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Tail probability via tube formula and Euler characteristic method when critical radius is zero

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Abstract

In Takemura and Kuriki (1999b) we have established that the tube formula and the Euler characteristic method give identical and valid asymptotic expansion of tail probability of the maximum of Gaussian random field when the random field has finite Karhunen-Loève expansion and the index set has positive critical radius. The purpose of this paper is to show that the positiveness of the critical radius is an essential condition. Namely, we prove that if the critical radius is zero, only the main term is valid and other higher order terms are generally not valid in the formal asymptotic expansion based on the tube formula or the Euler characteristic method. Our examples show that index sets with zero critical radius are commonly used in statistics.

1 Introduction

Let M be a closed subset of the unit sphere S^{n-1} in R^n. We consider upper tail probability of the maximum of a random field $Z(u)$, $u = (u_1, \ldots, u_n)' \in M$, defined by

$$Z(u) = u'z = \sum_{i=1}^{n} u_i z_i, \tag{1}$$

where $z = (z_1, \ldots, z_n)'$ is distributed according to n-dimensional standard multivariate normal distribution $N_n(0, I_n)$. This is the canonical form of Gaussian random field with finite Karhunen-Loève expansion and constant variance. Let $y = (y_1, \ldots, y_n)' = z/\|z\|$ be distributed according to the uniform distribution $\mathrm{Unif}(S^{n-1})$ on the unit sphere S^{n-1}. We also study upper tail probability of the maximum of

$$Y(u) = u'y. \tag{2}$$

In Takemura and Kuriki (1999b) we considered index set M which is locally approximated by convex cone. In this case M has positive critical radius and the tube method by Sun (1993) and the Euler characteristic method by Adler (1981) and Worsley (1995a, b) lead to identical valid asymptotic expansion of the upper tail probabilities. In a different setting, Adler (1998) showed that the Euler characteristic method for isotropic Gaussian random fields on piecewise smooth domain gives valid asymptotic expansion using the results by Piterbarg (1996).

These results might give an impression that the formal asymptotic expansion based on the Euler characteristic method or the tube formula is valid for practically all regular cases. However this is not the case if the critical radius of M is zero. The main purpose of this paper is to show that if the critical radius of M is zero, the asymptotic expansion based on the tube formula or the Euler characteristic method is generally incorrect except for the main term of the expansion. We also give some examples to demonstrate that index sets with zero critical radius are commonly used in statistics.

One advantage of the Euler characteristic method over the tube formula is that it can be applied to non-Gaussian fields, whereas the tube formula is essentially restricted to Gaussian fields. See, e.g., Worsley (1994), Cao and Worsley (1998,99) for application of the Euler characteristic method to various non-Gaussian fields. However the validity of the Euler characteristic method for non-Gaussian fields has not been established in general. Indeed our example in Section 3.2 suggests that the validity of the Euler characteristic method for non-Gaussian fields is hard to prove in general. In Section 3.2 we first apply the tube formula to a Gaussian field whose index set has zero critical radius. However this Gaussian field can be transformed to equivalent χ^2 field with very regular index set and we apply the Euler characteristic method to the resulting χ^2 field. It will be shown that the Euler characteristic method for this χ^2 field leads to an invalid asymptotic expansion, which is identical to the asymptotic expansion obtained by the formal tube formula for the original Gaussian field.

The organization of this paper is as follows. In Section 2, after preliminary discussion on the properties of index sets with zero critical radius, we give some theoretical results on asymptotic expansion based on the formal tube formula. In Section 3 we study some relevant examples in detail. Some proofs and mathematical details are given in Appendix.

2 General results

In this section we first define a class of index sets M for which the tube formula can be defined. Then in Section 2.2 the difference between the formal tube formula and the exact tube formula for these index sets is clarified. Invalidity of asymptotic expansion based on the formal tube formula or the formal Euler characteristic methods is stated in Section 2.3.

2.1 A class of sets to be considered

We consider a class of index sets M with the following property. At each point $x \in M$, M can be locally approximated by a cone but the cone is not necessarily convex. We call M with this property *locally conic.* This class contains boundary of a polyhedron and the union of submanifolds of S^{n-1} which intersect themselves on S^{n-1}. More complicated but statistically natural example is treated in Section 3.2. Unfortunately the class of locally conic sets can not be defined by standard terminology of manifolds because we allow self-intersection of the index set. Precise definitions of this class and other notions of this subsection are given in Appendix A.

The approximating cone of M at $x \in M$ is called *support cone* of M at x and is denoted by $S_x(M)$. Let $C(S_x(M))$ denote the convex hull of $S_x(M)$. The dual cone of $C(S_x(M))$

in R^n is called *normal cone* of M at x and is denoted by $N_x(M)$. As we shall show, the critical radius of M is zero if $S_x(M)$ is non-convex at some $x \in M$ because of the singularity of projection onto M around x.

We discuss several simple examples to illustrate the above notions. Note that in our definition in Appendix A the support cone $S_x(M)$ and the normal cone $N_x(M)$ are defined with their vertices located at the origin.

Example 2.1 *On the sphere $S^2 \subset R^3$ consider the union of two great circles:*

$$M = \{(x_1, x_2, x_3) \in S^2 \mid x_2 = 0\} \cup \{(x_1, x_2, x_3) \in S^2 \mid x_3 = 0\}.$$

Except for two points $(\pm 1, 0, 0)$ M is a regular one-dimensional manifold. However at these two points M can not be considered as a manifold in standard terminology because of the self-intersection. At $x = (\pm 1, 0, 0)$, $S_x(M) = \{(0, x_2, 0) \mid x_2 \in R\} \cup \{(0, 0, x_3) \mid x_3 \in R\}$, $C(S_x(M)) = \{(0, x_2, x_3) \mid (x_2, x_3) \in R^2\}$ and $N_x(M) = \mathrm{span}\{x\}$.

Example 2.2 *On S^2 consider*

$$M = \{(x_1, x_2, x_3) \in S^2 \mid x_2 x_3 \geq 0\}$$

whose boundary is M of Example 2.1. At $x = (x_1, x_2, x_3)$ with $x_2 x_3 > 0$, $S_x(M) = C(S_x(M))$ is the tangent plane $T_x(S^2)$ of S^2 at x and $N_x(M) = \mathrm{span}\{x\}$. At $x = (x_1, x_2, 0)$ with $|x_1| < 1$ and $x_2 > 0$, $S_x(M) = C(S_x(M)) = T_x(S^2) \cap \{(y_1, y_2, y_3) \mid y_3 \geq 0\}$ and $N_x(M) = \mathrm{span}\{x\} \oplus \{(0, 0, y_3) \mid y_3 \leq 0\}$, where "$\oplus$" is the orthogonal direct sum. At $x = (\pm 1, 0, 0)$, $S_x(M) = \{(0, x_2, x_3) \mid x_2 x_3 \geq 0\}$, $C(S_x(M)) = \{(0, x_2, x_3) \mid (x_2, x_3) \in R^2\}$, and $N_x(M) = \mathrm{span}\{x\}$.

Example 2.3 *Again on S^2 let M be the union of two half circles $M = \{(x_1, x_2, 0) \in S^2 \mid x_2 \geq 0\} \cup \{(x_1, 0, x_3) \in S^2 \mid x_3 \geq 0\}$. At $x = (\pm 1, 0, 0)$, $S_x(M) = \{(0, x_2, 0) \mid x_2 \geq 0\} \cup \{(0, 0, x_3) \mid x_3 \geq 0\}$, $C(S_x(M)) = \{(0, x_2, x_3) \mid x_2 \geq 0,\ x_3 \geq 0\}$ and $N_x(M) = \{(0, x_2, x_3) \mid x_2 \leq 0,\ x_3 \leq 0\} \oplus \mathrm{span}\{x\}$.*

In the above three examples the points $x = (\pm 1, 0, 0)$ exhibit certain singularity. However from the viewpoint of spherical tube around M in S^2, $x = (\pm 1, 0, 0)$ in Example 2.3 contribute to the volume of the tube just as other points in the sense that the points in the direction of $N_x(M)$ from x are projected to x when projected onto M. On the other hand in Examples 2.1 and 2.2, $x = (\pm 1, 0, 0)$ do not contribute to the volume of the spherical tube around M, because no point (other than x itself) is projected to x when projected onto M. In general consider a spherical tube around M in S^{n-1}. $x \in M$ does not contribute to the volume of the tube around M if the dimension of $N_x(M)$ and the dimension of M around x do not add up to n. From this consideration we call $x \in M$ *proper* d-dimensional boundary point of M if $S_x(M)$ contains a linear subspace L of dimension $d = n - \dim N_x(M)$. We define the dimension of M by the maximum value of d such that there exists a proper d-dimensional boundary point of M. Note that we use the term "boundary" even if x belongs to the relative interior of M.

Let ∂M_d, $d = 0, \ldots, m = \dim M$, denote the set of proper d-dimensional boundary points of M. We now make the following assumption on locally conic M.

Assumption 2.1 *For $d = 0, \ldots, m$, ∂M_d is a relatively open d-dimensional C^2-submanifold of R^n. Let $I(M)$ denote the set of improper boundary points of M. The Lebesgue measure of $\cup_{u \in I(M)} N_u(M)$ is zero.*

Here we are assuming that ∂M_d is an open manifold embedded in R^n. We call M satisfying this assumption "set with piecewise smooth proper boundary". In summary, we assume that the index set $M \subset S^{n-1}$ is locally conic closed set with piecewise smooth proper boundary.

We now consider spherical projection onto M. For $x, y \in S^{n-1}$ let

$$\mathrm{dist}(x, y) = \arccos(x'y) \in [0, \pi]$$

be the geodesic distance and define

$$\mathrm{dist}(x, M) = \mathrm{dist}(x, x_M) = \min_{y \in M} \mathrm{dist}(x, y),$$

where x_M is the spherical projection of x onto M. Although x_M may not be unique, $\mathrm{dist}(x, M)$ is uniquely determined because M is closed. We are interested in the geometry of the set of points with unique projection onto M:

$$R(M) = \{x \mid x_M \text{ is unique}\}. \tag{3}$$

Let

$$K(M) = \bigcup_{c \geq 0} cM$$

denote the smallest cone containing M and for $u \in M$ and $v \in N_u(K(M)) \cap S^{n-1}$ let

$$l = \{u \cos\theta + v \sin\theta \mid 0 \leq \theta < \pi\}$$

denote the half circle starting from $u \in M$ in the direction of v. In Appendix B it is shown that l is divided into two segments. The points on the first segment have u as the unique projection and the points on the other interval do not. More precisely define

$$\bar\theta(u, v) = \sup\{0 \leq \theta < \pi \mid u\cos\theta + v\sin\theta \in R(M),\ (u\cos\theta + v\sin\theta)_M = u\},$$

then u is not a unique projection of $u\cos\theta + v\sin\theta$ if and only if $\theta \geq \bar\theta(u, v)$. Now we have the following basic proposition concerning $R(M)$ of (3).

Proposition 2.1 *For $0 \leq \theta \leq \pi$ let*

$$[u, u\cos\theta + v\sin\theta) = \begin{cases} \{u\cos t + v\sin t \mid 0 \leq t < \theta\}, & \text{if } \theta > 0, \\ \{u\}, & \text{if } \theta = 0, \end{cases}$$

denote the segment of great circle joining u and $u\cos\theta + v\sin\theta$, which includes u and excludes $u\cos\theta + v\sin\theta$. For a locally conic closed set M

$$R(M) = \bigcup_{u \in M} \bigcup_{v \in N_u(K(M)), \|v\|=1} [u, u\cos\bar\theta(u, v) + v\sin\bar\theta(u, v)). \tag{4}$$

From this basic proposition we can prove the following property of $R(M)$.

Proposition 2.2 *For locally conic closed $M \subset S^{n-1}$ with piecewise smooth proper boundary, almost all $x \in S^{n-1}$ have unique projection onto M, i.e., the complement of $R(M)$ has zero spherical volume.*

Proofs of Propositions 2.1 and 2.2 are given in Appendix B.

2.2 Exact tube formula and formal tube formula

The open spherical tube of radius θ around a closed set $M \subset S^{n-1}$ is defined by

$$M_\theta = \{x \mid \mathrm{dist}(x, M) < \theta\}.$$

Classifying the points of tube by the projection onto M and the direction of the projection, M_θ can be written as

$$M_\theta = \bigcup_{u \in M} \bigcup_{v \in N_u(M), \|v\|=1} [u, u\cos\theta + v\sin\theta).$$

Note that here the cross section

$$C_u(\theta) = \bigcup_{v \in N_u(K(M)), \|v\|=1} [u, u\cos\theta + v\sin\theta)$$

may overlap for different u's. If we only count points with unique projection onto M we obtain

$$\tilde{M}_\theta = \bigcup_{u \in M} \bigcup_{v \in N_u(K(M)), \|v\|=1} [u, u\cos\theta' + v\sin\theta') \quad \subset \; M_\theta,$$

where $\theta' = \min(\theta, \bar\theta(u, v))$. Note that by Proposition 2.2 $M_\theta - \tilde{M}_\theta$ is a null set.

Writing the tube $\tilde{M}_\theta$ as above we see that $R(M)$ of (4) is a generalization of of tube, where the radius of the depends on $u \in M$ and on $v \in N_u(K(M))$. Define

$$\bar\theta(u) = \inf_{v \in N_u(K(M)), \|v\|=1} \bar\theta(u, v).$$

The critical radius (or angle) of $M \subset R^n$ is

$$\bar\theta = \inf\{\bar\theta(u) \mid u \in \cup_{d=0}^{n-2} \partial M_d\}.$$

In this definition we omit the interior ∂M_{n-1} of $M \subset S^{n-1}$, when M contains non-empty interior in S^{n-1}. In the case of positive critical radius $\bar\theta > 0$, the constant radius tube $\bigcup_{u \in M} C_u(\bar\theta)$ was essential for obtaining asymptotic expansion of tail probability of maximum of $Z(u)$ of (1) and $Y(u)$ of (2).

As mentioned already, we have the following simple lemma concerning the critical radius $\bar\theta$ of M.

Lemma 2.1 *The critical radius of M is zero if for some $x \in M$ the support cone $S_x(M)$ is not convex.*

Proof is given in Appendix B.

Now we study the volume of the tube M_θ, when M is a locally conic closed set with piecewise smooth proper boundary. From Lemma 2.2 of Takemura and Kuriki (1999b) the volume element dy of S^{n-1} at $y = x\cos\theta + v\sin\theta$, $x \in \partial M_d$, $v \in N_x(K(M))$, is written as

$$dy = \det(I_d \cos\theta + H(x, v)\sin\theta) \sin^{n-d-2}\theta \, d\theta \, dx \, dv.$$

Note that for $\theta < \bar{\theta}(x,v)$ the matrix $(I_d \cos\theta + H(x,v)\sin\theta)$ is positive definite. Therefore by the standard derivation of the tube formula, the spherical volume of the tube M_θ is written as

$$
\begin{aligned}
&V(M_\theta)\\
&= \int_{M-I(M)} dx \int_{v\in N_x(K(M))\cap S^{n-1}} dv \int_0^{\min(\theta,\bar{\theta}(x,v))} d\tau \det(I_d\cos\tau + H(x,v)\sin\tau)\sin^{n-d-2}\tau\\
&= \sum_{d=0}^m \int_{\partial M_d} dx \int_{v\in N_x(K(M))\cap S^{n-1}} dv \sum_{j=0}^d \mathrm{tr}_j H(x,v) \int_0^{\min(\theta,\bar{\theta}(x,v))} \cos^{d-j}\tau \sin^{n-d+j-2}\tau d\tau,
\end{aligned}
$$

where $\mathrm{tr}_j H$ denotes the j-th elementary symmetric function of the characteristic roots of H. Using the fact that for $0 \le \theta \le \pi/2$

$$
\int_0^\theta \cos^a\tau \sin^b \tau d\tau = \frac{\Omega_{a+b+2}}{\Omega_{a+1}\Omega_{b+1}} \bar{B}_{\frac{1}{2}(a+1),\frac{1}{2}(b+1)}(\cos^2\theta),
$$

where $\bar{B}_{k,l}$ denotes the upper probability function of beta distribution with parameter (k,l) and

$$
\Omega_c = V(S^{c-1}) = \frac{2\pi^{c/2}}{\Gamma(c/2)}
$$

is the volume of S^{c-1}, we have established the following theorem.

Theorem 2.1 *For locally conic closed set $M \subset S^{n-1}$ with piecewise smooth proper boundary the spherical volume of the tube M_θ, $\theta \le \pi/2$, is given by*

$$
\begin{aligned}
V(M_\theta) &= \Omega_n \sum_{d=0}^m \int_{\partial M_d} dx \int_{v\in N_x(K(M))\cap S^{n-1}} dv\\
&\quad \cdot \sum_{j=0}^d \frac{\mathrm{tr}_j H(x,v)}{\Omega_{d-j+1}\Omega_{n-d+j-1}} \bar{B}_{\frac{1}{2}(d-j+1),\frac{1}{2}(n-d+j-1)}(\cos^2 \min(\theta,\bar{\theta}(x,v))). \qquad (5)
\end{aligned}
$$

Theorem 2.1 can be generalized to the case $\min(\theta,\bar{\theta}(x,v)) > \pi/2$ as in Proposition 2.1 of Takemura and Kuriki (1999b). The formal tube formula for $\theta \le \pi/2$ is obtained by setting $\bar{\theta}(x,v) = \pi/2$:

$$
\begin{aligned}
\tilde{V}(M_\theta) &= \Omega_n \sum_{d=0}^m \int_{\partial M_d} dx \int_{v\in N_x(K(M))\cap S^{n-1}} dv\\
&\quad \cdot \sum_{j=0}^d \frac{\mathrm{tr}_j H(x,v)}{\Omega_{d-j+1}\Omega_{n-d+j-1}} \bar{B}_{\frac{1}{2}(d-j+1),\frac{1}{2}(n-d+j-1)}(\cos^2\theta). \qquad (6)
\end{aligned}
$$

Exact and formal tube formulas for the tube of M in R^n are given in Appendix C.

Since $V(M_\theta)/\Omega_n$ gives the exact tail probability of $\max_{u\in M} Y(u)$ we have the following.

Corollary 2.1 *For $t \ge 0$*

$$
\begin{aligned}
P\Big(\max_{u\in M} Y(u) \ge t\Big) &= \sum_{d=0}^m \int_{\partial M_d} dx \int_{v\in N_x(K(M))\cap S^{n-1}} dv\\
&\quad \cdot \sum_{j=0}^d \frac{\mathrm{tr}_j H(x,v)}{\Omega_{d-j+1}\Omega_{n-d+j-1}} \bar{B}_{\frac{1}{2}(d-j+1),\frac{1}{2}(n-d+j-1)}(\max(t^2,\bar{t}(x,v)^2)), \quad (7)
\end{aligned}
$$

where $\bar{t}(x,v) = \cos\bar{\theta}(x,v)$.

We can also derive exact tail probability for the maximum of $Z(u)$ in (1). Let g_k and G_k denote the density and the cumulative distribution function of χ^2 distribution with k degrees of freedom and write

$$Q_{k,l}(a,b) = \int_a^\infty g_k(x) G_l(bx) dx.$$

Theorem 2.2 *Let $M \subset S^{n-1}$ be a locally conic closed set with piecewise smooth proper boundary. For $t \geq 0$*

$$\begin{aligned} P\Big(\max_{u\in M} Z(u) \geq t\Big) &= \sum_{d=0}^{m} \int_{\partial M_d} dx \int_{N_x(K(M))\cap S^{n-1}} dv \\ &\quad \cdot \sum_{j=0}^{d} \frac{\mathrm{tr}_j\, H(x,v)}{\Omega_{d-j+1}\Omega_{n-d+j-1}} Q_{d-j+1,n-d+j-1}(t^2, \tan^2\bar\theta(x,v)). \end{aligned} \qquad (8)$$

Proof. Since for $z \sim N_n(0, I_n)$ $y = z/\|z\|$ and $\|z\|$ are independent, $P(\max_{u\in M} Z(u) \geq t) = P(\max_{u\in M} u'z \geq t)$ is calculated by substituting $t := t/\|z\|$ in (7) and taking expectation with respect to $\|z\|^2 \sim \chi^2(n)$. Let B be a random variable distributed as $B(a,b)$, the beta distribution with parameter (a,b). Then for $a+b=n$

$$\begin{aligned} E[\bar B_{a,b}(\max(t^2/\|z\|^2, \bar t^2))] &= P(\|z\|^2 B \geq t^2,\ B \geq \bar t^2) \\ &= P(\|z\|^2 B \geq t^2,\ \|z\|^2 B(1-\bar t^2)/\bar t^2 \geq \|z\|^2(1-B)) \\ &= Q_{a,b}(t^2, (1-\bar t^2)/\bar t^2), \end{aligned}$$

since $\|z\|^2 B$ and $\|z\|^2(1-B)$ are independently distributed according to $\chi^2(a)$ and $\chi^2(b)$, respectively. ∎

The formal asymptotic expansion by tube formula is obtained by letting $\bar\theta(x,v) = \pi/2$. In this case

$$Q_{d-j+1,n-d+j-1}(t^2, \infty) = \bar G_{d-j+1}(t^2) = 1 - G_{d-j+1}(t^2)$$

and the formal expansion is given by

$$\tilde P\Big(\max_{u\in M} Z(u) \geq t\Big) = \sum_{d=0}^{m} \int_{\partial M_d} dx \int_{N_x(K(M))\cap S^{n-1}} dv \sum_{j=0}^{d} \frac{\mathrm{tr}_j\, H(x,v)}{\Omega_{d-j+1}\Omega_{n-d+j-1}} \bar G_{d-j+1}(t^2), \qquad (9)$$

where $\bar G_k$ denotes the upper probability function of χ^2 distribution with k degrees of freedom.

So far we have discussed asymptotic expansion based on the formal tube formula. Here we briefly discuss its relation to the Euler characteristic method. Suppose that $x \in M$ is a point of self-intersection, such as $x = (\pm 1, 0, 0)$ in Example 2.1. The difficulty concerning the Euler characteristic method arises when x is contained in an excursion set of the random field. In this case usual form of Morse's theorem can not be used for justifying the Euler characteristic method. However we could formally apply Morse's theorem and ignore the points of self-intersection. This type of formal Euler characteristic method and the formal tube formula give identical asymptotic expansion as shown in Takemura and Kuriki (1999b).

2.3 Invalidity of formal expansion

In this subsection we show that when the critical radius $\bar{\theta}$ is zero, formal tube formula only gives valid main term and other higher order expansion terms are not valid in general. Concerning tail probability of $\max_{u\in M} Y(u)$ we let $\theta \downarrow 0$ and compare Taylor expansion of (5) and (6). Similarly we let $t \to \infty$ and compare (8) and (9).

First we consider the main terms of the expansions. In (5) the main term is given by $d = m, j = 0$. The case $m = n-1$ is trivial, because in this case (5) and (6) converge to $V(M) = V(\partial M_{n-1}) > 0$. Therefore let $m < n-1$. Then

$$V(M_\theta) \sim \int_{\partial M_m} dx \int_{v\in N_x(K(M))\cap S^{n-1}} dv \, \frac{\Omega_n}{\Omega_{m+1}\Omega_{n-m-1}} \bar{B}_{\frac{1}{2}(m+1),\frac{1}{2}(n-m-1)}(\cos^2 \min(\theta, \bar{\theta}(x,v))).$$

Write $\theta' = \min(\theta, \bar{\theta}(x,v))$, $a = (m+1)/2$, $b = (n-m-1)/2$. Ignoring the constant, which is common for (5) and (6), consider

$$\begin{aligned}
&\int_{\partial M_m} dx \int_{N_x(K(M))\cap S^{n-1}} dv \int_{\cos^2\theta'}^1 \xi^{a-1}(1-\xi)^{b-1} d\xi \\
&\sim \int_{\partial M_m} dx \int_{N_x(K(M))\cap S^{n-1}} dv \int_{\cos^2\theta'}^1 (1-\xi)^{b-1} d\xi \\
&= \frac{1}{b}\int_{\partial M_m} dx \int_{N_x(K(M))\cap S^{n-1}} dv \, \sin^{2b}\theta' \\
&= \frac{\theta^{2b}}{b}\int_{\partial M_m} dx \int_{N_x(K(M))\cap S^{n-1}} dv \, \frac{\sin^{2b}\min(\theta,\bar{\theta}(x,v))}{\theta^{2b}}.
\end{aligned}$$

Now for each fixed (x,v), $\bar{\theta}(x,v) > 0$, because M is locally conic and $N_x(K(M)) \neq \{0\}$. Therefore

$$\frac{\sin^{2b}\min(\theta,\bar{\theta}(x,v))}{\theta^{2b}} \to 1 \quad (\theta \to 0),$$

and by dominated convergence theorem we have

$$\int_{\partial M_m} dx \int_{N_x(K(M))\cap S^{n-1}} dv \, \frac{\sin^{2b}\min(\theta,\bar{\theta}(x,v))}{\theta^{2b}} \to \int_{\partial M_m} dx \int_{N_x(K(M))\cap S^{n-1}} dv.$$

Writing the constant again we obtain

$$V(M_\theta) \sim \frac{\theta^{n-m-1}}{n-m-1}\int_{\partial M_m} dx \int_{N_x(K(M))\cap S^{n-1}} dv \quad (\theta \to 0).$$

However this is the main term of $\tilde{V}(M_\theta)$ as well. Therefore we have shown that (5) and (6) have the same main term.

Proving that (8) and (9) have the same main term

$$P\Big(\max_{u\in M} Z(u) \geq t\Big) \sim \frac{\Gamma((n-m-1)/2)}{2^{(m+3)/2}\pi^{n/2}} t^{m-1} e^{-t^2/2} \int_{\partial M_m} dx \int_{N_x(K(M))\cap S^{n-1}} dv \quad (t \to \infty)$$

is entirely similar, by noting that for each (x,v)

$$\frac{Q_{a,n-a}(t^2, \tan^2\bar{\theta}(x,v))}{\bar{G}_a(t^2)} \to 1 \quad (t \to \infty).$$

We proceed to show that in general higher order terms of (5) and (6) or (8) and (9) are not equal. The arguments for these two cases are entirely similar. Here we discuss only the difference between (8) and (9). In order to show the discrepancy we only consider expansion terms arising from the term $d = m$, $j = 0$, in the summation of (8) and (9). Ignoring $1/(\Omega_{m+1}\Omega_{n-m-1})$ the difference of these two terms is written as

$$\int_{\partial M_m} dx \int_{N_x(K(M))\cap S^{n-1}} dv \int_{t^2}^{\infty} g_{m+1}(\xi)\, \bar{G}_{n-m-1}(\xi \tan^2 \bar{\theta}(x,v))\, d\xi. \qquad (10)$$

Define

$$A(t) = \{(x,v) \mid x \in \partial M_m,\ v \in N_x(K(M)) \cap S^{n-1},\ \tan\bar{\theta}(x,v) \le 1/t\}.$$

Now assume that there exists some $k > 0$ such that

$$\int_{A(t)} dxdv = O(t^{-k}). \qquad (11)$$

Fix $c > 1$. Then (10) is bounded below as

$$\begin{aligned}
(10) \ &\ge \int_{\partial M_m} dx \int_{N_x(K(M))\cap S^{n-1}} dv \int_{t^2}^{ct^2} g_{m+1}(\xi)\, \bar{G}_{n-m-1}(\xi \tan^2 \bar{\theta}(x,v))\, d\xi \\
&\ge \int_{A(t)} dxdv \int_{t^2}^{ct^2} g_{m+1}(\xi)\, \bar{G}_{n-m-1}(\xi \tan^2 \bar{\theta}(x,v))\, d\xi \\
&\ge \bar{G}_{n-m-1}(c) \int_{A(t)} dxdv \int_{t^2}^{ct^2} g_{m+1}(\xi)\, d\xi \\
&= O(t^{-k}\bar{G}_{m+1}(t^2)) = O(\bar{G}_{m+1-k}(t^2)).
\end{aligned}$$

However the term of order $O(\bar{G}_{m+1-k}(t^2))$ is not distinguishable from higher order expansion terms of (8) or (9). Therefore we have shown that higher order terms of (8) and (9) are not equal in general when (11) holds.

It may be the case that k in (11) is large and many terms of formal asymptotic expansion are correct. In this case we may want to approximate the tail probability using only the correct terms of the asymptotic expansion. Therefore it is important to determine the value of k in (11) for a given problem. In Appendix D we argue that in certain regular cases k is simply the difference of $m = \dim M$ and the dimension of the set of points with non-convex support cone.

3 Examples

The formulas for exact tail probabilities in Section 2.2 are of theoretical importance. However they may be difficult to explicitly evaluate for a given problem. Therefore in this section we investigate some simple examples in detail, where the exact tail probability as well as the formal expansion by the tube formula and the Euler characteristic method can be explicitly evaluated and the discrepancy between them can be clearly understood.

3.1 Boundary of polyhedral cone

Here we consider a simple example of the tail probability of the maximum of $Y(u)$ in (2). Consider the uniform distribution Unif(S^2) on the sphere S^2 in R^3. For simplicity of notation we avoid subscripts and let (x, y, z) denote a vector on R^3 or on S^2. Note that $\Omega_3 = 4\pi$, $\Omega_2 = 2\pi$, $\Omega_1 = 2$.

Let

$$K(M) = \{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$$

be the union of 3 coordinate planes and $M = K(M) \cap S^2$. Then

$$\begin{aligned} \max_{u \in M} Y(u) &= \max(\sqrt{y^2 + z^2}, \sqrt{x^2 + z^2}, \sqrt{x^2 + y^2}) \\ &= \max(\sqrt{1 - x^2}, \sqrt{1 - y^2}, \sqrt{1 - z^2}) \end{aligned}$$

and

$$P\Big(\max_{u \in M} Y(u) \ge \cos\theta\Big) = \frac{V(M_\theta)}{4\pi}.$$

$Y(u)$ corresponds to the maximum of 3 correlated beta variables. This type of statistics is commonly used in change point analysis or multiple comparisons. M consists of 12 arcs of length $\pi/2$ and 6 points of crossings. The 6 points are improper points and do not contribute to the volume of the tube. 12 arcs form one-dimensional proper boundary of M. Without loss of generality consider points on the arc $u = (\cos\tau, \sin\tau, 0)$, $0 \le \tau \le \pi/2$. $N_u(K(M)) \cap S^2 = (0, 0, \pm 1)$ and the cross section at u is the arc

$$C_u(\theta) = \cos\xi(\cos\tau, \sin\tau, 0) + \sin\xi(0, 0, 1), \quad |\xi| < \theta.$$

u is the unique projection of points in $C_u(\theta)$ if and only if

$$|\sin\xi| < \min(\cos\xi\cos\tau, \cos\xi\sin\tau).$$

Therefore for $v = (0, 0, \pm 1)$

$$\bar\theta(u, v) = \min(\arctan(\cos\tau), \arctan(\sin\tau)).$$

Now in (5) and (6) $m = d = 1$, $\int_{N_u(K(M)) \cap S^2} dv = 2$, $H = 0$, $\mathrm{tr}_0 H = 1$, $\mathrm{tr}_1 H = 0$, and

$$\bar B_{1, \frac{1}{2}}(t^2) = \frac{1}{2}\int_{t^2}^1 (1 - \xi)^{-1/2} d\xi = (1 - t^2)^{1/2}.$$

The largest angle from M is achieved by $(1, 1, 1)/\sqrt{3}$ with the angle $\arcsin(1/\sqrt{3})$ from M.

We first consider the formal tube formula, because it is simpler.

$$\begin{aligned} \tilde V(M_\theta) &= 2\,\Omega_3 \int_{12\,\text{arcs}} du \frac{1}{\Omega_1 \Omega_2} \bar B_{1, \frac{1}{2}}(\cos^2\theta) \\ &= 24 \sin\theta \int_0^{\pi/2} d\tau = 12\pi\sin\theta. \end{aligned}$$

We now consider the true volume $V(M_\theta)$. We only consider $\theta \le \arcsin(1/\sqrt{3})$. Even in this rather simple example the exact integration in (5) is somewhat complicated. Write $\theta = \arctan(\sin\tau_0)$ or $\tau_0 = \arcsin(\tan\theta)$. Then

$$\min(\theta, \bar\theta(u, v)) = \begin{cases} \arctan(\sin\tau), & \text{if } 0 \le \tau \le \tau_0, \\ \theta, & \text{if } \tau_0 < \tau < \pi/2 - \tau_0, \\ \arctan(\cos\tau), & \text{if } \pi/2 - \tau_0 \le \tau \le \pi/2. \end{cases}$$

The contribution of the middle case to the volume is

$$24\sin\theta\,(\pi/2 - 2\tau_0) = 12\pi\sin\theta - 48\sin\theta\arcsin(\tan\theta).$$

The contribution from the region where $\bar\theta(u,v) < \theta$ is

$$48\int_0^{\tau_0}\sin(\arctan(\sin\tau))\,d\tau = 48\int_0^{\tau_0}\frac{\sin\tau}{\sqrt{1+\sin^2\tau}}\,d\tau.$$

Let $w = \sin^2\tau$, with $dw = 2\sin\tau\cos\tau\,d\tau$. Then

$$\begin{aligned}
48\int_0^{\tau_0}\frac{\sin\tau}{\sqrt{1+\sin^2\tau}}\,d\tau &= 24\int_0^{\sin^2\tau_0}\frac{1}{\sqrt{1+w}\sqrt{1-w}}\,dw \\
&= 24\int_0^{\tan^2\theta}\frac{1}{\sqrt{1-w^2}}\,dw \\
&= 24\arcsin(\tan^2\theta).
\end{aligned}$$

Therefore we obtain

$$V(M_\theta) = 12\pi\sin\theta - 48\sin\theta\arcsin(\tan\theta) + 24\arcsin(\tan^2\theta).$$

Note that both $V(M_\theta)$ and $\tilde V(M_\theta)$ are $O(\theta)$ and they differ in the term of order $O(\theta^2)$:

$$V(M_\theta) = \tilde V(M_\theta) - 24\theta^2 + o(\theta^2).$$

$\bar\theta(u,v)$ tends to zero around the 6 crossing points of M. Note that in this example the conditions of Appendix D are satisfied with $c = 1$ in (26). The volume (actually the length in this example) of points $u \in \partial M_1$ with $\bar\theta(u) \le 1/t$ is $O(1/t)$. Therefore $k = 1$ in (11). This corresponds to the difference of $\dim M = 1$ and 0, which is the dimension of these 6 points.

3.2 Sum of several roots of Wishart matrix

In the example of the last subsection, $K(M)$ was a union of planes and there were no curvature involved in the example. Here we consider an example involving nonzero curvature. The statistic we consider is sum of several largest roots of Wishart matrix. Let Z be an $n\times p$ $(n \ge p)$ random matrix consisting of i.i.d. standard normal variables. Then $W = Z'Z$ is a $p\times p$ Wishart random matrix with n degrees of freedom. We consider the distribution of

$$T^2 = \lambda_1 + \cdots + \lambda_q, \qquad 1 < q < p,$$

where $\lambda_1 \ge \cdots \ge \lambda_p$ are characteristic roots of W. In Kuriki and Takemura (1998) we obtain a valid asymptotic expansion of tail probability for T^2 when $q = 1$. In this subsection we consider formal asymptotic expansion of the upper tail probability of T^2 when $q \ge 2$. There are two approaches we can take: One is the formal tube formula to the Gaussian random field based Z. The other is the Euler characteristic method to the χ^2 field based on W. We begin by showing that these two methods lead to identical asymptotic expansion of tail probability of T. Then we demonstrate the discrepancy between correct asymptotic expansion and the formal asymptotic expansion of the upper tail probability of T for the simple case of $p = 3$, $q = 2$ and $n = 4$.

Let $R^{n\times p}$ denote the set of $n\times p$ real matrices endowed with the inner product $\mathrm{tr}(u'v)$, $u,v\in R^{n\times p}$. Let

$$M=\{u\in R^{n\times p}\mid \mathrm{rank}(u)\le q,\ \mathrm{tr}(u'u)=1\}$$

and consider the Gaussian field $Z(u)=\mathrm{tr}(u'Z)$, $u\in M$. It is well known that $\max_{u\in M}Z(u)=T$. The properties of projection onto $K(M)$ were studied in Takemura and Kuriki (1999a), from which we summarize some results below. Let $x\in M$ be a matrix having a singular value decomposition $x=G_1L_1H_1'=l_1g_1h_1'+\cdots+l_qg_qh_q'$, where $L_1=\mathrm{diag}(l_1,\ldots,l_q)$, $G_1=(g_1,\ldots,g_q)$ and $H_1=(h_1,\ldots,h_q)$. Note that $l_1^2+\cdots+l_q^2=1$ since $\mathrm{tr}(x'x)=1$. Let $M(k)$ denote the set of matrices $M\in R^{n\times p}$ of rank k, $k=1,\ldots,q$. At $x=l_1g_1h_1'+\cdots+l_qg_qh_q'\in M(q)$,

$$\begin{aligned}S_x(K(M)) &= S_x(M)\oplus\mathrm{span}\{x\}=\mathrm{span}\{g_ih_j'\mid\min(i,j)\le q\},\\ N_x(K(M)) &= N_x(M)\cap\mathrm{span}\{x\}^{\perp}=\mathrm{span}\{g_ih_j'\mid\min(i,j)>q\},\end{aligned}$$

and

$$\bar\theta(x,v)=\arctan\Bigl(\frac{\min(l_1,\ldots,l_q)}{\max(l_{q+1},\ldots,l_p)}\Bigr),\quad v\in N_x(K(M)),\ \mathrm{tr}(v'v)=1, \tag{12}$$

where $g_{q+1},\ldots,g_n$ and $h_{q+1},\ldots,h_p$ are chosen so that $(g_1,\ldots,g_n)$ and $(h_1,\ldots,h_q)$ are orthogonal matrices and $l_{q+1},\ldots,l_p$ are singular values of v. Note that $l_{q+1}^2+\cdots+l_p^2=1$ because of $\mathrm{tr}(v'v)=1$. We see that $M(q)$ is the proper boundary (actually the relative interior) of M, since $N_x(M)=S_x^{\perp}(M)=T_x^{\perp}(M)$.

If $q=1$ then by (12) $\bar\theta(x)=\inf_v\bar\theta(x,v)=\inf_v\arctan(1/\max(l_2,\ldots,l_p))=\arctan(1)=\pi/4$. Hence the critical radius is $\bar\theta=\pi/4$, as obtained in Kuriki and Takemura (1998). On the other hand, for $q\ge 2$ by considering the case $l_q\doteq 0$, we see that $\bar\theta(x,v)$ can be arbitrarily close to zero on $M(q)$. Therefore the critical radius $\bar\theta$ of M is zero. This property $\bar\theta=0$ can be clarified by studying the structure of $M(k)$ for $k<q$. Let $x=l_1g_1h_1'+\cdots+l_kg_kh_k'\in M(k)$. It is easy to show that at x

$$\begin{aligned}S_x(K(M))&=\{c_{k+1}g_{k+1}h_{k+1}'+\cdots+c_qg_qh_q'\mid c_{k+1},\ldots c_q\in R,\\ &\qquad\qquad g_l'g_j=h_l'h_j=0,\ l\le k<j\le q\}\oplus T_x(M(k)),\\ C(S_x(K(M)))&=R^{n\times p},\quad N_x(K(M))=\{0\},\end{aligned}$$

where $T_x(M(k))=\mathrm{span}\{g_ih_j'\mid\min(i,j)\le k\}$. We see that x is an improper boundary point of M. It is somewhat difficult to imagine the geometry of M at $x\in M(k)$, $k<q$. One possible explanation of the fact $C(S_x(K(M)))=R^{n\times p}$ is as follows. Since $g_{k+1},\ldots,g_q$ and $h_{k+1},\ldots,h_q$ can have arbitrary direction in $M(q)$ around $x\in M(k)$, $K(M)$ intersects itself at x from sufficiently many directions along $S_x(K(M))$, such that the convex hull of $S_x(K(M))$ is the whole space $R^{n\times p}$. In any case it is again proved that the critical radius of M is zero by Lemma 2.1.

In the following let $l_1,\ldots,l_q$ (≥ 0) be unordered singular values of the random matrix Z, corresponding to any of $\binom{p}{q}$ ways of choosing q out of p unordered singular values of Z. Put $L_1=\mathrm{diag}(l_1,\ldots,l_q)$ as before. Similarly let $G_1=(g_1,\ldots,g_q)$ and $H_1=(h_1,\ldots,h_q)$ be the $n\times q$ and $p\times q$ matrices where g_i and h_i are the right and left singular vectors of Z with respect to the singular value l_i. $G_2=(g_{q+1},\ldots,g_n)$ and $H_2=(h_{q+1},\ldots,h_p)$ are suitably defined matrices such that $G=(G_1,G_2)$ and $H=(H_1,H_2)$ are orthogonal matrices.

Write $Z_1=G_1L_1H_1'$ and define a matrix $\tilde Z=(\tilde z_{ij})\in R^{(n-q)\times(p-q)}$ by

$$Z=Z_1+G_2\tilde ZH_2'.$$

Then $Z_1 \in K(M)$ and $Z - Z_1 = G_2 \tilde{Z} H_2' \in N_{Z_1}(K(M))$. Z_1 is a critical point of the height function $f_Z(u) = -\operatorname{tr}(u'Z)$, $u \in M$. Note that the projection of Z onto M, that is, the point $u \in M$ which maximizes $\operatorname{tr}(u'Z)$, corresponds to L_1 consisting of q largest singular values. However in the formal tube formula not only the projection but all critical points are counted. The volume element $dZ = \prod_{i=1}^n \prod_{j=1}^p dz_{ij}$ at $Z = (z_{ij})$ is decomposed as

$$dZ = dZ_1 \, d\tilde{Z} \, | \det(I_{q(p-q)} - L_1{}^{-2} \otimes \tilde{Z}'\tilde{Z})|, \tag{13}$$

where $d\tilde{Z} = \prod_{i=1}^{n-q} \prod_{j=1}^{p-q} d\tilde{z}_{ij}$ is the volume element of the orthogonal complement $T_{Z_1}^{\perp}(K(M))$ of $T_{Z_1}(K(M))$ at $Z - Z_1$, and "$\otimes$" denotes Kronecker's product. dZ_1 is the volume element of $K(M)$ at Z_1 defined by

$$dZ_1 = \prod_{1 \le i < j \le q} (l_i^2 - l_j^2) \det(L_1)^{n+p-2q} dL_1 \, dG_1 \, dH_1,$$

where $dL_1 = \prod_{i=1}^q dl_i$, $dG_1 = \bigwedge_{i=1}^q \bigwedge_{j=i+1}^n g_j' dg_i$, and $dH_1 = \bigwedge_{i=1}^q \bigwedge_{j=i+1}^p h_j' dh_i$.

By (13) the upper tail probability of T based on the formal tube formula is given by

$$\tilde{P}(T^2 > c^2) = \frac{1}{(2\pi)^{np/2}} \int_{T^2 > c^2} e^{-(T^2+S^2)/2} \det(I_{q(p-q)} - L_1{}^{-2} \otimes \tilde{Z}'\tilde{Z}) \, dZ_1 \, d\tilde{Z}, \tag{14}$$

where $S^2 = \operatorname{tr}(\tilde{Z}'\tilde{Z})$.

We now consider χ^2 field based on the Wishart matrix $W = Z'Z$. Let $\operatorname{Sym}(p)$ denote the set of $p \times p$ real symmetric matrices, endowed with the inner project $\operatorname{tr}(uv)$, $u, v \in \operatorname{Sym}(p)$. Let $\tilde{M}$ denote the set of $p \times p$ orthogonal projectors of rank q:

$$\tilde{M} = \{u \in \operatorname{Sym}(p) \mid u^2 = u, \ \operatorname{rank}(u) = q\},$$

and consider a χ^2 field defined by $\operatorname{tr}(uW)$, $u \in \tilde{M}$. The index set $\tilde{M}$ is the Grassmann manifold, which is a regular smooth submanifold without boundary. The sum of q largest roots of W can be written as the maximum of the χ^2 field $T^2 = \max_{u \in \tilde{M}} \operatorname{tr}(uW)$.

Let $D_1 = L_1^2 = \operatorname{diag}(l_1^2, \ldots, l_q^2)$ be the $q \times q$ diagonal matrix consisting of (unordered) q eigenvalues of W. Then

$$W_K = \bar{d} H_1 H_1', \quad \bar{d} = \operatorname{tr}(D_1)/q,$$

is the orthogonal projection of W onto the cone $K(\tilde{M})$.

Let $W_1 = H_1 D_1 H_1'$ and define a matrix $\tilde{W} \in \operatorname{Sym}(p-q)$ by

$$W = W_1 + H_2 \tilde{W} H_2'.$$

The Jacobian of this decomposition is given by Kuriki and Takemura (1999) as

$$dW = dW_1 \, d\tilde{W} \, | \det(I_{q(p-q)} - D_1{}^{-1} \otimes \tilde{W})|, \tag{15}$$

where $dW = 2^{p(p-1)/4} \prod_{i \le j} dw_{ij}$ and $d\tilde{W} = 2^{(p-q)(p-q-1)/4} \prod_{i \le j} d\tilde{w}_{ij}$ are the volume elements of $\operatorname{Sym}(p)$ and $\operatorname{Sym}(p-q)$ at $W = (w_{ij})$ and $\tilde{W} = (\tilde{w}_{ij})$, respectively, and

$$dW_1 = 2^{q(q-1)/4 + q(p-q)/2} \prod_{1 \le i < j \le q} (l_i^2 - l_j^2) \det(D_1)^{p-q} dD_1 \, dH_1 \tag{16}$$

with $dD_1 = \prod_{i=1}^q d(l_i^2)$.

For a parameter $t > 0$ let

$$\begin{aligned} W(t) &= W_K + t(W - W_K) \\ &= H_1 D_1(t) H_1' + H_2(t\tilde{W})H_2', \end{aligned}$$

where

$$D_1(t) = \bar{d}I_q + t(D_1 - \bar{d}I_q).$$

Note that $W(1) = W$. For $t > 0$, $W(t)$ has the identical projection point W_K on $K(\tilde{M})$. Since $d(t\tilde{W}) = t^{(p-q)(p-q+1)/2} d\tilde{W}$ and $dD_1(t)\,(= \prod_{i=1}^{q} d(\bar{d}+t(l_i^2-\bar{d}))) = t^{q-1}dD_1$, the volume element of $\mathrm{Sym}(p)$ at $W(t)$ is shown to be

$$\begin{aligned} dW(t) \;\propto\; & d\tilde{W}\, dH_1 \prod_{i<j}(l_i^2 - l_j^2)\, dD_1\, t^{(p-q)(p-q+1)/2+q(q-1)/2+q-1} \\ & \times |\det(\bar{d}I_{q(p-q)} + t((D_1 - \bar{d}I_q)\otimes I_{p-q} - I_q \otimes \tilde{W}))|. \end{aligned} \qquad (17)$$

On the other hand, by the general theory by Weyl (1939), the volume element has the form of

$$\begin{aligned} dW(t) = \; & (\text{volume element of } K(\tilde{M}) \text{ at } W_K) \\ & \times t^{\mathrm{codim}\, K} (\text{volume element of } T^{\perp}_{W_K}(K(\tilde{M})) \text{ at } W - W_K) \\ & \times |\det(I + tH(W_K, W - W_K))|, \end{aligned} \qquad (18)$$

where $H(W_K, W - W_K)$ denotes the second fundamental form of $K(\tilde{M})$ at W_K with respect to the normal direction $W - W_K$. (See also Lemma 2.1 of Kuriki and Takemura (1997).) By comparing (17) and (18), we have that under an appropriate coordinate system

$$H(W_K, W - W_K) = (1/\bar{d})\,((D_1 - \bar{d}I_q)\otimes I_{p-q} - I_q \otimes \tilde{W}).$$

Therefore

$$\begin{aligned} I_{q(p-q)} + H(W_K, W - W_K) &= (1/\bar{d})\,(D_1 \otimes I_{p-q} - I_q \otimes \tilde{W}) \\ &= (1/\bar{d})\,(D_1 \otimes I_{p-q})\,(I_{q(p-q)} - D_1^{-1} \otimes \tilde{W}). \end{aligned}$$

These relations can also be derived directly from the definition of the second fundamental form by introducing local coordinates on the Grassmann manifold.

By taking the height function $f_W(u) = -\mathrm{tr}(uW)$ as a Morse function, by virtue of Morse's theorem we get the Euler characteristic of the excursion set

$$A(W, c) = \{u \in \tilde{M} \mid \mathrm{tr}(uW) > c\}$$

as

$$\begin{aligned} \chi(A(W,c)) &= \sum I(\mathrm{tr}(W_1) > c)\,\mathrm{sgn}\det(I_{q(p-q)} + H(W_K, W - W_K)) \\ &= \sum I(\mathrm{tr}(W_1) > c)\,\mathrm{sgn}\det(I_{q(p-q)} - {D_1}^{-1} \otimes \tilde{W}), \end{aligned} \qquad (19)$$

where I denotes indicator function, and the summation is over $\binom{p}{q}$ ways of choosing q out of p characteristic roots of W. Note that density of the Wishart distribution is given by

$$\frac{1}{c_{n,p}} \det(W)^{(n-p-1)/2}\, e^{-\mathrm{tr}(W)/2}\, dW, \qquad (20)$$

where

$$c_{n,p} = 2^{p(p-1)/4+np/2}\,\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma((n-i+1)/2).$$

Combining (15), (19) and (20), we see that

$$\begin{aligned} E[\chi(A(W,c))] = {}& \frac{1}{c_{n,p}}\int_{\operatorname{tr}(W_1)>c} \det(I_{q(p-q)} - D_1{}^{-1}\otimes \tilde W) \\ & \times \det(D_1)^{(n-p-1)/2}\, e^{-\operatorname{tr}(W_1)/2}\, dW_1 \times \det(\tilde W)^{(n-p-1)/2}\, e^{-\operatorname{tr}(\tilde W)/2}\, d\tilde W. \qquad (21)\end{aligned}$$

Now in the right hand side of (14) consider the singular value decomposition $\tilde Z = \tilde G_2 L_2 \tilde H_2'$ with $L_2 = \operatorname{diag}(l_{q+1},\ldots,l_p)$. Then it is straightforward to show that (14) is equivalent to (21) by integrating (14) out with respect to G_1 and $\tilde G_2$. This proves the equivalence of the formal tube method based on the normal matrix Z and the (apparently very regular) Euler characteristic method based on the Wishart matrix W.

We now show that the formal tube method and the Euler characteristic method lead to incorrect asymptotic expansion in general by studying the simple case of $p=3$, $q=2$ and $n=4$ in detail.

The joint density of λ_i, $i=1,2,3$, is given by

$$\begin{aligned} f_n(\lambda_1,\lambda_2,\lambda_3) &= (1/d_n)\, e^{-(\lambda_1+\lambda_2+\lambda_3)/2}\,(\lambda_1\lambda_2\lambda_3)^{(n-4)/2}\,(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3), \\ &\qquad \lambda_1>\lambda_2>\lambda_3>0, \end{aligned}$$

where n is the degrees of freedom and

$$d_n = \frac{2^{3n/2}}{2\sqrt{\pi}}\,\Gamma(n/2)\Gamma((n-1)/2)\Gamma((n-2)/2)$$

(e.g., Chapter 13 of Anderson (1984)). For $n=4$ the density is

$$(1/16)\, e^{-(\lambda_1+\lambda_2+\lambda_3)/2}(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3).$$

Let $T^2 = \lambda_1+\lambda_2$. The exact tail probability in (8) simply leads to the following integration

$$P(T^2\ge t^2) = \int_{\substack{\lambda_1+\lambda_2\ge t^2\\ \lambda_1>\lambda_2>\lambda_3>0}} f(\lambda_1,\lambda_2,\lambda_3)\, d\lambda_1 d\lambda_2 d\lambda_3, \qquad (22)$$

whereas (9) is obtained by ignoring the range of λ_3 as

$$\tilde P(T^2\ge t^2) = \int_{\substack{\lambda_1+\lambda_2\ge t^2\\ \lambda_1>\lambda_2>0}} d\lambda_1 d\lambda_2 \int_0^\infty d\lambda_3\, f(\lambda_1,\lambda_2,\lambda_3). \qquad (23)$$

Using Mathematica the integration in (22) is evaluated as

$$P(T^2\ge t^2) = \Big(\frac{t^8}{128} - \frac{t^6}{16} + \frac{9t^4}{8} - \frac{15t^2}{2} + 33\Big)e^{-t^2/2} - 32\, e^{-3t^2/4},$$

where the main term is $t^8 e^{-t^2/2}$ corresponding to $\bar G_{m+1}(t^2) = \bar G_{10}(t^2)$. On the other hand (23) is evaluated as

$$\begin{aligned} \tilde P(T^2\ge t^2) &= \frac{1}{16}\int_{\substack{\lambda_1+\lambda_2\ge t^2\\ \lambda_1>\lambda_2>0}} e^{-(\lambda_1+\lambda_2)/2}(\lambda_1-\lambda_2)(2\lambda_1\lambda_2 - 4(\lambda_1+\lambda_2)+16) \\ &= \Big(\frac{t^8}{128} - \frac{t^6}{16} + \frac{t^4}{8} + \frac{t^2}{2} + 1\Big)e^{-t^2/2}. \end{aligned}$$

Note that $P(T \geq t)$ and $\tilde{P}(T \geq t)$ differ in the term of order $O(t^4 e^{-t^2/2})$. This corresponds to the case $k = 4$ in (11). This value of $k = 4$ can be confirmed by the argument of Appendix D. In our example

$$\dim M = \dim M(q) = q(n + p - q) - 1.$$

The main singularity comes from $M(q-1)$ with the dimension

$$\dim M(q-1) = (q-1)(n + p - q + 1) - 1.$$

Now

$$\dim M(q) - \dim M(q-1) = n + p - 2q + 1 = 4 + 3 - 2 \times 2 + 1 = 4,$$

which coincides with the value of $k = 4$. Note that the conditions of Appendix D is satisfied with $c = \sqrt{p-q}$ in (26), because $\mathrm{tr}(v'v) = 1$ in (12) is equivalent to $l_{q+1}^2 + \cdots + l_p^2 = 1$ and hence $\max(l_{q+1}, \ldots, l_p) \geq 1/\sqrt{p-q}$.

A Definition of locally conic set and related notions

Here we give precise definitions of various notions in Section 2.1. Throughout Section 2 we considered spherical tubes around $M \subset S^{n-1}$. In Appendix we prefer to consider $M \subset R^n$ and the volume of tubes in R^n for simplicity. Once the idea of a proof is clarified for the tubes in R^n, it is straightforward to adapt it to the spherical tube.

Let M be a closed subset of R^n. For each $x \in M$ we assume that M is locally approximated by a cone in the following definition.

Definition A.1 *A closed subset M of R^n is locally conic (of class $r = 2$) if for each $x \in M$ there exist an open neighborhood $U(x) \subset R^n$ of x, $\epsilon > 0$, closed cone K of R^n, and a C^2-diffeomorphism $\phi_x : (-\epsilon, \epsilon)^n \to U(x)$ with $\phi_x(0) = x$ such that $M \cap U(x)$ is the image of $K \cap (-\epsilon, \epsilon)^n$ by ϕ_x :*

$$M \cap U(x) = \phi_x(K \cap (-\epsilon, \epsilon)^n).$$

Furthermore if $V = U(x) \cap U(x') \neq \emptyset$ for $x, x' \in M$, then $\phi_{x'}^{-1} \circ \phi_x : \phi_x^{-1}(V) \to \phi_{x'}^{-1}(V)$ is a C^2-diffeomorphism.

In Definition A.1 we are following the standard definition of differentiable manifold. However M may not be a standard manifold because we allow self-intersections in M. Definition of the locally conic set is the same for M which is a subset of S^{n-1}.

For locally conic M we define the supporting cone and the normal cone at each $x \in M$ as follows. The support cone (or the tangent cone) of M at $x \in M$ is the image of K by the differential $d\phi$ at the origin:

$$S_x(M) + x = d\phi_{|(0,\ldots,0)} K. \tag{24}$$

Note that "+" on the left hand side of (24) is the vector sum and hence $S_x(M)$ is defined with its vertex located at the origin. Let $C(S_x(M))$ be the convex hull of $S_x(M)$. The normal cone $N_x(M)$ of M at x is the dual cone of $C(S_x(M))$ in R^n:

$$N_x(M) = \{y \mid y'v \leq 0,\ \forall v \in S_x(M)\} = \{y \mid y'v \leq 0,\ \forall v \in C(S_x(M))\}.$$

For $x \in M$ let

$$d = n - \dim N_x(M)$$

be the codimension of $N_x(M)$. Note that d is the dimension of the largest linear subspace contained in $C(S_x(M))$:

$$L = C(S_x(M)) \cap (-C(S_x(M))).$$

If L is contained in $S_x(M)$, then clearly L is the unique largest linear subspace contained in $S_x(M)$ and in this sense L is the tangent space $T_x(M)$ of M at x. On the other hand, if L is not contained in $S_x(M)$, then there are two non-nested linear subspaces contained in $S_x(M)$ and M does not possess a tangent space at x. In the tube formula the n-dimensional volume of the tube is obtained by integrating the product of the volume element of $N_x(M)$, the volume element of the tangent space $T_x(M)$ and the Jacobian containing the second fundamental form at x. This implies that if L is not contained in $S_x(M)$, then there should be no contribution to the volume of tube from x. This is the motivation for the definition of proper boundary in Section 2.1. For convenience we here give a formal definition of proper boundary.

Definition A.2 *Let M be locally conic and for $x \in M$ let $d = n - \dim N_x(M)$. x is a proper d-dimensional boundary point if $L = C(S_x(M)) \cap (-C(S_x(M)))$ is contained in $S_x(M)$.*

B Proofs of Propositions 2.1, 2.2 and Lemma 2.1

Here we give proofs for some results in Sections 2.1 and 2.2. Again we mainly consider versions of these results for tubes in R^n. For $x \in R^n$, let x_M denote the projection onto M with respect to the Euclidean distance and let $R(M)$ be defined by (3). For $x \in R(M)$ with $x_M \neq x$ consider the line segment joining x_M and x and let $u = ax_M + (1-a)x$, $0 < a < 1$, be an interior point of this line segment. We claim that $u \in R(M)$ and the projection of u coincides with x_M. Assume the contrary. Then there exists $\tilde{y} \neq x_M$, $\tilde{y} \in M$, such that

$$\|u - \tilde{y}\| \leq \|u - x_M\|.$$

By the triangular inequality

$$\begin{aligned} \|x - \tilde{y}\| &\leq \|x - u\| + \|u - \tilde{y}\| \\ &\leq \|x - u\| + \|u - x_M\| \\ &= \|x - x_M\|. \end{aligned}$$

However this contradicts the assumption that x_M is the unique projection of x onto M. Therefore u has unique projection x_M onto M.

Let $\|v\| = 1$ and let $l = \{y + rv \mid r \geq 0\}$ be the half line starting from $y \in M$ in the direction v. Then the above argument shows that l is divided into two intervals. The points on the first interval have unique projection y and the points on the other interval do not. More precisely define

$$\bar{r}(y, v) = \sup\{r \geq 0 \mid y + rv \in R(M),\ (y + rv)_M = y\}$$

then y is not a unique projection of $y+rv$ for $r > \bar{r}(y,v)$. Note that $\bar{r}(y,v) = 0$ corresponds to the case where no point other than y itself has y as the unique projection. $\bar{r}(y,v) = \infty$ corresponds to the case where all the points on the half line l has y as the unique projection, which is equivalent to

$$v'(x-y) \le 0, \qquad \forall x \in M. \tag{25}$$

Namely, M is entirely contained in one side of the hyperplane defined by the normal v.

For the case $0 < \bar{r}(y,v) < \infty$, we claim that $y + \bar{r}(y,v)v \notin R(M)$, hence the two intervals are $[0, \bar{r}(y,v))$, $[\bar{r}(y,v), \infty)$. This can be shown by proving the continuity of the map $x \mapsto x_M$ on the domain $R(M)$. Consider $x \in R(M)$ with $x_M \in M$. We show that

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \|\tilde{x} - x\| < \delta \Rightarrow \tilde{x} \in R(M),\ \|\tilde{x}_M - x_M\| < \epsilon.$$

Otherwise, there exists $\epsilon > 0$ and sequence of points $\{x_n\}$ such that $x_n \to x$ $(n \to \infty)$ and for each x_n one of its projections $y_n = (x_n)_M$ satisfies

$$\|y_n - x_M\| \ge \epsilon, \qquad n = 1, 2, \ldots$$

By considering a large enough closed ball centered at x, we can without loss of generality assume that $\{y_n\}$ lie in a compact region and have an accumulation point y_0 with $\|y_0 - x_M\| \ge \epsilon$. Then $\|x - y_0\| \le \|x - x_M\|$ and this contradicts the uniqueness of x_M. This argument proves that $x \mapsto x_M$ is a continuous map and also shows that $R(M)$ is an open set. From the openness of $R(M)$, it trivially follows that $y + \bar{r}(y,v)v \notin R(M)$. This proves the above claim. Recall that M is assumed to be locally conic. Therefore y is a projection of $x = y + rv$ for sufficiently small $r > 0$ if and only if $v \in N_y(K(M))$. Combining this observation with the above argument we have proved the following proposition.

Proposition B.1 *For $r \ge 0$, let*

$$[y, y+rv) = \begin{cases} \{y + tv \mid 0 \le t < r\}, & \text{if } r > 0, \\ \{y\}, & \text{if } r = 0. \end{cases}$$

For a locally conic closed set M

$$R(M) = \bigcup_{y \in M} \ \bigcup_{v \in N_y(M), \|v\|=1} [y, y + v\bar{r}(y,v)).$$

The proof of Proposition 2.1 is entirely the same as long as the geodesic distance is restricted to be less than π.

We proceed to prove a version of Proposition 2.2 for R^n. Let $x \notin R(M)$. Then there are at least two projections y_1, y_2 of x onto M. By Assumption 2.1 it suffices to consider the case where both y_1 and y_2 are proper boundary points of M. We need to distinguish two cases of non-uniqueness of projection. One case is that y is the "focal point" of y_1 or y_2 in the sense of p.33 of Milnor (1963). Corollary 6.2 of Milnor (1963) shows that the set of focal points is of Lebesgue measure zero. In the other case there exists neighborhoods $V(y_1)$, $V(y_2)$ of y_1 and y_2, respectively, such that y_i is the locally unique projection of x on $V(y_i)$. We can now repeat the same argument as above that there exists a neighborhood $U(x)$ of x such that projections $\pi_i : U(x) \to V(y_i)$ are continuous maps. Furthermore by introducing local coordinates it is easily shown that π_i, $i = 1, 2$, are of class C^2. Let

$$E(x) = \{z \in U(x) \mid \|z - \pi_1(z)\| = \|z - \pi_2(z)\|\}$$

then $E(x)$ is the set of points in $U(x)$ which is equidistant from $V(y_1)$ and $V(y_2)$. Let $g(z) = \|z - \pi_1(z)\|^2 - \|z - \pi_2(z)\|^2$. Then

$$\mathrm{grad}\, g = 2(z - \pi_1(z)) - 2(z - \pi_2(z)) = 2(\pi_2(z) - \pi_1(z)) \neq 0.$$

Therefore by implicit function theorem $E(x)$ is a $n-1$ dimensional submanifold of class C^2 in $U(x)$ and hence has Lebesgue measure zero. We have now shown that the set of x with two equidistant local unique projections $y_1, y_2 \in M$ has Lebesgue measure zero. We have proved a version of Proposition 2.2:

Proposition B.2 *For locally conic closed $M \subset R^n$ with piecewise smooth proper boundary, almost all $x \in R^n$ have unique projection onto M, i.e., the Lebesgue measure of the complement of $R(M)$ is zero.*

The proof of Proposition 2.2 is the same and omitted.

Finally we give an outline of proof of a version of Lemma 2.1 for R^n. Suppose that $x \in M$ has a non-convex support cone $S_x(M)$. It suffices to show that $\inf_{y \in U(x)} \bar{r}(y) = 0$, where $U(x)$ is a neighborhood of x and $\bar{r}(y) = \inf_{v \in N_y(M), \|v\|=1} \bar{r}(y, v)$. By taking $U(x)$ sufficiently small, the essential point of the proof is to show Lemma 2.1 for $M = K = S_x(M)$, which is a non-convex cone in R^n. Consider $y \in K \cap S^{n-1}$. Using (25) it can be easily shown that

$$\inf_{y \in K \cap S^{n-1}, v \in N_y(M), \|v\|=1} \bar{r}(y, v) = \infty$$

if and only if K is a convex cone. Since K is assumed to be non-convex, there exists $y \in K \cap S^{n-1}$ and $v \in N_y(M)$ such that $\bar{r}(y, v) < \infty$. By the proof of Proposition B.1 $x = y + \bar{r}(y, v)v$ has at least two equidistant projection onto M. Because of the scale invariance of the geometry of the cone, $\epsilon x = \epsilon y + \epsilon \bar{r}(y, v)v = \epsilon y + \bar{r}(\epsilon y, v)v$ has the same property for every $\epsilon > 0$. Therefore $\lim_{\epsilon \downarrow 0} \bar{r}(\epsilon y, v) = 0$ and this proves that the critical radius of M is zero.

C Exact and formal tube formula in R^n

Let M be a compact locally conic set with piecewise smooth proper boundary. We derive the exact and formal tube formula for the tube

$$M_r = \{x \mid \mathrm{dist}(x, M) < r\}, \qquad \mathrm{dist}(x, M) = \min_{y \in M} \|x - y\|,$$

around M. Let $x \in M$ be a proper boundary point and let $H(x, v)$ denote the second fundamental form at x with respect to the direction $v \in N_x(M)$, $\|v\| = 1$. Note that $I + rH(x, v)$ is positive definite for $r < \bar{r}(x, v)$. By the standard derivation of tube formula for compact M, the exact volume of the tube M_r is given as

$$\begin{aligned} V(M_r) &= V\Big(\bigcup_{x \in M - I(M)} \bigcup_{v \in N_x(M), \|v\|=1} [x, x + \min(r, \bar{r}(x, v))v) \Big) \\ &= \sum_{d=0}^{m} \int_{\partial M_d} dx \int_{v \in N_x(M) \cap S^{n-1}} dv \int_0^{\min(r, \bar{r}(x,v))} dt \, \det(I + tH(x, v)) \\ &= \sum_{d=0}^{m} \int_{\partial M_d} dx \int_{v \in N_x(M) \cap S^{n-1}} dv \sum_{j=0}^{d} \mathrm{tr}_j H(x, v) \int_0^{\min(r, \bar{r}(x,v))} t^j \, dt \\ &= \sum_{d=0}^{m} \int_{\partial M_d} dx \int_{v \in N_x(M) \cap S^{n-1}} dv \sum_{j=0}^{d} \frac{\mathrm{tr}_j H(x, v)}{j+1} \min(r, \bar{r}(x, v))^{j+1}. \end{aligned}$$

The formal tube formula is obtained by putting $\bar{r}(x,v)=\infty$:

$$\tilde{V}(M_r)=\sum_{d=0}^{m}\int_{\partial M_d}dx\int_{v\in N_x(M)\cap S^{n-1}}dv\sum_{j=0}^{d}\frac{\mathrm{tr}_j\,H(x,v)}{j+1}r^{j+1}.$$

D Characterization of k of (11)

Here we present an argument that in certain regular cases k of (11) is simply the difference of $m=\dim M$ and the dimension of the set of points with non-convex support cone.

As the first technical assumption we require that there exists $c\geq 1$ such that on ∂M_m

$$\liminf_{t\to 0}\ \inf_{x:\,\tan\bar{\theta}(x)\leq 1/(ct)}\int_{v\in N_x(K(M))\cap S^{n-1},\,\tan\bar{\theta}(x,v)\leq 1/t}dv>0. \tag{26}$$

This condition implies that for sufficiently small t the angle of $N_x(K(M))\cap\{v\mid\bar{\theta}(x,v)\leq 1/t\}$ is bounded away from 0 for all $x\in\partial M_m$ with $\bar{\theta}(x)\leq 1/(ct)$. The constant c is needed for the example of Section 3.2. Now for $c\geq 1$

$$\int_{x:\,\tan\bar{\theta}(x)\leq 1/(ct)}dx\int_{v\in N_x(K(M))\cap S^{n-1},\,\tan\bar{\theta}(x,v)\leq 1/t}dv\leq\int_{A(t)}dxdv\leq\Omega_{n-m}\int_{x:\,\tan\bar{\theta}(x)\leq 1/t}dx.$$

Therefore under the assumption (26)

$$\int_{A(t)}dxdv=O(t^{-k})\quad\Leftrightarrow\quad\int_{x:\,\tan\bar{\theta}(x)\leq 1/t}dx=O(t^{-k})$$

and k can be evaluated from the volume of the set $\{x\in\partial M_m\mid\tan\bar{\theta}(x)\leq 1/t\}$.

Let $\bar{M}$ denote that the set of points on the relative boundary of ∂M_m with non-convex support cone. We now make the second assumption that $\bar{M}$ forms a C^2-submanifold of R^n of dimension l. Finally we assume that for $x\in\partial M_m$, $\bar{\theta}(x)=O(1/t)$ if and only if $\mathrm{dist}(x,\bar{M})=O(1/t)$. Under these assumptions the set $\{x\in\partial M_m\mid\bar{\theta}(x)\leq 1/t\}$ is basically a tube around $\bar{M}$ in ∂M of radius $O(1/t)$. Therefore the m-dimensional volume of this tube is proportional to $O(t^{-k})$ with

$$k=m-l=\dim M-\dim\bar{M}.$$

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