CIRJE-F-50

On Simultaneous Switching Autoregressive Model

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June 1999

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August 1998

June 1999

(Revised Version)

Abstract

The simultaneous switching autoregressive (SSAR) model is a non-linear Markovian time series model, which was originally introduced by Kunitomo and Sato (1996a). This paper gives some conditions for the geometrical ergodicity of the SSAR models and discuss the estimation methods of unknown parameters. Also we shall mention to the relation between the SSAR models in the non-linear time series analysis and one type of disequilibrium econometric models, which is a tobit type in the class of limited dependent variables models. The latter class of econometric models has been extensively investigated and discussed by Amemiya (1985).

Key Words

Asymmetry, Non-linear Time Series, Simultaneous Switching Autoregressive (SSAR) Model, Geometrical Ergodicity, Disequilibrium Econometric Model.

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*This paper is dedicated to Professor Takeshi Amemiya of Stanford University. It is a revised version of Discussion Paper No.97-F-31 by the first author, Research Institute for the Japanese Economy, Faculty of Economics, University of Tokyo. We thank Professor Cheng Hsiao and two referees for their critical comments to the first version.

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1. Introduction

In the past two decades, several non-linear time series models have been proposed and investigated in the statistical time series analysis. The general class of non-linear times series models is written as

$$y_t = f(y_{t-1}, v_t) \ , \tag{1.1}$$

where $f(\cdot)$ is a measurable function, $\{y_t; 1 \leq t \leq T\}$ are a sequence of $m \times 1$ vectors of observable time series, and $\{v_t; 1 \leq t \leq T\}$ are a sequence of $n \times 1$ vectors of independently and identically distributed (i.i.d.) random variables. The initial value y_0 is either a fixed vector or a random vector in general. This class is called the Markovian time series model and many existing time series models have the Markovian representations as (1.1). More often in the statistical time series analysis, the sub-class of the Markovian models in the form of (1.1) has been used. If we assume the additivity of the past information of time series and the current disturbance term, the general form can be written as

$$y_t = f^*(y_{t-1}) + v_t \ , \tag{1.2}$$

where we take $m = n$ and $f^*(\cdot)$ is a measurable function. Two important examples of this type are the exponential autoregressive models and the threshold autoregressive models in the nonlinear time series analysis, which have been systematically discussed by Tong (1990).

On the other hand, Kunitomo and Sato (1996a) have introduced a simple stationary simultaneous switching autoregressive (SSAR) time series model. For the expository simplicity [1], let $\{y_t\}$ be a sequence of scalar time series satisfying

$$y_t = \begin{cases} a_1 y_{t-1} + \sigma_1 v_t & \text{if} \quad y_t \geq y_{t-1} \\ b_1 y_{t-1} + \sigma_2 v_t & \text{if} \quad y_t < y_{t-1} \end{cases} , \tag{1.3}$$

where a_1, b_1, σ_i ($\sigma_i > 0; i = 1, 2$) are unknown parameters, and $\{v_t\}$ are a sequence of i.i.d. random variables with $E(v_t) = 0$ and $E(v_t^2) = 1$. The initial value y_0 is either a fixed number or a random variable, but the latter is used for the stationary time series model. By imposing the condition given by

$$\frac{1 - a_1}{\sigma_1} = \frac{1 - b_1}{\sigma_2} = r \ , \tag{1.4}$$

this time series model can be rewritten in a more meaningful way as

$$y_t = \begin{cases} a_1 y_{t-1} + \sigma_1 v_t & \text{if} \quad v_t \geq r y_{t-1} \\ b_1 y_{t-1} + \sigma_2 v_t & \text{if} \quad v_t < r y_{t-1} \end{cases} ,$$

and the Markovian representation in a compact form is given by

$$y_t = y_{t-1} + [\sigma_1 I(v_t \geq r y_{t-1}) + \sigma_2 I(v_t < r y_{t-1})][-r y_{t-1} + v_t] \ , \tag{1.5}$$

[1] We have omitted the constant terms in (1.3) for instance. See (2.2) for the general p-th order SSAR model.

where $I(\cdot)$ is the indicator function. When $\sigma_1 = \sigma_2 = \sigma$, then the SSAR model becomes the standard $AR(1)$ model by re-parametrizing $a_1 = b_1 = 1 - \sigma r$. From this Markovian representation we immediately know that the SSAR model given by (1.3) is well-defined in the form of (1.1), but not in the form of (1.2). Also it is clear that given the past information $\{y_s; s \leq t-1\}$ there is an uncertainty whether the next phase is $I(v_t \geq r y_{t-1})$ or $I(v_t < r y_{t-1})$.

As we have shown (Kunitomo and Sato (1996a)), even this simplest univariate SSAR model defined by (1.3) (called SSAR(1)) gives us some explanations and descriptions on an important aspect of the asymmetrical movement of time series in two different (up-ward and down-ward) phases. For an illustration, we give some sample paths of the stationary SSAR(1) process in Figure 1. Also some stationary distributions satisfying the SSAR(1) model are shown in Figure 2. These figures suggest that we can produce asymmetrical patterns of sample paths and skewed stationary distributions with flexible moment properties even if the underlying disturbances follow the Gaussian distribution.

$< Figures$ 1 *and* 2 $>$

The simple SSAR model has been introduced from an econometric application and there are some intuitive reasons why the SSAR models are useful for econometric applications as we shall mention to in Section 2.2 and Section 5. Also it should be noted that the SSAR time series models are different from the threshold autoregressive (TAR) models, which have been extensively discussed in the non-linear time series analysis. Although there are many variants of the TAR models, the simplest form (often denoted as TAR(1)) without constant terms can be written as

$$y_t = \begin{cases} a_1 y_{t-1} + \sigma v_t & \text{if} \quad y_{t-1} \geq r \\ \\ b_1 y_{t-1} + \sigma v_t & \text{if} \quad y_{t-1} < r \end{cases} , \tag{1.6}$$

where r, a_1, b_1, σ are unknown parameters and $\{v_t\}$ are a sequence of i.i.d. random variables with $E(v_t) = 0$ and $E(v_t^2) = 1$. We notice that the TAR model given by (1.6) is well-defined in the form of (1.2). Although there are two phases in the TAR(1) model, given the past information $\{y_s; s \leq t-1\}$ there is no uncertainty on the next phase of y_t, because it is a function of the realized time series in the past.

In this paper we shall discuss some statistical properties of the SSAR models and their estimation methods. In Section 2, we introduce the SSAR models in the general form and discuss the relation between the SSAR models and some disequilicruim econometric models. Then we give some statistical properties of the SSAR models in Section 3. In Section 4, two estimation methods for the SSAR models will be discussed. Some concluding remarks will be given in Section 5.

2. The SSAR Models

2.1 A Class of the SSAR Models

Let $\boldsymbol{y}_t$ be an $m \times 1$ vector of the endogenous variables. The SSAR model we consider in this paper is represented by

$$(2.1) \qquad \boldsymbol{y}_t = \begin{cases} \boldsymbol{\mu}_1 + \boldsymbol{A}\boldsymbol{y}_{t-1} + \boldsymbol{D}_1\boldsymbol{v}_t & \text{if} \quad \boldsymbol{e}_1'\boldsymbol{y}_t \geq \boldsymbol{e}_1'\boldsymbol{y}_{t-1} \\ \boldsymbol{\mu}_2 + \boldsymbol{B}\boldsymbol{y}_{t-1} + \boldsymbol{D}_2\boldsymbol{v}_t & \text{if} \quad \boldsymbol{e}_1'\boldsymbol{y}_t < \boldsymbol{e}_1'\boldsymbol{y}_{t-1} \end{cases},$$

where $\boldsymbol{e}_1' = (1, 0, \cdots, 0)$ and $\boldsymbol{\mu}_i'$ $(i = 1, 2)$ are $1 \times m$ vectors of constants, and $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{D}_i$ $(i = 1, 2)$ are $m \times m$ matrices. The initial value $\boldsymbol{y}_0$ is either a fixed vector or a random vector, but the latter should be used for the stationary SSAR models. The SSAR model given by (2.1) is denoted by $SSAR_m(1)$.
We note that the condition $\boldsymbol{e}_1'\boldsymbol{y}_t \geq \boldsymbol{e}_1'\boldsymbol{y}_{t-1}$ in (2.1) has been used instead of the condition $\boldsymbol{e}_m'\boldsymbol{y}_t \geq \boldsymbol{e}_m'\boldsymbol{y}_{t-1}$ with $\boldsymbol{e}_m' = (0, 0, \cdots, 1)$ in Kunitomo and Sato (1996a), for instance. This change in our formulation does not harm any essential argument below.
The disturbance terms in (2.1) satisfy $E(\boldsymbol{v}_t) = 0$ and the variance-covariance matrix of $\boldsymbol{D}_i\boldsymbol{v}_t$ is denoted by $\boldsymbol{\Omega}_i$ $(= \boldsymbol{D}_i\boldsymbol{D}_i'\,; i = 1, 2)$.

We assume either
(i) $\{\boldsymbol{v}_t\}$ are absolutely continuous (mutually) independent random variables with the density function $g(\boldsymbol{v})$ which is continuous and everywhere positive in $\boldsymbol{R}^m$,
or
(ii) $\boldsymbol{D}_i\boldsymbol{v}_t = \sigma_i\boldsymbol{e}_1 v_t$ and $\{v_t\}$ are absolutely continuous (mutually) independent scalor random variables with the density function $g(v)$, which is continuous and everywhere positive in $\boldsymbol{R}$.

In the first case the disturbance terms $\{\boldsymbol{v}_t\}$ are distributed with $E(\boldsymbol{v}_t\boldsymbol{v}_t') = I_m$ and we assume that $\boldsymbol{\Omega}_i$ $(i = 1, 2)$ are positive definite matrices. In the second case the disturbance terms $\{v_t\}$ are distributed with $E(v_t) = 0$ and $E(v_t^2) = 1$. This corresponds to the Markovian representation of the univariate SSAR model given by

$$(2.2) \qquad y_t = \begin{cases} a_0 + \sum_{j=1}^{p} a_j y_{t-j} + \sigma_1 v_t & \text{if} \quad y_t \geq y_{t-1} \\ b_0 + \sum_{j=1}^{p} b_j y_{t-j} + \sigma_2 v_t & \text{if} \quad y_t < y_{t-1} \end{cases},$$

where $\{a_j; j = 0, \cdots, p\}$, $\{b_j; j = 0, \cdots, p\}$, and $\{\sigma_i; i = 1, 2\}$ are unknown parameters with the condition $\sigma_i > 0$ $(i = 1, 2)$. The initial conditions $y_{-p+1}, y_{-p+2}, \cdots, y_0$ are fixed numbers or random variables, but the latter should be used for the stationary SSAR models. The univariate SSAR model given by (2.2) is denoted by $SSAR(p)$.
If we define $p \times 1$ vectors $\boldsymbol{y}_t$ and $\boldsymbol{\mu}_i$ $(i = 1, 2)$ by

$$(2.3) \qquad \boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{pmatrix}, \quad \boldsymbol{\mu}_1 = \begin{pmatrix} a_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \boldsymbol{\mu}_2 = \begin{pmatrix} b_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and $p \times p$ matrices

$$
(2.4) \qquad \boldsymbol{A} = \begin{pmatrix} a_1 & \cdots & \cdots & a_p \\ 1 & & \mathbf{0} & \\ & \ddots & & \\ \mathbf{0} & & 1 & 0 \end{pmatrix} , \quad \boldsymbol{B} = \begin{pmatrix} b_1 & \cdots & \cdots & b_p \\ 1 & & \mathbf{0} & \\ & \ddots & & \\ \mathbf{0} & & 1 & 0 \end{pmatrix} ,
$$

then the resulting model can be regarded as a special case of (2.1) if we set $\boldsymbol{D}_i \boldsymbol{v}_t = \sigma_i \boldsymbol{e}_1 v_t$ and $m = p$. The SSAR model in (1.3) is a special case of (2.2) when $a_0 = b_0 = 0$ and $p = 1$.

We note that in (2.1) there are two phases (or regimes) at time t given $\mathcal{F}_{t-1}$, where $\mathcal{F}_{t-1}$ is the σ-field generated by $\{\boldsymbol{y}_s; s \leq t-1\}$. Then there is a basic question that the simultaneity among two phases and the values of the endogenous variables do not cause any logical inconsistency as a statistical model. This problem has been called the coherency problem in some econometric literature and the condition for the logical consistency has been called the coherency condition in the context of the disequilibrium econometric models (see Quandt (1988) for instance), which will be illustrated by an example in Section 2.2. The conditions of $\boldsymbol{e}_1' \boldsymbol{y}_t \geq \boldsymbol{e}_1' \boldsymbol{y}_{t-1}$ and $\boldsymbol{e}_1' \boldsymbol{y}_t < \boldsymbol{e}_1' \boldsymbol{y}_{t-1}$ can be rewritten as

$$
(2.5) \qquad \boldsymbol{e}_1' \boldsymbol{D}_1 \boldsymbol{v}_t \geq \boldsymbol{e}_1' (\boldsymbol{I}_m - \boldsymbol{A}) \boldsymbol{y}_{t-1} - \boldsymbol{e}_1' \boldsymbol{\mu}_1 ,
$$

and

$$
(2.6) \qquad \boldsymbol{e}_1' \boldsymbol{D}_2 \boldsymbol{v}_t < \boldsymbol{e}_1' (\boldsymbol{I}_m - \boldsymbol{B}) \boldsymbol{y}_{t-1} - \boldsymbol{e}_1' \boldsymbol{\mu}_2 ,
$$

respectively. A set of the coherency conditions for (2.1) can be summarized by a $1 \times (m+1)$ vector of unknown parameters

$$
(2.7) \qquad \begin{aligned} \frac{1}{\sigma_1} [\boldsymbol{e}_1' (\boldsymbol{I}_m - \boldsymbol{A}), \ -\boldsymbol{e}_1' \boldsymbol{\mu}_1] &= \frac{1}{\sigma_2} [\boldsymbol{e}_1' (\boldsymbol{I}_m - \boldsymbol{B}), \ -\boldsymbol{e}_1' \boldsymbol{\mu}_2] \\ &= [\boldsymbol{r}', r_0] , \end{aligned}
$$

where $\boldsymbol{r}' = (r_i)$ is a $1 \times m$ vector, r_0 is a scalar, and the scale parameters σ_j $(j = 1, 2)$ satisfy $\sigma_j^2 = \boldsymbol{e}_1' \boldsymbol{\Omega}_j \boldsymbol{e}_1 = \boldsymbol{e}_1' \boldsymbol{D}_j \boldsymbol{D}_j' \boldsymbol{e}_1$. For the normalizaion of the scale parameters, we may use a $1 \times m$ vector

$$
(2.8) \qquad \frac{1}{\sigma_1} \boldsymbol{e}_1' \boldsymbol{D}_1 = \frac{1}{\sigma_2} \boldsymbol{e}_1' \boldsymbol{D}_2 = \boldsymbol{d}'
$$

where we take $\boldsymbol{d}'\boldsymbol{d} = 1$. It is apparent from our formulation that the condition given by (2.8) is automatically satisfied for the p–th order univariate SSAR model.

2.2 The SSAR models and a Disequilibrium Econometric Model

In order to explain the motivation of introducing the SSAR models, we shall give a simple econometric example, which is a modified version of the disequilibrium econometric model originally investigated by Laffont and Garcia (1977). The disequilibrium econometric model of our concern here was first developed by Fair and Jaffee (1972), but since then a number of different econometric models have been proposed.

The standard econometric model consists of the demand and supply functions in a small market. Let D_t and S_t be the demand and supply of a commodity at time t. By assuming that they are linear, these two equations are written as

$$(2.9) \qquad \begin{cases} D_t &= \beta_1 p_t + \gamma_1' z_{1t} + u_{1t} \\ \\ S_t &= \beta_2 p_t + \gamma_2' z_{2t} + u_{2t} \end{cases} ,$$

where p_t is the price level, z_{1t} and z_{2t} are the predetermined variables appearing in the demand and supply equations, respectively. The demand shocks and the supply shocks are described by the disturbance terms u_{1t} and u_{2t}, respectively. The coefficients β_1, β_2, γ_1, and γ_2 are unknown parameters. For the expository simplicity, we set the number of predetermined variables is 2, γ_1, γ_2 being scalors, and we take $z_{1t} = p_{t-1}$ and $z_{2t} = p_{t-2}$ as an example.

The equilibrium condition explained by economics is given by $q_t = D_t = S_t$, where q_t is the quantity of the commodity traded in the market at time t. Instead of the equilibrium condition, however, Fair and Jaffee (1972) introduced the short-side condition

$$(2.10) \qquad q_t = \min(D_t, S_t) \quad .$$

We note that when we substitute (2.10) for the equilibrium condition, the econometric model consisting of (2.9) and (2.10) is not complete in the proper statistical sense. The quantity variable q_t is determined by (2.9) and (2.10) once the price variable p_t is given. There should be some dynamic process for the price level (or the quantity traded) and there have been several formulations to make the disequilibrium econometric model complete. In this section we shall adopt one simple formulation by Laffont and Garcia (1977). If $D_t > S_t$ at t in the market, there is an excess demand, which raises the price variable p_t . On the other hand, if $S_t > D_t$ at t in the market, there is an excess supply, causing p_t to go down. This consideration leads to the linearized price adjustment process [2]

$$(2.11) \qquad \Delta p_t = \begin{cases} \delta_1 (D_t - S_t) & \text{if } D_t \geq S_t \\ \delta_2 (D_t - S_t) & \text{if } D_t < S_t \end{cases} ,$$

where $\Delta p_t = p_t - p_{t-1}$. Since the coefficients δ_1 and δ_2 represent the adjustment speeds in the up-ward phase (or regime) and in the down-ward phase (or regime), we assume that $\delta_i > 0$ $(i = 1, 2)$ and they do not necessarily take the same value. Possibly there could be some economic justifications for these differences including the market behaviors of economic agents and the market microstructures.

We now consider the disequilibrium econometric model consisting of (2.9), (2.10), and (2.11). Our investigation aims to shed some new light on the time series aspects of this type of disequilibrium model, as this is an area that has largely been ignored in the econometric literature.

Let the 1×2 vector of endogenous variables $y_t' = (p_t, q_t)$ and the 1×2 vector of predetermined variables $z_t' = (p_{t-1}, p_{t-2})$. If the price variable p_t is in the up-ward phase, then we have the condition that $\Delta p_t \geq 0$, $q_t = S_t$ and

$$D_t = q_t + (D_t - S_t) = q_t + (1/\delta_1)\Delta p_t ,$$

[2] Alternatively, we can use the condition for Δp_{t+1} instead of (2.11) as discussed in Kunitomo and Sato (1996a). Then we have a slightly different SSAR model as the result.

provided that $\delta_1 > 0$. Hence the system of demand and supply functions can be rewritten as

$$(2.12)\qquad \begin{pmatrix} -\beta_1+\delta_1^{-1} & 1 \\ -\beta_2 & 1 \end{pmatrix} \boldsymbol{y}_t = \begin{pmatrix} \delta_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} \boldsymbol{y}_{t-1} + \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \boldsymbol{z}_t + \boldsymbol{u}_t \ ,$$

where $\boldsymbol{u}_t' = (u_{t1}, u_{t2})$ is a 1×2 vector of the disturbance terms, and γ_i $(i = 1, 2)$ are scalor coefficients in this example. If we assume $\delta_1 > 0$, we can solve (2.12) with respect to $\boldsymbol{y}_t$. The reduced form equations for (2.12) becomes

$$(2.13)\quad \boldsymbol{y}_t = \frac{1}{d_1}\begin{pmatrix} -\delta_1^{-1} & 0 \\ -\beta_2\delta_1^{-1} & 0 \end{pmatrix} \boldsymbol{y}_{t-1} + \frac{1}{d_1}\begin{pmatrix} -\gamma_1 & \gamma_2 \\ -\beta_2\gamma_1 & (\beta_1-\delta_1^{-1})\gamma_2 \end{pmatrix} \boldsymbol{z}_t + \boldsymbol{v}_t^{(1)} \ ,$$

where the disturbance vector of the reduced form equation is given by

$$(2.14)\qquad \boldsymbol{v}_t^{(1)} = \frac{1}{d_1}\begin{pmatrix} -1 & 1 \\ -\beta_2 & \beta_1-\delta_1^{-1} \end{pmatrix} \boldsymbol{u}_t \ ,$$

and $d_1 = \beta_1 - \beta_2 - \delta_1^{-1}$. In this phase the first component of $\boldsymbol{y}_t$, the price level p_t at t, follows

$$(2.15)\qquad p_t = [\frac{-1-\gamma_1\delta_1}{d_1\delta_1}]p_{t-1} + [\frac{\gamma_2}{d_1}]p_{t-2} + \sigma_1 v_t \ ,$$

where we take $\sigma_1^2 = (1/d_1^2)(-1,1)\boldsymbol{\Sigma}(-1,1)'$ and $v_t = [-u_{1t} + u_{2t}]/\sigma_1$.

Similarly, if the price variable p_t is in the down-ward phase, we have the condition that $\Delta p_t < 0$, $q_t = D_t$ and

$$S_t = q_t - \frac{1}{\delta_2}\Delta p_t \ ,$$

provided that $\delta_2 > 0$. Hence the system of the demand and supply functions in this phase can be written as

$$(2.16)\qquad \begin{pmatrix} -\beta_1 & 1 \\ -\beta_2-\delta_2^{-1} & 1 \end{pmatrix} \boldsymbol{y}_t = \begin{pmatrix} 0 & 0 \\ -\delta_2^{-1} & 0 \end{pmatrix} \boldsymbol{y}_{t-1} + \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} \boldsymbol{z}_t + \boldsymbol{u}_t \ .$$

By the same arguments to (2.12) and (2.13), the reduced form equations in the second phase are given by

$$(2.17)\quad \boldsymbol{y}_t = \frac{1}{d_2}\begin{pmatrix} -\delta_2^{-1} & 0 \\ -\beta_1\delta_2^{-1} & 0 \end{pmatrix} \boldsymbol{y}_{t-1} + \frac{1}{d_2}\begin{pmatrix} -\gamma_1 & \gamma_2 \\ -(\beta_2+\delta_2^{-1})\gamma_1 & \beta_1\gamma_2 \end{pmatrix} \boldsymbol{z}_t + \boldsymbol{v}_t^{(2)} \ ,$$

where the disturbance vector of the reduced form in this case is given by

$$(2.18)\qquad \boldsymbol{v}_t^{(2)} = \frac{1}{d_2}\begin{pmatrix} -1 & 1 \\ -(\beta_2+\delta_2^{-1}) & \beta_1 \end{pmatrix} \boldsymbol{u}_t \ ,$$

and $d_2 = \beta_1 - \beta_2 - \delta_2^{-1}$. In the second phase the price level p_t at t follows

$$(2.19)\qquad p_t = [\frac{-1-\delta_2\gamma_1}{d_2\delta_2}]p_{t-1} + [\frac{\gamma_2}{d_2}]p_{t-2} + \sigma_2 v_t \ ,$$

where we take $\sigma_2^2 = (1/d_2^2)(-1,1)\boldsymbol{\Sigma}(-1,1)'$.

By taking 1×2 vector $\boldsymbol{e}_1' = (1, 0)$, the condition that the market is in the excess demand $(\Delta p_t \geq 0)$ is equivalent to

$$(2.20)\qquad \boldsymbol{e}_1'\boldsymbol{y}_t \geq \boldsymbol{e}_1'\boldsymbol{y}_{t-1} \ .$$

The coherency conditions given by (2.7) are automatically satisfied in this example because there are two phases and the structural equations in each phase are well defined. Hence by noting that

$$z_t = e_1 e_1' y_{t-1} + e_2 e_1' y_{t-2} ,$$

we have the 2-dimensional 2nd order SSAR model (denoted as $SSAR_2(2)$) without constant terms as the solution of the disequilibrium econometric model given by (2.9), (2.10), and (2.11). It has also a Markovian representation as the $SSAR_4(1)$ model without constant terms in the form of (2.1) by defining a 4×1 state vector $\boldsymbol{Y}_t = (\boldsymbol{y}_t', \boldsymbol{y}_{t-1}')'$ and using the corresponding matrix representations of parameters which are similar to the scalor case as (2.3) and (2.4). The price level p_t follows the univariate SSAR(2) model without constant terms in this case. We further note that when $\delta_1 = \delta_2$, the vector $\boldsymbol{y}_t$ follows the $SSAR_2(2)$ model while the price level p_t follows the standard AR(2) model. It is evident that if there were more exogenous variables in (2.9) such as the constant terms, the time trends, and the lagged endogenous variables, for instance, the resulting reduced form would have been in the form of a more complicated SSAR model.

The most important feature of this representation is that the endogenous variables may take quite different values in two different phases or regimes. This type of statistical time series model could be regarded as ***threshold model*** in the nonlinear time series literature. However, we should note that there is no simultaneity in the standard threshold models in the non-linear time analysis. Since in the time series model given by (2.1) the endogenous variables and two phases at time t are determined simultaneously in a particular way, we have called it a ***simultaneous switching time series model***. This simultaneity has not only an important economic interpretation, but also a new aspect in the non-linear time series modelling.

3. Some Statistical Properties

3.1 The SSAR Models and Ergodicity

The first important property of a statistical time series model is whether it is ergodic or not. For the Markovian time series models, the geometrical ergodicity and the related concepts have been developed in the nonlinear time series analysis. For the sake of completeness, we mention to its definition and the drift criterion. For the more precise definitions of related concepts including irreducibility, aperiodicity, and ergodicity of the Markov chains with the general state space, see Tong (1990), or Nummelin (1984). Lemma 1 was taken from Appendix 1.3 of Tong (1990).

Definition 1 : (i) $\{\mathrm{y_t}\}$ is geometrically ergodic if there exists a probability measure π on $(\boldsymbol{R}^m, \mathcal{B}(\boldsymbol{R}^m))$, a positive constant $\rho < 1$, and π-integrable non-negative measurable function $h(\cdot)$ such that

(3.1) $$\|P^n(\boldsymbol{x}, \cdot) - \pi(\cdot)\|_\tau \leq \rho^n h(\boldsymbol{x}) ,$$

where $\| \cdot \|_\tau$ denotes the total variation norm, $\boldsymbol{x} = (x_i)$ is an $m \times 1$ vector, and $P(\boldsymbol{x}, \cdot)$ is the transition probability.

(ii) $\{\mathbf{y_t}\}$ is ϕ-irreducible if for any $\boldsymbol{x} \in \boldsymbol{R}^m$ and $A \in \mathcal{B}(\boldsymbol{R}^m)$ with $\phi(A) > 0$ ($\phi(\cdot)$ is a σ-finite measure),

$$\sum_{n=1}^{\infty} P^n(\boldsymbol{x}, A) > 0 . \tag{3.2}$$

Lemma 1 : *Let $\{\boldsymbol{y}_t\}$ be ϕ-irreducible and aperiodic Markov chain. Suppose that there exists a compact set $\boldsymbol{C}$, a non-negative measurable function $G(\cdot)$, and constants $r > 1, \gamma > 0$, and $K > 0$ such that for $\boldsymbol{y} \in \boldsymbol{R}^m$*

$$E[rG(\boldsymbol{y}_t)|\boldsymbol{y}_{t-1} = \boldsymbol{y}] < G(\boldsymbol{y}) - \gamma \quad (\boldsymbol{y} \notin \boldsymbol{C}) , \tag{3.3}$$

and

$$E[G(\boldsymbol{y}_t)|\boldsymbol{y}_{t-1} = \boldsymbol{y}] < K \quad (\boldsymbol{y} \in \boldsymbol{C}) . \tag{3.4}$$

Then $\{\boldsymbol{y}_t\}$ is geometrically ergodic.

A probability measure $\pi(\cdot)$ in (3.1) satisfies the stationarity condition

$$\pi(\boldsymbol{A}) = \int_{\boldsymbol{x}} P(\boldsymbol{x}, \boldsymbol{A})\pi(d\boldsymbol{x}) \tag{3.5}$$

for any $\boldsymbol{A} \in \mathcal{B}(\boldsymbol{R}^m)$. Then if we take the initial distribution as the same as $\pi(\cdot)$, the process $\{\boldsymbol{y}_t\}$ is strictly stationary.

Now we consider the $SSAR_m(1)$ model given by (2.1). When $m = 1$, we have the necessary and sufficient condition on the geometric erogodicity for the SSAR model. It is also a sufficient condition on the existence of moments if we assume the existence of moments for the disturbance terms. For the sake of completeness and an illustration, we state this result and its proof in a formal way.

Proposition 1 : *In the SSAR model (2.1) when $m = 1$, assume (i) the coherency conditions given by*

$$\begin{aligned} r_0 &= -\frac{\mu_1}{\sigma_1} = -\frac{\mu_2}{\sigma_2} , \\ r_1 &= \frac{1-A}{\sigma_1} = \frac{1-B}{\sigma_2} , \end{aligned} \tag{3.6}$$

where $\boldsymbol{D}_i = \sigma_i$ $(i = 1, 2)$.
Then the necessary and sufficient conditions for the geometric ergodicity are given by

$$A < 1, B < 1, AB < 1 . \tag{3.7}$$

(ii) In addition to the above conditions, assume that $E[|v_t|^k] < +\infty$ for any positive integer $k \geq 1$ and $E[|y_0|^k] < +\infty$ for any positive integer $k \geq 1$.
Then

$$E[|y_t|^k] < +\infty . \tag{3.8}$$

Proof : [1] We first prove the second part. When $m = 1$, we can take the criterion function

$$G(x) = \begin{cases} k_1^k x^k + c_1 & x > 0 \\ k_2^k |x|^k + c_1 & x \leq 0 \end{cases}, \tag{3.9}$$

where k is any positive integer and c_1 is a positive constant, and positive constants k_1, k_2, are defined shortly.

We first consider the case when $y_{t-1} = x > M > 0$. Then

$$\begin{aligned} E[G(y_t)|y_{t-1} = x] &\leq [\sum_{i=0}^{k-1} c_i^* x^i] + k_1^k A^k x^k P\{v_t \geq r_0 + rx\} \\ &+ k_1^k B^k x^k P\{r_0 + (r_1 - \frac{1}{\sigma_2})x < v_t < r_0 + r_1 x\} \\ &+ k_2^k B^k x^k P\{v_t \leq r_0 + (r_1 - \frac{1}{\sigma_2})x\} , \end{aligned} \tag{3.10}$$

where c_i^* $(i = 0, \cdots, k-1)$ are positive constants. Because $A < 1, B < 1$, and $AB < 1$, we can take $k_1 > 0$ and $k_2 > 0$ such that $1 > A > -k_2/k_1$ and $1 > B > -k_1/k_2$ and then $k_2^k > (-A)^k k_1^k$ for $A \leq 0$ and $k_1^k > (-B)^k k_2^k$ for $B \leq 0$ for $k \geq 1$.

We note that the conditions $B < 0$ and $0 \leq B < 1$ correspond to the cases when $1/\sigma_2 < r_1 < 1/\sigma_2 + 1/\sigma_1$ and $0 < r_1 \leq 1/\sigma_2$, respectively. When $0 < r_1 \leq 1/\sigma_2$ $(0 \leq B < 1)$, the coefficients of second and fourth terms on the right-hand side of (3.10) can be small. Then by taking a sufficiently large M, we have

$$E[G(y_t)|y_{t-1} = x] \leq c_2(M) + \delta_1 k_1^k x^k , \tag{3.11}$$

where $0 < \delta_1 < 1$ and $c_2(M)$ is a positive constant depending M. When $1/\sigma_2 < r_1 \leq 1/\sigma_2 + 1/\sigma_1$ $(B < 0)$, the coefficients of the second and third terms on the right-hand side of (3.10) can be small. Because $k_2^k |B|^k < k_1^k$ in this case, we can take a sufficiently large M and we also have

$$E[G(y_t)|y_{t-1} = x] \leq c_3(M) + \delta_2 k_1^k x^k,$$

where $0 < \delta_2 < 1$ and $c_3(M)$ depending on M is a positive constant. By taking $\max\{\delta_1, \delta_2\} < \delta_3 < 1$, we have

$$E[G(y_t)|y_{t-1} = x] \leq c_4(M) + \delta_3 G(x),$$

where $c_4(M)$ is a positive constant. We can also use the similar arguments for the case when $y_{t-1} = x < -M < 0$. Then we can take positive constants $0 < \delta < 1$ and $c_5(M)$ depending on M for any $y_{t-1} = x$ such that

$$E[G(y_t)|y_{t-1} = x] \leq c_5(M) + \delta G(x) . \tag{3.12}$$

Because

$$\begin{aligned} E[G(y_t)|y_0 = x] &= E\{E[G(y_t)|y_{t-1}]|y_0\} \\ &\leq c_5(M)[1 + \delta + \cdots + \delta^{t-1}] + \delta^t G(y_0) , \end{aligned} \tag{3.13}$$

is bounded, we have the desired result for any positive integer k.
[2] (Sufficiency) Next we consider the assertion (i) on the geometrical ergodicity. When $m = 1$, we immediately know that $\{y_t\}$ is ϕ-irreducible and aperiodic Markov chain. As in [1] we take $k = 1, M^*$ $(> M > 0)$, and a compact set $C = [-M^*, M^*]$, where M has been taken to satisfy (3.12). Then by taking a sufficiently large M^* and using Lemma 1, we obtain that $\{y_t\}$ is geometrically ergodic under the conditions given by (3.7).
(Necessity) The necessity for the geometrical ergodic conditions (3.7) is based on the arguments which are similar to the proof of Theorem 2.1 of Chan et. al. (1985). However, there is one important difference. Because of the coherency conditions given by (3.6), we can easily lead to the contradictions to the geometrical ergodicity when we have the boundary conditions $A = 1, B = 1$, or $AB = 1$. They are straightforward, but quite tedious and we omit the details. (Q.E.D.)

We give some sufficient conditions for the existence of higher order moments and their boundedness when $m \geq 1$. All of them are sufficient, but they are often too strong and we do not necessarily need those conditions. (See the conditions in Proposition 1.) The proofs of the following two propositions are straightforward and so brief.

Proposition 2 *In the SSAR model when $m \geq 1$, assume (i) the coherency conditions given by (2.7), (ii) a sufficient condition for the geometric ergodicity $0 < \rho_1 < 1$, where*

$$\rho_1 = \max\{\lambda_{\max}(\boldsymbol{A}'\boldsymbol{A}), \lambda_{\max}(\boldsymbol{B}'\boldsymbol{B})\},$$

and $\lambda_{\max}(\boldsymbol{C})$ is the maximum characteristic root of a symmetric matrix $\boldsymbol{C}$ in its absolute value, (iii) $E[\|\boldsymbol{v}_t\|^k] < +\infty$ for any positive integer $k \geq 1$, and (iv) $E[\|\boldsymbol{y}_0\|^k] < +\infty$ for any positive integer $k \geq 1$.
Then

$$E[\|\boldsymbol{y}_t\|^k] < +\infty \,. \tag{3.14}$$

Proof : When $m \geq 1$, we can take the criterion function

$$G(\boldsymbol{x}) = \|\boldsymbol{x}\|^k \,, \tag{3.15}$$

where $\boldsymbol{x} = (x_i)$. We only show (3.14) for $k = 1$. When $k = 1$, we have

$$\begin{aligned} E[G(\boldsymbol{y}_t)|\boldsymbol{y}_{t-1} = \boldsymbol{x}] &\leq c_6 + E[\|\boldsymbol{A}(t)\|]\|\boldsymbol{x}\| + E[\|\boldsymbol{D}(t)\boldsymbol{u}_t\|] \\ &\leq c_7 + \sqrt{\rho_1} G(\boldsymbol{x}) \,, \end{aligned} \tag{3.16}$$

where $\boldsymbol{A}(t) = \boldsymbol{A}I_t^{(1)} + \boldsymbol{B}I_t^{(2)}$ and c_i $(i = 6, 7)$ are positive constants. The rest of arguments and those for $k \geq 2$ are essentially the same as the proof of Proposition 1. (Q.E.D.)

Proposition 3 *In the SSAR model when $m \geq 1$, assume (i) the coherency conditions given by (2.7), (ii) a sufficient condition for the geometric ergodicity $0 < \rho_2 < 1$ or $0 < \rho_3 < 1$, where*

$$\rho_2 = \max_{1 \leq j \leq m} \{\sum_{i=1}^{m} |a_{ij}|, \sum_{i=1}^{m} |b_{ij}|\},$$

and

$$\rho_3 = \max_{1\le i\le m}\{\sum_{j=1}^{m}|a_{ij}|,\sum_{j=1}^{m}|b_{ij}|\}$$

for $\boldsymbol{A} = (a_{ij}), \boldsymbol{B} = (b_{ij})$, *(iii)* $E[\|\boldsymbol{v}_t\|^k] < +\infty$ *for any positive integer* $k \ge 1$, *and (iv)* $E[\|\boldsymbol{y}_0\|^k] < +\infty$ *for any positive integer* $k \ge 1$.
Then

$$E[\|\boldsymbol{y}_t\|^k] < +\infty \,. \tag{3.17}$$

Proof : For $\boldsymbol{x} = (x_i)$, we take the criterion function

$$G(\boldsymbol{x}) = (\sum_{i=1}^{m}|x_i|)^k \tag{3.18}$$

for the first condition in (ii) and

$$G(\boldsymbol{x}) = (\max_{i=1,\cdots,m}|x_i|)^k \tag{3.19}$$

for the second condition in (ii), respectively. Then we use the same arguments as the proofs of Proposition 1 and Proposition 2. (Q.E.D.)

3.2 The Univariate SSAR(p) Model

We give some sufficient conditions for the geometric ergodicity and existence of higher order moments for the $p-$th order univariate SSAR model. We need some special consideration because the disturbance term in the Markovian representation is degenerate in a sense. From the Markovian representation of (2.4), we know that $\rho_2 = \rho_3 = 1$ for the univariate SSAR(1) models when $p \ge 2$. Hence Proposition 3 is useless for these SSAR models. The following conditions we shall give are sufficient, but often too strong and we do not necessarily need those conditions. The proof of Proposition 4 is based on the method used in Chan and Tong (1985) for the threshold autoregressive models with minor modifications.

Proposition 4 *In the $p-$th order SSAR model given by (2.2), assume (i) the coherency conditions given by*

$$\begin{aligned} r_0 &= -\frac{a_0}{\sigma_1} = -\frac{b_0}{\sigma_2}\,, \\ r_1 &= \frac{1-a_1}{\sigma_1} = \frac{1-b_1}{\sigma_2}\,, \\ r_j &= -\frac{a_j}{\sigma_1} = -\frac{b_j}{\sigma_2}\ (j=2,\cdots,p)\,, \end{aligned} \tag{3.20}$$

(ii) a sufficient condition $0 < \rho_4 < 1$, *where*

$$\rho_4 = \max\{\sum_{j=1}^{p}|a_j|,\sum_{j=1}^{p}|b_j|\},$$

(iii) $\{v_t\}$ has an absolutely continuous distribution with respect to the Lebesgue measure on $\boldsymbol{R}$, and its density function $g(v)$ is continuous and strictly positive almost everywhere, and (iv) $E[\|v_t\|] < +\infty$.
Then $\{y_t\}$ is geometrically ergodic.

Proof :
(i) Because of the assumption $\sum_{j=1}^{p}|a_j| < 1$, we can take $\xi_j^{(1)}$ $(j = 1, \cdots, m = p)$ such that

$$1 < \frac{\xi_1^{(1)}}{\xi_2^{(1)}} < \frac{\xi_1^{(1)}}{\xi_3^{(1)}} < \cdots < \frac{\xi_1^{(1)}}{\xi_p^{(1)}} \tag{3.21}$$

and

$$1 > \sum_{j=1}^{p} |a_j| \frac{\xi_1^{(1)}}{\xi_j^{(1)}} \ . \tag{3.22}$$

Then we have $1 > \xi_{j+1}^{(1)}/\xi_j^{(1)}$ $(j = 1, \cdots, p-1)$. By the same token for the condition $\sum_{j=1}^{p}|b_j| < 1$, we can also take $\xi_j^{(2)}$ $(j = 1, \cdots, p)$ such that $1 > \xi_{j+1}^{(2)}/\xi_j^{(2)}$ $(j = 1, \cdots, p-1)$ and

$$1 > \sum_{j=1}^{p} |b_j| \frac{\xi_1^{(2)}}{\xi_j^{(2)}} \ . \tag{3.23}$$

By rearranging $\nu_j = \min\{\frac{\xi_1^{(1)}}{\xi_j^{(1)}}, \frac{\xi_1^{(2)}}{\xi_j^{(2)}}\}$ $(j = 1, \cdots, p)$, under the assumptions $(0 < \rho_4 < 1)$ we can take ξ_j $(j = 1, \cdots, p)$ and θ such that $1 > \theta > \xi_{j+1}/\xi_j$ $(j = 1, \cdots, p-1)$,

$$1 < \frac{\xi_1}{\xi_2} < \frac{\xi_1}{\xi_3} < \cdots < \frac{\xi_1}{\xi_p} \ , \tag{3.24}$$

and

$$1 > \theta > \max\{\sum_{j=1}^{p} |a_j| \frac{\xi_1}{\xi_j}, \sum_{j=1}^{p} |b_j| \frac{\xi_1}{\xi_j}\} \ . \tag{3.25}$$

We take the criterion function

$$G(\boldsymbol{x}) = 1 + \max_{1 \le j \le p} |x_j| \xi_j \ . \tag{3.26}$$

Let a vector process $\boldsymbol{y}_t' = (y_t, y_{t-1}, \cdots, y_{t-p+1})$ and consider a Markovian representation for $\{\boldsymbol{y}_t\}$. Then it is straightforward to show

$$\begin{aligned}
&E[G(\boldsymbol{y}_t)|\boldsymbol{y}_{t-1} = \boldsymbol{x}] \\
\le\ & c_8 + E[\max\{\sum_{j=1}^{p} |A_j(t)||y_{t-j}|\xi_1, |x_1|\xi_2, \cdots, |x_{p-1}|\xi_p\}|\boldsymbol{y}_{t-1} = \boldsymbol{x}] \\
\le\ & c_9 + E[\max\{\sum_{j=1}^{p} |A_j(t)||y_{t-j}|\xi_1, \theta|x_1|\xi_1, \cdots, \theta|x_{p-1}|\xi_{p-1}\}|\boldsymbol{y}_{t-1} = \boldsymbol{x}] \\
\le\ & c_{10} + \max\{[\max\{\sum_{j=1}^{p} |a_j| \frac{\xi_1}{\xi_j}, \sum_{j=1}^{p} |b_j| \frac{\xi_1}{\xi_j}\}] \\
& \quad \times [\max\{|x_1|\xi_1, \cdots, |x_p|\xi_p\}], \theta|x_1|\xi_1, \cdots, \theta|x_{p-1}|\xi_{p-1}\} \\
\le\ & c_{11} + \theta G(\boldsymbol{x}),
\end{aligned} \tag{3.27}$$

where $A_j(t) = a_j I_t^{(1)} + b_j I_t^{(2)}$, c_i $(i = 8, \cdots, 11)$ are some positive constants, and $I_t^{(1)} = I(y_t \geq y_{t-1})$ and $I_t^{(2)} = I(y_t < y_{t-1})$ are defined by the indicator function $I(\cdot)$.

(ii) We consider the case when $p = m$. We define a function by

$$\sigma(x) = x[\frac{1}{\sigma_1} I_{\{x \geq 0\}} + \frac{1}{\sigma_2} I_{\{x<0\}}] \tag{3.28}$$

for any x .

Then given $\boldsymbol{y}_{t-1}' = (y_{t-1}, \cdots, y_{t-m}) = (x_1, \cdots, x_m)$ and a set $\boldsymbol{A} = (a_1, b_1) \times x_1 \times \cdots \times x_{m-1} \in \boldsymbol{R}^m$,

$$P(\boldsymbol{x}, \boldsymbol{A}) = \int_{\sigma(a_1 - x_1)}^{\sigma(b_1 - x_1)} g[v_{t+1} - r_0 - \sum_{i=1}^{m} r_i x_i] dv_{t+1} . \tag{3.29}$$

Let $\eta = \min\{1/\sigma_1, 1/\sigma_2\} > 0$. Because $g(\cdot)$ is continuous and everywhere positive in $\boldsymbol{R}$, we can take a sufficiently small $\epsilon > 0$ such that for some compact set $(x_1, \cdots, x_m) \in \boldsymbol{C}_1$ and $g[v_{t+1} - r_0 - \sum_{i=1}^{m} r_i x_i] > \epsilon$ for any $(x_1, \cdots, x_m) \in \boldsymbol{C}_1$. Hence given $\boldsymbol{y}_{t-1} = \boldsymbol{x}$ we have

$$P(\boldsymbol{x}, \boldsymbol{A}) > \epsilon \eta (b_1 - a_1) > 0 .$$

Also we take a set $\boldsymbol{A} = (a_1, b_1) \times (a_2, b_2) \times x_1 \times \cdots \times x_{m-2} \in \boldsymbol{R}^m$ with $a_i < 0 < b_i$ $(i = 1, 2)$. Then

$$\begin{aligned} P^2(\boldsymbol{x}, \boldsymbol{A}) \quad = \quad & \int_{\sigma(a_2 - x_1)}^{\sigma(b_2 - x_1)} \{ \int_{\sigma(a_1 - y_{t+1})}^{\sigma(b_1 - y_{t+1})} g[v_{t+1} - r_0 - \sum_{i=1}^{m} r_i x_i] \\ & \times g[v_{t+2} - r_0 - r_1 y_{t+1} - \sum_{i=2}^{m} r_i x_{i-1}] dv_{t+1} dv_{t+2} , \end{aligned} \tag{3.30}$$

where $y_{t+1} = x_1 + \sigma(v_{t+1} - r_0 - \sum_{i=1}^{m} r_i x_i)$.

Then by the same argument, we can take a sufficiently small $\epsilon > 0$, and

$$P^2(\boldsymbol{x}, \boldsymbol{A}) > \epsilon^2 \eta (b_1 - a_1)(b_2 - a_2) > 0 .$$

Hence by using the assumptions in (iii), for $A = (a_1, b_1) \times \cdots \times (a_m, b_m) \in \boldsymbol{R}^m$ with $a_i < 0 < b_i$ $(i = 1, \cdots, m)$, we have

$$\inf_{\boldsymbol{x} \in \boldsymbol{C}_m} P^m(\boldsymbol{x}, \boldsymbol{A}) > 0 \tag{3.31}$$

for a compact set $\boldsymbol{C}_m$ including the origin in $\boldsymbol{R}^m$. This shows that the Markov chain for $\{\boldsymbol{y}_t\}$ is ϕ-irreducible.

(iii) By using the result in (i), we already know the growth condition in Lemma 1 is satisfied if we set $\boldsymbol{C} = \{\|\boldsymbol{x}\| \leq M\}$ for a large M. Because the Markov chain for $\{\boldsymbol{y}_t\}$ is aperiodic and $\phi-$irreducible as we have shown in (ii), we can apply Lemma 1. Hence we establish that $\{y_t\}$ is geometrically ergodic. (Q.E.D.)

Proposition 5 *In the $p-$th order univariate SSAR model (2.2) assume (i) the coherency conditions given by (3.20), (ii) a sufficient condition for the geometric ergodicity $\rho_4 < 1$, (iii) $E[|v_t|^k] < +\infty$ for any positive integer $k \geq 1$, and (iv) $E[|y_0|^k] < +\infty$ for any positive integer $k \geq 1$.*

Then

$$E[|y_t|^k] < +\infty . \tag{3.32}$$

Proof : The method of proof is similar to the first part of the proof of Propposition 4. We take the criterion function

$$G(\boldsymbol{x}) = 1 + (\max_{1 \le j \le p} |x_j|\xi_j)^k \tag{3.33}$$

for $\boldsymbol{x} = (x_i)$, where ξ_j $(j = 1, \cdots, p)$ are defined as in the proof of Proposition 4. Then we consider the Markovian representation for $\boldsymbol{y}_t^{'} = (y_t, y_{t-1}, \cdots, y_{t-p+1})$. For $k \ge 1$, we have

$$E[G(\boldsymbol{y}_t)|\boldsymbol{y}_{t-1} = \boldsymbol{x}] \le c_{12} + \theta G(\boldsymbol{x}), \tag{3.34}$$

for some positive c_{12} and $0 < \theta < 1$. The rest of our arguments is the same as the proof of Proposition 1. (Q.E.D.)

We should mention again that the above conditions given in this paper are quite strong and sufficient, but they are not necessary and could be improved. Also some of the results can be extended to more general cases easily. For an illustration, we will show the existence of moments for the univariate SSAR(p) model with the MA error.

Let $\{w_t\}$ be the i.i.d disturbance terms satisfying the condition (iii) with $k = 1$ in Proposition 5. We assume that the disturbance terms $\{v_t\}$ in the SSAR(p) model given by (2.2) are a sequence of correlated random variables such that

$$v_t = \sum_{j=0}^{q} c_j^* w_{t-j} \ , \tag{3.35}$$

where $\{c_j^*\}$ are constants with $c_0^* = 1$ for the normalization. If we use a vector process $\boldsymbol{y}_t^{'} = (y_t, y_{t-1}, \cdots, y_{t-p+1}, w_t, w_{t-1}, \cdots, w_{t-q+1})$, then we have a Markovian representation for the vector process $\{\boldsymbol{y}_t\}$. By taking the criterion function

$$G(\boldsymbol{x}) = 1 + \max_{1 \le j \le p+q} |x_j|\xi_j \ , \tag{3.36}$$

where ξ_j $(j = 1, \cdots, m)$ are defined as in the proof of Proposition 4 and $m = p + q$. Then we have an inequality

$$E[G(\boldsymbol{y}_t)|\boldsymbol{y}_{t-1} = \boldsymbol{x}] \le \ c_{13}[1 + \sum_{j=1}^{q} |w_{t-j}|] + \theta G(\boldsymbol{x}), \tag{3.37}$$

where $0 < \theta < 1$ and c_{13} is some constant.
By repeating the above procedure and taking the conditional expectations, we have

$$E[G(\boldsymbol{y}_t)|\boldsymbol{y}_0 = \boldsymbol{x}] \le \ c_{13} \sum_{k=0}^{t-1} \theta^k E[1 + \sum_{j=1}^{q} |w_{t-k-j}||\boldsymbol{y}_0 = \boldsymbol{x}] + \theta^t G(\boldsymbol{x}) \ . \tag{3.38}$$

Then by taking the expectation with respect to the initial distribution, we finally have

$$E[|y_t|] < +\infty \ , \tag{3.39}$$

provided that we assume the condition (iv) with $k = 1$ in Proposition 5 and the condition $E[|w_s|] < \infty$ for $-q \le s \le 0$. We can use the similar arguments to obtain

$$E[|y_t|^k] < +\infty \tag{3.40}$$

for an arbitrary integer $k \ge 1$.

4. Estimation of the SSAR Models

4.1 Maximum Likelihood Estimation

Sato and Kunitomo (1996) have proposed to use the maximum likelihood (ML) estimation for the SSAR models. Given the initial condition $\boldsymbol{y}_0$ and $|\boldsymbol{\Omega}_i| \neq 0$ $(i = 1,2)$, the ML estimator for the vector SSAR models under the Gaussian disturbances is defined by maximizing the conditional log-likelihood function

$$
(4.1) \quad \log L_T(\boldsymbol{\theta}) = -\frac{m(T-1)}{2}\log(2\pi) - \frac{1}{2}\sum_{t=2}^{T}\sum_{i=1}^{2} I_t^{(i)} \log(|\boldsymbol{\Omega}_i|)
$$

$$
-\frac{1}{2}\sum_{t=2}^{T}\sum_{i=1}^{2}(\boldsymbol{y}_t - \boldsymbol{\mu}_i - \boldsymbol{A}_i\boldsymbol{y}_{t-1})'\boldsymbol{\Omega}_i^{-1}(\boldsymbol{y}_t - \boldsymbol{\mu}_i - \boldsymbol{A}_i\boldsymbol{y}_{t-1})I_t^{(i)} ,
$$

where $\boldsymbol{A}_1 = \boldsymbol{A}$, $\boldsymbol{A}_2 = \boldsymbol{B}$, $I_t^{(1)} = I(\boldsymbol{e}_1'\boldsymbol{y}_t \geq \boldsymbol{e}_1'\boldsymbol{y}_{t-1})$, and $I_t^{(2)} = I(\boldsymbol{e}_1'\boldsymbol{y}_t < \boldsymbol{e}_1'\boldsymbol{y}_{t-1})$ with the indicator function $I(\cdot)$. In the above notation we denote the vector of structural parameters $\boldsymbol{\theta} = (\theta_i)$.

We note that the numerical maximization in the maximum likelihood estimation should be done by using the coherency conditions given by (2.7). Because the restrictions imposed by these conditions are not highly nonlinear, the surfaces of the likelihood functions are standard and smooth in most cases. As an illustration for the numerical optimization problem we give the concentrated likelihood function for the first order SSAR model with constant terms in Figure 3. In the SSAR(1) model $\boldsymbol{\theta}' = (r_0, r_1, \sigma_1, \sigma_2)$ and we concentrate the likelihood function with respect to r_0 and r_1 such that $r_0 = r_0(\sigma_1, \sigma_2)$ and $r_1 = r_1(\sigma_1, \sigma_2)$ by using the likelihood equations.

< *Figure* 3 >

As for the asymptotic properties of the ML estimation method when the underlying process is (geometrically) ergodic, we expect that under a set of regularity conditions and the Gaussian disturbances the ML estimator $\hat{\boldsymbol{\theta}}_{ML}$ of unknown parameter $\boldsymbol{\theta}$ is consistent and asymptotically normally distributed as

$$
(4.2) \quad \sqrt{T}\left(\hat{\boldsymbol{\theta}}_{ML} - \boldsymbol{\theta}_0\right) \xrightarrow{d} N\left[0,\ I(\boldsymbol{\theta}_0)^{-1}\right] ,
$$

where

$$
(4.3) \quad I(\boldsymbol{\theta}_0) = \lim_{T\to\infty}\frac{1}{T}E\left[-\frac{\partial^2 \log L_T(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\right] ,
$$

which is a non-singular matrix and $\boldsymbol{\theta}_0$ is the vector of true parameters.

The rigorous proof for the asymptotic properties of the ML estimator is the result of lengthy arguments on the (geometrically) ergodic SSAR models. We also expect the corresponding results on the ML estimator for the the scalor SSAR(p) models when they are (geometrically) ergodic.

4.2 An Instrumental Variables Estimation

Because the estimation problem for the SSAR models is quite similar to the estimation problem of the structural equations in the nonlinear simultaneous equations, the alternative estimation method would be the nonlinear instrumental variables (IV) estimation.

Actually it is a special case of the Generalized Method of Moments (GMM) proposed by Hansen (1982). Given the initial condition $\boldsymbol{y}_0$ and the observations of $\{\boldsymbol{y}_t\}$, one type of the IV estimators is defined by minimizing the criterion function

$$Q_T(\boldsymbol{\theta}) = \boldsymbol{F}_T(\boldsymbol{\theta})' \boldsymbol{H}_T^{-1} \boldsymbol{F}_T(\boldsymbol{\theta}) \ , \tag{4.4}$$

where

$$\boldsymbol{F}_T(\boldsymbol{\theta}) = \frac{1}{T}\sum_{t=2}^{T} \begin{pmatrix} v_{1,t}(\boldsymbol{\theta}) \\ \vdots \\ v_{m,t}(\boldsymbol{\theta}) \\ v_{1,t}^2(\boldsymbol{\theta}) - 1 \\ \vdots \\ v_{m,t}^2(\boldsymbol{\theta}) - 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ y_{1,t-1} \\ \vdots \\ y_{m,t-1} \end{pmatrix} , \tag{4.5}$$

$$\boldsymbol{H}_T = \boldsymbol{W} \otimes \frac{1}{T}\sum_{t=2}^{T} \begin{pmatrix} 1 \\ y_{1,t-1} \\ \vdots \\ y_{m,t-1} \end{pmatrix} \begin{pmatrix} 1 \\ y_{1,t-1} \\ \vdots \\ y_{m,t-1} \end{pmatrix}' ,$$

and $\boldsymbol{W}$ is a $2m \times 2m$ nonsingular matrix.
In this notation the random variables $\boldsymbol{v}_t(\boldsymbol{\theta}) = (v_{i,t}(\boldsymbol{\theta}))$ and $\boldsymbol{y}_{t-1} = (y_{i,t-1})$ are obtained from

$$\boldsymbol{v}(\boldsymbol{\theta}) = I_t^{(1)} \boldsymbol{D}_1^{-1}[\boldsymbol{y}_t - \boldsymbol{\mu}_1 - \boldsymbol{A}\boldsymbol{y}_{t-1}] + I_t^{(2)} \boldsymbol{D}_2^{-1}[\boldsymbol{y}_t - \boldsymbol{\mu}_2 - \boldsymbol{B}\boldsymbol{y}_{t-1}]$$

under the assumption of $|\boldsymbol{D}_i| \neq 0$ $(i = 1,2)$. We denote the nonlinear instrumental variables estimator for the vector of the structural parameters $\boldsymbol{\theta}$ as $\hat{\boldsymbol{\theta}}_{IV}$.
Let

$$\boldsymbol{\Omega} = E[\begin{pmatrix} v_{1,t}(\boldsymbol{\theta}_0) \\ \vdots \\ v_{m,t}(\boldsymbol{\theta}_0) \\ v_{1,t}^2(\boldsymbol{\theta}_0) - 1 \\ \vdots \\ v_{m,t}^2(\boldsymbol{\theta}_0) - 1 \end{pmatrix} \Big(v_{1,t}(\boldsymbol{\theta}_0), \cdots, v_{m,t}(\boldsymbol{\theta}_0), v_{1,t}^2(\boldsymbol{\theta}_0) - 1, \cdots, v_{m,t}^2(\boldsymbol{\theta}_0) - 1\Big)]$$

be the variance-covariance matrix. Then it may be desirable to use an estimator $\hat{\boldsymbol{\Omega}}_T$ of $\boldsymbol{\Omega}$ for the matrix $\boldsymbol{W}$ in $\boldsymbol{H}_T$. But it seems that we should not use an iteration procedure of the minimization of the criterion function $\boldsymbol{Q}_T$ and $\hat{\boldsymbol{\Omega}}_T$ in the estimation. It is partly because we can hardly obtain an initial consistent estimator for $\boldsymbol{\Omega}$ and we do not necessarily have a good numerical convergence. As an illustration of the numerical optimization problem, we give the surface of the criterion function for the SSAR(1) model with constant terms when

$$\boldsymbol{W} = \boldsymbol{\Omega} = \begin{pmatrix} 1 & 0 \\ 0 & \kappa_4 + 2 \end{pmatrix}$$

as if it were known in Figure 4. The parameter κ_4 represents the kurtosis of the disturbances ($\kappa_4 = E(v_t^4) - 3$) and we did concentrate the criterion function with respect to the parameters such that $r_0 = r_0(\sigma_1, \sigma_2)$ and $r_1 = r_1(\sigma_1, \sigma_2)$ by using its components. The surfaces of the criterion function could be flat in a direction depending on the parameter κ_4, which may result in a difficulty in the numerical convergence property in the estimation.

< *Figure* 4 >

Also we can use other orthogonal conditions such as $E[v_{it}v_{jt}] = 0$ for $i \neq j$. Then the computation of the IV estimation would be more complicated.

As for the asymptotic properties of the ML estimation method when the underlying process is (geometrically) ergodic, we expect that under a set of regularity conditions and fairly general distributions for the disturbances the nonlinear instrumental variables estimator $\hat{\boldsymbol{\theta}}_{IV}$ of unknown parameter $\boldsymbol{\theta}$ is consistent and asymptotically normally distributed as

(4.6)
$$\sqrt{T}\left(\hat{\boldsymbol{\theta}}_{IV} - \boldsymbol{\theta}_0\right) \xrightarrow{d} N\left[0,\ \boldsymbol{V}(\boldsymbol{\theta}_0)\right] \quad ,$$

where

(4.7)
$$V(\boldsymbol{\theta}_0) = (\boldsymbol{G}'\boldsymbol{H}^{-1}\boldsymbol{G})^{-1}\boldsymbol{G}'\boldsymbol{H}^{-1}(\boldsymbol{\Omega} \otimes \boldsymbol{M})\boldsymbol{H}^{-1}\boldsymbol{G}(\boldsymbol{G}'\boldsymbol{H}^{-1}\boldsymbol{G})^{-1} ,$$

where $\boldsymbol{H} = \text{plim}_{T\to\infty} \boldsymbol{H}_T$,

$$\boldsymbol{G} = \operatorname*{plim}_{T\to\infty}\left[-\frac{\partial \boldsymbol{F}_T(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0}\right] ,$$

and

$$\boldsymbol{M} = \operatorname*{plim}_{T\to\infty} \frac{1}{T}\sum_{t=2}^{T} \begin{pmatrix} 1 \\ y_{1,t-1} \\ \vdots \\ y_{m,t-1} \end{pmatrix} \begin{pmatrix} 1 \\ y_{1,t-1} \\ \vdots \\ y_{m,t-1} \end{pmatrix}' ,$$

provided the quantities appeared are well-defined.

The rigorous proof would be a set of lengthy arguments on the (geometrically) ergodic SSAR models. We also expect the corresponding results on the nonlinear instrumental variables estimator for the univariate SSAR(p) models when they are (geometrically) ergodic.

5. Concluding Remarks

In this paper we have given several conditions on the geometric ergodicity and the existence of moments for the simultaneous switching autoregressive (SSAR) models. Some of our derivations and discussions in this paper are rather straightforward and we hope that they could be well-understood for econometricians as well as statisticians.

The SSAR model has been introduced by a disequlibrium econometric model, which can be classified as a Tobit type in the class of limited dependent variables models [3]. Amemiya (1974) has investigated the estimation method of a Fair-Jaffee model and the estimation problem of the SSAR model can be regarded as an extension in its dynamic aspects. Kunitomo and Sato (1996a,b) have discussed this issue in a systematic way. An

[3] Chapter 10 of Amemiya (1985) has been a survey on this class of econometric applications including some disequilibrium econometric models.

interesting aspect of the SSAR modelling in econometrics may be to shed new lights on the possible applicability of the limited dependent variables models mainly developed for the cross section data analyses to the non-linear time series analysis.

Finally, we should mention that the standard SSAR process discussed in this paper has been recently extended to a class of non-stationary SSAR processes by Kunitomo and Sato (1996b,c). For an illustration, in Figure 5 we give two sample paths of the homoskedastic SSARI(1) (first-order simultaneous switching integrated autoregressive) process and the SSARI(1) process with the first order ARCH (autoregressive conditional heteroskedasticity) model. They are the I(1) processes defined by the SSAR model given by (2.2) with $p = 1$ and

(5.1) $$v_t = v_{t-1} + w_t\sqrt{h_t}\ ,$$

where $\{w_t\}$ are i.i.d. random variables with $E(w_t) = 0, E(w_t^2) = 1$ and $h_t = 1 + \alpha w_{t-1}^2$. The conditional heteroskedasticity is modeled by the volatility function h_t, and we have the homoskedastic SSARI(1) model when $\alpha = 0$ in this framework [4]. This type of time series modelling shall be useful for econometric applications of financial time series data. This is because they give simple ways to produce integrated processes with the asymmetrical paths as well as the conditional heteroskedasticity, which have been often observed in many financial time series including asset price processes.

$< Figure\ 5 >$

[4]The ARCH modelling for the conditional volatility functions of asset variables has been introduced by Engle (1982).

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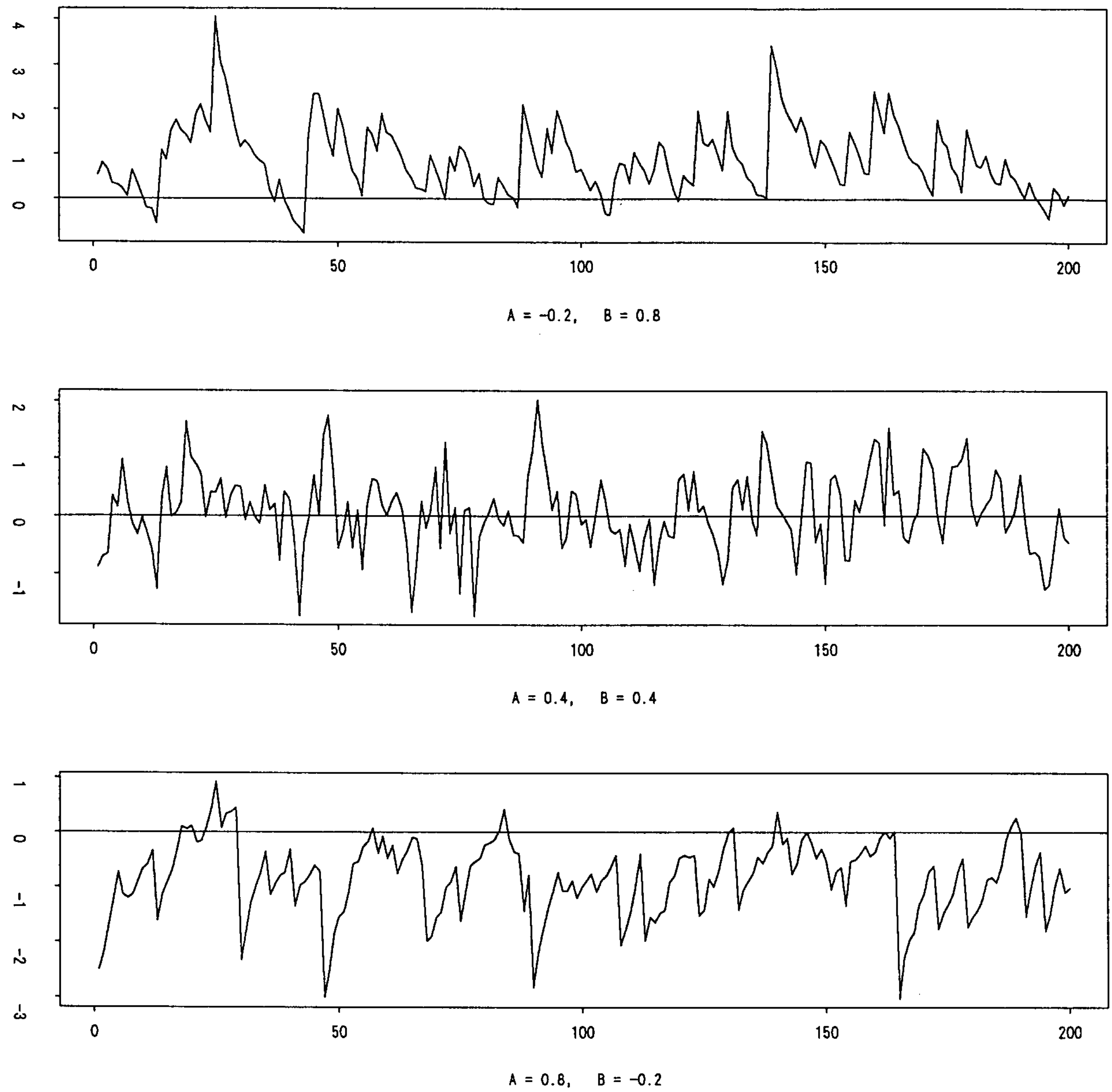


Figure 1

Some of sample paths generated by the ergodic SSAR(1) processes without constant terms are given. When A=B, the SSAR(1) model is the standard AR(1) model. When A $\neq$ B, the sample paths in the up-ward phase are significantly different from those in the down-ward phase. We set r=1 in all simulations.

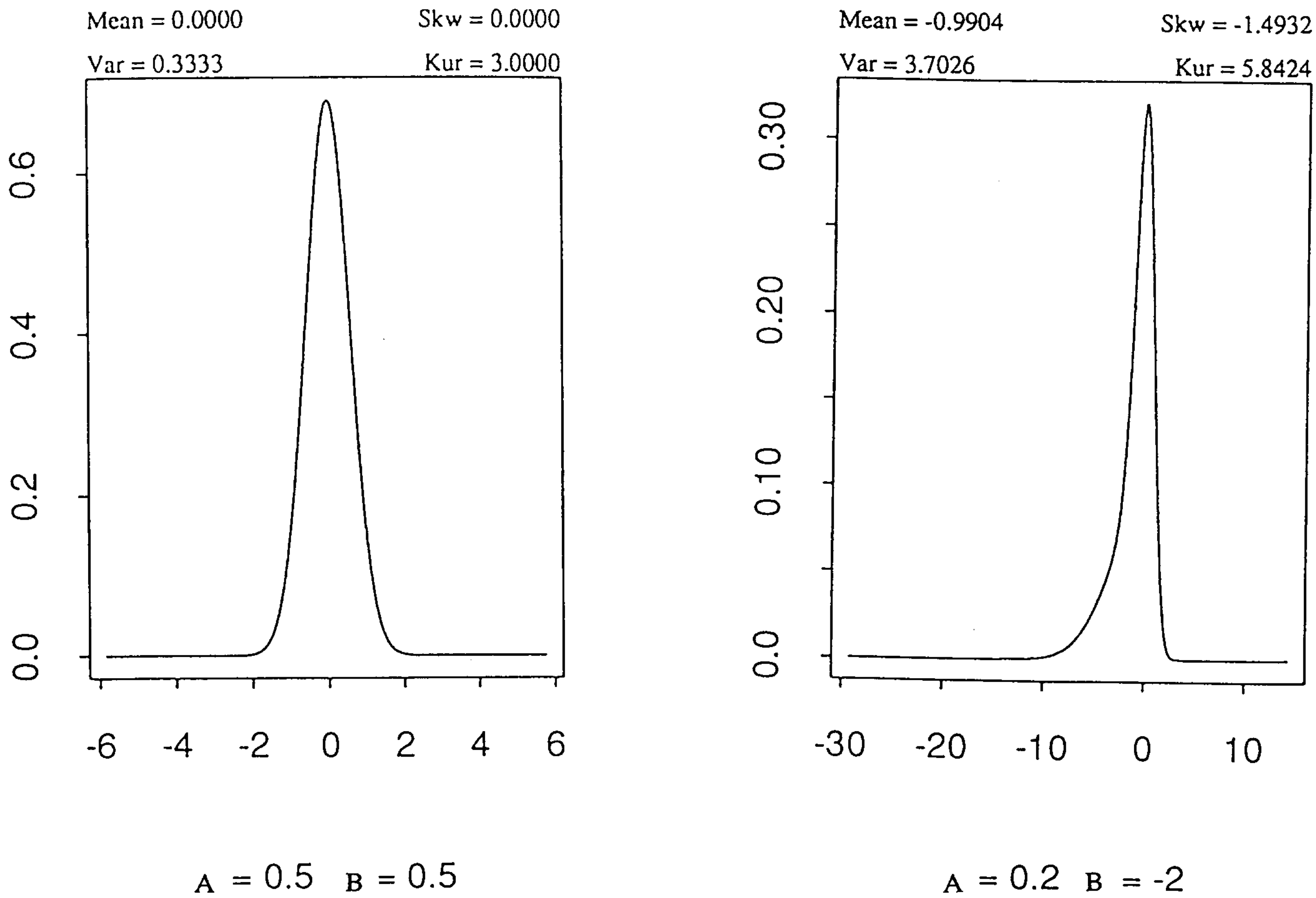


Figure 2

Some of the sattionary distributions for the SSAR(1) processes are given. For each distribution the first four moments (Mean, Var, Skw, Kur) are calculated numerically under the normal disturbances where Skw is the skewness and Kur is $3+\kappa_4$ (where κ_4 is kurtosis). When $A \neq B$, we have the stationary distributions which are quite different from the normal distribution.

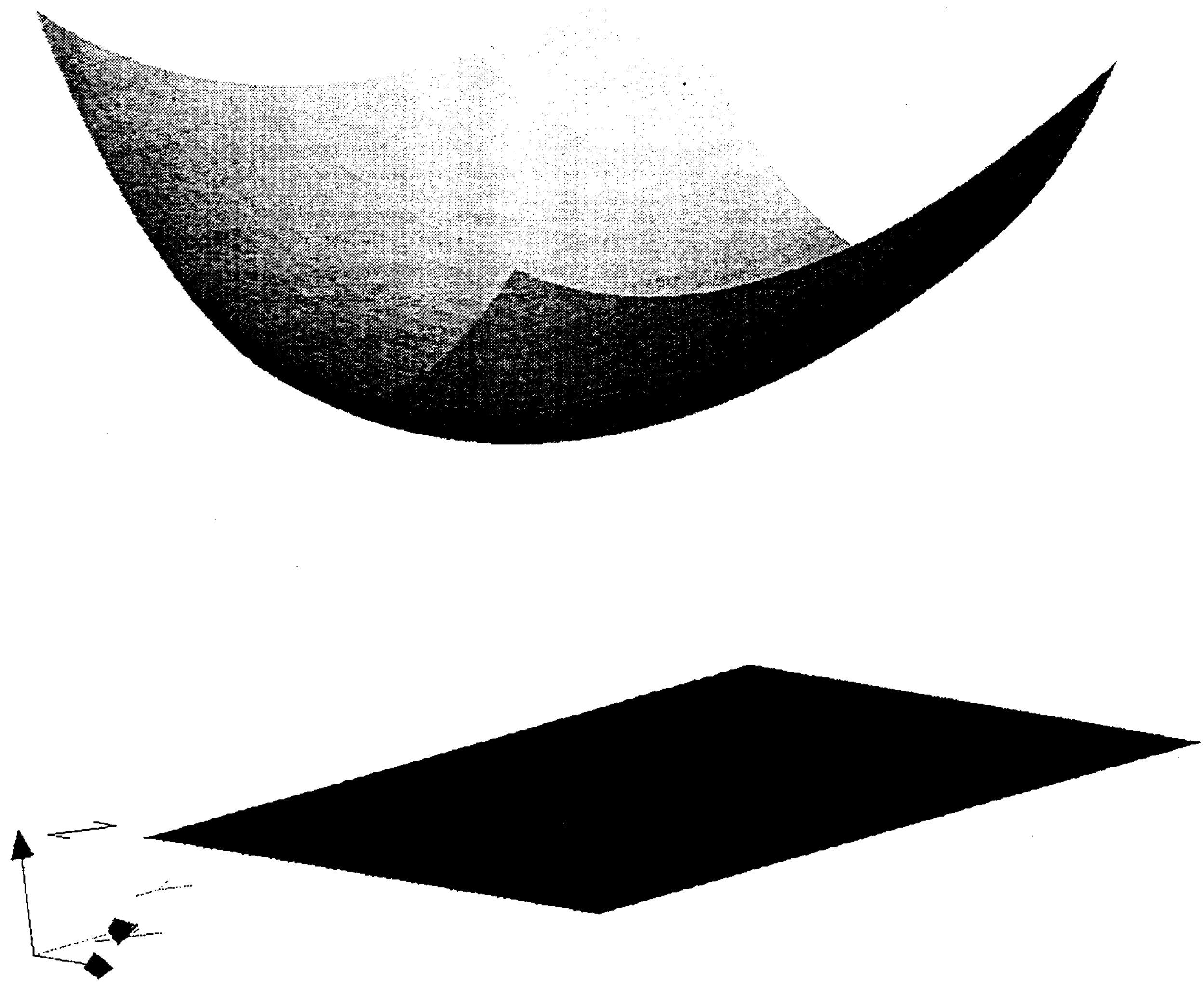


Figure 3

The surface of (-1) times the concentrated likelihood function of the SSAR(1) model with constant terms when A=0.2 and B=0.8 is drawn. It is constructed by the simulations under the Gaussian noises and T=20,000. Y-axis is the σ_1 variable while X-axis is the σ_2 variable.

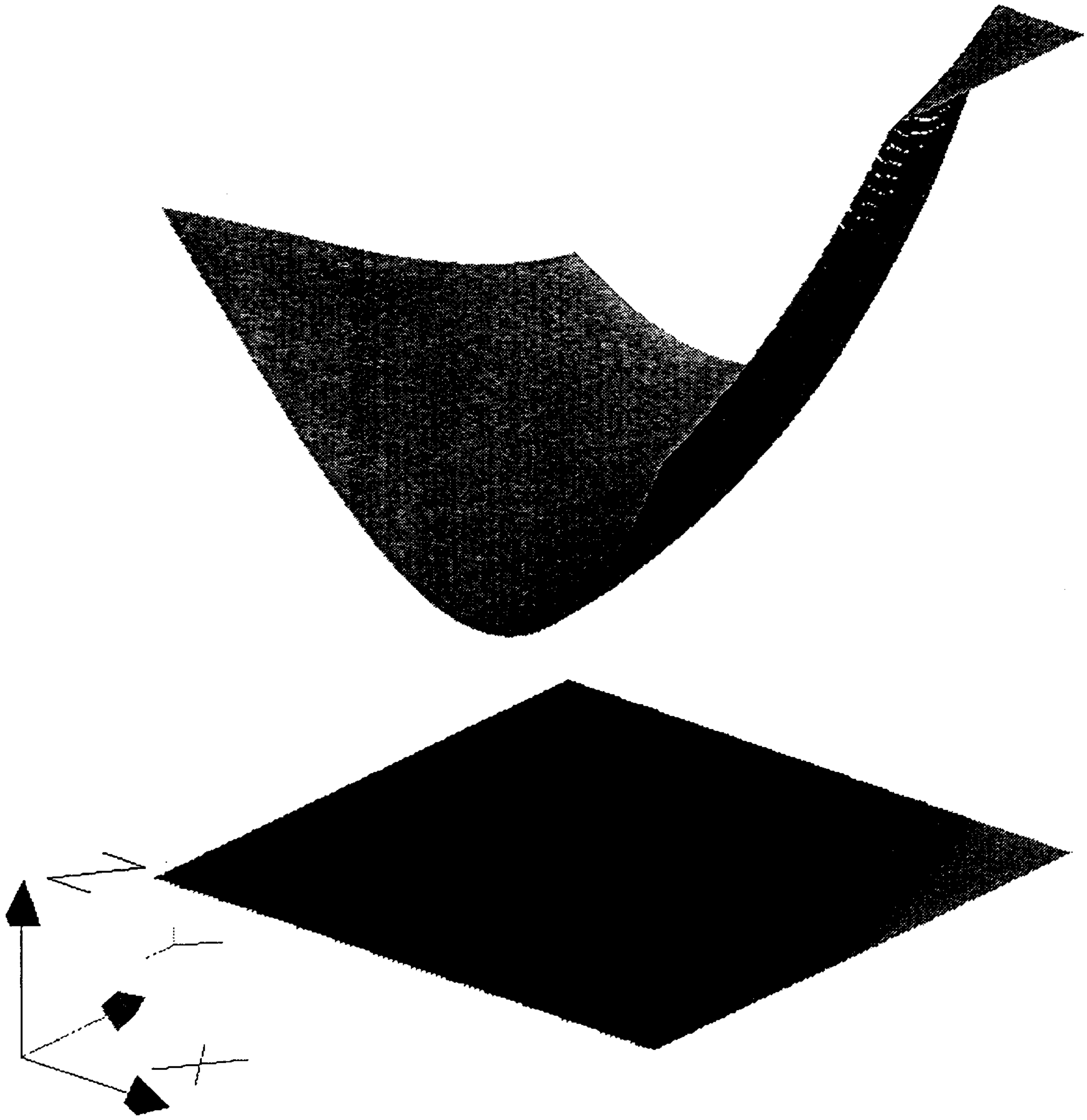
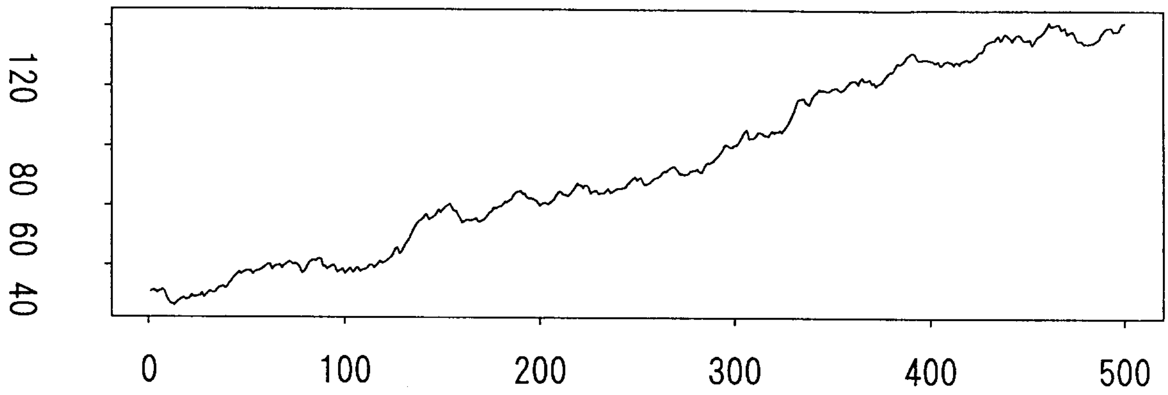
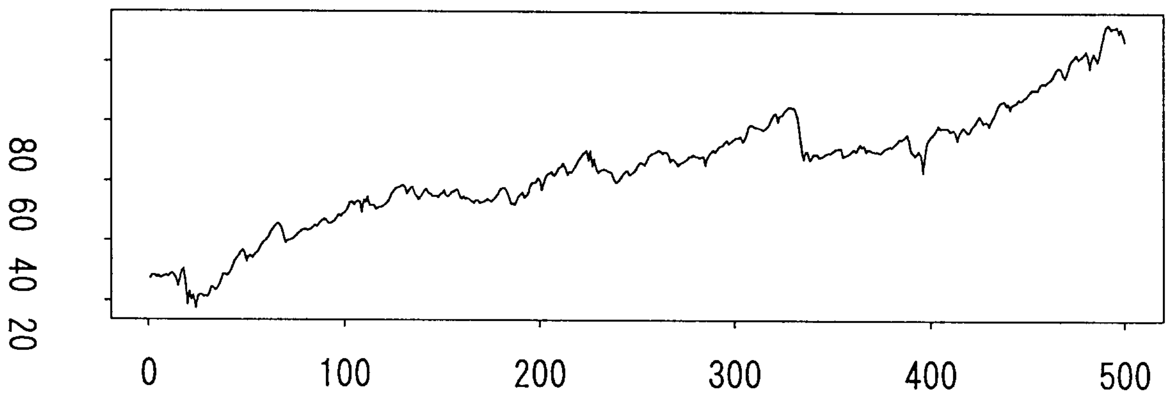


Figure 4

The surface of the concentrated criterion function of the SSAR(1) model with constant terms when A=0.2, B=0.8, and κ_4 =-2.0 is drawn. It is constructed by the simulations under the Gaussian noises and T=20,000. Y-axis is the σ_1 variable while X-axis is the σ_2 variable.



A = 0.4, B = -0.4



A = 0.4, B = -0.4, Alpha = 0.8

Figure 5

Some of sample paths generated by the non-ergodic SSAR(1) processes are given. The first one is a sample path of the SSARI(1) model given by Kunitomo and Sato (1996b) while the second one is a sample path of the SSARI(1)-ARCH(1) model given by Kunitomo and Sato (1996c) where Aplha is the coefficient of ARCH(1) model for the disturbance terms. We have set a linear time trend function in the simulations.