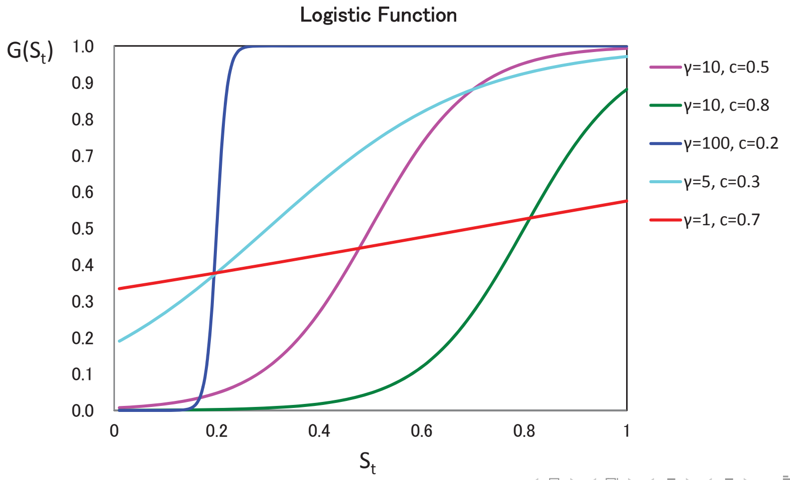
6. Transition function $G(s_t; c, \gamma)$ is expressed by a logistic function $G(s_t; c, \gamma)$

$$G(s_t; c, \gamma) = \frac{1}{1 + \exp(-\gamma(s_t - c))}, \quad \gamma > 0$$

   1. s_t : transition variable
   2. c: location parameter
   3. γ: smoothness parameter
7. Adopt $s_t = t/T$ as a transition variable to capture dominant trends (Lin and Teräsvirta, 1994, JoE)
   1. $\mathbf{R}^{(1)}$: $\mathbf{R}$ around the beginning of the sample
   2. $\mathbf{R}^{(2)}$: $\mathbf{R}$ around the end of the sample

STC model

8. Can describe a wide variety of patterns of regime transition depending on the values of γ and c



STC model

9. Can extend to the three state STC model
$$\mathbf{R}_t = \mathbf{R}^{(1)} + G_1(s_t;\gamma_1,c_1)(\mathbf{R}^{(2)} - \mathbf{R}^{(1)}) + G_2(s_t;\gamma_2,c_2)(\mathbf{R}^{(3)} - \mathbf{R}^{(2)})$$
10. Assume $0.05 \leq c_1 < c_2 \leq 0.95$ to detect the correlation transition within the sample period
11. $\mathbf{R}_t$ changes from $\mathbf{R}^{(1)}$ via $\mathbf{R}^{(2)}$ to $\mathbf{R}^{(3)}$ with time
12. γ_i and c_i as well as $\mathbf{R}^{(i)}$ are estimated from the data
13. Can select the best pattern for the long-run trends in integration in East Asian Equity markets
14. Can examine the increasing integration by testing the null of $r_{ij}^{(k)} = r_{ij}^{(k+1)}, \ \forall i \neq j, k = 1, 2$

Data

1. Sample period: from Jan 1995 to Jan 2013
2. Opening and closing price of stock index for four East Asian countries (CH, HK, JP, KR)
    1. Shanghai Stock Exchange Composite Index (SSEC)
    2. Hang Seng Index (HSI)
    3. Nikkei Stock Average 225 Index (Nikkei225)
    4. Korea Composite Stock Price Index (KOSPI)
3. Korean market was open on Saturday until 1998
4. Obtain qualitatively similar results using data after 1998

Decomposition of equity returns

1. East Asian equity markets open at almost the same time
2. Define three returns to examine the extent to which returns contribute more to the changes in integration
3. Equity returns: close-to-close returns or RCC
$$RCC_t = \log PC_t - \log PC_{t-1}$$
4. Trading-hours returns: open-to-close returns or ROC
$$ROC_t = \log PC_t - \log PO_t$$
5. After-trading-hours returns: close-to-open returns or RCO
$$RCO_t = \log PO_t - \log PC_{t-1}$$

Decomposition of equity returns

6. Decomposition of equity returns
$$\begin{aligned} RCC_t &= \log PC_t - \log PC_{t-1} \\ &= \log PC_t - \log PO_t + \log PO_t - \log PC_{t-1} \\ &= RCO_t + ROC_t \end{aligned}$$
7. Examine the extent to which returns contribute more to the changes in integration in East Asian markets
8. Many days when the data for the four countries cannot be obtained for daily data
9. Calculate weekly returns by summing the returns for one week from the closing price of the Wednesday
10. Decomposition is still valid for weekly returns

Results for equity returns

1. Estimation results

		CH-HK	CH-JP	CH-KR	HK-JP	HK-KR	JP-KR
Regime 1	Estimate	-0.283	-0.078	-0.210	0.006	0.005	-0.193
	Std. Error	0.248	0.137	0.185	0.212	0.261	0.276
Regime 2	Estimate	0.419	0.222	0.262	0.619	0.723	0.662
	Std. Error	0.047	0.043	0.048	0.030	0.031	0.030
Test of equality	Wald stat	6.44	3.51	4.92	7.17	6.43	8.56
	P-value	0.011	0.061	0.027	0.007	0.011	0.003

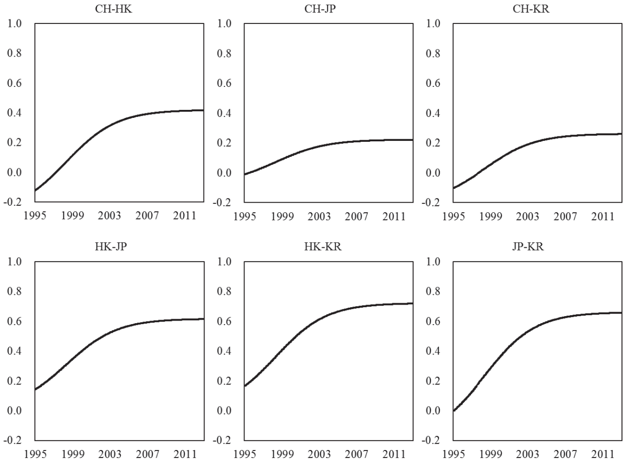
Results for equity returns

2. Correlations in regime 1 are generally low
   1. Average: -0.125
   2. Max: 0.006 (HK-JP pair)
   3. Min: -0.283 (CH-HK pair)
3. No significant correlation exists for any country pairs
4. No international integration in the East Asian equity market around the year 1995

Results for equity returns

5. Correlations in regime 2 are significantly positive for all pairs
    1. Average: 0.485
    2. Max: 0.723 (HK-KR pair)
    3. Min: 0.222 (CH-JP pair)
6. Correlations in regime 2 are uniformly higher than those in the first regime
7. International integration has increased in recent years
8. When and how has the international integration of equity returns in East Asian equity markets increased?

Results for equity returns



9. International integration in East Asian equity markets increased greatly between 1995 and 2003

Results for trading-hours returns

1. Estimation results

		CH-HK	CH-JP	CH-KR	HK-JP	HK-KR	JP-KR
Regime 1	Estimate	-0.287	0.075	-0.127	-0.015	0.040	-0.239
	Std. Error	0.320	0.105	0.162	0.182	0.203	0.364
Regime 2	Estimate	0.287	0.069	0.127	0.308	0.381	0.459
	Std. Error	0.047	0.044	0.051	0.039	0.038	0.044
Test of equality	Wald stat	2.771	0.002	1.885	2.656	2.462	3.417
	P-value	0.096	0.962	0.170	0.103	0.117	0.065

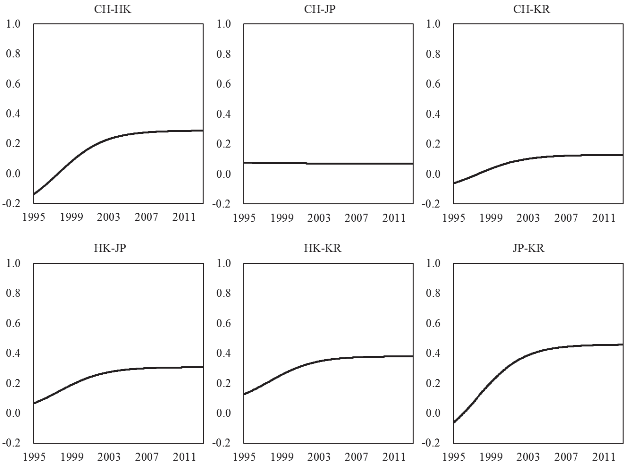
Results for trading-hours returns

2. Correlations in regime 1 are gnerally low
    1. Average: -0.092
    2. Max: 0.076 (CH-JP pair)
    3. Min: -0.287 (CH-HK pair)
3. No significant correlation exists for any country pairs
4. No international integration in the East Asian equity market around the year 1995

Results for trading-hours returns

5. Correlations in regime 2 are significantly positive for all pairs except CH-JP pair
    1. Average: 0.272
    2. Max: 0.459 (JP-KR pair)
    3. Min: 0.06 (CH-JP pair)
6. Correlations in regime 2 become higher for CH-HK and JP-KR pairs
7. International integration has increased in recent years, but not much compared to equity returns

Results for trading-hours returns



9. Correlation has increased greatly between 1995 and 2003 for CH-HK and JP-KR pairs

Results for after-trading-hours returns

1. Estimation results

		CH-HK	CH-JP	CH-KR	HK-JP	HK-KR	JP-KR
Regime 1	Estimate	-0.082	-0.158	-0.142	0.558	0.334	0.298
	Std. Error	0.062	0.066	0.060	0.047	0.114	0.119
Regime 2	Estimate	0.519	0.456	0.450	0.723	0.832	0.798
	Std. Error	0.106	0.103	0.116	0.023	0.024	0.026
Test of equality	Wald stat	28.57	33.10	22.60	8.90	19.55	17.33
	P-value	0.000	0.000	0.000	0.003	0.000	0.000

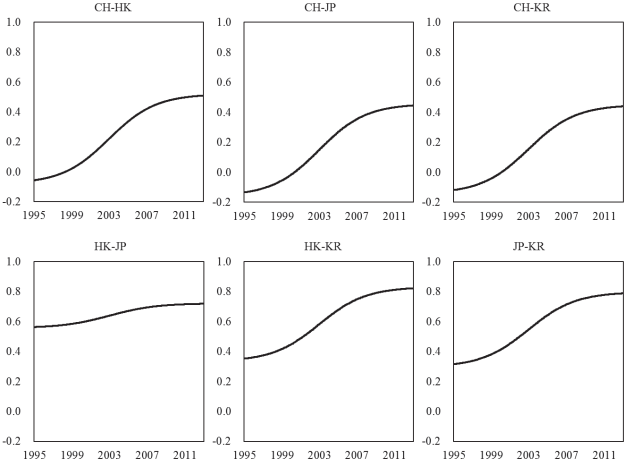
Results for after-trading-hours returns

2. Correlations in regime 1 are low for China-related pairs and relatively high for other pairs
    1. Average: 0.135
    2. Max: 0.558 (HK-JP pair)
    3. Min: -0.158 (CH-JP pair)
3. Significant negative correlation for the CH-JP and CH-KR pairs
4. Positive significant correlation for all pairs that do not include China

Results for after-trading-hours returns

5. Correlations in regime 2 are significantly positive for all pairs
   1. Average: 0.630
   2. Max: 0.832 (HK-KR pair)
   3. Min: 0.450 (CH-KR pair)
6. Correlations in regime 2 are uniformly higher than those in the first regime
7. International integration has increased in recent years

Results for after-trading-hours returns



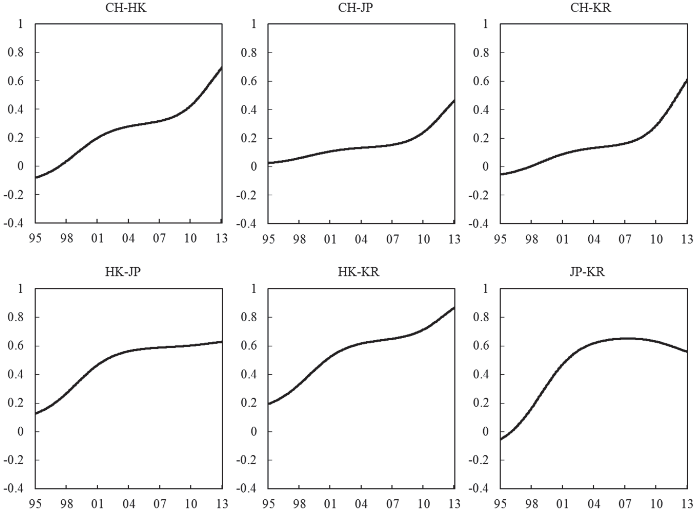
9. Increase in integration of equity returns in East Asian equity markets is largely attributable to RCO

Results of three-regime STC model

1. Results so far are based on the two-regime assumption
2. Assume a monotonic trend in the international integration of East Asian equity markets
3. Estimate the three-regime STC model to assess the possible non-monotonic trend
4. Results are qualitatively same as those of the two-regime model
5. China-related pairs international integration increased significantly after 2007
6. Integration for the pairs excluding China increased significantly between 1995 and 2001

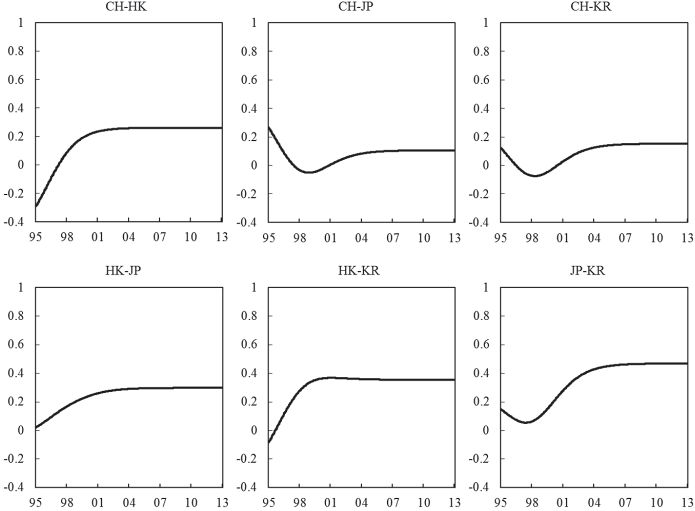
Results of three-regime STC model

7. Dynamics of equity return correlation



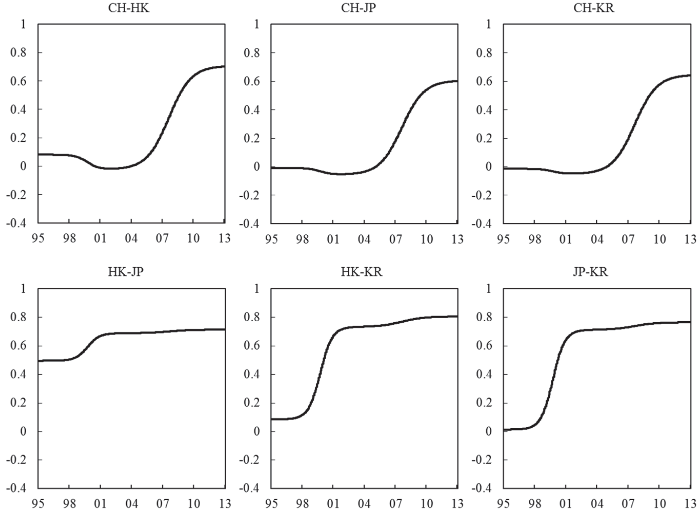
Results of three-regime STC model

8. Dynamics of trading-hours return correlation



Results of three-regime STC model

9. Dynamics of after-trading-hours return correlation

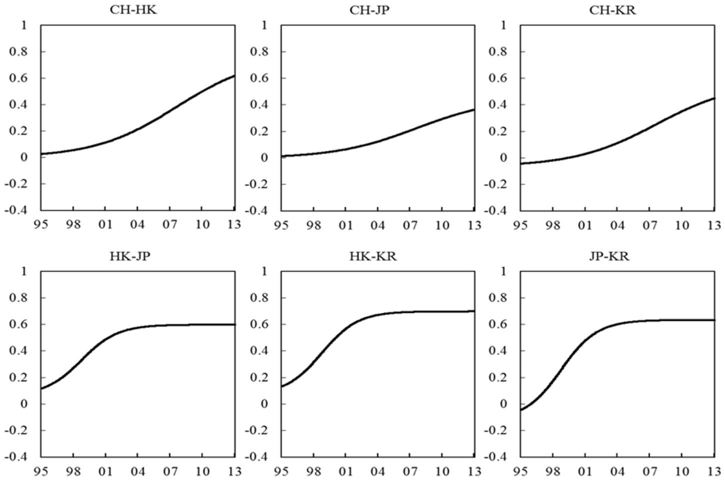


Results of constrained three-regime STC model

1. Difference between China-related pairs and pairs excluding China in terms of the timing of the increase in international integration
2. Impose constraints on the three-regime STC model
   1. $r_{ij}^{(1)} = r_{ij}^{(2)}$ for China-related pairs
   2. $r_{ij}^{(2)} = r_{ij}^{(3)}$ for pairs excluding China
3. China-related pairs' international integration increased significantly after 2007
4. Integration for the pairs excluding China increased significantly between 1998 and 2001
5. Increase in integration is largely attributable to after-trading-hours returns

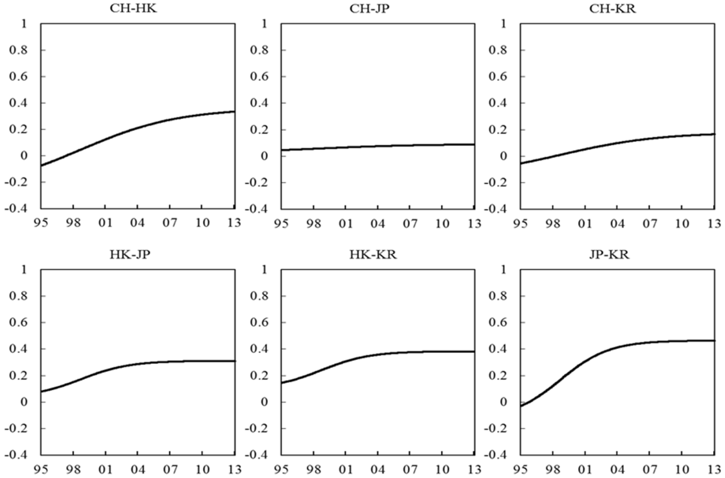
Results of constrained three-regime STC model

6. Dynamics of equity return correlation



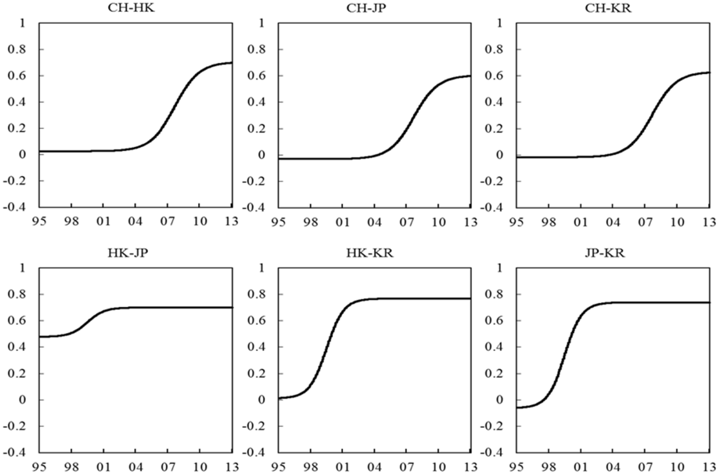
Results of constrained three-regime STC model

7. Dynamics of trading-hours return correlation



Results of constrained three-regime STC model

8. Dynamics of after-trading-hours return correlation



Possible reasons for increase in integration

1. More aggressive investments by foreign investors
2. China-related pairs' international integration increased significantly after 2007
    1. Circulation problem of non-tradable shares
    2. Split-share structure (non-tradable share) reform has been implemented since April 2005
    3. 98% of listed companies had implemented reforms by the end of 2006
3. Integration for the pairs excluding China increased significantly between 1998 and 2001
    1. Letup in the Asian currency crisis
    2. Deregulation of the equity market in South Korea
    3. Reform of the equity market exchange in HK

Impact on diversified investment

1. Examines the impact of the increase in international integration on asset allocation
2. Calculate the weights for the minimum variance portfolio based on the correlation in 1995 and 2013

		CH	HK	JP	KR
Equity returns	1995	0.242	0.210	0.346	0.202
	2013	0.287	0.118	0.595	0.000
Trading-hours returns	1995	0.222	0.284	0.297	0.197
	2013	0.232	0.294	0.390	0.085
After-trading-hours returns	1995	0.250	0.024	0.538	0.188
	2013	0.073	0.000	0.927	0.000

Impact on diversified investment

3. Effects of international asset allocation are vastly reduced for equity returns and after-trading-hours returns
4. International asset allocation still has a slight effect on trading-hours returns
5. Investing in each country's leading index is not enough to get diversification effects
6. Important to consider further diversification, such as industrial sector diversification

Conclusion

1. Examine the dynamics of correlation (integration) in East Asian Equity Markets
2. No international integration in the East Asian equity market around 1995
3. China-related pairs' international integration increased significantly after 2007
4. Integration for the pairs excluding China increased significantly between 1998 and 2001
5. Increase in integration is largely attributable to after-trading-hours returns
6. Effects of international asset allocation in East Asian markets are vastly reduced

References

[1] Bekaert, G., Hodrick, R.J., Zhang, X. (2009). "International stock return comovements," *Journal of Finance* 64(6), 2591-2626.

[2] Berben, R.-P., and Jansen, W.J. (2005). "Comovement in international equity markets: A sectoral view," *Journal of International Money and Finance* 24, 832-857.

[3] Christoffersen, P., Errunza, V., Jacobs, K., and Langlois, H. (2012). "Is the potential for international diversification disappearing? A dynamic copula approach," *Review of Financial Studies* 25(12), 3711-3751.

[4] Kumar, M.S., and Okimoto, T. (2011). "Dynamics of international integration of government securities' markets." *Journal of Banking and Finance* 35, 142-154.

[5] Lin, C.-F.J., and Teräsvirta, T. (1994). “Testing the constancy of regression parameters against continuous structural change,” *Journal of Econometrics* 62, 211-228.

[6] Longin, F. and Solnik, B. (1995). “Is the correlation in international equity returns constant: 1960-1990?,” *Journal of International Money and Finance* 14, 3-26.

[7] Ohashi, K. and Okimoto, T. (2016). “Increasing trends in the excess comovements of commodity prices,” *Journal of Commodity Markets* 1, 48-64.

[8] Okimoto, T. (2014), “Asymmetric Increasing Trends in Dependence in International Equity Markets,” *Journal of Banking and Finance* 46, 219-232.

[9] Teräsvirta, T. (1994). “Specification, estimation, and evaluation of smooth transition autoregressive models,” *Journal of the American Statistical Association* 89(425), 208-218.

SMPP models with an application to causality analysis of financial markets

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研究集会
「経済リスクの統計学の新展開：稀な事象と再帰的事象」

概要

- 背景
- SMPP モデル
- SHPP モデル
- SMPP モデルの Granger 因果性
- 漸近理論
- 数値実験
- 実データ分析
- まとめ

背景

- 点過程モデルがよく利用される（高頻度) 金融データ, 神経スパイクデータ, 地震による複数地点の震度データなどでは同時点において複数のイベント (co-jump) が観測されることがある.
- しかし, 多くの点過程のモデルにおいて仮定される条件 → co-jump は起こらない: d 次元点過程 $N = (N_1, \dots, N_d)$ に対して,

$$P(N_j(t+\Delta) - N_j(t) = 1 | \mathcal{H}_t) = \lambda_j(t|\mathcal{H}_t)\Delta + o_P(\Delta), \quad 1 \le j \le d,$$

$$P(N_j(t+\Delta) - N_j(t) \ge 2 | \mathcal{H}_t) = o_P(\Delta), \quad 1 \le j \le d,$$

$$P(N_g(t+\Delta) - N_g(t) \ge 2 | \mathcal{H}_t) = o_P(\Delta), \quad N_g = \sum_{j=1}^{d} N_j,$$

ここで $\mathcal{H}$ は N の過去の情報を含む filtration.

背景

- Co-jump (simultaneous event) の存在を考慮する場合, 従来の点過程モデルの仮定から外れる.
- Co-jump を持つような点過程モデルをどう考えればよいか？
- Solo (2007) は同時点での複数のイベントの発生を考慮した点過程のモデル (simultaneous event multivariate point process model) を提案. このアイデアを利用+マーク付きのケースに拡張.

SMPP モデル (構成)

- d 次元点過程 N を考える．各時点において d 個成分のうち少なくとも 1 つの成分のイベントが発生する場合の組み合わせは $2^d - 1$ 通り ((各成分について jump する or しない)-(どの成分も jump しない)).
- この $2^d - 1$ 通りのイベントに対してそれぞれのイベントをカウントする点過程 N^* を考える

$$N^*(t) = (N_1^*(t), \ldots, N_{2^d-1}^*(t)).$$

- このとき, 点過程 N^* は co-jump を持たない.

SMPP モデル (構成)

- 例として $d = 3$ のケースを考える．このとき，N^* は $2^3 - 1 = 7$ 次元の点過程で，
$$N^* = (N_1^*, \ldots, N_7^*).$$
- 各成分の点過程として例えば，

N_j^*, $j = 1, 2, 3$ は第 j 成分のみの jump をカウント，

N_j^*, $j = 4, 5, 6$ は第 $(1, 2)$, $(1, 3)$, $(2, 3)$ 成分の (co-)jump をカウント，

N_7^* は第 1, 2, 3 成分すべてが jump するイベントをカウント，

などと考える．

SMPP モデル (構成)

- この場合, 元の 3 次元の点過程との対応は以下の通り:

$$
\begin{aligned}
N_1 &= N_1^* + N_4^* + N_5^* + N_7^*, \\
N_2 &= N_2^* + N_4^* + N_6^* + N_7^*, \\
N_3 &= N_3^* + N_5^* + N_6^* + N_7^*.
\end{aligned}
$$

- N と N^* は一対一対応.
- Co-jump を持つ点過程モデルの構成にはこの関係を使う.

SMPP モデル (強度関数)

- N^* に対しては従来の点過程の議論が適用できるので, N^* に対して強度関数を考える:

$$\lambda^*(t) = (\lambda_1^*(t), \ldots, \lambda_{2^d-1}^*(t)).$$

- 例えば $d = 3$ の時は N と N^* の関係と同様に以下の関係が成り立つ:

$$\begin{aligned}
\lambda_1 &= \lambda_1^* + \lambda_4^* + \lambda_5^* + \lambda_7^*, \\
\lambda_2 &= \lambda_2^* + \lambda_4^* + \lambda_6^* + \lambda_7^*, \\
\lambda_3 &= \lambda_3^* + \lambda_5^* + \lambda_6^* + \lambda_7^*.
\end{aligned}$$

SHPP モデル

これまでの議論を多次元 Hawkes 過程に適用
(simultaneous Hawkes point process (SHPP)).

- 観測期間 $[0, T]$ を n 個の区間に分割: $I_i^n = (t_{i-1}^n, t_i^n]$,
  $0 = t_0^n < t_1^n < \cdots < t_{n-1}^n < t_n^n = T$.
- $Y(s) = (Y_1(s), \ldots, Y_d(s))$: d-次元確率過程 (ex. log-return).
- 各成分に対して閾値 $u = (u_1, \ldots, u_d)$ を設定.
- 各観測時刻 t_i^n において $Y = (Y_1, \ldots, Y_d)$ の少なくとも 1 つの成分が閾値を超過していればイベント発生とみなし, これを jump と定義する. この jump をカウントする d 次元計数過程 N^n を, N^{n*} を用いてモデル化.

SHPP モデル

- 計数過程 N^n の jump は各区間 $I_i^n = (t_{i-1}^n, t_i^n]$ の端点 t_i^n で発生.
- 高頻度観測 $\max_{1\le i\le n} |t_i^n - t_{i-1}^n| \stackrel{n\to\infty}{\longrightarrow} 0$ のとき, $N^{n*} \to N^*$.
- 連続時間確率過程の離散観測の問題と関連.
- N^{n*} がある点過程の実現値であると考え, N^{n*} のモデリングを考える (以下 n は省略).

SHPP モデル

N^* 各成分の閾値超過レート (強度関数) が次の形で与えられるとする.

$$\lambda_j^*(t, x|\mathcal{H}_t^*) = \left(\lambda_{j,0}^* + \sum_{i=1}^{2^d-1} \int_{-\infty}^{t} c_{ji}^*(x) g_{ji}^*(t-s) N_i^*(ds \times dx)\right) \times \left(-\frac{\partial}{\partial x_{(j)}} F_\theta(x_1, \ldots, x_d|\mathcal{H}_t^*)\Big|_{x_{(j)}=u_{(j)}, x_{-(j)}=u_{-(j)}}\right).$$

$j = 1, \ldots, 2^d - 1$, $\lambda_{j,0}^*$ は正定数,
$\mathcal{H}^*$: N^* の history,
$F^{(j)}$: 時刻 t で閾値 $u = (u_1, \ldots, u_d)$ を超過した成分の同時分布.
$((j) \subset \{1, \ldots, d\})$,
$c_{ji}^*(\cdot) : \mathbb{R} \to \mathbb{R}_+$: インパクト関数,
$g_{ji}^*(\cdot) : \mathbb{R} \to \mathbb{R}_+$: 減衰関数.

SHPP モデル

(Co-jump を考慮しない場合では) このようなモデルは金融時系列分析ではよく利用される.

- Grothe et al.(2014)

$$c_{ji}(x) = 1 + G_{ji}^{\leftarrow}(F_{i,t}(x_i)), \quad g_i(t) = e^{-\gamma_i t}, \quad X_j(t) \sim \mathrm{GPD}(\xi_j, \sigma_j(t)),$$

  $G^{\leftarrow}$ は平均 δ_{ji} の指数分布の分布関数の逆関数, $\gamma_i > 0$.

- 国友・江原・栗栖 (2016)

$$c_{ji}(x) = A_{ji} x_i^{\delta_{ji}} + B_{ji}, \quad g_i(t) = e^{-\gamma_i t}, \quad X_j(t) \sim \mathrm{GPD}(\xi_j, \sigma_j),$$

  $A_{ji}, B_{ji}, \gamma_i > 0$, $0 \leq \delta_{ji} \leq 1$.

以下の強度関数により定義されるマーク付き多次元 Hawkes 過程を考える:

$$\lambda_j^*(t|\mathcal{H}_t^*) = \left(\lambda_{j,0}^* + \sum_{i=1}^{2^d-1} \int_{-\infty}^{t} A_{ji}^* \max(X_{(i)})^{\delta_{ji}^*} e^{-\gamma_{ji}^*(t-s)} N_i^*(ds \times dx) \right),$$

ただし, 各マークは $X_{i,s} \overset{i.i.d.}{\sim} \mathrm{GPD}(\sigma_i, \xi_i)$, $X_i \perp\!\!\!\perp X_j$, $i \neq j$.

$C^* = (C_{ji}^*)_{1 \leq i,j \leq 2^d-1}$, $C_{ji}^* = E[c_{ji}^*(X_1)]$,
$\Gamma^* = diag(\gamma_1^*, \ldots, \gamma_{2^d-1}^*)$ とおくと,

SHPP モデルの定常性

SHPP モデルが定常
$\Leftrightarrow$ $\mathrm{spr}(C^*(\Gamma^*)^{-1}) < 1$, $\max_{1 \leq i,j \leq 2^d-1} |E[(c_{ji}^*(X_1))^2]| < \infty$.

SMPPモデルの因果性

- 点過程 (連続確率過程) に対する (Granger) 因果性をどう考える？
- Simultaneous event 無し
  - 離散時系列
    Granger(1969).
  - 連続時間確率過程
    Comte and Renault(1996), Florence and Fougére(1996).
  - グラフィカルモデリング
    Eichler et al.(2016).
- Simultaneous event 有り
  - Simultaneous multivariate point process (SMPP)
    本研究.

SMPPモデルの因果性

連続時間確率過程 (X_t, Y_t, Z_t) が定義されるフィルター付き確率空間を $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, P)$ とする．また，

$$\mathcal{F}_t^{(Z)} = \sigma(Z_s : s \le t), \quad \mathcal{F}_t^{(Y,Z)} = \sigma((Z_s, Y_s) : s \le t).$$

とする．以下は大域的な因果性 (Florence and Fougére(1996)):

Weak global non-causality

$\mathcal{F}^{(Y,Z)}$ を所与として $\mathcal{F}$ は Z に対して weakly globally non-causal
$\overset{def.}{\Leftrightarrow}$ 任意の s, t に対して，$E[Z_t|\mathcal{F}_s] = E[Z_t|\mathcal{F}_s^{(Y,Z)}]$

Strong global non-causality

$\mathcal{F}^{(Y,Z)}$ を所与として $\mathcal{F}$ は Z に対して strongly globally non-causal
$\overset{def.}{\Leftrightarrow}$ 任意の s, t に対して，$\mathcal{F}_t^{(Z)} \perp\!\!\!\perp \mathcal{F}_s|\mathcal{F}_s^{(Y,Z)}$

SMPP モデルの因果性

Z をセミマルチンゲールとし, そのセミマルチンゲール分解を考える：

$$Z_t = Z_0 + A_t + M_t,$$

A: 可予測過程, M: 局所マルチンゲール.
以下は瞬間的な因果性 (Florence and Fougére(1996)):

Weak instantaneous non-causality

$\mathcal{F}^{(Y,Z)}$ を所与として $\mathcal{F}$ は Z に対して weakly instantaneously non-causal $\overset{def.}{\Leftrightarrow}$ 任意の t に対して, $\mathcal{F}^{(Y,Z)}$-semimart. Z が $\mathcal{F}$ に関しても同一の分解をもつ.

Strong instantaneous non-causality

$\mathcal{F}^{(Y,Z)}$ を所与として $\mathcal{F}$ は Z に対して strongly instantaneously non-causal $\overset{def.}{\Leftrightarrow}$ 任意の $\mathcal{F}^{(Z)}$-adapted な $\mathcal{F}^{(Y,Z)}$-semimart. が, $\mathcal{F}$ に関しても同一の分解をもつ.

SMPPモデルの因果性

以下の命題は Florence et al.(1996) の Theorem 1 の簡単な拡張.

Proposition 1

N^* に関して上記の 4 つの因果性の概念は同値.

- Co-jump があると Florence and Fougére(1996) や Eichler et al.(2016) とは異なる因果性の概念 (同時点での相関) が必要.

  → instantaneous Granger-non-causality (IGNC).

- パラメトリックモデル (SHPP モデル) に対しては GNC (or IGNC) は簡単になる.

SHPP モデルの因果性

2 次元 $N = (N_1, N_2)$ の場合.

- GNC ($N = N^*$, co-jump 無し)

  N_2 が N_1 に対して G-非因果的 $\Leftrightarrow c_{12}(x) = 0 \Leftrightarrow \alpha_{12} = 0$.

- IGNC ($N \neq N^*$, co-jump 有り)

  N_2 が N_1 に対して 瞬時 G-非因果的 (Type 1)
  $\Leftrightarrow c^*_{12}(x) = 0, c^*_{13}(x) \neq 0$.

  N_2 が N_1 に対して 瞬時 G-非因果的 (Type 2)
  $\Leftrightarrow c^*_{12}(x) \neq 0, c^*_{13}(x) = 0$.

  N_2 が N_1 に対して 瞬時 G-非因果的 (Type 3)
  $\Leftrightarrow c^*_{12}(x) = c^*_{13}(x) = 0$.

漸近理論

- 多次元点過程の MLE の CLT に関する理論的結果は整理されてない?
- 1 次元 Hawkes 過程の MLE の CLT は Ogata(1978).

漸近理論

- $N = (N_i)_{1 \le i \le p}$: $\mathbb{R}_+$ 上の p-次元 $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$-適合 simple 点過程.
- $A = (A_i)_{1 \le i \le p}$: N の連続な compensator.
- $g_T(t) = (g_T^{(i,j)}(t))$: $d \times p$ predictable process.
- $\Sigma = (\sigma_{i,j})_{1 \le i,j \le d}$: $\mathcal{F}_0$-可測非負定値ランダム行列.

漸近理論

以下の条件を仮定: $T \to \infty$ として,

$$\max_{1\le i,j\le d}\max_{1\le k\le p}\left(E\left[\frac{1}{T}\int_0^T |g_T^{(i,k)}(t)g_T^{(j,k)}(t)|dA_k(t)\right]\right) < \infty, \tag{1}$$

$$\frac{1}{T^{1+\delta}}\max_{1\le k\le p} A_k(T) \xrightarrow{p} 0, \quad \forall\delta > 0, \tag{2}$$

$$\frac{1}{T}\int_0^T \sum_{k=1}^{p} g_T^{(i,k)}(t)g_T^{(j,k)}(t)dA_k(t) \xrightarrow{p} \sigma_{i,j}, \quad 1 \le i \le d, \tag{3}$$

$$\max_{1\le k\le p} E\left[\frac{1}{T}\int_0^T \|g_T^{(\cdot,k)}(t)\|^2 I(\|g_T^{(\cdot,k)}(t)\| > c)dA_k(t)|\mathcal{F}_0\right] \xrightarrow{p} 0, \forall c > 0. \tag{4}$$

$g_T^{(\cdot,k)}(t) = (g_T^{(1,k)}(t), \ldots, g_T^{(d,k)}(t))^\top$.

漸近理論

Theorem 1(Ogata(1978) の改良)

条件 (1), (2), (3), (4) を仮定, このとき,

$$X_T = \frac{1}{\sqrt{T}} \int_0^T \sum_{k=1}^{p} g_T^{(\cdot,k)}(t)[dN_k(t) - dA_k(t)] \overset{\mathcal{F}_0-stably}{\longrightarrow} N_d(0, \Sigma).$$

- 多次元点過程の最尤推定では $X_T = \frac{1}{\sqrt{T}} \frac{\partial L_T(\theta)}{\partial \theta}$.
- 極限に現れる Fisher 情報行列はランダム.
- N^* でなく, 最初から co-jump をもつ N そのものに対する一般の安定収束も考えられるが, 統計的モデリング, 数値計算の際に co-jump 無し, 線形強度関数の仮定は現状欠かせない.
- この場合でも Wilks' theorem (LR test の漸近的性質) は成り立つ.

漸近理論

- $L_T^*(\theta)$: N^* の対数尤度
- θ_0: N^* の真のパラメータ, $\theta_0 \in \Theta(\subset \mathbb{R}^d)$: compact.
- $\hat{\theta}_{ML}$: MLE of N
- $\hat{\theta}^*_{ML}$: $(d-r) \times 1$ MLE of N^*, $0 \leq r \leq d$. $\hat{\theta}^*_{ML} \subset \Theta_1(\subset \Theta)$.

漸近理論

尤度比検定統計量の漸近論を考えるため, 以下の条件を仮定:
$I(\theta_0)$: $\mathcal{F}_0$-m'ble p.d., $T \to \infty$ として,

$$\frac{1}{T}\int_0^T \sum_{k=1}^{p} \left(\frac{\partial \log \lambda_k(t,\theta)}{\partial \theta}\right)\left(\frac{\partial \log \lambda_k(t,\theta)}{\partial \theta}\right)^{\top} \lambda_k(t,\theta)dt \xrightarrow{P} I(\theta_0), \tag{5}$$

$$\frac{1}{T}\int_0^T \sum_{k=1}^{p} \frac{1}{\lambda_k(t,\theta)} \frac{\partial^2 \lambda_k(t,\theta)}{\partial \theta \partial \theta^{\top}} [dN_k(t) - \lambda_k(t,\theta)dt] \xrightarrow{P} 0. \tag{6}$$

$$\frac{1}{\sqrt{T}}\int_0^T \sum_{k=1}^{p} \frac{\partial \log \lambda_k(t,\theta)}{\partial \theta} [dN_k(t) - \lambda_k(t,\theta)dt] \xrightarrow{\mathcal{F}_0-stably} N_d(0, I(\theta_0)), \tag{7}$$

Theorem 2(LR test の漸近的性質)

(i) Co-jump が存在するとき, 正則条件 (例えば Ogata(1978) の条件) の下で $\hat{\theta}_{ML}$ は一致性をもたない.

(ii) 条件 (5), (6), (7) を仮定する. このとき, 以下が成立:

$$2(L_T^*(\theta_0) - L_T^*(\hat{\theta}_{ML}^*)) \xrightarrow{d} \chi(r),$$

ここで $\chi(r)$ は自由度 r のカイ 2 乗分布.

- 実データ解析では上記の結果を (因果性の検定に) 利用する.

数値実験

- 多次元点過程の MLE の数値実験をしっかりやっている研究も (ほとんど) ない?

  Chavez-Demoulin et al.(2005) や Grothe et al.(2014) では数値実験の結果は報告されていない.

- 数値実験によりマーク付き多次元点過程の MLE の漸近的性質を確認.

数値実験

DGP: $N^* = (N_1^*, N_2^*, N_3^*)$,

$$\begin{aligned}\lambda_j^*(t|\mathcal{H}_t^*) = \lambda_{j,0}^* &+ \alpha_{j,1}^* \int_{-\infty}^{t} x_1 e^{-\gamma^* s} N_1^*(ds \times dx_1) \\ &+ \alpha_{j,2}^* \int_{-\infty}^{t} x_2 e^{-\gamma^* s} N_2^*(ds \times dx_2) \\ &+ \alpha_{j,3}^* \int_{-\infty}^{t} (x_1 \vee x_2) e^{-\gamma^* s} N_3^*(ds \times dx), \quad j = 1, 2, 3,\end{aligned}$$

$X_j \overset{i.i.d.}{\sim} \mathrm{GPD}(\sigma_j, \xi_j)$, $j = 1, 2$, $X_1 \perp\!\!\!\perp X_2$.

数値実験

	α^*_{11}	α^*_{12}	α^*_{13}	α^*_{21}	α^*_{22}	α^*_{23}	
True	0.57000	0.00000	0.19000	0.00010	0.71000	0.09500	
Mean	0.63641	0.00259	0.12387	0.03994	0.76318	0.07905	
RMSE	0.01045	0.00426	0.00913	0.00568	0.01004	0.00557	
	α^*_{31}	α^*_{32}	α^*_{33}	γ^*	$\lambda^*_{1,0}$	$\lambda^*_{2,0}$	$\lambda^*_{3,0}$
True	0.05900	0.12000	0.20000	0.02700	0.00930	0.00530	0.00084
Mean	0.06748	0.13922	0.11315	0.02859	0.00853	0.00427	0.00107
RMSE	0.00272	0.00380	0.00963	0.00033	0.00019	0.00017	0.00007

Table: Summary of numerical experiments. Simulation size $N = 100$. For GPD(σ_j, ξ_j), we set $(\sigma_1, \xi_1) = (0.007, 0.22)$, $(\sigma_2, \xi_2) = (0.008, 0.15)$.

数値実験 $(I(\hat{\theta}^*_{ML})^{1/2}(\hat{\theta}^*_{ML} - \theta_0))$

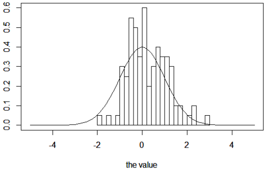

Figure: α^*_{12}

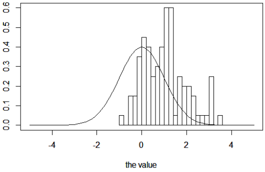

Figure: α^*_{21}

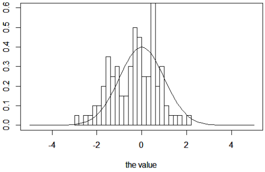

Figure: α^*_{23}

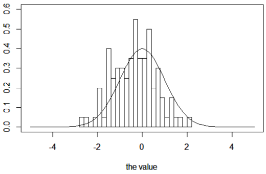

Figure: α^*_{31}

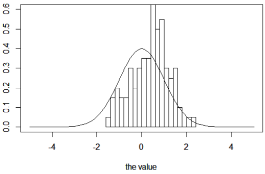

Figure: γ^*

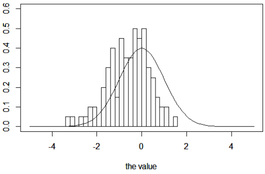

Figure: $\lambda^*_{3,0}$

実データ分析

- SHPP モデルを利用して実データに対して GNC, IGNC の検定を行う.
- 国友・江原・栗栖 (2016) では, マーク付き多次元 Hawkes 過程を用いて (東京/ニューヨーク), (東京/ロンドン) に対してマーク付き多次元 Hawkes モデルによる因果性の検証を行っている.
- この場合, 東京市場の open〜close とニューヨーク（ロンドン）市場の open〜close は時間が重なっていないので co-jump は起こりえない.

実データ分析

例 1： 東京, 香港, シンガポール,

例 2： ロンドン, パリ, フランクフルト

各市場の open～close の時間が重なっているため co-jump が起こりうる. この場合, 例えば国友・江原・栗栖 (2016) のモデルは使えない.

→ Co-jump を考慮したマーク付き多次元 Hawkes 過程（SHPP モデル）を使う.

実データ分析

- 市場指数: Nikkei225, S&P500, FTSE100, HSI.
- 期間: 1990/1/2 $\sim$ 2015/8/25.
- モデル: Bivariate SHPP model.
- 閾値: $u = -2\%$.
- 検定手法: MLE $\rightarrow$ LR test.

実データ分析（Jump Size）

Markets	σ_i	ξ_i
Tokyo	0.00806(0.00065)	0.16874(0.06431)
New York	0.00765(0.00076)	0.21538(0.08082)
London	0.00850(0.00084)	0.10799(0.07717)
Hong Kong	0.00861(0.00055)	0.15773(0.05076)

Table: Estimated GPD parameters.

実データ分析（Model Selection）

Intensity function の形を AIC で選択.

- Model 1: $c^*_{ji}(x) = 1$,
- Model 2: $c^*_{ji}(x) = a^*_{ji} \max(x_{(j)})$,
- Model 3: $c^*_{ji}(x) = a^*_{ji} \max(x_{(j)})^{\delta^*}$.

	T-NY	T-L	T-HK
Model 1	4902.27	4856.04	7935.47
Model 2	4897.19	4851.55	7915.58
Model 3	4899.54	4853.48	–

Table: AICs of each model.

Causality test では Model 2 を利用.

実データ分析（Causality Test）

Null	T-NY	T-L
$c_{21}(x) = 0$	accept	accept
$c_{12}(x) = 0$	reject	reject

Table: GNC test at the 5% significant level. T: Tokyo (N_1), NY: New York (N_2), L: London (N_2).

Null		T-HK
Type 1	$c^*_{12}(x) = 0$	accept
	$c^*_{21}(x) = 0$	accept
Type 2	$c^*_{13}(x) = 0$	reject
	$c^*_{23}(x) = 0$	accept
Type 3	$c^*_{12}(x) = 0,\ c^*_{13}(x) = 0$	reject
	$c^*_{21}(x) = 0,\ c^*_{23}(x) = 0$	accept

Table: IGNC test at the 5% significant level, T: Tokyo(N_1), HK: Hong Kong(N_2).

まとめ

- 同時点のイベント発生を考慮した点過程モデルを考察.
- SHPP モデルを提案.
- Co-jump をもつ点過程の MLE, LR test の漸近的性質を理論的に考察.
- 多次元点過程における MLE の漸近的性質を数値実験により確認.
- 提案モデルにおける GNC, IGNC を定義し, 実データ分析を行った.

今後の課題

- **self(mutually)-exciting**: 線形 Hawkes 過程 (内生的イベントのみに依存) 以外のモデリング (+外生的イベントに依存)

  例 1: 非線形 Hawkes 過程

  例 2: 多次元 Hawkes+Cox 過程 (Dynamic Contagion Process)

- **self(mutually)-damping**: (例) $g_{ij}(t) < 0, \exists t \in \mathbb{R}_+$.

$$\mathrm{Cov}(N_i((s,t]), N_j((t,u])) < 0, \exists s,t,u \in \mathbb{R}, s < t < u.$$

- **self(mutually)-correcting**: (例) $\lambda(t|\mathcal{H}_t) = \exp(\alpha + \beta(t - \rho N(t)))$

$$\mathrm{Cov}(N_i((s,t]), N_j((t,u])) < 0, \forall s,t,u \in \mathbb{R}, s < t < u.$$

- SMPP モデルに対する Causality measure

- Bayesian modeling

- Chavez-Demoulin, V., Davison, A.C. and McNeil, A.J. (2005), Estimating value-at-risk: a point process approach. *Quant. Finance*, 5-2, 227-234.
- Comte, F. and Renault, E. (1996). Noncausality in continuous time models. *Econometric Theory*, 12, 215-256.
- Eichler, M., Dahlhaus, R. and Dueck, J.(2016), Graphical modeling for multivariate Hawkes processes with nonparametric link functions. To appear in *J. Time. Ser. Anal.*
- Florens, J.P. and Fougére, D. (1996), Noncausality in continuous time. *Econometrica*, 64, 1195-1212.
- Granger, C.W. (1969), Investigating Causal Relations by Econometric Models and Cross-Spectral Methods. *Econometrica*, 37, 161-194.

参考文献 2

- Grothe, O., Korniichuk, V. and Manner, H. (2014), Modeling multivariate extreme events using self-exciting point processes. *J. Econometrics* 182, 269-289.
- 国友・江原・栗栖 (2016), 多次元ホークス型モデルによる金融市場の因果性分析. CIRJE J-278.
- Ogata, Y. (1978), The asymptotic behavior of maximum likelihood estimators for stationary point processes. *Ann. Inst. Statist. Math.* 30, 243-261.
- Solo, V. (2007), Likelihood function for multivariate point processes with coincidences. *Proceedings of the 46th IEEE Conference on Decision and Control New Orleans, LA, USA, Dec. 12-14, 2007*

Greeks の数値計算について

楠岡成雄

$(\Omega, \mathcal{F}, P)$ 確率空間

$B(t) = (B^1(t), \ldots, B^d(t))$, $t \geq 0$, d-次元 Wiener 過程

$B^0(t) = t$, $t \geqq 0$.

$V_k \in C_b^\infty(\mathbf{R}^N \times \mathbf{R}^M; \mathbf{R}^N)$, $k = 0, 1, \ldots, d$,

Stratonovich 型 SDE on $\mathbf{R}^N$

$$dX(t, x, \theta) = \sum_{k=0}^{d} V_k(X(t, x), \theta) \circ dB^k(t)$$

$$X(0, x, \theta) = x \in \mathbf{R}^N$$

$b \in C_b^\infty(\mathbf{R}^N \times \mathbf{R}^M; \mathbf{R}^N)$

$$b(x,\theta) = V_0(x;\theta) + \frac{1}{2}\sum_{k=1}^{d}\sum_{i=1}^{N} V_k^i(x,\theta)\frac{\partial}{\partial x^i}V_k(x;\theta)$$

伊藤型 SDE on $\mathbf{R}^N$

$$dX(t,x,\theta) = \sum_{k=1}^{d} V_k(X(t,x),\theta)dB^k(t) + b(X(t,x,\theta),\theta)dt$$

$$X(0,x,\theta) = x \in \mathbf{R}^N$$

(1) Calibration: 市場を説明する θ の選択

$E[f_k(X(T_k, x; \theta))]$, $k = 1, 2, \ldots, K$, の数値計算の問題

θ を変えながら何度も計算する必要がある

(2) θ_0 が選択された後 Greeks の計算

$$\frac{\partial}{\partial \theta^i} E[f(X(T, x, \theta))]|_{\theta=\theta_0}$$

$$\frac{\partial^2}{\partial \theta^i \partial \theta^j} E[f(X(T, x, \theta))]|_{\theta=\theta_0}$$

Euler-丸山近似

$$X_h : [0,\infty) \times \mathbf{R}^N \times \mathbf{R}^M \times C([0,\infty);\mathbf{R}^N) \to \mathbf{R}^N,\ h > 0,$$

$$X_h(0,x,\theta;w) = x,$$

$$X_h(t,x,\theta;w)$$

$$= X_h(nh,x,\theta;w) + \sum_{k=1}^{d} V_k(X_h(nh,x,\theta))(w^k(t) - w^k(nh))$$

$$+ b(X_h(nh,x,\theta))(t-nh),$$

$$t \in (nh,(n+1)h],\ \ n = 0,1,\ldots$$

Theorem 1　(丸山) $\forall T > 0\ \exists C \in (0, \infty)$ such that

$$\sup_{(x,\theta)\in \mathbf{R}^N \times \mathbf{R}^M} E[\sup_{t\in[0,T]} |X(t,x,\theta) - X_{T/n}(T,x,\theta;B(\cdot))|^2]^{1/2}$$

$$\leqq \frac{C}{n^{1/2}}, \qquad n \geqq 1$$

$f : \mathbf{R}^N \to \mathbf{R}$ が Lipsitz 連続ならば

$$\sup_{(x,\theta)\in \mathbf{R}^N \times \mathbf{R}^M} |E[f(X(T,x,\theta))] - E[f(X_{T/n}(T,x,\theta;B(\cdot)))]| = O(n^{-1/2})$$

多くの場合、経験上

$$|E[f(X(T,x,\theta))] - E[f(X_{T/n}(T,x,\theta;B(\cdot)))]| = O(n^{-1})$$

となる

Euler-丸山近似の数値計算

モンテカルロ法

$\{B_m(t);\ t \geqq 0\}$, $m = 1, 2, \ldots,$ 　独立な d-次元 Wiener 過程の族

$m \geqq 1$

$$E_{f,m}(T, x, \theta) = \frac{1}{m} \sum_{r=1}^{m} f(X_{T/n}(T, x, \theta; B_r(\cdot))$$

$$|E[f(X_{T/n}(T, x, \theta; B(\cdot)))] - E_{f,m}(T, x, \theta)| = O(m^{-1/2})$$

Greeks の数値計算

$$(\Delta_{i,\delta} g)(\theta) = \frac{1}{\delta}(g(\theta + \delta e_i) - g(\theta))$$

$$(\Delta_{i,j,\delta} g)(\theta) = \frac{1}{\delta^2}(g(\theta + \delta e_i + \delta e_j) + g(\theta) - g(\theta + \delta e_i) - g(\theta + \delta e_j))$$

$$\frac{\partial}{\partial \theta^i} E[f(X(T,x,\theta))]|_{\theta=\theta_0} \Leftarrow (\Delta_{i,\delta} E_m(x,\cdot))_{\cdot=\theta_0}$$

$$\frac{\partial^2}{\partial \theta^i \partial \theta^j} E[f(X(T,x,\theta))]|_{\theta=\theta_0} \Leftarrow (\Delta_{i,j,\delta} E_m(x,\cdot))_{\cdot=\theta_0}$$

で計算することが多い

この方法の正当化

$\mathbf{R}^N$ 上のベクトル場 $V_k^{(\theta)}$, $k = 0, 1, \ldots, d$,

$$V_k^{(\theta)} = \sum_{i=1}^{N} V_k^i(x;\theta)\frac{\partial}{\partial x^i}, \qquad k = 0, 1, \ldots, d$$

$\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0, 1, \ldots, d\}^n$, $n \geqq 1$, に対して
ベクトル場 $V_{[\alpha]}^{(\theta)}$ を以下で定義

$$V_{[\alpha]}^{(\theta)} = [V_{\alpha_1}^{(\theta)}, [V_{\alpha_2}^{(\theta)}, \cdots [V_{\alpha_{n-1}^{(\theta)}}, V_{\alpha_n}^{(\theta)}] \cdots]$$

$$\mathcal{A}_{00} = (\bigcup_{n=1}^{\infty} \{0, 1, \ldots, d\}^n) \setminus \{0\}$$

(UH) $\exists\ \ L \subset \mathcal{A}_{00}$ 有限集合 $\exists c > 0$

$$\sum_{\alpha\in L}(V_\alpha^{(\theta)}(x),\xi)^2_{\mathbf{R}^N} \geqq c|\xi|^2, \qquad \xi, x \in \mathbf{R}^N, \theta \in \mathbf{R}^M$$

Theorem 2　条件 (UH) が成立するならば $\forall T > 0\ \exists C \in (0,\infty)$

$$\sup_{(x,\theta)\in\mathbf{R}^N\times\mathbf{R}^M} |E[f(X(T,x,\theta))] - E[f(X_{T/n}(T,x,\theta;B(\cdot)))]|$$

$$\leqq \frac{C}{n}||f||_\infty, \qquad f \in C_b^\infty(\mathbf{R}^N),\ n \geqq 1$$

条件 (UH) の下では任意の有界可測関数 $f : \mathbf{R}^N \to \mathbf{R}$　に対して

$$|E[f(X(T,x,\theta)] - E_{f,m}(T,x,\theta)| = O(\frac{1}{n} + \frac{1}{m^{1/2}})$$

が成立

Theorem 3　条件 (UH) が成立するならば $\forall T > 0$ $\forall i, j = 1, \ldots, N$ $\forall \gamma > 0$, $\exists C \in (0, \infty)$

$$\sup_{(x,\theta)\in\mathbf{R}^N\times\mathbf{R}^M} |\frac{\partial}{\partial\theta^i}E[f(X(T,x,\theta))] - \Delta_{i,\delta}E[f(X_{T/n}(T,x,\theta))]|$$

$$\leqq C(\frac{1}{n} + \frac{1}{\delta n^\gamma} + \delta)||f||_\infty, \qquad f \in C_b^\infty(\mathbf{R}^N),\ n \geqq 1,\ \delta \in (0,1]$$

$$\sup_{(x,\theta)\in\mathbf{R}^N\times\mathbf{R}^M} |\frac{\partial^2}{\partial\theta^i\partial\theta^j}E[f(X(T,x,\theta))] - \Delta_{i,j,\delta}E[f(X_{T/n}(T,x,\theta))]|$$

$$\leqq C(\frac{1}{n} + \frac{1}{\delta^2 n^\gamma} + \delta)||f||_\infty, \qquad f \in C_b^\infty(\mathbf{R}^N),\ n \geqq 1,\ \delta \in (0,1]$$

条件 (UH) の下で $f : \mathbf{R}^N \to \mathbf{R}$ が Lipshitz 連続ならば

$$|\frac{\partial}{\partial\theta^i}E[f(X(T,x,\theta))] - \Delta_{i,\delta}E_{f,m}(x,\theta)|$$

$$= O(\frac{1}{n} + \frac{1}{\delta n^\gamma} + \delta + \frac{1}{m^{1/2}})$$

(異なる θ に対して同じ乱数で計算していることが重要)

条件 (UH) の下で $f : \mathbf{R}^N \to \mathbf{R}$ が Lipshitz 連続ならば

$$|\frac{\partial^2}{\partial\theta^i\partial\theta^j}E[f(X(T,x,\theta))] - \Delta_{i,j,\delta}E_{f,m}(x,\theta)|$$

$$= O(\frac{1}{n} + \frac{1}{\delta n^\gamma} + \delta + \frac{1}{\delta m^{1/2}})$$

Gaussian K-scheme (special KLNV method)

二宮-Victoir, 二宮-二宮　でも同様なことが示される

(UFG*) $\exists$ $L \subset \mathcal{A}_{00}$ 有限集合 $\exists \varphi_{\alpha,\beta} \in C_b^\infty(\mathbf{R}^N \times \mathbf{R}^M))$, $\exists \varphi_{i,\alpha,\beta} \in C_b^\infty(\mathbf{R}^N \times \mathbf{R}^M))$, $\alpha \in A_{00}$, $\beta \in L$, $i = 1, \ldots, M$,

$$V_\alpha^{(\theta)} = \sum_{\beta \in L} \varphi_{\alpha,\beta}(\cdot, \theta) V_\beta^{(\theta)} \qquad \alpha \in A_{00}$$

$$\frac{\partial}{\partial \theta^i} V_\alpha^{(\theta)} = \sum_{\beta \in L} \varphi_{i,\alpha,\beta}(\cdot, \theta) V_\beta^{(\theta)} \qquad \alpha \in A_{00},\ i = 1, \ldots, M.$$

(UH) $\Rightarrow$ (UFG*)

Gaussian K-scheme

$\tilde{X}_h(t, x; \theta) = \tilde{X}_h(t, x; \theta, w, \eta)$

Theorem 4　条件 (UFG*) が成り立つならば

$\forall T > 0, \forall i, j = 1, \ldots, N, \exists C \in (0, \infty)$

$$\sup_{(x,\theta)\in\mathbf{R}^N\times\mathbf{R}^M} |E[f(X(T,x,\theta))] - E[f(X_{T/n}(T,x,\theta))]|$$

$$\leqq C\frac{1}{n^2}||f||_\infty, \qquad f \in C_b^\infty(\mathbf{R}^N),\ n \geqq 1$$

Theorem 5　条件 (UFG*) が成り立つならば

$\forall T > 0,\ \forall i, j = 1, \ldots, N,\ \forall \gamma > 0,\ \exists C \in (0, \infty)$

$$\sup_{(x,\theta)\in\mathbf{R}^N\times\mathbf{R}^M} |\frac{\partial}{\partial\theta^i} E[f(X(T,x,\theta))] - \Delta_{i,\delta} E[f(X_{T/n}(T,x,\theta))]|$$

$$\leqq C(\frac{1}{n^2} + \frac{1}{\delta n^\gamma} + \delta)||f||_\infty, \qquad f \in C_b^\infty(\mathbf{R}^N),\ n \geqq 1,\ \delta \in (0,1]$$

$$\sup_{(x,\theta)\in\mathbf{R}^N\times\mathbf{R}^M} |\frac{\partial^2}{\partial\theta^i\partial\theta^j} E[f(X(T,x,\theta))] - \Delta_{i,j,\delta} E[f(X_{T/n}(T,x,\theta))]|$$

$$\leqq C(\frac{1}{n^2} + \frac{1}{\delta^2 n^\gamma} + \delta)||f||_\infty, \qquad f \in C_b^\infty(\mathbf{R}^N),\ n \geqq 1,\ \delta \in (0,1]$$

Interacting Particle Methods for Conditional Distributions

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Problem

Let us consider the canonical space $\Omega := D([0,T];\mathbb{R}^n)$ of càdlàg (right continuous with left limits) sample paths $X := \{X_t = X(t,\omega) = \omega(t)\,,\ 0 \le t \le T\}$, $\omega \in \Omega$, and a probability measure $\mathbb{P}$ on it.
Given a small positive number $\varepsilon > 0$ we consider rare events of the form $\{\omega \in \Omega : X_t \in A\}$, $A \in \mathcal{A}_\varepsilon$ for

$$\mathcal{A}_\varepsilon := \{A \in \mathcal{B}(\mathbb{R}^n) : 0 < \mathfrak{p} := \mathbb{P}(X_T \in A) < \varepsilon\}\,.$$

We want to compute numerically the conditional moments

$$\mathbb{E}[f(X) \,|\, X_T \in A]\,,$$

for $A \in \mathcal{A}_\varepsilon$ and $f : D([0,T];\mathbb{R}^n) \to \mathbb{R}$.

Examples: variance, probability of first passage time/default times, expectations of occupation times, ...

Example: mean-field interbank-lending market

On $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \ge 0}, \mathbb{P})$ let us consider a banking system $X := (X(t) := (X_1(t), \ldots, X_n(t)), 0 \le t < \infty)$ of $n \, (\ge 2)$ banks. $X_i(t)$: monetary reserve of bank i at time t with SDE

$$X_i(t) = X_i(0) + \int_0^t \Big[c_i + \sum_{j=1}^{n} (X_j(u) - X_i(u)) \cdot p_{i,j}(X(u)) \Big] \mathrm{d}u + \int_0^t \sum_{k=1}^{n} \sigma_{ik}(X(u)) \sqrt{X_i(u)} \, \mathrm{d}W_k(u) \, ; \quad i = 1, \ldots, n \, , \;\; 0 \le t < \infty \, ,$$

where $(W_1, \ldots, W_n)$ is the n-dimensional standard Brownian motion, c_i 's are constants, $p_{i,j}(\cdot)$ and $\sigma_{ik}(\cdot)$ are some smooth functions.

Given $k = 1, \ldots, n$ and $b_1, b_2 > 0$, what is the conditional probability

$$\mathbb{P}\Big(\sum_{i=1}^{n} \mathbf{1}_{\{\min_{0 \le t \le T} X_i(t) > b_2\}} = k \mid \sum_{i=1}^{n} X_i(T)/n < b_1 \Big) \, ?$$

(survival probabilities when systemic events may occur)

- When $p_{i,j}(\cdot) \equiv a_{i,j}$ (constant), it reduces to a system of Ornstein-Uhlenbeck type:

$$\mathrm{d}X_i(t) = \sum_{j=1}^{n} a_{i,j}(X_j(t) - X_i(t))\mathrm{d}t + \mathrm{d}W_i(t), \quad t \ge 0 \, .$$

In particular, we are interested in the case of large n.
CARMONA, FOUQUE & SUN ('12), FOUQUE & ICHIBA ('13), ICHIBA & SHKOLNIKOV ('13), FOUQUE, ICHIBA, DETERING ('16)

Another example but with defaults

With initial configuration $X_0 := (X_0^1, \dots, X_0^N) \in (0,\infty)^N$ consider

$$X_t^i = X_0^i + \int_0^t b(X_s^i, \overline{X}_s)\mathrm{d}s + W_t^i + \int_0^t \overline{X}_{s-}\Big(\mathrm{d}M_s^i - \frac{1}{N}\sum_{j\neq i}\mathrm{d}M_s^j\Big)\,; \quad t \ge 0\,,$$

$$\tau_k^i := \inf\Big\{ s > \tau_{k-1}^i : X_{s-}^i - \frac{\overline{X}_{s-}}{N}\sum_{j\neq i}(M_s^j - M_{s-}^j) \le 0 \Big\}\,; \quad k \in \mathbb{N}\,,$$

$$M_t^i := \sum_{k=1}^{\infty} \mathbf{1}_{\{\tau_k^i \le t\}}\,,$$

for $i = 1, \dots, N$, where
$W_t := (W_t^1, \dots, W_t^N)$, $t \ge 0$ is a standard Brownian motion,
$\overline{X}_t$ is the average of $X_t := (X_t^1, \dots, X_t^N)$ and
M_t^i is the cumulative number of defaults by time $t \ge 0$,
τ_k^i is the k-th default time with $\tau_0^i = 0$ for each i.

ELIE, ICHIBA & LAURIÉRE ('16).

Conditional simulation for small probability

Let us generate I.I.D. copies $X^{(1)}, \ldots, X^{(J)}$ of the sample path X from $\mathbb{P}$, and approximate

$$\mathbb{E}[f(X) \,|\, X_T \in A] = \frac{1}{\mathfrak{p}} \mathbb{E}[f(X) \cdot \mathbf{1}_{\{X_T \in A\}}]$$

by the estimator

$$\frac{1}{\mathfrak{p}} \cdot \frac{1}{J} \sum_{j=1}^{J} f(X^{(j)}) \cdot \mathbf{1}_{\{X_T^{(j)} \in A\}} \,,$$

where $\mathfrak{p} = \mathbb{P}(X_T \in A)$, $A \in \mathcal{A}_\varepsilon$.

Problem: by the Strong Law of Large Numbers the sample proportion

$$\frac{1}{J} \sum_{j=1}^{J} \mathbf{1}_{\{X_T^{(j)} \in A\}} = \frac{1}{J} \sharp\{j : 1 \le j \le J \,, X_T^{(j)} \in A\} \xrightarrow[J \to \infty]{\mathbb{P}-a.s.} \mathfrak{p} \,,$$

is small (since $\mathfrak{p} \in (0, \varepsilon)$ is small), and hence it is expected that this estimator fluctuates in the finite sample.

Remedy: Change of measure

Let us take a new measure $\widetilde{\mathbb{P}}$ such that by a RADON-NYKODIM derivative

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_T} = L_T^{-1}\,, \quad \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}}\Big|_{\mathcal{F}_T} = L_T\,, \quad \text{and} \quad \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}}\Big|_{\mathcal{F}_T^{(j)}} = L_{T,j}\,,$$

where $\mathcal{F}_t := \sigma(X(s), 0 \le s \le t)$, $\mathcal{F}_t^{(j)} := \sigma(X^{(j)}(s), 0 \le s \le t)$, $0 \le t \le T$, $j = 1, \ldots, J$. Then

$$\mathbb{P}(A) = \widetilde{\mathbb{E}}\Big[\mathbf{1}_A \cdot \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}}\Big|_{\mathcal{F}_T}\Big] = \widetilde{\mathbb{E}}\big[\mathbf{1}_A \cdot L_T\big]\,,$$

and hence the small probability $\mathbb{P}(A)$ in the original probability measure $\mathbb{P}$ can be approximated by

$$\frac{1}{J}\sum_{j=1}^{J}\big(\mathbf{1}_{\{X_T^{(j)} \in A\}} \cdot L_{T,j}\big) =: \varphi_A(\widetilde{\mathbb{P}})\,,$$

where $\widetilde{\mathbb{P}}(A)$ is not necessarily so small, that is,

$$\frac{1}{J}\,\sharp\{j\,:\,1\le j\le J\,,X_T^{(j)}\in A\}\ \xrightarrow[J\to\infty]{\widetilde{\mathbb{P}}-a.s.}\ \widetilde{\mathbb{P}}(A)\ =\ \mathbb{E}\big[\mathbf{1}_A\cdot L_T^{-1}\big]\,.$$

Then $\mathbb{E}[f(X)\,|\,X_T\in A]$ is estimated by

$$\Big(\sum_{j=1}^{J}L_{T,j}\,\mathbf{1}_{\{X_T^{(j)}\in A\}}\Big)^{-1}\Big(\sum_{j=1}^{J}f\big(X^{(j)}\big)\,L_{T,j}\,\mathbf{1}_{\{X_T^{(j)}\in A\}}\Big)\ =:\ \varphi_f\big(\widetilde{\mathbb{P}}\big)\,.$$

Note that these estimators depend on the choice of the new probability measures $\widetilde{\mathbb{P}}$.

• What should we choose among possible collection $\mathcal{P}$ of such probability measures $\widetilde{\mathbb{P}}$?

Minimum variance estimator for probability estimation

The probability estimator $\varphi_A(\widetilde{\mathbb{P}})$ is unbiased, in the sense that $\widetilde{\mathbb{E}}[\varphi_A(\widetilde{\mathbb{P}})] = \mathbb{P}(A) = \mathfrak{p}$. Thus its variance is given by

$$\widetilde{\mathbb{E}}\big[(\varphi_A(\widetilde{\mathbb{P}}) - \mathbb{P}(A))^2\big] = \cdots = \frac{1}{J}\mathbb{E}\big[\mathbf{1}_{\{X_T \in A\}} \cdot L_T\big] - \mathfrak{p}^2 .$$

A choice of $\widetilde{\mathbb{P}}$ is minimizing the variance or equivalently solving

$$\min_{\widetilde{\mathbb{P}} \in \mathcal{P}} \mathbb{E}\big[\mathbf{1}_{\{X_T \in A\}} \cdot L_T\big] .$$

For example, if we take

$$L_T^{(\alpha)} := \big(\mathbb{E}[e^{\alpha X_T}]\big)^{-1} \cdot e^{\alpha X_T} ,$$

then we may choose α^* by solving the minimization problem

$$\min_{\alpha} \mathbb{E}\big[\mathbf{1}_{\{X_T \in A\}} \cdot e^{\alpha X_T - \kappa_T(\alpha)}\big] ,$$

where $\kappa_T(\alpha) := \log \mathbb{E}[e^{\alpha X_T}]$, the cumulant of X_T under $\mathbb{P}$.

• Note that the estimator $\varphi_f(\widetilde{\mathbb{P}})$ is not unbiased, in general, so that the optimization problem does not become so simple.

Empirical Importance Sampling (Del Moral ('04))

Let us take the time grids $\mathcal{T} := \{0 = t_0 < t_1 < \cdots < t_K = T\}$, the subpath $X_{[0,t_k]} := \{X_t : 0 \le t \le t_k\}$ in $\mathcal{D}_k := D([0,t_k];\mathbb{R})$, Feynman-Kac potentials $G_k : D_k \to \mathbb{R}_+$ for $k = 1, \ldots, K$. Let us define

$$L_T^{-1} = \frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_T} := c_T^{-1} \prod_{k=1}^{K} G_k(X_{[0,t_k]}) ,$$

where c_T is a normalizing constant such that $\mathbb{E}[L_T] = 1$.
We also define the sample counterparts:

$$L_{T,j}^{-1} = c_T^{-1} \prod_{k=1}^{K} G_k(X_{[0,t_k]}^{(j)}) ,$$

Then we may write

$$\mathbb{E}\big[f(X)\cdot\mathbf{1}_{\{X_T\in A\}}\big] \;=\; \mathbb{E}\Big[f(X)\cdot\mathbf{1}_{\{X_T\in A\}}\Big(\prod_{k=1}^{K} G_k(X_{[0,t_k]})\Big)^{-1}\prod_{k=1}^{K} G_k(X_{[0,t_k]})\Big]$$

$$=\; \eta_K(\widetilde{f})\cdot\prod_{k=1}^{K}\eta_k(G_k) \qquad \text{schematically,}$$

where

$$\widetilde{f} \;:=\; \widetilde{f}(X) \;:=\; f(X)\cdot\mathbf{1}_{\{X_T\in A\}}\Big(\prod_{k=1}^{K} G_k(X_{[0,t_k]})\Big)^{-1},$$

and

$$\eta_k(g) \;:=\; \Big(\mathbb{E}\Big[\prod_{\ell=1}^{k-1} G_\ell(X_{[0,t_\ell]})\Big]\Big)^{-1}\cdot\mathbb{E}\Big[g(X)\prod_{\ell=1}^{k-1} G_\ell(X_{[0,t_\ell]})\Big]$$

$$=:\; \big[\gamma_k(1)\big]^{-1}\,\gamma_k(g)$$

for $k \;=\; 1,\ldots,K$.

• The collection $\{X^{(j)}, j = 1, \ldots, J\}$ can be seen as an interacting particle system through a resampling procedure based on the potentials $G_k, k = 1, \ldots, K$.

At the k-th step, each particle $X^{(j)}_{[0,t_{k-1}]}$ is
(i) extended to $X^{(j)}_{[0,t_k]}$ by independent simulation under $\mathbb{P}$

(Mutation Scheme)

and then (ii) resampled with weights (or importance)

$$w_k^{(j)} = G_k(X^{(j)}_{[0,t_k]}), \qquad j = 1, \ldots, J,$$

with average wegiths $w_k^{(J)} := J^{-1} \sum_{j=1}^{J} G_k(X^{(j)}_{[0,t_k]})$.

(Selection Scheme)

Thus $\mathbb{E}\big[f(X) \cdot \mathbf{1}_{\{X_T \in A\}}\big]$ can be estimated from its sample counterpart:

$$\frac{1}{J} \sum_{j=1}^{J} \widetilde{f}(X^{(j)}) \prod_{k=1}^{K} w_k^{(J)}.$$

- An important case consists of *multiplicative* potentials of the form

$$G_k(X_{[0,t_k]}) \;=\; \exp\big(V(X_{t_k}) - V(X_{t_{k-1}})\big)$$

for some function $V(\cdot)$.

In particular, if $V(\cdot)$ is chosen as $V(X_0) \;=\; 0$,

$$V(X_T) \;=\; \Big(\prod_{k=1}^{K-1} G_k(X_{[0,t_k]})\Big)^{-1} \quad \text{on} \quad \{X_T \in A\}\,,$$

then the above estimator for $\mathbb{E}[f(X)|A]$ reduces to

$$\frac{1}{\sharp\{j : X_T^{(j)} \in A\}} \sum_{\{j: X_T^{(j)} \in A\}} f(X^{(j)})\,.$$

Example: mean-field interbank-lending market

On $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ let us consider a banking system $X := (X(t) := (X_1(t), \ldots, X_n(t)), 0 \leq t < \infty)$ of $n \, (\geq 2)$ banks. $X_i(t)$: monetary reserve of bank i at time t with SDE

$$X_i(t) = X_i(0) + \int_0^t \Big[c_i + \sum_{j=1}^n (X_j(u) - X_i(u)) \cdot p_{i,j}(X(u)) \Big] \mathrm{d}u$$

$$+ \int_0^t \sum_{k=1}^n \sigma_{ik}(X(u)) \sqrt{X_i(u)} \mathrm{d}W_k(u) \, ;$$

$$i = 1, \ldots, n \, , \quad 0 \leq t < \infty \, .$$

Given $k = 1, \ldots, n$ and $b_1, b_2 > 0$, what is the conditional probability

$$\mathbb{P}\Big(\sum_{i=1}^n \mathbf{1}_{\{\min_{0 \leq t \leq T} X_i(t) > b_2\}} = k \mid \sum_{i=1}^n X_i(T)/n < b_1 \Big) \ ?$$

$$X_i(t) = X_i(0) + \int_0^t \Big[c_i + \sum_{j=1}^{n}(X_j(u) - X_i(u)) \cdot p_{i,j}(X(u))\Big]\mathrm{d}u$$

$$+ \int_0^t \sum_{k=1}^{n} \sigma_{ik}(X(u))\sqrt{X_i(u)}\mathrm{d}W_k(u) \,;$$

$$i = 1, \ldots, n\,, \;\; 0 \le t < \infty\,.$$

• Here $W := ((W_1(t), \ldots, W_d(t)),\, 0 \le t < \infty)$ is the standard d-dimensional Brownian motion, c_i is a nonnegative constant, $\mathrm{x} := (X_1(0), \ldots, X_n(0)) \in [0, \infty)^n$ is an initial reserve and

• $p_{i,j} : [0, \infty)^n \to [0, 1]$ is bounded, α-Hölder continuous on compact sets in $(0, \infty)^n$ for some $\alpha \in (0, 1]$.

• $a(\cdot) := (a_{ij}(\cdot)) = \sum_{k=1}^{n}(\sigma_{ik}\sigma_{jk})(\cdot)$ is strictly positive definite, α-Hölder continuous on compact sets for some $\alpha \in (0, 1]$.

Proposition[FOUQUE & ICHIBA (2013)]. Let us assume that the lending preferences $\{p_{i,j}(\cdot), 1 \le i,j \le n\}$ that for some indexes $(1,\ldots,k)$

$$\sup_{x \in [0,\infty)^n} |x_i - x_j| \cdot p_{i,j}(x) < \frac{1}{k(n-1)} \Big(2 - \sum_{i=1}^{k} c_i\Big) =: 2\boldsymbol{c}_0$$

for $1 \le i \le k, 1 \le j \le n$. Then these k banks are bankrupt together at some time $t \in (0,\infty)$ almost surely, i.e.,

$$\mathbb{P}_{\mathrm{x}}\big(\tau < \infty\big) = 1 ,$$

where $\tau := \inf\{t : X_1(t) = X_2(t) = \cdots = X_k(t) = 0\}$ for $\mathrm{x} \in [0,\infty)^n$.

Proof is based on comparison theorem by IKEDA & WATANABE ('77). The partial sum $\mathcal{X}_k(\cdot) = \sum_{i=1}^{k} X_i(\cdot)$ is dominated by a squared Bessel process with dimension strictly less than 2.

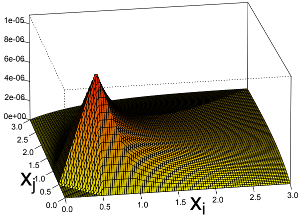
Note that there are possibly many choices of the lending preference that satisfy the above inequality. For example,

$$p_{i,j}(\cdot) \equiv 0\,; \quad 1 \le i,j \le n\,.$$

Another example is

$$\frac{p_{i,j}(x)}{c_1} = \begin{cases} 2(x_i \wedge x_j) \,/\, (x_i + x_j)^2 & \text{if } x_i + x_j \ge 1\,, \\ 1 - 2(x_i \wedge x_j) & \text{if } x_i \wedge x_j \ge 1/2\,,\ 1/2 \le x_i + x_j \le 1\,, \\ 2(x_i + x_j) - 1 & \text{if } x_i \wedge x_j \le 1/2,\ 1/2 < x_i + x_j < 1\,, \\ 0 & \text{otherwise}, \end{cases}$$

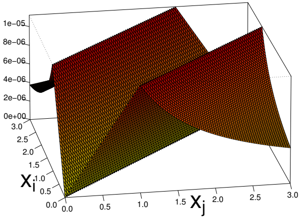
where the constant c_1 is less than c_0.



Similarly, given a nonnegative function $h : [0, \infty) \to [0, 1]$ which is α-Hölder continuous on compact sets in $(0, \infty)$ for some $\alpha \in (0, 1]$, we can take

$$p_{i,j}(x) = h(|x_i - x_j|) \, ; \quad x = (x_1, \dots, x_n) \in [0, \infty)^n \, , \; 1 \le i, j \le n \, .$$

The condition holds if we choose $c_1 < c_0$ and $h(x) = c_1 \, / \, x$ for $x \ge 1$ and $h(x) = c_1 x$ for $x \le 1$.



Interacting particle system algorithm [Del Moral and Garnier ('05)]

Intuition: consider a background MC $(\xi_k)_{k\geq 0}$ with transition kernel $K_k(\xi_{k-1},\xi_k)$, and its history $\eta_k := (\xi_0,\dots,\xi_k)$, $k \geq 0$. Given $f_k : \mathbb{R}^{k+1} \to \mathbb{R}$, define

$$\gamma_k(f_k) \;=\; \mathbb{E}\Big(f_k(\eta_k)\cdot \prod_{1\leq \ell<k} G_k(\eta_\ell)\Big)$$

with a multiplicative potential function, and its normalized measure

$$\nu_k(f_k) \;=\; \frac{\gamma_k(f_k)}{\gamma_k(1)}\,.$$

Since $\gamma_{k+1}(1) = \gamma_k(G_k) = \nu_k(G_k)\gamma_k(1) = \cdots = \prod_{\ell=1}^{n} \nu_\ell(G_\ell)$,

$$\mathbb{E}(f_k(\eta_k)) = \gamma_k(f_k \prod_{1\leq\ell<k} (G_\ell)^{-1}) = \nu_k(f_k \prod_{1\leq\ell<k} (G_\ell)^{-1}) \prod_{1\leq\ell<n} \nu_\ell(G_\ell)\,.$$

Here we can use a recursion: $\eta_1(\cdot) \;=\; K_1(\xi_0,\cdot)$,

$$\nu_k(\cdot) \;=\; \int \nu_{k-1}(d\,\eta_{k-1})\frac{G_{k-1}(\eta_{k-1})}{\nu_{k-1}(G_{k-1})}K_k(\eta_{k-1},\cdot)\,.$$

Dividing the time interval $[0, T]$ into L equal subintervals $[(\ell - 1)T/L, \ell T/L]$ with $\ell = 1 \ldots, L$, we simulate M random chains

$$\{Y_\ell^{(j)} = (\widehat{X}^{(j)}(\ell T/L), \widehat{m}^{(j)}(\ell T/L))\}_{1 \leq \ell \leq L}\,; \quad j = 1, \ldots, M\,,$$

where $\widehat{X}^{(j)}(\cdot)$ is the jth simulation of $X(\cdot)$ and $\widehat{m}^{(j)}$ is the jth simulation of the vector $m(\cdot) := (m_1(\cdot), \ldots, m_n(\cdot))$ of the running minimum
$m_i(t) = \min_{0 \leq s \leq t} X_i(s)$, for $1 \leq i, j \leq n$, $0 \leq t \leq T$.

After initializing the chain, for each $\ell = 1, \ldots, L$, repeat the following selection and mutation stages

- (Selection Stage). Sampling M new particles from $\{Y_\ell^{(j)}\}_{1 \le j \le M}$ with Gibbs weights

$$\Big(\prod_{i=1}^{n} \gamma_{i,\ell}^{(j)}\Big)\Big(\sum_{j=1}^{M}\prod_{i=1}^{n} \gamma_{i,\ell}^{(j)}\Big)^{-1} \quad \text{where}$$

$$\gamma_{i,\ell}^{(j)} := \Big[\frac{\min(\widehat{m}_i^{(j)}((\ell-1)T/L), \widehat{X}_i^{(j)}(\ell T/L))}{\widehat{m}_i^{(j)}((\ell-1)T/L)}\Big]^{-\alpha},$$

  for each $j = 1, \ldots, M$ with some $\alpha > 0$.
- (Mutation Stage). Running Euler scheme to get the new value $Y_{\ell+1}^{(j)}$, $j = 1, \ldots, M$, starting from the new particles sampled in the above.

The probability estimate of $\mathbb{P}_{\mathrm{x}}(\mathfrak{n} = k)$ is given by

$$\widehat{\mathbb{P}}_{\mathrm{x}}(\mathfrak{n} = k) = \frac{1}{M} \sum_{j=1}^{M} \Big(\mathbf{1}_{\{\widehat{\mathfrak{n}}^{(j)} = k\}} \prod_{i=1}^{n} \Big[\frac{m_i^{(j)}(T)}{m_i^{(j)}(0)} \Big]^{\alpha} \Big) \cdot \Big[\prod_{\ell=0}^{L-1} \Big(\frac{1}{M} \sum_{a=1}^{M} \prod_{i=1}^{n} \gamma_{i,\ell}^{(a)} \Big) \Big] ;$$

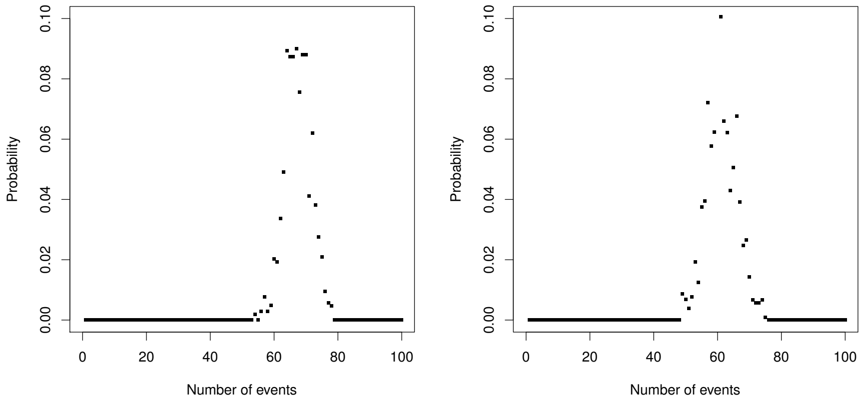
for $k = 1, \ldots, n$, where $\widehat{\mathfrak{n}}^{(j)}$ is the corresponding number to $\mathfrak{n}$ in the jth simulation for $j = 1, \ldots, M$.

Extreme examples

Set $\mathrm{x} = (1, \ldots, 1)$, $\delta = 2$ and $p_{i,j}(\cdot)$ specified as in the first picture, $T = 1$, $n = 100$, $M = 1000$ (# copies), $L = 10$ (# subintervals of Time), $\alpha = 0.0001$, and run the system with the sub-subinterval for the Euler scheme in the mutation stage as 0.001 to compute

$$\mathbb{P}_{\mathrm{x}}(\mathfrak{n} = k) = \mathbb{P}_{\mathrm{x}}\Big(\sum_{i=1}^{n} \mathbf{1}_{\{\min_{0\le s\le T} X_i(s) \le b\}} = k\Big)\,; \quad k = 1, 2, \ldots$$

$b = 0.1$ (left), $b = 0.001$ (right).



For error analysis and comparisons with other methods in this context see CARMONA & CRÉPY (2009). □

Brownian motion plus Poisson jumps

To illustrate conditional distributions for heavy-tailed systems, we shall consider the rare event

$$A \;:=\; \{X_T \in [a-\delta, a+\delta]\}$$

for the process

$$X_t \;=\; W_t - k\,N_t\,, \qquad X_0 \;=\; 0\,,$$

where $W_.$ is Brownian motion and $N_.$ is an independent Poisson process with intensity $\lambda > 0$.
Note that $A \;=\; \cup_{\ell=0}^{\infty}\{N_T \;=\; \ell\,, W_T \in [a+k\ell-\delta, a+k\ell+\delta]\}$, and hence with the cumulative distribution function $\Phi(\cdot)$ of the standard normal, we obtain

$$\mathbb{P}(A) \;=\; \sum_{\ell=0}^{\infty} e^{-\lambda T}\frac{(\lambda T)^{\ell}}{\ell!}\cdot\left[\Phi\Big(\frac{a+k\ell+\delta}{\sqrt{T}}\Big) - \Phi\Big(\frac{a+k\ell+\delta}{\sqrt{T}}\Big)\right].$$

A suitable empirical change of measure is of the bivariate form

$$L_t(\alpha_W, \alpha_N) \;:=\; \exp\Big(\alpha_W W_t - \alpha_N k N_t - \kappa_T(\alpha_W, \alpha_N)\Big)\,,$$

where

$$\kappa_T(\alpha_W, \alpha_N) \;=\; -\frac{1}{2}\alpha_W^2 t + \lambda t\big(e^{-\alpha_N k} - 1\big)\,; \quad 0 \le t \le T\,.$$

Under the resulting importance sampling measure $\mathbb{P}^{(\alpha_W,\alpha_N)}$, (W_t) has drift α_W and (N_t) has intensity $\lambda e^{-\alpha_N k}$. This implies

$$\mathbb{E}^{(\alpha_W,\alpha_N)}\big[X_T\big] \;=\; \alpha_W T + \lambda T k e^{-k\alpha_N}\,.$$

- One can compute the variance of the estimator for $\mathbb{E}[f(X)\,\mathbf{1}_{X_T\in A}]$ and minimizing it.
- One can compute the conditional intensity given the set $\{X_T \in A\}$.

One-dimensional case: use of Markov Bridge

Let us consider the first passage time $\tau_0 := \inf\{t : X_t = 0\}$ for one-dimensional diffusion

$$\mathrm{d}X_t = a(X_t)\mathrm{d}t + \mathrm{d}W_t \, ; \quad X_0 = x \, , \quad 0 \le t \le T$$

where a is assumed to be continuously differentiable, and

$$\int_0^\infty \exp\Big(-2 \int_0^w a(z)\mathrm{d}z \Big)\mathrm{d}w = \infty \, .$$

By the GIRSANOV-MARUYAMA change of measure

$$\mathbb{P}_x(\tau_0 \le t) = \mathbb{E}^{\mathbb{Q}_x}\big[Z_{t\wedge\tau_0} \mathbf{1}_{\{\tau_0 \le t\}} \big] \, ,$$

where $\mathrm{d}\mathbb{Q}_x / \mathrm{d}\mathbb{P}_x |_{\mathcal{F}_t} = Z_t$, $0 \le t \le T$ with

$$Z_\cdot = \exp\Big(-\int_0^\cdot a(X_u)\mathrm{d}W_u - \frac{1}{2}\int_0^\cdot a^2(X_u)\mathrm{d}u \Big) \, ,$$

moreover, under $\mathbb{Q}_x$, $X.$ is a Brownian motion. Hence the density function of τ_0 is given by

$$q_x(t) := \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}},$$

and the density $p_x(\cdot)$ of τ_0 under $\mathbb{P}_x$ is given by

$$p_x(t) = q_x(t) \exp\left(-\int_0^x a(v)\mathrm{d}v\right) \mathbb{E}_{\mathbb{Q}_x}\left[\exp\left(-\int_0^{\tau_0} \gamma(X_u)\mathrm{d}t\right) \Big| \tau_0 = t\right],$$

where $\gamma(x) = (a^2(x) + a'(x))\,/\,2$ for $x > 0$.

Under $\mathbb{Q}_x$, given $\tau_0 = t$, the time reversal $X_{\tau_0 - s}$, $0 \le s \le t$ is the three dimensional BESSEL bridge from the level 0 to x (REVUZ & YOR (1999)).

This implies that

$$p_x(t) = q_x(t) \exp\left(-\int_0^x a(v)\mathrm{d}v\right) \mathbb{E}_{BB^3}\left[\exp\left(-\int_0^1 \gamma\left(\left|ux\boldsymbol{e}_1 + \sqrt{t}\beta_u\right|\right)\mathrm{d}u\right)\right],$$

where $\beta.$ is the three dimensional Brownian bridge, and $\boldsymbol{e}_1 = (1, 1, 1)$.

When $A \subset (-\infty, 0]$, with this idea we may compute the conditional expectation

$$\mathbb{E}[f(X) | X_T \in A] = \int_0^T \mathbb{E}\big[f(X) | \tau_0 = t, X_T \in A\big] p_x(t) \mathrm{d}t \,,$$

where $\mathbb{E}[f(X) | \tau_0 = t, X_T \in A]$ may be estimated by the empirical importance sampling as we have seen before, and

$$p_x(t) = q_x(t) \exp\Big(- \int_0^x a(v) \mathrm{d}v \Big) \mathbb{E}_{BB^3}\Big[\exp\Big(- \int_0^1 \gamma\Big(\big| u x \boldsymbol{e}_1 + \sqrt{t} \beta_u \big| \Big) \mathrm{d}u \Big)$$

is approximated by its sample counterpart

$$\widehat{p}_x^{(J)}(t) := q_x(t) \exp\Big(- \int_0^x a(v) \mathrm{d}v \Big) \frac{1}{J} \sum_{j=1}^{J} \exp\Big(- \int_0^1 \gamma\Big(\big| u x \boldsymbol{e}_1 + \sqrt{t} \beta_u^{(j)} \big| \Big) \mathrm{d}u \Big) \,,$$

where $\beta_{\cdot}^{(j)}$, $j = 1, \ldots, J$ is the I.I.D. copy of three dimensional Brownian bridge.

- If the functional f on the sample paths depends only on the negative paths after τ_0, e.g.,

$$f(X) := \min_{0 \le s \le T} (-X(s))^+ ,$$

(local time for the reflected process in the sense of SKOROKHOD), and

$$f(X) := \int_0^T (-X_s) \mathbf{1}_{\{X_s \le 0\}} \mathrm{d}s ,$$

(cumulative deficit amounts)

then $\mathbb{E}[f(X) | \tau_0 = t, X_T \in A]$ can be computed by the empirical importance sampling for X on the time interval $(\tau_0, T]$.

- Note that one can extend our consideration for the diffusion

$$\mathrm{d}Y_t = b(Y_t)\mathrm{d}t + \sigma(Y_t)\mathrm{d}W_t ; \quad Y_0 = y, 0 \le t \le T$$

and the first passage time of level $\ell(< y)$, where $1/\sigma(\ge 0)$ is locally integrable. This is because the transformation

$$X_t := \int_\ell^{Y_t} \frac{1}{\sigma(z)} \mathrm{d}z .$$

- The approximation of $p_x(\cdot)$ by $\widehat{p}_x^{(J)}(\cdot)$ converges in the order of $1/\sqrt{J}$. Indeed, one can show that

$$\sqrt{J}|\widehat{p}_x^{(J)}(t) - p_x(t)|$$

converges to the centered Gaussian process with covariance

$$\Gamma(s,t) \;=\; p_x(s)p_x(t)e^{-2\int_0^x a(v)\mathrm{d}v}\mathrm{Cov}_{BB^3}(e^{-sI(s)}, e^{-tI(t)})\,,$$

where $I(s) \;:=\; \int_0^1 \gamma(|ux\boldsymbol{e}_1 + \sqrt{s}\beta_u|)\mathrm{d}u$, and in particular,

$$\sqrt{J}\max_{0\le t\le T}|\widehat{p}_x^{(J)}(t) - p_x(t)|$$

is bounded in probability (cf. ICHIBA & KARDARAS ('09)).

• For the multidimensional case of the form

$$\mathrm{d}X_t \;=\; b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t \,, \quad X_0 \;=\; x \,,$$

we may consider DOOB's h-transform and the (time-reversed) Markov bridge law : given $X_t \;=\; a$,

$$\mathrm{d}X^0_t \;=\; b^0(t, X^0_t)\mathrm{d}t + \sigma(t, X^0_t)\mathrm{d}W_t \,,$$

where

$$b^0(t,x) \;:=\; b(t,x) + [\sigma\sigma^*](t,x)\, \nabla_x \log p(t,x,T,a) \,,$$

for the transition probability $p(t,x,T,a)$.

• Similar calculation can be done for LÉVY processes, e.g., $X_t \;=\; W_t - kN_t$, where $W.$ is the Brownian motion and $N.$ is Poisson process, independent of $W.$.

(see PRIVAULT & ZAMBRINI (2004) for Markovian bridges).

Summary:

Computation of conditional expectation, given the condition that has only a small probability.

- Use of Empirical Importance Sampling and Interacting Particle System
- Use of Time-reversal and Markov bridge

- Further topics:

The mean field approximation (MCKEAN-VLASOV type limit) of the system as $n \to \infty$ and estimation of conditional probability.

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When $n \to \infty$, ...

For example, with $a_{i,j} = -1_{\{j=i+1\}} + \mathbf{1}_{\{i=j\}}$ for $i,j \in \mathbb{N}$, the distribution of the first k components $(X_1(\cdot), \ldots, X_k(\cdot))$ of the large system of OU processes $(X_1(\cdot), \ldots, X_n(\cdot)$:

$$\mathrm{d}X_i(t) = (X_{i+1}(t) - X_i(t))\mathrm{d}t + \mathrm{d}W_i(t)\,; \quad i = 1, 2, \ldots, n-1\,,$$

$$\mathrm{d}X_n(t) = -X_n(t)\mathrm{d}t + \mathrm{d}W_n(t)\,; \quad t \geq 0\,,$$

becomes, in the limit as $n \to \infty$, the Gaussian process represented as

$$Y_i(t) = \int_0^t e^{-(t-s)} Y_{i+1}(s)\mathrm{d}s + \int_0^t e^{-(t-s)}\mathrm{d}B_i(s)\,, \quad i = 1, \ldots, k-1\,,$$

$$Y_k(t) = e^{-t} \sum_{i=0}^{\infty} \int_0^t \frac{e^s (t-s)^i}{i!} \mathrm{d}\beta_i(s)\,; \quad t \geq 0\,,$$

where $(B_1(\cdot), \ldots, B_{k-1}(\cdot), \beta_0(\cdot), \beta_1(\cdot), \ldots)$ are independent Brownian motions.

The covariance structure of $Y_k(\cdot)$ is

$$\mathrm{Cov}(Y_k(s), Y_k(t)) \;=\; e^{-(t-s)} \int_0^t e^{-2u} I_0(\sqrt{(t-s+u)u})\mathrm{d}u\,,$$

$$I_\nu(x) \;:=\; \sum_{k=0}^{\infty} (x/2)^{2k+\nu} / (k!\Gamma(\nu+k+1))\,; \quad x > 0\,, \nu > -1\,.$$

In particular, the variance

$$\mathrm{Var}(Y_k(t)) \;=\; te^{-2t}(I_0(2t) + I_1(2t))\,; \quad t \geq 0\,,$$

and hence for large $t \to \infty$ $\mathrm{Var}(Y_k(t)) \;=\; O(\sqrt{t})$; $k \in \mathbb{N}_0$.

It suggests some connection to fractional OU and fractional BM.