CIRJE-F-54

Maximum of Gaussian field on piecewise smooth domain:

Equivalence of tube method and Euler characteristic method

Akimichi Takemura
University of Tokyo

Satoshi Kuriki
The Institute of Statistical Mathematics

July 1999

Maximum of Gaussian field on piecewise smooth domain: Equivalence of tube method and Euler characteristic method

Akimichi Takemura
Faculty of Economics, University of Tokyo
and
Satoshi Kuriki
The Institute of Statistical Mathematics

July, 1999

Abstract

Consider a Gaussian random field with finite Karhunen-Loève expansion of the form $Z(u) = \sum_{i=1}^{n} u_i z_i$, where z_i, $i = 1, \ldots, n$, are independent standard normal variables and $u = (u_1, \ldots, u_n)'$ ranges over an index set M, which is a subset of the unit sphere S^{n-1} in R^n. Under a very general assumption that M is a manifold with piecewise smooth boundary, we prove the validity and the equivalence of two currently available methods for obtaining the asymptotic expansion of tail probability of the maximum of $Z(u)$. One is the tube method, where the volume of tube around the index set M is evaluated. The other is the Euler characteristic method, where the expectation for the Euler characteristic of excursion set is evaluated. In order to show this equivalence we prove a version of the Morse's theorem for a manifold with piecewise smooth boundary. These results on the tail probabilities are generalizations of those of Takemura and Kuriki (1997), where M was assumed to be convex.

Key words: chi-bar-squared distribution, Gauss-Bonnet theorem, Karhunen-Loève expansion, kinematic fundamental formula, manifold with boundary, Morse function.

1 Introduction

Let M be a closed subset of the unit sphere S^{n-1} in R^n. We consider a random field $\{Z(u),\ u=(u_1,\ldots,u_n)'\in M\}$ defined by

$$Z(u)=u'z=\sum_{i=1}^{n}u_iz_i, \tag{1.1}$$

where $z=(z_1,\ldots,z_n)'$ is distributed according to the n-dimensional standard multivariate normal distribution $N_n(0,I_n)$. The covariance function is given by

$$r(u,v)=E[Z(u)Z(v)]=u'v.$$

The variance of $Z(u)$ is $r(u,u)=\|u\|^2=1$ since $u\in S^{n-1}$. (1.1) is the canonical form of Gaussian random field with finite Karhunen-Loève expansion and constant variance. In this paper we study the asymptotic behavior of the upper tail probability

$$P\Big(\max_{u\in M}Z(u)\geq x\Big) \tag{1.2}$$

as x goes to infinity.

As a related random field to (1.1) we define

$$Y(u)=u'y=\sum_{i=1}^{n}u_iy_i, \tag{1.3}$$

where $y=(y_1,\ldots,y_n)'=z/\|z\|$ is distributed according to the uniform distribution $\mathrm{Unif}(S^{n-1})$ on the unit sphere S^{n-1}. We also study the upper tail probability

$$P\Big(\max_{u\in M}Y(u)\geq x\Big). \tag{1.4}$$

Once formulated in the canonical form (1.1), the upper tail probabilities (1.2) and (1.4) depend on the geometry of the index set M. Although in our setting we are restricted to random fields with finite Karhunen-Loève expansion, we want to consider a class of index sets M which is as general as possible. This class should include polyhedral regions, (geodesically) convex regions, and manifolds with or without boundaries. In our previous works we studied convex regions in Takemura and Kuriki (1997) and manifolds without boundary in Kuriki and Takemura (1998). Unifying these cases we make the following assumption on M:

Assumption 1.1 *M is a compact m-dimensional C^2-manifold with piecewise smooth boundary in the sense of Definition A.1.*

A manifold M with piecewise smooth boundary can be approximated locally by the support cone (tangent cone) $S_u(M)$ at any point $u\in M$. See Appendix A.1 for the precise definition of support cone. In addition to Assumption 1.1 we make another assumption:

Assumption 1.2 *At each point $u \in M$, the support cone $S_u(M)$ of M is convex.*

Roughly speaking Assumption 1.2 means that M is locally convex. This assumption is essential for the validity of asymptotic expansion of the upper tail probability (1.2). In our subsequent work we will discuss in detail that the tube method as well as the Euler characteristic method leads to incorrect asymptotic expansion when Assumption 1.2 fails.

In order to derive asymptotic expansion of the upper tail probability (1.2) for $Z(u)$, two methods are currently available. One is the "tube method" developed by Sun (1993). She showed that given an expression for the upper probability (1.4) for $Y(u)$ valid for $x \in [x_c, 1]$ ($x_c < 1$ is a constant), the asymptotic expansion of the upper probability (1.2) for $Z(u)$ is obtained automatically from the expression for (1.4). As will be explained in Section 2, the upper probability for $Y(u)$ is exactly the ratio of volume of tube (tubular neighborhood) around M to the volume of the unit sphere S^{n-1}. Therefore the problem is reduced to obtaining the formula for volume of tube (tube formula). The tube formula for a manifold of general dimension without boundary was obtained by Weyl (1939). For a manifold with piecewise smooth boundary, the tube formula for $\dim M = 1$ was given in Hotelling (1939) and for $\dim M = 2$ it was given in Knowles and Siegmund (1989). When M is a geodesically convex domain with piecewise smooth boundary, Takemura and Kuriki (1997) gave a formula which is essentially equivalent to the tube formula. In this paper we present the tube formula for a manifold with piecewise smooth boundary of general dimension under Assumption 1.2 of local convexity.

The other method for obtaining the asymptotic expansion of the tail probability (1.2) is the "Euler characteristic method" developed by Adler (1981) and Worsley (1995a, b). As we will see in Section 3, the Euler characteristic method is applicable in principle to any random fields. However, differently from the tube method, the Euler characteristic method is a heuristic approach and its validity in general setting has not been proved. Recently, Adler (1998) showed that the Euler characteristic method for isotropic Gaussian random fields on piecewise smooth domain gives the valid asymptotic expansion using the results by Piterbarg (1996). In this paper in the case where the Gaussian field is of the form (1.1) but not assumed isotropic, we give a proof that the Euler characteristic method is equivalent to the tube method and hence gives a valid asymptotic expansion. In order to show this equivalence we prepare a generalized version of the Morse's theorem for manifold with piecewise smooth boundary. Moreover our geometric consideration gives us an alternative proof of Naiman's inequality (Naiman (1986), Johnstone and Siegmund (1989)).

The outline of this paper is as follows. In Section 2 we define the tube on the sphere, and give a tube formula for manifold with piecewise smooth boundary. We also discuss how to calculate the critical radius of tube, which is essential for determining the order of the remainder term of the asymptotic expansion. In Section 3 we explain the Euler characteristic method for the Gaussian random field (1.1). Then the equivalence of the tube method and the Euler characteristic method is proved using a generalized version of the Morse's theorem. Furthermore we will give an alternative simplified proof of Naiman's inequality. In Section 4 as an example we discuss the distribution of the maximum of cosine field treated in Piterbarg (1996). In Appendix we summarize geometric preliminaries about manifold with piecewise smooth boundary. A generalized version of the Morse's theorem is also given there.

2 Tube method

In this section we derive tube formula for the tube around piecewise smooth $M \subset S^{n-1}$ and derive asymptotic expansion of probabilities (1.2) and (1.4) based on the tube formula.

2.1 Tube and its critical radius

Let

$$M_\theta = \{y \in S^{n-1} \mid u'y \ge \cos\theta,\ \exists u \in M\}.$$

Since y in (1.4) is distributed uniformly on S^{n-1}, the probability (1.4) for $x = \cos\theta$ is written as

$$P\Big(\max_{u \in M} Y(u) \ge \cos\theta\Big) = \frac{1}{\Omega_n} \operatorname{Vol}(M_\theta),$$

where $\operatorname{Vol}(\cdot)$ denotes the spherical volume on S^{n-1} and

$$\Omega_n = \operatorname{Vol}(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

is the total volume of S^{n-1}.

Let

$$\operatorname{dist}(u,v) = \cos^{-1}(u'v) \in [0,\pi], \quad u,v \in S^{n-1},$$

be the distance on the unit sphere S^{n-1}, and let

$$\operatorname{dist}(u,M) = \min_{v \in M} \operatorname{dist}(u,v).$$

Then the set M_θ can be written as

$$M_\theta = \{y \in S^{n-1} \mid \operatorname{dist}(y,M) \le \theta\},$$

i.e., M_θ is the set of points whose distance from M are less than or equal to θ. We call M_θ spherical *tube* around M with radius θ. Therefore the evaluation of the tail probability (1.4) is reduced to the evaluation of the volume of tube M_θ.

Since M is closed, for each $y \in S^{n-1}$ there exists a closest point y_M in M such that

$$\operatorname{dist}(y, y_M) = \operatorname{dist}(y, M).$$

Although y_M might not be unique, the distance $\operatorname{dist}(y,M)$ is uniquely determined.

Define a subset of S^{n-1} by

$$C_u(\theta) = M_\theta \cap (u + N_u(M)), \quad u \in M,$$

where $N_u(M)$ is the normal cone at u, which is the dual cone of $S_u(M)$ in R^n. $C_u(\theta)$ is the cross section of M_θ crossing M at $u \in M$ and consists of points $y \in M_\theta$ such that $u = y_M \in M$. Since each $y \in M_\theta$ belongs to $C_{y_M}(\theta)$, M_θ can be written as the union of cross sections:

$$M_\theta = \bigcup_{u \in M} C_u(\theta).$$

By the Assumption 1.2 at each point $u \in M$, M is locally approximated by the convex support cone $S_u(M)$. Because of this, for each y sufficiently close to M the point y_M is uniquely defined. From the compactness of M it can be shown that there exists $\theta > 0$ such that for every $y \in M_\theta$ the point y_M is unique. The supremum θ_c of such θ is called *critical radius* of M:

$$\theta_c = \theta_c(M) = \sup\{\theta > 0 \mid y_M \text{ unique for all } y \in M_\theta\}.$$

It is easily shown that θ_c can also be defined by

$$\theta_c = \sup\{\theta > 0 \mid C_u(\theta),\ u \in M, \text{ are disjoint}\}.$$

Let

$$K = K(M) = \bigcup_{c \geq 0} cM$$

denote the smallest cone containing M. The critical radius can be computed using the following formula.

Lemma 2.1

$$\inf_{u,v \in M} \frac{\|u-v\|^2}{2\|P_v^\perp(u-v)\|} = \begin{cases} \tan\theta_c & (\theta_c < \pi/2) \\ \infty & (\theta_c \geq \pi/2), \end{cases} \tag{2.1}$$

where $P_v^\perp$ is the orthogonal projection in R^n onto the normal cone $N_v(K)$ of K at v.

For the case of one-dimensional smooth manifold this result is given in Proposition 4.3 of Johansen and Johnstone (1990). Extension to smooth manifolds of higher dimension is stated in Lemma A.1 of Kuriki and Takemura (1998). We omit the proof of Lemma 2.1, since it is essentially the same as the proof given by Johansen and Johnstone (1990).

It can be proved that $\theta_c \geq \pi/2$ if and only if K ($\neq R^n$) is convex. If M is a geodesically convex region on S^n, then the critical radius $\theta_c(M)$ may be greater than $\pi/2$. In this case the denominator of the left hand side of (2.1) is 0 and (2.1) does not give the critical radius.

2.2 Tubal coordinates and Jacobian

Fix a relative interior point y of $M_{\theta_c} \cap M^C$. We introduce here the tubal coordinates of M_{θ_c} around y.

Suppose that y_M is a relative interior point of a component of d-dimensional boundary ∂M_d of M. Let $u = y_M$, $\theta = \cos^{-1}(u'y)$. If $\theta \le \pi/2$, then $u\cos\theta$ is the orthogonal projection in R^n of y onto $K = K(M)$. Put

$$v = \frac{y - u\cos\theta}{\|y - u\cos\theta\|} = \frac{y - u\cos\theta}{\sin\theta} \in S^{n-1}.$$

Considering the two-dimensional plane spanned by y and u we see that y is uniquely written as

$$y = u\cos\theta + v\sin\theta, \quad 0 < \theta < \theta_c, \quad u \in \partial M_d, \quad v \in N_u(K) \cap S^{n-1}.$$

We call the coordinates (θ, u, v) tubal coordinates.

The Jacobian of the transformation $y \leftrightarrow (\theta, u, v)$ is given as follows. The second fundamental form $H(u, v)$ of M at u in the direction v is defined as in Section 2.3 of Takemura and Kuriki (1997).

Lemma 2.2 *Let* $\mathrm{d}y$ *be the volume element of* M_{θ_c} *(or* S^{n-1}*) at* y*,* $\mathrm{d}u$ *be the volume element of* ∂M_d *at* $u = y_M$*,* $\mathrm{d}v$ *be the volume element of* $S^{n-d-2} = N_u(K(M)) \cap S^{n-1}$ *at* v*. Then*

$$\mathrm{d}y = \det(\cos\theta I_d + \sin\theta H(u, v))\ \sin^{n-d-2}\theta\, \mathrm{d}\theta\, \mathrm{d}u\, \mathrm{d}v. \tag{2.2}$$

Proof. Introduce a parameter $t > 0$. Let $z = ty \in R^n$ and put $r = t\cos\theta$, $s = t\sin\theta$. Then $z = ru + sv$, which gives a one-to-one correspondence between z and (r, s, u, v). The Jacobian of this transformation is essentially given by Weyl (1939) as

$$\mathrm{d}z = \det(rI_d + sH(u, v))\, s^{n-d-2}\, \mathrm{d}r\, \mathrm{d}s\, \mathrm{d}u\, \mathrm{d}v. \tag{2.3}$$

See Appendix A of Kuriki and Takemura (1997) for a proof. Here note that the Lebesgue measure $\mathrm{d}z$ at z is decomposed as

$$\mathrm{d}z = t^{n-1}\mathrm{d}t\, \mathrm{d}y, \tag{2.4}$$

where $\mathrm{d}y$ is the volume element of S^{n-1} at $y = z/\|z\|$. Note also that

$$\mathrm{d}r\, \mathrm{d}s = t\mathrm{d}t\, \mathrm{d}\theta. \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.3), and comparing the coefficients of $t^{n-1}\mathrm{d}t$, we have the lemma. ∎

Note that for $\theta < \theta_c$ the determinant in (2.2) is nonnegative.

2.3 Tube formula and tail probabilities

Here we present the tube formula for the spherical volume of tube around M. The tube formula of this section unifies the tube formula in the sense of Weyl (1939) and the Steiner formula for the convex sets discussed in Takemura and Kuriki (1997).

Let $u \in \partial M_d$ and let $v \in N_u(K(M))$, $\|v\| = 1$. The l-th symmetric function of the principal curvatures of M, i.e., the eigenvalues of the second fundamental form $H(u,v)$, is denoted by $\mathrm{tr}_l H(u,v)$. The tube formula $\mathrm{Vol}(M_\theta)$ for M_θ is given as follows.

Proposition 2.1 *For $e = 0, \ldots, m$, let*

$$w_{m+1-e} = \frac{1}{\Omega_{m+1-e}\Omega_{n-m-1+e}} \sum_{d=m-e}^{m} \int_{\partial M_d} \mathrm{d}u \int_{N_u(K(M))\cap S^{n-1}} \mathrm{d}v\, \mathrm{tr}_{d-m+e} H(u,v), \qquad (2.6)$$

where for each $0 \le d \le m$, $\mathrm{d}u$ and $\mathrm{d}v$ are the volume elements defined in Lemma 2.2. Then for $\theta \le \theta_c(M)$ the spherical volume of M_θ is given by

$$\mathrm{Vol}(M_\theta) = \begin{cases} \Omega_n \sum_{e=0}^{m} w_{m+1-e}\Big(1 - B_{\frac{1}{2}(m+1-e),\frac{1}{2}(n-m-1+e)}(\cos^2\theta)\Big) & (0 \le \theta \le \pi/2) \\ \Omega_n \sum_{e=0}^{m} w_{m+1-e}\Big(1 + (-1)^{m-e} B_{\frac{1}{2}(m+1-e),\frac{1}{2}(n-m-1+e)}(\cos^2\theta)\Big) & (\pi/2 < \theta < \theta_c), \end{cases} \qquad (2.7)$$

where $B_{a,b}(\cdot)$ denotes the cumulative distribution function of beta distribution with parameter (a,b).

Proof. By virtue of the Jacobian given in Lemma 2.2, the spherical volume of $M_{\theta'}$ for $\theta' \le \theta_c$ is given by

$$\mathrm{Vol}(M_{\theta'}) = \sum_{d=0}^{m} \int_0^{\theta'} \mathrm{d}\theta \int_{\partial M_d} \mathrm{d}u \int_{N_u(K)\cap S^{n-1}} \mathrm{d}v\, \det(\cos\theta I_d + \sin\theta H(u,v))\, \sin^{n-d-2}\theta.$$

Using the expansion formula for the determinant $\det(I_d + A) = \sum_{l=0}^{d} \mathrm{tr}_l A$, we obtain the result by straightforward integration. ∎

Corollary 2.1 *For $x \ge \cos\theta_c(M)$,*

$$P\Big(\max_{u\in M} Y(u) \ge x\Big) = \begin{cases} \sum_{e=0}^{m} w_{m+1-e}\Big(1 - B_{\frac{1}{2}(m+1-e),\frac{1}{2}(n-m-1+e)}(x^2)\Big) & (0 \le x \le 1) \\ \sum_{e=0}^{m} w_{m+1-e}\Big(1 + (-1)^{m-e} B_{\frac{1}{2}(m+1-e),\frac{1}{2}(n-m-1+e)}(x^2)\Big) & (\cos\theta_c < x < 0), \end{cases} \qquad (2.8)$$

where w_{m+1-e} is given in (2.6).

Note that in (2.7) and (2.8) the second cases are needed only when $\theta_c > \pi/2$.

Now consider the maximum of $Z(u)$. Let $G_k(\cdot)$ and $g_k(\cdot)$ denote the cumulative distribution function and the density function of χ^2 distribution with k degrees of freedom, respectively. Using the techniques of Sun (1993) and Kuriki and Takemura (1998), we obtain the following result by the tube method:

Proposition 2.2 *If $\theta_c < \pi/2$, then as $x \to \infty$*

$$P\Big(\max_{u\in M} Z(u) \geq x\Big) = \sum_{e=0}^{m} w_{m+1-e}(1 - G_{m+1-e}(x^2)) + O(g_n(x^2(1+\tan^2\theta_c))). \qquad (2.9)$$

If $\theta_c \geq \pi/2$, then for each $x \geq 0$

$$P\Big(\max_{u\in M} Z(u) \geq x\Big) = \sum_{e=0}^{m} w_{m+1-e}(1 - G_{m+1-e}(x^2)). \qquad (2.10)$$

Here w_{m+1-e} is given in (2.6).

Note that the remainder term in (2.9) is of the order of $o(1 - G_1(x^2))$.

Remark 2.1 *When $\theta_c(M) \geq \pi/2$, all of the coefficients w_{m+1-e} in (2.10) are nonnegative since $K(M)$ is convex and hence the second fundamental form $H(u,v)$ in (2.6) is nonnegative definite. This distribution is a finite mixture of χ^2 distributions referred to as $\bar{\chi}^2$-distribution (e.g., Shapiro (1988)).*

3 Euler characteristic method

In order to approximate tail probabilities of random fields such as (1.2) or (1.4), Adler (1981) and Worsley (1995a, b) have developed a technique based on the Euler characteristic of excursion set. In this paper we call their method *Euler characteristic method*. We begin by a brief elucidation of the idea of their method in the case of $Z(u)$ in (1.1).

3.1 Excursion set and its expectation

The excursion set is a subset of the index set M consisting of u for which (a realization of) the random field $Z(u) = u'z$ is greater than or equal to a threshold. That is,

$$A(z,x) = \{u \in M \mid u'z \geq x\}$$

is the excursion set for $Z(u) = u'z$. It holds by definition that

$$P\Big(\max_{u\in M} Z(u) \geq x\Big) = P(A(z,x) \neq \emptyset).$$

Let $\chi(A(z,x))$ denote the Euler characteristic (Euler-Poincaré characteristic) of the excursion set $A(z,x)$. The Euler characteristic method approximates the tail probability (1.2) for large x by

$$P\Big(\max_{u\in M} Z(u) \ge x\Big) \approx E[\chi(A(z,x))]. \tag{3.1}$$

A rationale for the approximation (3.1) is as follows. The Euler characteristic is an integer-valued topological invariant. In particular it takes the values

$$\chi(A(z,x)) = \begin{cases} 1 & (A(z,x) \text{ is homotopy equivalent to a point}) \\ 0 & (A(z,x) \text{ is empty}). \end{cases}$$

Suppose that the threshold x is large. If $\max_{u\in M} u'z < x$, then $A(z,x) = \emptyset$. Now consider the case $\max_{u\in M} u'z > x$. Note that the maximizing point u^*, i.e., $\max_{u\in M} u'z = (u^*)'z$, is uniquely determined with probability one. Therefore given $\max_{u\in M} u'z > x$, with a conditional probability nearly equal to 1, $A(z,x)$ will be some neighborhood of u^*, which is homotopic to a point set $\{u^*\}$.

Summarizing the discussions above, it is expected that for large x

$$I(A(z,x) \ne \emptyset) \approx \chi(A(z,x)) \quad \text{(with a probability nearly equal to 1)}, \tag{3.2}$$

where $I(\cdot)$ is the indicator function. By taking the expectation for (3.2), we have $P(A(z,x) \ne \emptyset) \approx E[\chi(A(z,x))]$, and (3.1) follows.

Differently from the tube method in Section 2, the Euler characteristic method is applicable to any random field. However this method as described above is heuristic; The meaning of the symbol "$\approx$" in (3.1) has to be examined in each case. Recently, Adler (1998), Theorem 4.5.2, showed that in the case of isotropic Gaussian random field the Euler characteristic method gives the valid asymptotic expansion for (1.2) as x goes to infinity under mild regularity conditions. Adler (1998) proved this by checking that all terms of expansions are the same as a formula obtained earlier by Piterbarg (1996), Theorem 5.1. (See §2.5 of Adler (1981) for the definition of isotropic field.) In the following subsection we will prove that the Euler characteristic method for the Gaussian random field $Z(u)$ in (1.1) is reduced to the tube method of Section 2. This implies that the Euler characteristic method is valid for the case of $Z(u)$ in (1.1).

3.2 Equivalence to the tube method

We prove the equivalence of the tube method and the Euler characteristic method first for $Y(u)$ in (1.3) and then for $Z(u)$ in (1.1). Let

$$A(y,x) = \{u \in S^{n-1} \mid u'y \ge x\} \cap M$$

be the excursion set of the random field $Y(u) = u'y$, $y \sim \mathrm{Unif}(S^{n-1})$. In order to evaluate the expectation of the Euler characteristic of $A(y,x)$ we need a version of the Morse theorem, which is given in Appendix A.3. The following result together with Proposition 2.1 establishes the equivalence of two methods for $Y(u)$ in (1.3).

Proposition 3.1 *Let y be distributed uniformly on S^{n-1}. Then*

$$
\begin{aligned}
&E[\chi(A(y,x))] \\
&= \int_{S^{n-1}} \chi(A(y,x))\mathrm{d}y/\Omega_n \\
&= \begin{cases} \sum_{e=0}^{m} w_{m+1-e}\Bigl(1 - B_{\frac{1}{2}(m+1-e),\frac{1}{2}(n-m-1+e)}(x^2)\Bigr) & (0 \le x \le 1) \\ \sum_{e=0}^{m} w_{m+1-e}\Bigl(1 + (-1)^{m-e} B_{\frac{1}{2}(m+1-e),\frac{1}{2}(n-m-1+e)}(x^2)\Bigr) & (-1 \le x < 0), \end{cases}
\end{aligned}
\tag{3.3}
$$

where $\mathrm{d}y$ *denotes the volume element of* S^{n-1} *and* w_{m+1-e} *is given in (2.6).*

Proof. Let $y \in S^{n-1}$. The key idea of the proof is to consider

$$f_y(u) = -u'y$$

as a Morse function. Using the same line of argument as Theorem 6.6 of Milnor (1963), we see that $f_{y|M}$ is a Morse function on M in the sense of Appendix A.3 for almost all y. Since the gradient of $f_y(u)$, $u \in R^n$, is $-y$, the gradient of $f_{y|M}$ is given by the $T_u(M)$ component of $-y$. Using this fact it is easily shown that $u \in \partial M$ is an extended inward critical point of $f_{y|M}$ if and only if $y \in N_u(M)$.

We now consider $\chi(A(y,x))$ using $f_{y|A(y,x)}$. If u is on the relative boundary of $A(y,x)$, then either $-u'y = -x$ or $u \in \partial M$. Suppose that u_0 with $-u_0'y = -x$ is a critical point of $f_{y|A(y,x)}$. Because u_0 is an inner point of some relative neighborhood $\tilde{M}$ of M and $-u'y$ is increasing as we leave $A(y,x)$ outward at u_0, the gradient of $f_{y|A(y,x)}$ is directed outward on u_0. Hence u_0 is not counted in the Euler characteristic $\chi(A(y,x))$. On the other hand suppose that $u_0 \in \partial M$, $-u'y < -x$, is a critical point of $f_{y|A(y,x)}$. This u_0 is counted in $\chi(A(y,x))$ exactly as it is counted in $\chi(M)$. Also note that if $u_0 \in M^O$, $-u_0'y < -x$, is a critical point of $f_{y|A(y,x)}$, it is counted in $\chi(A(y,x))$ exactly as it is counted in $\chi(M)$. We see that the Euler characteristic $\chi(A(y,x))$ is written as Proposition A.4, where augmented type numbers are obtained by counting critical points u of $f_{y|M}$ with $-u'y < -x$.

Consider the index of $f_{y|\partial M_d}$ at the critical point $u \in \partial M_d$, $y \in N_u(M)$. Let $H(u,y)$ denote the second fundamental form of ∂M_d at u with respect to the vector y. Then by the same line of argument as stated on page 36 of Milnor (1963), the Hessian matrix of $f_{y|\partial M_d}$ at u is given by $(u'y)I_d + H(u,y)$ and hence the index of the critical point u is the number of negative characteristic roots of $(u'y)I_d + H(u,y)$. In the tubal coordinates, this matrix is written as

$$\cos\theta I_d + H(u, u\cos\theta + v\sin\theta) = \cos\theta I_d + \sin\theta H(u,v),$$

where $\theta = \cos^{-1}(u'y)$ and $v = (y - u\cos\theta)/\sin\theta$. It follows that u is counted in $\chi(M)$ or $\chi(A(y,x))$ with the sign $\operatorname{sgn}\det(\cos\theta I_d + \sin\theta H(u,v))$. That is, we have

$$\chi(A(y,x)) = \sum_{u\,:\,y\in N_u(M)} I(\cos\theta > x)\operatorname{sgn}\det(\cos\theta I_d + \sin\theta H(u,v)) \quad \text{a.s.}$$

By Lemma 2.2 the Jacobian of the correspondence between the volume element of S^{n-1} and tubal coordinates (in the sense of unsigned measures) is written as

$$dy = |\det(\cos\theta I_d + \sin\theta H(u,v))| \sin^{n-d-2}\theta \, d\theta \, du \, dv,$$

where $|\cdot|$ is the absolute value. (Although Lemma 2.2 treats only the case $y \in M_{\theta_c}$ and $u = y_M$, it can be extended to the case $y \in S^{n-1}$ and $u \in M$ such that $y \in N_u(M)$ by taking the absolute value of determinant.) Since

$$\operatorname{sgn}\det(\cos\theta I_d + \sin\theta H(u,v)) \times |\det(\cos\theta I_d + \sin\theta H(u,v))| = \det(\cos\theta I_d + \sin\theta H(u,v)),$$

we have

$$\int_{S^{n-1}} \chi(A(y,x)) \, dy = \sum_{d=0}^{m} \int_0^{\cos^{-1}(x)} d\theta \int_{\partial M_d} du \int_{N_u(K(M))\cap S^{n-1}} dv \det(\cos\theta I_d + \sin\theta H(u,v)) \sin^{n-d-2}\theta.$$

As in the proof of Proposition 2.1 this yields (3.3). ∎

Remark 3.1 *As stated in the proof of Proposition 3.1, $\chi(A(y,\cos\theta))$ is the degree of many-valued map $y \in M_\theta \mapsto u \in M$ such that $y \in N_u(M)$ and the orientation of $N_u(M)$ is taken into account. In this sense the integral of the Euler characteristic $\int_{S^{n-1}} A(y,\cos\theta)\,dy$ for θ greater than the critical radius $\theta_c(M)$ can be regarded as the signed volume of tube.*

Remark 3.2 *Let D_0 and D_1 be a pair of domains of S^{n-1}. Suppose D_0 fixed and D_1 moving. Let dK_1 denote the kinematic density of D_1, i.e., an invariant measure for the group of motion in S^{n-1}. The integral of this type*

$$\int_{D_0\cap D_1\neq\emptyset} \chi(D_0 \cap D_1) \, dK_1$$

is called kinematic fundamental formula. The formula when both ∂D_0 and ∂D_0 are smooth (of class C^2) is given in IV.18.3 of Santaló (1976). Our Proposition 3.1 is a version of kinematic fundamental formula for $D_0 = M$, $D_1 = \{u \in S^{n-1} \mid u'y \geq x\}$ but $\partial D_0 = \partial M$ is not necessarily smooth.

It is now easy to translate the above equivalence of two methods for $Y(u)$ to the equivalence for $Z(u)$. The expectation of the Euler characteristic for the excursion set

$$A(z,x) = \{u \in S^{n-1} \mid u'z \geq x\} \cap M$$

of $Z(u) = u'z$ is given in the following proposition.

Proposition 3.2 *Let z be distributed according to the standard multivariate normal distribution $N_n(0, I_n)$. Then*

$$E[\chi(A(z,x))] = \begin{cases} \sum_{e=0}^{m} w_{m+1-e}\Big(1 - G_{m+1-e}(x^2)\Big) & (x \geq 0) \\ \sum_{e=0}^{m} w_{m+1-e}\Big(1 + (-1)^{m-e} G_{m+1-e}(x^2)\Big) & (x < 0). \end{cases}$$

Proof. Note that $A(z, x) = A(y, x/\|z\|)$ with $y = z/\|z\|$. Since y and $\|z\|$ are independent, the expectation $E[A(z, x)]$ can be calculated by substituting $x^2 := x^2/\|z\|^2$ in (3.3) and taking expectation with respect to $\|z\|^2 \sim \chi^2(n)$. ∎

The above proposition and Proposition 2.2 show that the asymptotic expansion obtained by the tube method and the Euler characteristic method are the same.

Now consider the special case of $x = -1$ in (3.3). Noting that $A(y, -1) = M$ we have the following corollary.

Corollary 3.1

$$\chi(M) = 2 \sum_{\substack{e=0 \\ m-e:\text{even}}}^{m} w_{m+1-e} = \begin{cases} 2(w_1 + w_3 + \cdots + w_{m+1}) & (m\text{: even}) \\ 2(w_1 + w_3 + \cdots + w_m) & (m\text{: odd}), \end{cases} \tag{3.4}$$

where w_{m+1-e} is given in (2.6).

Remark 3.3 *For piecewise smooth $M \subset S^{n-1}$ it can be shown that (3.4) is equivalent to Proposition A.3. (See Remark A.1.) This is a version of Gauss-Bonnet theorem for a positive-reach manifold with boundary by Federer (1959), Theorem 5.19. See also Section IV.17.2 of Santaló (1976).*

Remark 3.4 *Suppose that $K(M)$ is a convex proper cone, which is the case considered in Takemura and Kuriki (1997). Then $\chi(M) = 1$ and Corollary 3.1 yields*

$$\frac{1}{2} = \begin{cases} w_1 + w_3 + \cdots + w_{m+1} & (m\text{: even}) \\ w_1 + w_3 + \cdots + w_m & (m\text{: odd}). \end{cases}$$

This is exactly the Shapiro's conjecture (Shapiro (1987)) on the weights of $\bar{\chi}^2$ distribution. Therefore Corollary 3.1 is a generalization of Shapiro's conjecture.

3.3 Alternative proof of Naiman's inequality

In this subsection we give an alternative proof of Naiman's inequality. It is based on the following characterization of the critical radius $\theta_c(M)$.

Lemma 3.1

$$\theta_c(M) = \sup\{\theta > 0 \mid I(A(y, \cos\theta) \neq \emptyset) = \chi(A(y, \cos\theta)) \text{ for all } y\}. \tag{3.5}$$

Proof. If $\theta < \theta_c(M)$ each $y \in M_\theta$ has a unique nearest point $y_M \in M$. As in the proof of Proposition 3.1 let $f_y(u) = -u'y$ and let $f_{y|M}$ denote its restriction on M. The index of $f_{y|M}$ at y_M is 1 and this is the only index counted in $\chi(A(y,\cos\theta))$. Therefore $\chi(A(y,\cos\theta)) = I(A(y,\cos\theta) \neq \emptyset)$. On the other hand if $\theta > \theta_c(M)$ it is easy to see that there exists an open set U such that to $y \in U$ correspond two u's such that $u'y > \cos\theta$ and $y \in N_u(M)$. Then $\chi(A(y,\cos\theta))$ is either 0 or 2. This proves (3.5). ∎

From this lemma we have

$$I(A(y,\cos\theta) \neq \emptyset) = \chi(A(y,\cos\theta)), \quad \theta < \theta_c.$$

On the other hand, when $\theta \geq \theta_c$, there is no general relation between $\chi(A(y,\cos\theta))$ and $I(A(y,\cos\theta) \neq \emptyset)$. However in the particular case where $M \subset S^{n-1}$ is one-dimensional and homotopic to the line segment $[0,1]$, then $\chi(A(y,\cos\theta))$ equals the number of connected components of $A(y,\cos\theta)$, and therefore the inequality

$$I(A(y,\cos\theta) \neq \emptyset) \leq \chi(A(y,\cos\theta)) \tag{3.6}$$

always holds.

By taking the expectations of the both sides of (3.6) with respect to $y \sim \mathrm{Unif}(S^{n-1})$, we have for $0 \leq \theta \leq \pi$ that

$$\begin{aligned}\frac{\mathrm{Vol}(M_\theta)}{\Omega_n} &\leq \frac{1}{\Omega_2\Omega_{n-2}}\mathrm{Vol}(M)\,\mathrm{Vol}(S^{(n-2)-1})(1 - B_{1,\frac{1}{2}(n-2)}(\cos^2\theta)) \\ &\quad + \frac{1}{\Omega_1\Omega_{n-1}}\mathrm{Vol}(\partial M)\frac{\mathrm{Vol}(S^{(n-1)-1})}{2}(1 \mp B_{\frac{1}{2},\frac{1}{2}(n-1)}(\cos^2\theta)) \\ &= \frac{1}{2\pi}\mathrm{Vol}(M)(1 - B_{1,\frac{1}{2}(n-2)}(\cos^2\theta)) + \frac{1}{2}(1 \mp B_{\frac{1}{2},\frac{1}{2}(n-1)}(\cos^2\theta)) \end{aligned}\tag{3.7}$$

by Proposition 3.1. Noting that

$$\begin{aligned} 1 - B_{1,\frac{1}{2}(n-2)}(x^2) &= (1-x^2)^{(n-2)/2}, \\ 1 \mp B_{\frac{1}{2},\frac{1}{2}(n-1)}(x^2) &= \frac{2\,\Gamma(n/2)}{\sqrt{\pi}\,\Gamma((n-1)/2)}\int_x^1 (1-x^2)^{(n-3)/2}\,\mathrm{d}x, \end{aligned}$$

we see that (3.7) is the same as the inequality (3.4) of Johnstone and Siegmund (1989).

Naiman's inequality states that the inequality (3.7) holds even when M is a piecewise C^1-curve. We can show this by taking a sequence of C^2-curves $\{M^i\}_{i=1,2,\ldots}$ such that

$$\mathrm{Vol}(M^i) \to \mathrm{Vol}(M), \qquad \mathrm{Vol}((M^i)_\theta) \to \mathrm{Vol}(M_\theta).$$

4 Maximum of cosine field: An example

In this section we study the cosine field in some length, because it is the building block for isotropic random fields in the sense of §2.5 of Adler (1998) and of basic importance.

4.1 Cosine field

The cosine field is defined as

$$Z(t) = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} (z_{2i-1} \cos t_i + z_{2i} \sin t_i)$$

with the index

$$t = (t_1, \dots, t_m) \in [0, T_1] \times \cdots \times [0, T_m], \quad 0 \le T_i < 2\pi,$$

where $z = (z_1, \dots, z_n)' \sim N_n(0, I_n)$, $n = 2m$. Piterbarg (1996), Lemmas 5.1, 5.2, derived an asymptotic expansion for the tail probability of the maximum of $Z(t)$. In this section we show that our tube formula gives another derivation of the asymptotic expansion. In addition we evaluate the remainder term of asymptotic expansion more precisely than Piterbarg (1996) by explicitly evaluating the critical radius.

$Z(t)$ is written as $Z(t) = \phi(t)'z$, where

$$\phi = \phi(t) = (\cos t_1, \sin t_1, \dots, \cos t_m, \sin t_m)'/\sqrt{m} \in S^{2m-1}.$$

$\phi(t)$ is injective and the index set M on S^{2m-1} is $\phi([0, T_1] \times \cdots \times [0, T_m])$.

Denote the partial differential of ϕ with respect to t_i by the subscript i, e.g.,

$$\begin{aligned} \phi_i &= \frac{\partial}{\partial t_i}\phi = (0, \dots, 0, -\sin t_i, \cos t_i, 0, \dots, 0)'/\sqrt{m}, \\ \phi_{ij} &= \frac{\partial^2}{\partial t_i \partial t_j}\phi = \delta_{ij}(0, \dots, 0, -\cos t_i, -\sin t_i, 0, \dots, 0)'/\sqrt{m}, \end{aligned}$$

where δ_{ij} is Kronecker's delta.

4.2 Distribution of maximum

The d-dimensional boundary ∂M_d consists of 2^{m-d} disjoint components. Note that the phase ψ_i of $(z_{2i-1}, z_{2i}) = (r_i \cos \psi_i, r_i \sin \psi_i)$, $r_i^2 = z_{2i-1}^2 + z_{2i}^2$, is uniformly distributed on $[0, 2\pi)$. Therefore without loss of generality, we may consider a component

$$\{\phi(t) \mid t_{d+1} = \cdots = t_m = 0\},$$

and fix $u = \phi(t)$ as a relative interior point of the component. The metric of ∂M_d at u is given by

$$g_{ij}(t) = \phi_i'\phi_j = (1/m)\delta_{ij}.$$

The support cone $S_u(M)$ at u is the convex cone spanned by the lines span$\{\phi_i\}$, $i = 1, \dots, d$, and the rays cone$\{\phi_i\} = \{c\phi_i \mid c \ge 0\}$, $i = d+1, \dots, m$. Note that for $i = d+1, \dots, m$, $\phi_i = e_{2i} = (0, \dots, 0, 1, 0, \dots, 0)'$, where 1 is the $2i$-th element.

The normal cone $N_u(K(M))$ is the dual cone of $S_u(K(M)) = \mathrm{span}\{u\} \oplus S_u(M)$. It is easily seen that

$$N_u(K(M)) = \{v = (a_1 \cos t_1, a_1 \sin t_1, \ldots, a_d \cos t_d, a_d \sin t_d, a_{d+1}, b_{d+1}, \ldots, a_m, b_m)' \\ \mid a_1 + \cdots + a_m = 0,\ b_{d+1}, \ldots, b_m \le 0\}. \qquad (4.1)$$

The squared length of v in (4.1) is $\|v\|^2 = a_1^2 + \cdots + a_m^2 + b_{d+1}^2 + \cdots + b_m^2$.

The second fundamental form of ∂M_d at u with respect v in (4.1) is given by $H(u,v)_{ij} = -v'\phi_{ij}(u) \times m = \sqrt{m} a_i \delta_{ij}$ or

$$H(u,v) = \sqrt{m}\ \mathrm{diag}(a_1, \ldots, a_d).$$

Now we proceed to evaluate the weights w_{m+1-e} of (2.6) for the cosine field. Write $w_{m+1-e} = \sum_{d=m-e}^{m} w_{m+1-e}^{(d)}$, where

$$w_{m+1-e}^{(d)} = \frac{1}{\Omega_{m+1-e}\Omega_{n-m-1+e}} \int_{\partial M_d} du \int_{N_u(K(M)) \cap S^{n-1}} dv\ \mathrm{tr}_{d-m+e} H(u,v). \qquad (4.2)$$

For convenience write $l = d - m + e$ and

$$J_1 = \int_{N_u(K(M)) \cap S^{n-1}} dv\ \mathrm{tr}_l H(u,v).$$

Let $R^2 \sim \chi^2(2m-d-1)$, and consider the expectation

$$J_2 = E\Big[\int_{N_u(K(M)) \cap S^{n-1}} dv\ \mathrm{tr}_l H(u, Rv)\Big]/\Omega_{2m-d-1} = J_1 \times \frac{E[(\chi^2_{2m-d-1})^{l/2}]}{\Omega_{2m-d-1}}. \qquad (4.3)$$

Since the degrees of freedom of R^2 is the dimension of the normal cone $\dim N_u(K(M)) = n-1-d = 2m-d-1$, J_2 can be calculated by taking the expectation

$$J_2 = E[I(b_{d+1}, \ldots, b_m \le 0)\, \mathrm{tr}_l(\sqrt{m}\ \mathrm{diag}(a_1, \ldots, a_d))],$$

where

$$(a_1, \ldots, a_m) \sim N_m(0, I_m - (1/m) 1_m 1_m'), \qquad 1_m = (1, \ldots, 1)' \in R^m,$$
$$b_{d+1}, \ldots, b_m \sim N(0,1),$$

and $(a_1, \ldots, a_m), b_{d+1}, \ldots, b_m$ are mutually independent.

Since $E[a_i a_j] = -1/m$ $(i \ne j)$, we have

$$E[a_1 a_2 \cdots a_k] = \begin{cases} (k-1)!!\,(-1/m)^{k/2} & \text{(for } k \text{ even)} \\ 0 & \text{(for } k \text{ odd),} \end{cases}$$

where $(k-1)!! = (k-1)(k-3)\cdots 3 \cdot 1$. Therefore J_2 for l even is

$$J_2 = (1/2)^{m-d} \times \binom{d}{l} (l-1)!!\,(-1/m)^{l/2} \times m^{l/2} = \frac{d!\,(-1)^{l/2}}{2^{m-d+l/2}\,(d-l)!\,(l/2)!} \qquad (4.4)$$

and $J_2 = 0$ for l odd. Combining (4.2), (4.3), and (4.4), and noting that $\int_{\partial M_d} du = 2^{m-d}\,\Sigma_d$, where

$$\Sigma_d = \sum_{i_1<\cdots<i_d} T_{i_1}\cdots T_{i_d}, \tag{4.5}$$

we get for l even

$$w_{m+1-e}^{(d)} = w_{d+1-l}^{(d)} = \frac{d!\,(-1)^{l/2}\,\Gamma((d+1-l)/2)}{2^{l+1}\,\pi^{(d+1)/2}\,(d-l)!\,(l/2)!}\,\Sigma_d = \frac{d!\,(-1)^{l/2}}{2^{d+1}\,\pi^{d/2}\,\Gamma((d+2-l)/2)\,(l/2)!}\,\Sigma_d$$

and $w_{d+1-l}^{(d)} = 0$ for l odd.

Let

$$H_k(x) = (-1)^k \frac{d^k}{dx^k} e^{-x^2/2} \Big/ e^{-x^2/2}$$

be the Hermitian polynomial of degree k. Denote the coefficient of x^{k-l} in $H_k(x)$ by $h(k;l)$. It is well known that

$$h(k;l) = \begin{cases} (-1)^{l/2}\binom{k}{l}(l-1)!! = (-1)^{l/2}\dfrac{k!}{(k-l)!\,2^{l/2}\,(l/2)!} & \text{(for } l \text{ even)} \\ 0 & \text{(for } l \text{ odd).} \end{cases}$$

Using this we have

$$w_{d+1-l}^{(d)} = h(d;l) \times \frac{2^{(d+1-l)/2}\,\Gamma((d+1-l)/2)}{2(2\pi)^{(d+1)/2}}\,\Sigma_d.$$

Multiplying this by $1 - G_{d+1-l}(x^2)$, and taking a summation over $l = 0, \ldots, d$ for fixed d we have

$$\begin{aligned} \sum_{l=0}^{d} w_{d+1-l}^{(d)}(1 - G_{d+1-l}(x^2)) &= \frac{\Sigma_d}{2(2\pi)^{(d+1)/2}} \sum_{l=0}^{d} h(d;l) \int_{x^2}^{\infty} y^{(d+1-l)/2-1} e^{-y/2}\mathrm{d}y \\ &= \frac{\Sigma_d}{(2\pi)^{(d+1)/2}} \sum_{l=0}^{d} h(d;l) \int_{x}^{\infty} z^{d-l} e^{-z^2/2}\mathrm{d}z \\ &= \frac{\Sigma_d}{(2\pi)^{(d+1)/2}} \int_{x}^{\infty} H_d(z) e^{-z^2/2}\mathrm{d}z \\ &= \frac{\Sigma_d}{(2\pi)^{(d+1)/2}} H_{d-1}(x) e^{-x^2/2}. \end{aligned}$$

By summing this up over $d = 0, \ldots, m$, the asymptotic expansion for the tail probability of $\max_t Z(t)$ is obtained. We will summarize the result at the end of this section as Proposition 4.1.

By Corollary 3.1 we have

$$\chi(M) \;=\; 2\sum_{d=0}^{m} \sum_{\substack{l=0 \\ d-l:\text{even}}}^{d} w_{d+1-l}^{(d)}$$

$$= 2\sum_{\substack{d=0\\ d:\text{even}}}^{m}\sum_{\substack{l=0\\ l:\text{even}}}^{d}\frac{d!}{2^{d+1}\pi^{d/2}(d/2)!}(-1)^{l/2}\binom{d/2}{l/2}\Sigma_d$$
$$= 2\sum_{\substack{d=0\\ d:\text{even}}}^{m}\frac{1}{2}\delta_{d,0} = 1,$$

which was expected since M is homotopic to a point $\phi(0) = (1,0,\ldots,1,0)'$.

4.3 critical radius

We here evaluate the critical radius θ_c by Lemma 2.1. We make the following additional assumption on the index set as is done in Piterbarg (1996):

$$0 \le T_i \le \pi, \quad i = 1,\ldots,m. \tag{4.6}$$

The orthogonal projection matrix onto the space span$\{\phi,\phi_1,\ldots,\phi_d\}$ is written as

$$Q_\phi = \phi\phi' + m\sum_{i=1}^{d}\phi_i\phi_i'.$$

Fix v in the relative interior of $\{\phi(t) \mid t_{d+1} = \cdots = t_m = 0\}$. The orthogonal projection onto the normal cone $N_v(K(M))$ is give by

$$P_v^\perp(w) = (I_{2m} - Q_v)w + \sum_{i=d+1}^{m} e_{2i}\min(0, w_{2i}),$$

where $w = (w_1,\ldots,w_{2m})' \in R^{2m}$.

By Lemma 2.1,

$$\tan^2\theta_c = \inf_{u,v\in M}\frac{\|u-v\|^4}{4\|P_v^\perp(u-v)\|^2} = \inf_{u,v\in M}\frac{(1-u'v)^2}{\|(I_{2m}-Q_v)(u-v)\|^2 + \sum_{i=d+1}^{m}\min(0,u_{2i}-v_{2i})^2}.$$

In the expression above we assumed that the infimum is attained when $v = \phi(t)$ with $t_{d+1} = \cdots = t_m = 0$ for a particular value of d. Put $u = \phi(s)$, $s = (s_1,\ldots,s_m)$.

Note that for $i = d+1,\ldots,m$, $v_{2i} = 0$, $u_{2i} = \sin s_i \ge 0$, and hence $\min(0, u_{2i} - v_{2i}) = 0$ by the assumption (4.6).

Put

$$u'v = \phi(s)'\phi(t) = \frac{1}{m}\sum_{i=1}^{m} x_i, \qquad x_i = (\cos s_i, \sin s_i)\begin{pmatrix}\cos t_i\\ \sin t_i\end{pmatrix}.$$

Noting

$$\|(I_{2m} - Q_v)(u-v)\|^2 = 1 - u'Q_v u = 1 - (\phi(s)'\phi(t))^2 - m\sum_{i=1}^{d}(\phi(s)'\phi_i(t))^2,$$

and

$$\phi(s)'\phi_i(t) = \frac{1}{m}(\cos s_i, \sin s_i)\begin{pmatrix}-\sin t_i\\ \cos t_i\end{pmatrix} = \pm\frac{1}{m}\sqrt{1-x_i^2},$$

the argument of the infimum is written as

$$\frac{(1-\frac{1}{m}\sum_{i=1}^m x_i)^2}{1-(\frac{1}{m}\sum_{i=1}^m x_i)^2-\frac{1}{m}\sum_{i=1}^d(1-x_i^2)} = \frac{(\sum_{i=1}^m y_i)^2}{m\sum_{i=1}^d y_i^2+2m\sum_{i=d+1}^m y_i-(\sum_{i=1}^m y_i)^2},$$

where we put $y_i = 1-x_i$. Note that $0 \le y_i \le 2$. By virtue of the inequality

$$\sum y_i^2 \le (\sum y_i)^2$$

(the equality holds iff $y_i = 0$ except for at most one index i), we see

$$\begin{aligned}\frac{(\sum_{i=1}^m y_i)^2}{m\sum_{i=1}^d y_i^2+2m\sum_{i=d+1}^m y_i-(\sum_{i=1}^m y_i)^2} &\ge \frac{(\sum_{i=1}^m y_i)^2}{m(\sum_{i=1}^d y_i)^2+2m\sum_{i=d+1}^m y_i-(\sum_{i=1}^m y_i)^2} \\ &\ge \frac{(\sum_{i=1}^d y_i)^2}{m(\sum_{i=1}^d y_i)^2-(\sum_{i=1}^d y_i)^2} = \frac{1}{m-1},\end{aligned}$$

where the equality of the second inequality holds iff $\sum_{i=d+1}^m y_i = 0$.

This infimum $1/(m-1)$ is attained in the case where

$$y_1 \to +0,\ y_2 = \cdots = y_d = y_{d+1} = \cdots = y_m = 0.$$

This is possible when at least T_1 is positive. Since the infimum $1/(m-1)$ is independent of d, we conclude that

$$\tan^2\theta_c = \frac{1}{m-1}$$

when $0 \le T_i \le \pi$ and $\exists i,\ T_i > 0$. The case $T_1 = \cdots = T_m = 0$ is trivial.

Proposition 4.1 *Assume that* $0 \le T_i \le \pi$, $i = 1,\ldots,m$. *Then*

$$P\Big(\max_t Z(t) \ge x\Big) = \sum_{d=1}^m \frac{\Sigma_d}{(2\pi)^{d/2}} H_{d-1}(x)\varphi(x) + \int_x^\infty \varphi(x)\,\mathrm{d}x + O\Big(g_{2m}\Big(\frac{m}{m-1}x^2\Big)\Big)$$

as $x \to \infty$, *where* $\varphi(x) = e^{-x^2/2}/\sqrt{2\pi}$ *and* Σ_d *is given in (4.5).*

This gives the same asymptotic expansion as Piterbarg (1996). In addition we have made the remainder term more precise.

A Appendix

A.1 Manifold with piecewise smooth boundary

We summarize our definition of manifold M of dimension m with piecewise smooth boundary. A cone is called *proper* if $K \cap (-K) = \{0\}$.

Definition A.1 *Let M be a topological m-dimensional manifold with boundary. M is called a manifold with piecewise smooth boundary of class C^r if each $x \in M$ has a neighborhood $W(x)$ which is C^r-diffeomorphic to the following set $\tilde{W}$ in R^m:*

$$\tilde{W} = (-\epsilon, \epsilon)^d \times (K \cap \epsilon B^O), \tag{A.1}$$

where $0 \le d \le m$, $\epsilon > 0$, K is a closed proper cone in R^{m-d}, $B^O = \{x \mid \|x\| < 1\} \subset R^{m-d}$ is the open ball in R^{m-d}, and "$\times$" denotes the direct product.

K need not be a polyhedral cone. We studied one important example of non-polyhedral cone in Kuriki and Takemura (1999).

For each $0 \le d < m$, let ∂M_d denote the set of points x having a neighborhood $W(x)$ which is C^r-diffeomorphic to $\tilde{W}$ of the form (A.1). By standard argument it can be shown that ∂M_d forms a d-dimensional manifold of class C^r. $\partial M = \bigcup_{d=0}^{m-1} \partial M_d$ forms the boundary of M. For convenience and notational consistency we also write $\partial M_m = M^O$, the interior of M, although ∂ symbol here might be somewhat confusing.

Definition A.1 is an intrinsic definition and M is not necessarily a submanifold of a Euclidean space. However for our purposes it suffices to consider submanifolds of a Euclidean space and we assume that all manifolds are submanifolds of R^n endowed with the standard inner product $\langle x, y\rangle = x'y$, where x, y are considered as n-dimensional column vector. As a submanifold of R^n the topology on M coincides with the relative topology induced from R^n. Therefore M^O denotes the relative interior of M and ∂M denotes the relative boundary of M in R^n.

Let $x \in \partial M_d \subset R^n$, $0 \le d \le m$. Take a local coordinate system $(w_1, \ldots, w_m)$ and write $x(w_1, \ldots, w_m)$ for points in a neighborhood of $x = x(0, \ldots, 0)$. We can take this local coordinate system in accordance with (A.1), i.e., for all $|w_i| < \epsilon$, $i = 1, \ldots, d$,

$$\frac{\partial x}{\partial w_j} = \frac{\partial x}{\partial w_j}(w_1, \ldots, w_d, 0, \ldots, 0) \in R^n, \quad j = 1, \ldots, d,$$

form a basis for the tangent space of $T_x(\partial M_d)$ of ∂M_d at $x = x(w_1, \ldots, w_d, 0, \ldots, 0)$. Furthermore we may assume that at every $x = x(w_1, \ldots, w_d, 0, \ldots, 0)$,

$$N_j = \frac{\partial x}{\partial w_j}(w_1, \ldots, w_d, 0, \ldots, 0), \quad j = d+1, \ldots, n,$$

form an orthonormal basis of $T_x(M) \cap T_x^{\perp}(\partial M_d)$, which is the orthogonal complement of $T_x(\partial M_d)$ in $T_x(M)$. Define the *support cone* $S_x(M)$ of M at $x = x(0, \ldots, 0)$ by

$$S_x(M) = T_x(\partial M_d) \oplus \{w_{d+1}N_{d+1} + \cdots + w_m N_m \mid (w_{d+1}, \ldots, w_m) \in K\}, \tag{A.2}$$

where "$\oplus$" is the orthogonal sum and $N_j = N_j(x)$, $j = d+1, \ldots, m$. Furthermore define the *normal cone* $N_x(M)$ of M at x as the dual cone of $S_x(M)$:

$$N_x(M) = \{y \in R^n \mid y'z \le 0, \ \forall z \in S_x(M)\}.$$

Using the above local coordinates $N_x(M)$ can be written as

$$N_x(M) = T_x(M)^{\perp} \oplus \{w_{d+1}N_{d+1} + \cdots + w_m N_m \mid (w_{d+1}, \ldots, w_m) \in K^*\}, \tag{A.3}$$

where K^* denotes the dual cone of K in R^{m-d}. The support cone $S_x(M)$ is a cone approximating M at x. For the case where M is a convex set the notions of support cone and normal cone given here coincide with the those in Section 2.2 of Schneider (1993). See also Section 2.3 of Takemura and Kuriki (1997).

A.2 Tube formula and integral of Euler characteristic: Euclidean case

In Sections 2 and 3 we have considered geometry of submanifolds of S^{n-1}. We summarize here corresponding results on tube formula and integral of the Euler characteristic in the case of Euclidean space R^n. Concepts and results for the Euclidean spaces are more transparent than those for the unit sphere S^{n-1}.

Let M be a compact m-dimensional submanifold of R^n with piecewise smooth boundary of class C^2 satisfying Assumptions 1.1 and 1.2. Let $x \in R^n$. Since M is closed there exists a closest point $x_M \in M$ from x. The tube M_r around M with radius r is defined by

$$M_r = \{x \in R^n \mid \|x - x_M\| \le r\}.$$

M_r can be alternatively written as

$$M_r = M + rB,$$

where $B = \{x \mid \|x\| \le 1\} \subset R^n$ is a closed unit ball and "+" denotes the vector sum.

The cross section $C_x(r)$ of M_r at $x \in M$ is defined by

$$C_x(r) = x + (N_x(M) \cap rB),$$

which is the set of points y such that $\|y - x\| \le r$ and $y - x \in N_x(M)$. We see that M_r can be written as the union of cross sections:

$$M_r = \bigcup_{x \in M} C_x(r).$$

From the compactness and the local convexity of M it can be shown that there exists $r > 0$ such that every $x \in M_r$ has unique projection point x_M. The critical radius r_c of M is the supremum of such r:

$$r_c = r_c(M) = \sup\{r > 0 \mid x_M \text{ unique for all } x \in M_r\}.$$

r_c can also be defined by

$$r_c = \sup\{r > 0 \mid C_x(r),\ x \in M,\ \text{are disjoint}\}.$$

In integral geometry literature critical radius of M is called *reach* of M (e.g., Federer (1959), Stoyan, et al. (1995)). The critical radius can be computed using the following formula.

Lemma A.1 *The critical radius r_c of M is given by*

$$r_c = \inf_{x,y \in M} \frac{\|x-y\|^2}{2\|P_y^\perp(x-y)\|},$$

where $P_y^\perp$ is the orthogonal projection onto the normal cone $N_y(M)$ of M at y.

As in the case of S^{n-1}, the positiveness of critical radius is assured by Assumption 1.2. From this reason the property of local convexity of Assumption 1.2 is called *positive-reach* in integral geometry literature.

Let

$$V_n(M_r) = \sum_{e=0}^{m} \frac{r^{n-m+e}}{n-m+e} \sum_{d=m-e}^{m} \int_{\partial M_d} \mathrm{d}y \int_{N_y(M)\cap S^{n-1}} \mathrm{d}v \ \mathrm{tr}_{d-m+e} H(y,v), \tag{A.4}$$

where for each $0 \le d \le m$ $\mathrm{d}y$ denotes the volume element of ∂M_d and $\mathrm{d}v$ denotes the volume element of $S^{n-d-1} = N_y(M) \cap S^{n-1}$. Then the following result holds.

Proposition A.1 *For $r \le r_c(M)$, $V_n(M_r)$ in (A.4) is equal to the n-dimensional volume* $\mathrm{Vol}(M_r)$ *of M_r.*

For $x \in R^n$ let

$$A(x,r) = M \cap (x + rB)$$

denote the intersection of M and the closed ball around x of radius r. The basic relation linking the tube method and the Euler characteristic method is given in the following proposition.

Proposition A.2

$$V_n(M_r) = \int_{R^n} \chi(A(x,r))\,\mathrm{d}x, \tag{A.5}$$

where $\mathrm{d}x$ *denotes the Lebesgue measure and $\chi(A(x,r))$ denotes the Euler characteristic of $A(x,r)$.*

As stated in Remark 3.2, (A.5) is a version of the kinematic fundamental formula for the case of Euclidean space (cf. III.15.4 of Santaló (1976)).

The following is a Gauss-Bonnet theorem for a positive-reach manifold with boundary. This is given by the coefficient of r^n in the signed tube formula (A.4). The notation is the same as in (A.4).

Proposition A.3 *The Euler characteristic of M is given by*

$$\chi(M) = \frac{1}{\Omega_n} \sum_{d=0}^{m} \int_{\partial M_d} \mathrm{d}y \int_{N_y(M)\cap S^{n-1}} \mathrm{d}v \;\; \det H(y,v). \tag{A.6}$$

Remark A.1 *Suppose that M is a subset of S^{n-1}. Then $v \in N_y(M) \cap S^{n-1}$ is uniquely written as*

$$v = y\cos\theta + w\sin\theta, \quad w \in N_y(K(M)) \cap S^{n-1}, \quad 0 \le \theta < \pi.$$

Correspondingly, the second fundamental form in (A.6) is written as

$$H(y,v) = \cos\theta I_d + \sin\theta H(y,w).$$

Also for y fixed

$$\mathrm{d}v = \sin^{n-d-2}\theta \mathrm{d}\theta \,\mathrm{d}w,$$

where dw *is the volume element of $N_y(K(M)) \cap S^{n-1}$. Therefore we have*

$$\chi(M) = \frac{1}{\Omega_n} \sum_{d=0}^{m} \int_0^{\pi} \mathrm{d}\theta \int_{\partial M_d} \mathrm{d}y \int_{N_u(K(M))\cap S^{n-1}} \mathrm{d}w \det(\cos\theta I_d + \sin\theta H(y,w)) \, \sin^{n-d-2}\theta,$$

which is equivalent to (3.4).

A.3 Morse theorem for manifold with piecewise smooth boundary

Here we generalize Theorem 10.2 of Morse and Cairns (1969) for M with piecewise smooth boundary of class C^2. For a real valued function f defined on X, $f_{|X'}$ denote its restriction to $X' \subset X$.

Let f be a real-valued C^2-function defined on some relatively open neighborhood $\tilde{M}$ of M. As in Morse and Cairns (1969) we assume the following conditions:

i) There is no critical point of f on the relative boundary ∂M of M.

ii) For each $0 \le d \le m$, $f_{|\partial M_d}$ is non-degenerate at its critical points.

We call f satisfying these conditions *Morse function* on M.

Note that f needs to be defined only on $\tilde{M}$. Therefore we can discuss Morse functions on M intrinsically without reference to R^n. However for our purposes it is convenient to consider M and its Morse function in R^n. Let f be a C^2-function defined on the whole R^n. As a Morse function on M we require that $f_{|\tilde{M}}$ satisfies the above conditions i) and ii). Note that the gradient of $f_{|\tilde{M}}$ at $x \in \tilde{M}$ coincides with the orthogonal projection of the gradient of f to the tangent space $T_x(\tilde{M})$ and condition i) requires that the gradient of f has non-zero $T_x(\tilde{M})$ component for each $x \in \partial M$.

Let f a Morse function on M. In the case of M with smooth $m-1$-dimensional boundary, the critical point $x \in \partial M$ of $f_{|\partial M}$ is counted in Theorem 10.2 of Morse and

Cairns (1969) if and only if the gradient of f, which is normal to the tangent space $T_x(M)$, is directed into the interior M^O of M. Noting that the normal cone $N_x(M)$ at x is the one-dimensional cone generated by the outward normal vector at x this condition can be expressed as $-\operatorname{grad} f \in N_x(M)$. We use this condition as a criterion for counting critical points on ∂M.

Definition A.2 *Let $0 \le d < m$ and let $x \in \partial M_d$ be a critical point of $f_{|\partial M_d}$. x is extended inward critical point if*

$$-\operatorname{grad} f \in N_x(M).$$

Let ν_k, $k = 0, \dots, m-1$, denote the number of extended inward critical points of index k on ∂M and let $\mu_k, k = 0, \dots, m$, denote the number of critical points on M^O of index k. The augmented type numbers μ'_k, $k = 0, \dots, m$, of f are

$$\mu_0 + \nu_0, \mu_1 + \nu_1, \dots, \mu_{m-1} + \nu_{m-1}, \mu_m.$$

Worsley (1995a) shows how the boundary critical points are counted in the Euler characteristic for the case of R^2 and R^3. Definition A.2 clarifies which critical points are counted in general dimension.

We are ready to state a generalization of Theorem 10.2 of Morse and Cairns (1969).

Proposition A.4 *Let M be a compact m-dimensional manifold with piecewise smooth boundary. The Euler characteristic $\chi(M)$ of M is given by*

$$\chi(M) = \mu'_0 - \mu'_1 + \cdots + (-1)^m \mu'_m.$$

Proof. We follow the line of argument given in Section 11 of Morse and Cairns (1969). We omit their discussion on "critical arc", because it is basically the same for the case of M with piecewise smooth boundary. The essential point of their argument is to modify f by some function ζ such that the gradient field of $\hat{f} = f + \zeta$ is directed outwards everywhere on the boundary on M. By doing this they shift all inward critical points into the interior of M. This operation reduces their Boundary Condition B to their Boundary Condition A. For our present setup we need to smoothly approximate ∂M in addition to shifting all extended inward critical points. For doing this we find it easier to shift extended inward critical points outward to the exterior of M (rather than shifting inward).

For our proof it is convenient to use a particular relative open neighborhood of M. Let $C_x^O(r) = x + (N_x(M) \cap rB^O)$ be the open r cross section at $x \in M$. For $r < r_c(M)$ let

$$\tilde{M} = \bigcup_{x \in M} C_x^O(r) \cap T_x(M).$$

This $\tilde{M}$ has the advantage that it is flat in the direction of $N_x(M)$ at $x \in \partial M$. Without loss of generality we can assume that f is defined on this $\tilde{M}$. In addition choose sufficiently small r' and let

$$\bar{M}_{r'} = \bigcup_{x \in M} C_x(r') \cap T_x(M).$$

Although the boundary of $\bar{M}_{r'}$ is only of class C^1, it can be arbitrarily closely approximated by a manifold with boundary of class C^∞. Note that $\bar{M}_{r'}$ is homotopic to M and hence $\chi(\bar{M}_{r'}) = \chi(M)$. We use the coordinate system as in (A.2) and (A.3). Our modifying function ζ is an increasing convex function of $r^2 = \|x - x_M\|^2$ with $\zeta(0) = 0$. Hence $\zeta(x) > 0$ only for $x \notin M$. For $x_M \in \partial M_d$

$$\zeta(x) = \zeta(w_{d+1}^2 + \cdots + w_m^2).$$

On the cross section $C_x^O(r)$ the gradient field of $\hat{f} = f + \zeta$ is given by

$$\operatorname{grad} \hat{f} = \operatorname{grad} f + 2\zeta'(r^2)(w_{d+1}N_{d+1} + \cdots + w_m N_m), \qquad (w_{d+1}, \ldots, w_m) \in K^*,$$

in the notation of (A.3). Note that by making $\zeta'(r^2)$, $r^2 > 0$, sufficiently large, we add strong outward vector field to the gradient field of f. Therefore by appropriate choice of ζ the gradient field of $\hat{f}$ is directed outwards at every $x \in \partial \bar{M}_{r'}$, thus reducing our case to Boundary Condition A of Morse and Cairns (1969). More explicit choice of ζ may be described as on page 78 of Morse and Cairns (1969).

Now suppose that $x \in \partial M_d$ is an extended inward critical point of $f_{|\partial M_d}$. Then $-\operatorname{grad} f(x) \in N_x(M)$ and in terms of the basis $\{N_{d+1}, \ldots, N_m\}$ we can write

$$-\operatorname{grad} f(x) = a_{d+1}N_{d+1} + \cdots + a_m N_m$$

for some constants $(a_{d+1}, \ldots, a_m) \in K^*$. By setting

$$2\zeta'(r^2)(w_{d+1}, \ldots, w_m) = (a_{d+1}, \ldots, a_m), \tag{A.7}$$

we see that the extended inward critical point is shifted outwards and becomes a critical point in the interior of $\bar{M}_{r'}$.

We need to check that the index of the Hessian matrix is not changed by above shifting. We follow the argument on page 81 of Morse and Cairns (1969). Since ζ depends only on $r^2 = w_{d+1}^2 + \cdots + w_m^2$, the Hessian matrix of $\hat{f}$ differs from that of f only in the lower-right $(m-d) \times (m-d)$ submatrix as follows:

$$\left(\frac{\partial^2 \hat{f}}{\partial w_i \partial w_j} \right) = \left(\frac{\partial^2 f}{\partial w_i \partial w_j} \right) + \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix},$$

where

$$M = 2\zeta'(r^2) I_{m-d} + 4\zeta''(r^2) \begin{pmatrix} w_{d+1} \\ \vdots \\ w_m \end{pmatrix} (w_{d+1}, \ldots, w_m).$$

Note that the second term on the right hand side is non-negative definite, whereas the first term is positive definite being a positive multiple of the identity matrix I_{m-d}. It follows that by letting $\zeta'(r^2)$ sufficiently large we can make the index of the Hessian matrix of $\hat{f}$ equal to the index of the Hessian matrix of f.

It is easy to see that by modification $f \to \hat{f}$, no critical point appears in the interior of $\bar{M}_{r'}$ other than those given in (A.7). Hence $\hat{f}$ satisfies Boundary Condition A of Morse and Cairns (1969) and has type numbers equal to the augmented type numbers of Definition A.2. This completes the proof of Theorem A.4. ∎

References

[1] Adler, R. J. (1981). *The Geometry of Random Fields.* John-Wiley & Sons, Chichester.

[2] Adler, R. J. (1998). On excursion sets, tube formulae, and maxima of random fields. *Technical report*, Technion – Israel Institute of Technology.

[3] Federer, H. (1959). Curvature measures. *Trans. Amer. Math. Soc.*, **93**, 418–491.

[4] Hotelling, H. (1939). Tubes and spheres in n-spaces, and a class of statistical problems. *Am. J. Math.*, **61**, 440–460.

[5] Johansen, S. and Johnstone, I. (1990). Hotelling's theorem on the volume of tubes: Some illustrations in simultaneous inference and data analysis. *Ann. Statist.*, **18**, 652–684.

[6] Johnstone, I. and Siegmund, D. (1989). On Hotelling's formula for the volume of tubes and Naiman's inequality. *Ann. Statist.*, **18**, 652–684.

[7] Knowles, M. and Siegmund, D. (1989). On Hotelling's approach to testing for a nonlinear parameter in regression. *Internat. Statist. Rev.*, **57**, 205–220.

[8] Kuriki, S. and Takemura, A. (1997). James-Stein type estimator by shrinkage to closed convex set with smooth boundary. *Discussion Paper 97-F-22*, Faculty of Economics, Univ. of Tokyo.

[9] Kuriki, S. and Takemura, A. (1998). Tail probabilities of the maxima of multilinear forms and their applications. *Discussion Paper CIRJE-F-4*, Faculty of Economics, Univ. of Tokyo.

[10] Kuriki, S. and Takemura, A. (1999). Some geometry of the cone of nonnegative definite matrices and weights of associated $\bar{\chi}^2$ distribution. *Ann. Inst. Stat. Math.*, to appear.

[11] Milnor, J. (1963). *Morse Theory.* Annals of Mathematical Studies, **51**, Princeton University Press, Princeton.

[12] Morse, M. and Cairns, S. S. (1969). *Critical Point Theory in Global Analysis and Differential Topology.* Academic Press, New York.

[13] Naiman, D. Q. (1986). Conservative confidence bands in curvilinear regression. *Ann. Statist.*, **14**, 896–906.

[14] Piterbarg, V. I. (1996). *Asymptotic Methods in the Theory of Gaussian Processes and Fields*. Translations of Mathematical Monographs, **148**, American Mathematical Society, Providence.

[15] Santaló, L. A. (1976). *Integral Geometry and Geometric Probability.* Addison-Wesley, London.

[16] Schneider, R. (1993). *Convex Bodies: The Brunn-Minkowski Theory*. Cambridge University Press, Cambridge.

[17] Shapiro, A. (1987). A conjecture related to chi-bar-squared distributions. *Amer. Math. Monthly*, **94**, 46–48.

[18] Shapiro, A. (1988). Towards a unified theory of inequality constrained testing in multivariate analysis. *Internat. Statist. Rev.*, **56**, 49–62.

[19] Stoyan, D., Kendall, W. S. and Mecke, J. (1995). *Stochastic Geometry and its Applications*, 2nd ed. John-Wiley & Sons, Chichester.

[20] Sun, J. (1993). Tail probabilities of the maxima of Gaussian random fields. *Ann. Probab.*, **21**, 34–71.

[21] Takemura, A. and Kuriki, S. (1997). Weights of $\bar{\chi}^2$ distribution for smooth or piecewise smooth cone alternatives. *Ann. Statist.*, **25**, 2368–2387.

[22] Weyl, H. (1939). On the volume of tubes. *Am. J. Math.*, **61**, 461–472.

[23] Worsley, K. J. (1995a). Estimating the number of peaks in a random field using the Hadwiger characteristic of excursion sets, with applications to medical images. *Ann. Statist.*, **23**, 640–669.

[24] Worsley, K. J. (1995b). Boundary corrections for the expected Euler characteristic of excursion sets of random fields, with an application to astrophysics. *Adv. Appl. Probab.*, **27**, 943–959.