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A Revealed Preference Theory for Nonexpected Utility on "Certain × Uncertain" Consumption Pairs

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ABSTRACT

In this paper, we consider a two period decision problem, where the feasible set is the set of "certain $\times$ uncertain" consumption pairs. That is the decision maker chooses (x, m) in a feasible set, where x is a certain first period consumption and m is a random second period consumption, a Borel probability measure on the set of real numbers. The purpose of this paper is to present revealed preference theory for nonexpected utility on "certain $\times$ uncertain" consumption pairs. We present necessary and sufficient conditions for the data to be consistent with some nonexpected utility functions. In contrast to the standard revealed preference theory, it is shown that the acyclicity of the binary relation naturally introduced from the decision is not sufficient for the existence of a utility function.

1 Introduction

These two decades have seen a growing interest in nonexpected utility theory both in atemporal and in intertemporal frameworks; see, for example Chew [1983], Dekel [1986], Epstein and Zin [1989], Kreps and Porteus [1978], and Selden [1978]. However, not much has been developed in revealed preference theory for nonexpected utility.[1] In this paper, we consider a two period decision problem, where the feasible set is the set of "certain × uncertain" consumption pairs. That is the decision maker chooses (x, m) in a feasible set, where x is a certain first period consumption and m is a random second period consumption, a Borel probability measure on the set of real numbers. Typical examples of such a problem are financial decision problems, see for example Fama and Miller [1972], general intertemporal decision problems, see for example Selden [1978], and so forth. The purpose of this paper is to present a revealed preference theory for nonexpected utility on "certain × uncertain" consumption pairs.

A typical approach to modeling an individual choice in this environment is to specify an additively separable utility function over time with a constant discount factor, and further hypothesize expected utility maximization in the second period, but this is not the only approach. Clearly, there is no intrinsic reason why utility functions should be additive over time and also there are behavioral hypotheses other than expected utility maximization. In fact Epstein and Zin [1989], Kreps and Porteus [1978], and others specify a recursive intertemporal utility function and consider nonexpected utility maximization behavior, for example. Thus there is no unique way to define a "rational consumer" whose preference we attempt to reveal from given data.

Therefore, we can define varieties of classes of utility functions and, for each class, can seek necessary and sufficient conditions for data to be consistent with a utility function in the class, i.e., the nonparametric restrictions on the data derived from the utility maximization. In this paper, we investigate several types of nonexpected utility functions.

In order to obtain necessary and sufficient conditions for the existence of a nonexpected utility functions, we follow the revealed preference tradition. In contrast to the standard revealed preference theory, it is shown that the acyclicity of the binary relation naturally introduced from the decision is not sufficient for the existence of our utility functions.

The plan of this paper is as follows. The next section specifies the economic environment in which the decision maker makes choices and the data we assume to observe. Moreover, we intuitively explain how a utility function is constructed. In section 3, we then proceed to provide necessary and sufficient conditions for a data set to be rationalized by a risk separable utility function. Then, in Section 4, we discuss the case that the second period utility has an expected

[1] As for expected utility theory, Border [1992], Green and Osband [1991], He and Huang [1994], Kamiya and Ichimura [1995], and Kim [1991] presented some conditions for expected utility maximization. On the other hand, Epstein and Melino [1995] recently investigated semiparametric restrictions of nonexpected utility rationalization.

utility representation and the case that the utility is not risk separable.

2 The Model and Examples

Let X be a compact subset of R, where R is the set of real numbers, and $M(X)$ be the set of all Borel probability measures over X. The domain of the agent's choice is $X \times M(X)$. The data consist of a finite number, N, of choices, $y^1 = (x^1, m^1), \ldots, y^N = (x^N, y^N)$, and corresponding feasible sets, $B^1, ..., B^N$, from which these choices are made, i.e., $y^n \in B^n, n = 1, \ldots, N$. We assume that $B^1, ..., B^N$ are all subsets of $X \times M(X)$. Let $D = \{(y^n, B^n)\}_{n=1}^N$, $D^d = \{(x^n, m^n)\}_{n=1}^N$, and X_D and M_D be the projections of D^d on X and $M(X)$, respectively, i.e., $X_D = \{x^n\}_{n=1}^N$ and $M_D = \{m^n\}_{n=1}^N$. Note that our "certain $\times$ uncertain" environment is the same as that of Selden [1978].

The following example illustrates a typical case we have in mind.

Example 1. At $t = 1$, the decision maker obtains an exogenously given income I_1, and consumes x_1 and holds security s_1 at prices p_1 and q_1, respectively.

There are two states u and d in the second period. The probability of u is denoted by $\alpha \in [0, 1]$. The price of security, the price of good, and the exogenously given income at state i are denoted by $q_2(i)$, $p_2(i)$, and $I(i)$, $i = u, d$, respectively. Note that at $t = 2$ the decision maker does not hold security and consumes $\frac{q_2(i)s_1 + I_2(i)}{p_2(i)}$, $i = u, d$.

Take X to be a compact subset of R_+. For $z_1, z_2 \in X$, let $\alpha \circ z_1 + (1 - \alpha) \circ z_2$ denotes the lottery which has rewards z_1 and z_2 with probability α and $1 - \alpha$ respectively. Then, in the case described above,

$$\begin{aligned} B^1 = \{(z_1, \alpha \circ z_2(u) + (1-\alpha) \circ z_2(d)) \in X \times M(X) \mid \\ & z_1, z_2(u), z_2(d) \in X, \\ & \exists s_1 \in R \text{ such that} \\ & p_1 z_1 + q_1 s_1 = I_1 \\ & p_2(u) z_2(u) = q_2(u) s_1 + I_2(u) \\ & p_2(d) z_2(d) = q_2(d) s_1 + I_2(d) \quad \}, \end{aligned}$$

and the decision maker chooses

$$y^1 = (x_1, \alpha \circ x_2(u) + (1-\alpha) \circ x_2(d)) \in B^1.$$

Suppose we have other data $B^n \subset X \times M(X)$ and $y^n \in B^n, n = 2, \ldots, N$.

As for the timing of the decision, there are two cases: (i) the decision maker chooses (x, m) at $t = 1$ and (ii) the decision maker chooses x at $t = 1$ and m at the beginning of $t = 2$. The following arguments can be applied to both cases.

There are several types of nonexpected utility functions. In this section, we only investigate the risk separable case. A pair of functions (W, μ), where $W : X \times R \to R$ and $\mu : M(X) \to R$, is said to be a risk separable rationalization for D if for all $n = 1, \ldots, N$

$$W(x^n, \mu(m^n)) > W(x, \mu(m)) \text{ for all } (x, m) \in B^n \setminus \{(x^n, m^n)\},$$ [2]

W is strictly increasing, and μ is strictly increasing with respect to first order stochastic dominance; for the precise definition, see Section 3.

μ can be considered as a utility function on uncertain second period consumption and W can be considered as a utility function of a pair of certain first period consumption and the utility of uncertain second period consumption. The binary relation derived from (W, μ) is risk separable, i.e., the ranking of the uncertainty about the second period consumption is independent of the level of the first period consumption. That is $\exists \overline{x}, W(\overline{x}, \mu(m)) > W(\overline{x}, \mu(m'))$ implies $\forall x, W(x, \mu(m)) > W(x, \mu(m'))$. It is also worthwhile noting that $W(\overline{x}, \mu(m)) > W(\overline{x}, \mu(m'))$ implies $\mu(m) > \mu(m')$ because W is strictly increasing.

Remark. We may consider the case that μ has an expected utility representation, i.e., $\mu(m) = \int u dm$ for some increasing function $u : X \to R$, and the case that the second period utility u depend also on the first period consumption, i.e., the case that the utility function is not risk separable. These cases are discussed in Section 4.

From the data, we can naturally define a binary relation:

$$(x^n, m^n) \succ (x, m) \text{ if } (x, m) \in B^n \text{ and } (x, m) \neq (x^n, m^n).$$

It seems that if the binary relation $\succ$ defined above does not have a cycle, then there exists a risk separable utility function which rationalizes the data. However, because of the special structure of our utility function, we need further conditions for the existence of the risk separable rationalization. The following example shows that there exists a data set for which we can construct a utility function on $X \times M(X)$ that rationalizes the data but there does *not* exist a risk separable rationalization.

Example 2.

Consider the economy in Example 1. The good's price is 1 in both periods. We have two observations. In the first observation security price in the first period is 1, and in the second period, it is either 1 or -1 with equal probability. The first period income I_1 is 6 and the second period income I_2 is always 3. We observe that the first period consumption x_1 and the amount

[2] Let $u : X \to R$ be $u(x) = x$. Then $\mu(m)$ can be written as $\mu(\tilde{U})$, where $\tilde{U}$ is the distribution of utility in the second period.

of security purchased s_1 are 4 and 2, respectively. In the second observation security price in the first period is 1 and both the security price and the income in the second period are stochastic: $(q_2, I_2) = (-1, 5)$ or $(1, 1)$ with equal probability. We assume that $I_1 = 2$. We observe that $(x_1, s_1) = (1, 1)$.

Clearly we can construct a utility function on $X \times M(X)$ that would rationalize the two observations. But the data caanot be rationalized by a risk separable utility function. To see this, first observe that the purchased security in the first observation implies a purchase of the lottery $.5 \circ 5 + .5 \circ 1$. Moreover, with even less purchase of the security the decision maker could have purchased the lottery $.5 \circ 4 + .5 \circ 2$. Next, note that the distribution of second period consumption in the second observation is equal to $.5 \circ 4 + .5 \circ 2$. In the second observation, the decision maker could get the cheaper lottery $.5 \circ 5 + .5 \circ 1$ which is equal to the chosen distribution in the first observation. Thus μ cannot be constructed in this case.

Below, we sketch how to obtain a necessary and sufficient condition for the existence of a risk separable rationalization.[3]

In Example 1, let $\alpha = .3$, $p_1 = p_2(u) = p_2(d) = q_1 = 1$, $I_1 = 10$, $I_2(u) = I_2(d) = 0$, $q_2(u) = 1.5$ and $q_2(d) = .5$. Suppose the decision maker chooses $x_1 = 4$ and $s_1 = 6$. Then, by the budget constraints, $x_2(u) = 4$ and $x_2(d) = 12$. We denote this consumption pair by $(4, (4[.3], 12[.7]))$, where the probabilities are in the brackets. On the other hand, the decision maker could choose $x_1 = 7$ and $s_1 = 3$. That is $(7, (2[.3], 6[.7]))$ is feasible. Then it can be considered that $(4, (4[.3], 12[.7]))$ is revealed preferred to $(7, (2[.3], 6[.7]))$. Thus if there exists a rationalization, the following binary relation $\succ$ must be consistent with the rationalization:

$$(4, (4[.3], 12[.7])) \succ (7, (2[.3], 6[.7])).$$

[3]We present some definitions used in this paper.

- For a set A, a set of ordered pairs, denoted by $\succ \subset A \times A$, is said to be a binary relation. Moreover, $(a_1, a_2) \in \succ$ is also written as $a_1 \succ a_2$.
- A binary relation $\succ \subset A \times A$ is said to be transitive if $\forall a_1, a_2, a_3 \in A, a_1 \succ a_2$ and $a_2 \succ a_3$ imply $a_1 \succ a_3$.
- For a binary relation $\succ \subset A \times A$, $\succ' \subset A \times A$ is said to be the transitive closure of $\succ$ if $\succ'$ is the smallest transitive binary relation such that $\succ \subset \succ'$. That is, $\succ'$ is the intersection of all transitive binary relations which include $\succ$.
- A binary relation $\succ \subset A \times A$ is said to be cyclic if there exist $a_1, \dots, a_n \in A$, such that $a_1 \succ a_2 \succ \dots \succ a_n \succ a_1$.
- A binary relation $\succ \subset A \times A$ is said to be acyclic if $\succ$ is not cyclic.
- A binary relation $\succ$ on a set A is said to be total if $\forall a, b \in A$ such that $a \neq b$,either $a \succ b$ or $b \succ a$ holds.
- $(a_1, a_2) \notin \succ$ is written as $\sim (a_1 \succ a_2)$.

That is, for any rationalization (W, μ) for D,

$$W(4, \mu(4[.3], 12[.7])) > W(7, \mu(2[.3], 6[.7]))$$

must hold. Similarly, for all feasible (x_1, m), we denote

$$(4, (4[.3], 12[.7])) \succ (x_1, m).$$

Since the number of data is N, similar binary relations can be defined for each data.

Further, we wish to require the monotonicity of utility functions. Thus we define the following binary relation which is also consistent with the utility function:

$$(x_1, m) \succ (x_1', m') \quad \text{if} \quad (i)\ x_1 \geq x_1', \ \text{and} \quad m \text{ first order stochastic dominates } m' \quad \text{or} \quad (ii)\ x_1 > x_1' \ \text{and} \ m = m'.$$

That is, for any rationalization (W, μ) for D,

$$W(x_1, \mu(m)) > W(x_1', \mu(m'))$$

must hold. For example, for any rationalization (W, μ),

$$W(7, \mu(10[.4], 12[.6])) > W(6, \mu(9[.4], 12[.6]))$$

holds, so that

$$(7, (10[.4], 12[.6])) \succ (6, (9[.4], 12[.6])$$

must be consistent with the existence of (W, μ).

If the binary relation introduced so far has a cycle, then clearly there does not exists a rationalization (W, μ). For example, if

$$(x_1, m) \succ (x_1', m') \succ (x_1, m),$$

holds, then for any realization (W, μ)

$$W(x_1, \mu(m)) > W(x_1', \mu(m')) > W(x_1, \mu(m)),$$

must hold. This is a contradiction.

Further, binary relation can be also naturally introduced. Suppose

$$(7, (10[.4], 12[.6])) \succ (7, (11[.4], 11[.6])$$

holds, then we need to introduce the following binary relation:

$$(*) \quad (x, (10[.4], 12[.6])) \succ (x, (11[.4], 11[.6])) \quad \text{for all } \ x \in X.$$

The reason is as follows. If there exists a rationalization (W, μ), then

$$W(7, \mu(10[.4], 12[.6])) > W(7, \mu(11[.4], 11[.6]))$$

holds. Thus, by the monotonicity of W,

$$\mu(10[.4], 12[.6]) > \mu(11[.4], 11[.6])$$

must hold so that

$$W(x, \mu(10[.4], 12[.6])) > W(x, \mu(11[.4], 11[.6])) \qquad \text{for all} \;\; x \in X$$

holds. Thus the existence of (W, μ) must be consistent with $(*)$. By this way, we extend the binary relation as large as possible. (The details are discussed in the next section.) By the construction, if the extended binary relation has a cycle, then clearly there does not exists a rationalization (W, μ).

In the standard revealed preference theory, the acyclicity of the binary relation naturally introduced is sufficient for the existence of a utility function; see, for example, Richter [1971]. However, in our case, the acyclicity of the extended binary relation is not sufficient for the existence of (W, μ). The following example shows that the acyclicity of the extended binary relation is not sufficient for the existence of a risk separable rationalization.

Example 3. Let $N = 16$. For $i = 1$, let

$$B^{ia} = \{(x^{ia}, m^1), (x^{ia} + 1, m^{ia})\}, \; y^{ia} = (x^{ia} + 1, m^{ia})$$
$$B^{ib} = \{(x^{ia} + 1, m^2), (x^{ia}, m^{ib})\}, \; y^{ib} = (x^{ia}, m^{ib})$$
$$B^{ic} = \{(x^{ic}, m^3), (x^{ic}, m^{ib})\}, \; y^{ic} = (x^{ic}, m^3)$$
$$B^{id} = \{(x^{id}, m^4), (x^{id}, m^{ia})\}, \; y^{ia} = (x^{id}, m^4),$$

where the probability measures of second period consumption in the above do not first order dominate each other, and $x^{ia}, x^{ia} + 1, x^{ic}$, and x^{id} are all different real numbers. The other feasible sets, $B^{ij}, i = 2, 3, 4, j = a, b, c, d$, choices, $y^{ij}, i = 2, 3, 4, j = a, b, c, d$, will be given later. Clearly, we obtain the following binary relations:

$$m^{1b} \succ m^2, m^3 \succ m^{1b}, m^4 \succ m^{1a}.$$

Then, by risk separability,

$$(x, m^{1b}) \succ (x, m^2), (x, m^3) \succ (x, m^{1b}), \text{ and } \; (x, m^4) \succ (x, m^{1a})$$

hold for all x. Note that risk separability does not create further binary relation. Suppose we have a rationalization (W, μ), then one and only one of the following cases holds:

$$
\begin{aligned}
&(1)\ \ \mu(m^1) > \mu(m^3) \ \text{ and } \ \mu(m^2) > \mu(m^4)\\
&(2)\ \ \mu(m^1) > \mu(m^3) \ \text{ and } \ \mu(m^2) \le \mu(m^4)\\
&(3)\ \ \mu(m^1) \le \mu(m^3) \ \text{ and } \ \mu(m^2) > \mu(m^4)\\
&(4)\ \ \mu(m^1) \le \mu(m^3) \ \text{ and } \ \mu(m^2) \le \mu(m^4).
\end{aligned}
$$

Suppose (1) holds. Then it must be consistent with $m^1 \succ m^3$ and $m^2 \succ m^4$. So, by $(x^{1c}, m^1) \succ (x^{1c}, m^3) \succ (x^{1c}, m^{1b})$, $m^1 \succ m^{1b}$ holds. Similarly, by $(x^{1d}, m^2) \succ (x^{1d}, m^4) \succ (x^{1d}, m^{1a})$, $m^2 \succ m^{1a}$ holds. Then, we have a cycle

$$(x^{1a}, m^1) \succ (x^{1a}, m^{1b}) \succ (x^{1a}+1, m^2) \succ (x^{1a}+1, m^{1a}) \succ (x^{1a}, m^1).$$

For the cases (2), (3), and (4), we construct $\{(B^{ia}, y^{ia}), \ldots, (B^{id}, y^{id})\}_{i=2}^{4}$ in which the roles of m^1, m^3, m^2, and m^4 are properly replaced, the probability measures of second period consumption do not first order dominate each other, and the first period consumptions are all different real numbers. It is easy to check that there is no cycle of the binary relation defined by the data. However, by the same argument as for (1), we obtain a cycle in the other three cases.

We should introduce further condition for the existence of (W, μ). In the next section, if there exists a total binary relation on D^d such that, (i) it is consistent with the extended binary relation, and (ii) when we further extend the total binary relation on D^d, it is still acyclic, then there exists a rationalization (W, μ). The condition is also shown to be necessary.

3 Risk Separable Rationalization

In this section, we present our results on risk separable rationalizations. Recall that, in our model, there are a finite number of sets of alternatives, $B^1, \ldots, B^N$, such that $B^n \subset X \times M(X)$ for $n = 1, \ldots, N$ and that the decision maker chooses $(x^n, m^n) \in B^n$. Thus the choices of the decision maker are represented by $D = \{(x^n, m^n), B^n\}_{n=1}^{N}$. Recall also that $D^d = \{(x^n, m^n)\}_{n=1}^{N}$, $X_D = \{x^n\}_{n=1}^{N}$, and $M_D = \{m^n\}_{n=1}^{N}$.

Next, we introduce the topology of weak* convergence on $M(X)$.[4] It is well-known that $M(X)$ endowed with the topology of weak* convergence is a compact separable metric space; see

[4]The topology of weak* convergence is often called the topology of weak convergence. In the topology, $m^q \to m$ if and only if $\int u dm^q \to \int u dm$ for all bounded continuous function $u : X \to R$.

Parthasarathy [1967]. The metric, called Prohorov metric, is denoted by $d^p : M(X) \times M(X) \rightarrow R_+$. The metric on $X \times M(X)$ is defined by

$$d((x,m),(x',m')) = |x - x'| + d^p(m,m').$$

DEFINITION **3.1** . *For $m,\ m' \in M(X)$, m first order stochastic dominates m', denoted mFm', if $m \neq m'$ and $\forall \alpha \in R,\ m(\{x|x \geq \alpha\}) \geq m'(\{x|x \geq \alpha\})$.*

Now we are ready to present the definition of a risk separable rationalization.

DEFINITION **3.2** . *The pair of functions (W,μ), where $W : X \times R \rightarrow R$ and $\mu : M(X) \rightarrow R$, is said to be a risk separable rationalization for D if, for all $n = 1,\ldots,N$,*

1. *$W(x^n,\mu(m^n)) > W(x,\mu(m))$ for all $(x,m) \in B^n \setminus \{(x^n,m^n)\}$,*[5]
2. *W is continuous and strictly increasing,*
3. *μ is continuous in the topology of weak* convergence and is strictly increasing with respect to first order stochastic dominance, i.e., $\mu(m) > \mu(m')$ for all $m,m' \in M(X)$ such that mFm'.*

If D has a risk separable rationalization (W,μ), then by the monotonicity of W and μ it needs to be consistent with the following binary relations.

DEFINITION **3.3** . *For all $(x,m) \in X \times M(X)$, we define the following binary relations.*

1. *$(x,m) \succ_F (x',m')$ if (i) $x \geq x'$ and mFm', or (ii) $x > x'$ and $m = m'$.*
2. *$(x,m) \succeq_F (x',m')$ if $(x,m) \succ_F (x',m')$ or $(x,m) = (x',m')$.*

Let $\widetilde{B}^n = \{(x,m) \in X \times M(X) \mid \exists(\overline{x},\overline{m}) \in B^n,\ (\overline{x},\overline{m}) \succeq_F (x,m)\}$. If D has a risk separable rationalization (W,μ), then it also needs to be consistent with the following binary relations.

DEFINITION **3.4** . *For all $(x,m) \in X \times M(X)$, we define the following binary relations.*

1. *$(x,m) \succ_R (x',m')$ if $(x,m) \neq (x',m')$, and $\exists n, (x,m) = (x^n,m^n) \in D^d$ and $(x',m') \in \widetilde{B}^n$.*
2. *$(x,m) \succeq_R (x',m')$ if $\exists n, (x,m) = (x^n,m^n) \in D^d$ and $(x',m') \in \widetilde{B}^n$.*

[5] Let $u : X \rightarrow R$ be $u(x) = x$. Then $\mu(m)$ can be written as $\mu(\tilde{U})$, where $\tilde{U}$ is the distribution of utility in the second period.

Note that $\succ_R$ is a revealed preference type binary relation.

In what follows, we assume the following condition for the data.

CONDITION 3.1 . *For all $n = 1, \ldots, N$, B^n is closed and there does not exist $(x', m') \in B^n$ such that $(x', m') \succ_F (x^n, m^n)$.*

Let $\succ^0 = \succ_R \bigcup \succ_F$ and $\succeq^0 = \succeq_R \bigcup \succeq_F$. The transitive closures of $\succ^0$ and $\succeq^0$ are denoted by $\overline{\succ}^0$ and $\overline{\succeq}^0$, respectively. Of course, for the existence of a risk separable rationalization, the binary relation introduced in $X \times M(X)$ through 1-4 above should be acyclic. But, as we have shown in Example 2, even when it is acyclic, the data may not have a risk separable rationalization.

The plan of the rest of this section is as follows. We first define risk separability and time consistency in terms of binary relations. Then we define the smallest risk separable, time consistent binary relation containing $\succ^0$. However, the acyclicity of the extended binary relation is not sufficient for the existence of (W, μ); see Example 3 in Section 2. We need further condition for the existence. If there exists a total binary relation on $X_D \times M_D$ such that (i) it is consistent with the extended binary relation, and (ii) when we further extend the total binary relation on $X_D \times M_D$ using risk separability and time consistency, it is still acyclic, then there exists a rationalization (W, μ). The condition is also shown to be necessary. It is not easy to check whether or not the binary relation has a cycle, since the domain has an infinite number of elements. Thus next we define a binary relation only on $X_D \times M_D$ and present a condition on the new binary relation which is equivalent to the acyclicity of the binary relation on $X \times M(X)$. Since $X_D \times M_D$ has a finite number of elements, it is easy to check the new condition.

Risk separability and time consistency can be defined in terms of binary relations as follows. Note that the following definition parallels those in Chew and Epstein [1989] and Johnsen and Donaldson [1985].

DEFINITION **3.5** .

- *A binary relation $\succ \subset (X \times M(X)) \times (X \times M(X))$ is said to be risk separable if $\exists \bar{x}, (\bar{x}, m) \succ (\bar{x}, m')$ implies $\forall x, (x, m) \succ (x, m')$.*
- *A pair of binary relations $(\succ, G)$, where $\succ \subset (X \times M(X)) \times (X \times M(X))$ and $G \subset M(X) \times M(X)$, is said to be time consistent if $\exists \bar{x}, (\bar{x}, m) \succ (\bar{x}, m')$ implies mGm'.*

The smallest risk separable, time consistent binary relation is defined as follows. Let $(\succ^\alpha, F^\alpha)$ be a pair of binary relations on $X \times M(X)$ and $M(X)$, respectively, such that (i) $\succ^0 \subset \succ^\alpha$ and $F^0 \subset F^\alpha$ and (ii) $(\succ^\alpha, F^\alpha)$ satisfies risk separability and time consistency. Then clearly $(\succ^I, F^I) = (\bigcap_\alpha \succ^\alpha, \bigcap_\alpha F^\alpha)$ satisfies risk separability and time consistency and it is the smallest binary relation satisfying (i) and (ii). As we have shown in Example 3, the acyclicity of this pair of binary relations is not sufficient for the existence of (W, μ).

ASSUMPTION 3.1 . *There exists a total binary relation* $\succ_D^*$ *on* D^d *satisfying the following conditions:*

(*) the smallest risk separable, time consistent binary relation which contains $\succ_D^{I*} = \succ_D^* \bigcup \succ^I$, *denoted by* $\succ^{\min}$, *is acyclic.*

Note that $\succ_D^*$ *can be naturally considered as a binary relation on* $X \times M(X)$. *Thus in the above,* $\succ_D^* \bigcup \succ^I$ *can be naturally defined.*

THEOREM **3.1** . *Suppose Condition 3.1 holds. Then there exists a risk separable rationalization if and only if Assumption 3.1 holds.*

The proof of the above theorem is given in the following manner. At the end of this section, we present a theorem (Theorem 3.3) which is equivalent to the above theorem. Then, in the appendix, we prove Theorem 3.3.

At first glance, the condition seems not to be necessary. Indeed, if there exists (W, μ), then there may exist $(x, m), (x', m') \in X_D \times M_D$ such that $W(x, \mu(m)) = W(x', \mu(m'))$, i.e., the elements of $X_D \times M_D$ are not totally ordered. However, in this case, we can generate a new binary relation, which is total and acyclic on $X_D \times M_D$, by modifying the binary relation generated by (W, μ).

It is not easy to check whether or not the given data satisfy Assumption 3.1, since $\succ^{\min}$ is not constructively defined and the domain contains an infinite number of elements. First, we construct $\succ^{\min}$ by a finite number of steps.

Step 1 For all $m, m' \in M(X)$, if there exists $\overline{x} \in X$ such that $(\overline{x}, m) \overline{\succ}^0 (\overline{x}, m')$, then we define mF^1m'. Moreover, we define $(x, m)T^1(x, m')$ for all $x \in X$. Let $\overline{F}^1$ be the transitive closure of F^1 and $\overline{T}^1$ be the transitive closure of the union of $\succ_F$ and T^1. Let $\succ^1$ be the union of $\succ^0$ and $\overline{T}^1$, and $\overline{\succ}^1$ be its transitive closure. Moreover, let $\succeq^1$ be the union of $\succeq^0$ and $\overline{T}^1$, and $\overline{\succeq}^1$ be its transitive closure.

⋮

Step h For all $m, m' \in M(X)$, if there exists $\overline{x} \in X$ such that $(\overline{x}, m) \overline{\succ}^{h-1} (\overline{x}, m')$, then we define mF^1m'. Moreover, we define $(x,m)T^1(x,m')$ for all $x \in X$. Let $\overline{F}^h$ be the transitive closure of F^h and $\overline{T}^h$ be the transitive closure of the union of $\succ_F$ and T^h. Let $\succ^h$ be the union of $\succ^0$ and $\overline{T}^h$, and $\overline{\succ}^h$ be its transitive closure. Moreover, let $\succeq^h$ be the union of $\succeq^0$ and $\overline{T}^h$, and $\overline{\succeq}^h$ be its transitive closure.

By the following lemma, after a finite number of steps, the process ends with binary relations with risk separability and time consistency.

LEMMA **3.1** . *There exists an integer* $\overline{h}$ *such that for all* $h \geq \overline{h}$, $\succ^h = \succ^{\overline{h}}$.

Proof. See the appendix.

Note that clearly $\overline{\succ}^{\overline{h}} = \succ^{\overline{h}} = \succ^I$ holds. Let $\succ^{I*} = \succ^I \bigcup \succ^*_D$. Then we extend it by using the same steps as the above. By the same argument as the proof of the above lemma, the binary relation obtained is equal to $\succ^{\min}$.

Since $X \times M(X)$ contains an infinite number of elements, we cannot directly verify Assumption 3.1. We will present an assumption which can be directly checked and is equivalent to Assumption 3.1.

Let

$$\widetilde{B}^n_D = \widetilde{B}^n \bigcap (X_D \times M_D).$$

We define some binary relations on $X_D \times M_D$ as follows.

DEFINITION **3.6** . *For all* $(x,m) \in X_D \times M_D$, *we define the following binary relations.*

1. $(x^i, m^j) \succ_{RD} (x^{i'}, m^{j'})$ *if* $(x^i, m^j) \neq (x^{i'}, m^{j'})$ *and* $\exists n, (x^i, m^j) = (x^n, m^n) \in D^d$ *and* $(x^{i'}, m^{j'}) \in \widetilde{B}^n_D$.
2. $(x^i, m^j) \succeq_{RD} (x^{i'}, m^{j'})$ *if* $\exists n, (x^i, m^j) = (x^n, m^n) \in D^d$ *and* $(x^{i'}, m^{j'}) \in \widetilde{B}^n_D$.
3. $(x^i, m^j) \succ_{FD} (x^{i'}, m^{j'})$ *if (i)* $x^i \geq x^{i'}$ *and* $m^j F m^{j'}$, *or (ii)* $x^i > x^{i'}$ *and* $m^j = m^{j'}$.
4. $(x^i, m^j) \succeq_{FD} (x^{i'}, m^{j'})$ *if* $(x^i, m^j) \succ_{FD} (x^{i'}, m^{j'})$ *or* $x^i = x^{i'}$ *and* $m^j = m^{j'}$.

Let $\succ^0_D = \succ_{RD} \bigcup \succ_{FD}$ and $\succeq^0_D = \succeq_{RD} \bigcup \succeq_{FD}$. The transitive closures of $\succ^0_D$ and $\succeq^0_D$ are denoted by $\overline{\succ}^0_D$ and $\overline{\succeq}^0_D$, respectively. By the following steps, $\overline{\succ}^0_D$ and $\overline{\succeq}^0_D$ are extended to satisfy risk separability and time consistency.

Step 1 For all $m, m' \in M_D$, if there exists $\overline{x} \in X_D$ such that $(\overline{x}, m)\overline{\succ}_D^0(\overline{x}, m')$, then we define mF_D^1m'. Let $\overline{F}_D^1$ be its transitive closure. Moreover, we define $(x, m)T_D^1(x, m')$ for all $x \in X_D$. Let $\overline{T}_D^1$ be the transitive closure of the union of $\succ_{FD}$ and T_D^1. Let $\succ_D^1$ be the union of $\succ_D^0$ and $\overline{T}_D^1$, and $\overline{\succ}_D^1$ be its transitive closure. Moreover, let $\succeq_D^1$ be the union of $\succeq_D^0$ and $\overline{T}_D^1$, and $\overline{\succeq}_D^1$ be its transitive closure.

$\vdots$

Step h For all $m, m' \in M_D$, if there exists $\overline{x} \in X_D$ such that $(\overline{x}, m)\overline{\succ}_D^{h-1}(\overline{x}, m')$, then we define mF_D^hm'. Let $\overline{F}_D^h$ be its transitive closure. Moreover, we define $(x, m)T_D^h(x, m')$ for all $x \in X_D$. Let $\overline{T}_D^h$be the transitive closure of the union of $\succ_{FD}$ and T_D^h. Let $\succ_D^h$ be the union of $\succ_D^0$ and $\overline{T}_D^h$, and $\overline{\succ}_D^h$ be its transitive closure. Moreover, let $\succeq_D^h$ be the union of $\succeq_D^0$ and $\overline{T}_D^h$, and $\overline{\succeq}_D^h$ be its transitive closure.

Since $X_D \times M_D$ contains only a finite number of elements, then, after a finite number of steps, the process yields a binary relation with risk separability and time consistency. Let h^* be the integer such that $\forall h \geq h^*$, $\succ_D^h = \succ_D^{h^*}$. Let $\succ_D^{I*} = \succ^{h^*} \bigcup \succ_D^*$. Then we extend it by the same steps as the above. Let the extension of $\succ_D^{I*}$ be $\succ_D^{\min}$.

ASSUMPTION 3.2 . *$\succ_D^{\min}$ is acyclic.*

THEOREM **3.2** . *Suppose Condition 3.1 holds. Then Assumption 3.1 holds if and only if Assumption 3.2 holds.*

Proof. Assumption 3.1 clearly implies Assumption 3.2. The converse immediately follows from Assumption 3.1 and Lemma 5.2 in the appendix.

Q.E.D.

THEOREM **3.3** . *Suppose Condition 3.1 holds. Then there exists a risk separable rationalization (W, μ) for D if and only if Assumption 3.2 holds.*

Proof. See the appendix.

Clearly, the acyclicity of $\succ_D^{\min}$ can be checked in a finite number of steps.
Under the same assumption, μ can be chosen as a certainty equivalent.[6]

[6]For the details of a certainty equivalent, see Chew and Epstein [1989] and Epstein and Zin [1989].

Corollary 3.1 . *Suppose Condition 3.1 holds. Then there exists a risk separable rationalization* (W', μ') *for* D *such that* $\forall x \in X, \mu'(\delta(x)) = x$ *if and only if Assumption 3.1 holds, where* $\delta(x)$ *is the probability measure which assigns unit mass to* $\{x\}$.

Proof. For (W, μ) obtained in Theorem 3.1, clearly, there exists a continuous, strictly increasing function $f : R \to R$ such that $\forall x \in X, f(\mu(\delta(x))) = x$. Let $\mu' : M(X) \to R$ be $\mu'(m) = f(\mu(m))$. Then we define $W'(x, a) = W(x, f^{-1}(a))$. (W', μ') is clearly a risk separable rationalization for D such that $\forall x \in X, \mu'(\delta(x)) = x$.

Q.E.D.

4 Other Rationalizations

Using the proof of the theorems in Section 3, we can obtain necessary and sufficient conditions for the existence of other rationalizations.

We first seek for the condition for the existence of an expected utility representation of μ.

Definition 4.1 . *The pair of functions* (W, u), *where* $W : X \times R \to R$ *and* $u : X \to R$, *is said to be a risk separable rationalization with an expected utility representation if, for all* $n = 1, \ldots, N$,

1. $W(x^n, \int u dm^n) \geq W(x, \int u dm)$ *for all* $(x, m) \in B^n$,
2. W *is continuous and strictly increasing,*
3. u *is a continuous, strictly increasing function.*

Considering $\succ_D^{\min}$ as a binary relation on $X \times M(X)$, $\overline{\succ}^{\natural} = \succ^{\bar{h}} \bigcup \overline{\succ}_D^{\min}$ can be defined, where $\overline{\succ}_D^{\min}$ is the transitive closure of $\succ_D^{\min}$ and $\succ^{\bar{h}}$ is defined in Lemma 3.1. Below, we extend it to a transitive, risk separable, time consistent binary relation on $X \times M(X)$. First, for all $\forall m, m' \in M(X)$, we define

$$mF^{\#}m' \quad \text{if } \exists \overline{x} \in X, \ (\overline{x}, m)\overline{\succ}^{\natural}(\overline{x}, m').$$

Let $\overline{F}^{\#}$ be its transitive closure. Then, for $m, m' \in M(X)$ such that $m\overline{F}^{\#}m'$, we define $(x, m)T^{\#}(x, m')$ for all $x \in X$. Let $\overline{T}^{\#}$ be the transitive closure of the union of $\succ_F$ and $T^{\#}$, $\succ^{\#} = \succ^0 \bigcup \overline{T}^{\#}$, and $\overline{\succ}^{\#}$ be the transitive closure of $\succ^{\#}$. $(\overline{\succ}^{\#}, \overline{F}^{\#})$ satisfies risk separability and time consistency; for the details, see the appendix.

Let, for $m^n \in M_D$,

$$B(m^n) = \{m \in M(X) \mid m^n\overline{F}^{\#}m\}$$

and

$$A = \left\{ m \in \overline{M}(X) \mid \forall u \in \overline{U}_c, \int udm \geq 0 \right\},$$

where $\overline{M}(X)$ is the set of finite countably additive Borel signed measures on X, and $\overline{U}_c$ is the set of continuous increasing functions on X. Also let

$$G = \left\{ \left(\sum_{n=1}^{N} \alpha^n \left(\overline{m}^n - m^n \right) \right) \mid \overline{m}^n \in \overline{co}B(m^n), n = 1, \ldots, N, \alpha = (\alpha^1, \ldots, \alpha^N) \in S^{N-1} \right\},$$

where $\overline{co}B(m^n)$ is the (weak*) closed convex hull of $B(m^n)$ and S^{N-1} is the $(N-1)$-dimensional unit simplex. Let $cn\ G$ be the cone generated by G and $cl\ (cn\ G)$ be the weak* closure of $cn\ G$.

Assumption 4.1 .

$$cl\ (cn\ G) \bigcap A = \{0\},$$

where 0 is the zero measure.

Theorem **4.1** . *Suppose Condition 3.1 holds. Then there exists a risk separable rationalization with an expected utility representation (W, u) if and only if Assumptions 3.2 and 4.1 hold.*

Proof. Clearly, the assumptions are necessary. By Assumption 4.1, there exists a strictly increasing function u on X such that $m^n \overline{F}^{\#} m$ implies $\int udm^n \geq \int udm$; for the proof, see Border [1982].[7] Then the existence of a risk separable rationalization with an expected utility representation follows from the same arguments as in Lemma 5.5 in the appendix and as in the subsection of the construction of W in the appendix.

Q.E.D.

Next, as in Selden [1979], we investigate the case that the utility is not risk separable.

Definition **4.2** . *The pair of functions (W, u), where $W : X \times R \to R$ and $u : X \times X \to R$, is said to be a rationalization with an expected utility representation if, for all $n = 1, \ldots, N$,*

1. $W(x^n, \int u(x^n, z)dm^n(z)) \geq W(x, \int u(x, z)dm(z))$ *for all* $(x, m) \in B^n$,
2. *W is continuous and strictly increasing,*
3. *u is continuous and is strictly increasing in the second argument.*

[7] In fact, in order to obtain u we need to strengthen Border's condition slightly; for the details, see Kamiya and Ichimura [1995].

Since the utility is not risk separable, we can construct the second period utility function for each first period consumption x. Let, for each $(x^n, m^n) \in D^d$,

$$B(x^n, m^n) = \{m \in M(X) \mid (x^n, m^n) \succeq_R (x^n, m)\}.$$

Then let $I(x^n) = \{i \in \{1, \dots, N\} \mid x^i = x^n\}$ and

$$G(x^n) = \left\{ \left(\sum_{i \in I(x^n)} \alpha^i \left(\overline{m}^i - m^i \right) \right) \mid \overline{m}^i \in \overline{co} B(x^i, m^i), i \in I(x^n), \alpha_i \geq 0, \sum_{i \in I(x^n)} \alpha_i = 1, \right\}.$$

Assumption 4.2 . *For all $n = 1, \dots, N$,*

$$cl\ (cn\ G(x^n)) \bigcap A = \{0\}.$$

Assumption 4.3 . *$\succ_R$ is acyclic.*

Theorem **4.2** . *Suppose that Condition 3.1 holds and that there exist $\underline{x}$ and $\overline{x}$ such that $X = [\underline{x}, \overline{x}]$. Then there exists a rationalization with an expected utility representation (W, u) if and only if Assumptions 4.3 and 4.2 hold.*

Proof. Clearly, the assumptions are necessary. By Assumption 4.2, there exists a function u on $X \times X$ such that u is strictly increasing in the second argument and $(x^n, m^n) \succeq_R (x^n, m)$ implies $\int u(x^n, z) dm^n(z) \geq \int u(x^n, z) dm(z)$. Without loss of generality, we can assume that $x^n \leq x^{n+1}, n = 1, \dots, N-1$. Then, we define

$$u(x, z) = u(x^1, z) \quad \text{if} \quad x \in [\underline{x}, x^1]$$

$$u(x, z) = tu(x^n, z) + (1-t)u(x^{n+1}, z) \quad \text{if } \exists t \in [0, 1], x = tx^n + (1-t)x^{n+1}.$$

Then, by Assumption 4.3, we can extend $\succ_R$ to a total, acyclic binary relation on D^d. Then the existence of a rationalization with an expected utility representation follows from the same arguments as in Lemma 5.5 in the appendix and as in the subsection of the construction of W in the appendix.

Q.E.D.

5 Appendix

The Proof of Lemma 3.1

We first prove Lemma 3.1. Since D^d has a finite number of elements, Lemma 3.1 follows from the next lemma. Indeed, if a new binary relation on $M(X)$ (or $X \times M(X)$) is introduced in Step h, then, by the following lemma, a new binary relation on D^d must have been introduced in Step $h-1$. Since D^d has only a finite number of elements, there exists $\overline{h}$ such that $\forall h \geq \overline{h}$, $\succ^h = \succ^{\overline{h}}$.

LEMMA **5.1** . *Suppose, for some $m, m' \in M(X)$, $mF^h m'$ and $\sim (m\overline{F}^{h-1} m')$ hold. Then there exist $(x^d, m^d), (\overline{x}^d, \overline{m}^d) \in D^d$ such that $(x^d, m^d)\overline{\succ}^{h-1}(\overline{x}^d, \overline{m}^d)$ and $\sim ((x^d, m^d) \succ^{h-2} (\overline{x}^d, \overline{m}^d))$.*

Proof. By $mF^h m'$, there exist $(x_1, m_1), \ldots, (x_t, m_t) \in X \times M(X)$ and $x \in X$ such that

$$(1) \quad (x, m) \succ^{h-1} (x_1, m_1) \succ^{h-1} \cdots \succ^{h-1} (x_t, m_t) \succ^{h-1} (x, m').$$

Suppose, in (1), each $\succ^{h-1}$ is T^{h-1} or $\succ_F$. Then clearly $x = x_1 = \cdots = x_k$ holds so that $mF^{h-1}m_1F^{h-1} \cdots F^{h-1}m_kF^{h-1}m'$ holds. Thus, by the transitivity of $\overline{F}^{h-1}$, $m\overline{F}^{h-1}m'$ holds. This is a contradiction. Thus at least one $\succ^{h-1}$ is $\succ_R$.

Suppose, in (1), just one $\succ^{h-1}$ is $\succ_R$, i.e., there exists just one i such that

$$(2) \quad (x_i, m_i) \succ_R (x_{i+1}, m_{i+1}).$$

Thus, by (1), $x_{i+1} \geq x \geq x_i$ holds so that $m_i F^1 m_{i+1}$. Since the other $\succ^{h-1}$ in (1) are T^{h-1} or $\succ_F$, then $m\overline{F}^{h-1}m'$ holds. This is a contradiction.

Suppose, in (1), more than one $\succ^{h-1}$ are $\succ_R$. Let the first one and the last one from the left of (1) be

$$(3) \quad (x_i, m_i) \succ_R (x_{i+1}, m_{i+1})$$

and

$$(4) \quad (x_j, m_j) \succ_R (x_{j+1}, m_{j+1}),$$

respectively. Note that, by the definition of $\succ_R$, $(x_i, m_i), (x_j, m_j) \in D^d$ holds. By (1),

$$(5) \quad (x_i, m_i)\overline{\succ}^{h-1}(x_j, m_j)$$

holds.

Suppose

$$(6) \quad (x_i, m_i)\overline{\succ}^{h-2}(x_j, m_j)$$

also holds. Then since by (1) $x_{j+1} \geq x \geq x_i$ holds, then, by (4) and (6), $m_i F^{h-1} m_{j+1}$ holds. Since all $\succ^{h-1}$ are T^{h-1} or $\succ_F$ on the left of (x_i, m_i) and the right of (x_{j+1}, m_{j+1}), then $m\overline{F}^{h-1}m'$. This is a contradiction. Thus $(x_i, m_i)\overline{\succ}^{h-2}(x_j, m_j)$ does not hold.

Q.E.D.

The steps of the Proof of Theorem 3.3

Below, we give the steps of the proof.

1. $(\succ_D^{\min}, F_D^{\min})$ on $X_D \times M_D$ is extended to the smallest binary relations $(\overline{\succ}^{\#}, \overline{F}^{\#})$ on $X \times M(X)$ which satisfy transitivity, risk separability, and time consistency, where $F_D^{\min}$ is the binary relation on M_D naturally defined with $\succ_D^{\min}$. They are also shown to be acyclic. (Lemma 5.3)
2. We construct a continuous function $\mu : M(X) \to R$ which is consistent with $\overline{F}^{\#}$. (Lemma 5.4)
3. μ defines a complete binary relation on $M(X)$. Thus, by time consistency and risk separability, we also obtain a new binary relation on $X \times M(X)$. It is proved that the new binary relation on $X \times M(X)$ is acyclic. (Lemma 5.5)
4. Then we construct a function W which is consistent with the new binary relation.
5. Finally, we prove the necessity of Assumption 3.1.

The Acyclicity of $(\overline{\succ}^{\#}, \overline{F}^{\#})$

Next, considering $\succ_D^{\min}$ as a binary relation on $X \times M(X)$, $\overline{\succ}^{\natural} = \succ^{\bar{h}} \bigcup \overline{\succ}_D^{\min}$ can be defined, where $\overline{\succ}_D^{\min}$ is the transitive closure of $\succ_D^{\min}$ and $\succ^{\bar{h}}$ is defined in Lemma 3.1. Below, we extend it to a transitive, risk separable, time consistent binary relation on $X \times M(X)$.

First, we define the following binary relation.

$$\forall m, m' \in M(X), \quad mF^{\#}m' \quad \text{if } \exists \overline{x} \in X, \ (\overline{x}, m)\overline{\succ}^{\natural}(\overline{x}, m').$$

Let $\overline{F}^{\#}$ be its transitive closure. Then, for $m, m' \in M(X)$ such that $m\overline{F}^{\#}m'$, we define $\forall x \in X$, $(x, m)T^{\#}(x, m')$. Let $\overline{T}^{\#}$ be the transitive closure of the union of $\succ_F$ and $T^{\#}$, $\succ^{\#} = \succ^0 \bigcup \overline{T}^{\#}$, and $\overline{\succ}^{\#}$ be the transitive closure of $\succ^{\#}$.

$(\overline{\succ}^{\#}, \overline{F}^{\#})$ satisfies risk separability and time consistency. Suppose the contrary. Then we can extend it to a transitive, risk separable, time consistent binary relation. Thus, by the same argument as of Lemma 5.1, a new binary relation on D^d must be introduced. However, this contradicts the fact that $\succ_D^{\min}$ is total on D^d.

Below, after proving some lemmas, we show the acyclicity of $(\overline{\succ}^{\#}, \overline{F}^{\#})$.

LEMMA **5.2** . $\forall (x,m),(x',m') \in X_D \times M_D$,

$$(a)\quad (x,m)\overline{T}^h(x',m') \quad \textit{implies} \quad (x,m)\overline{T}_D^h(x',m') \quad \textit{for all} \quad h=1,\dots,\bar{h},$$

$$(b)\quad (x,m)\ \overline{\succ}^h\ (x',m') \quad \textit{implies} \quad (x,m)\ \overline{\succ}_D^h\ (x',m') \quad \textit{for all} \quad h=1,\dots,\bar{h},$$

and

$$(c)\quad (x,m)\ \overline{\succ}^{\#}\ (x',m') \quad \textit{implies} \quad (x,m)\ \overline{\succ}_D^{\min}\ (x',m').$$

Proof. We first prove (a) and (b) by induction.

(i) Suppose there exist $(x,m),(x',m') \in X_D \times M_D$ such that $(x,m)\overline{T}^1(x',m')$. By the definition of $\overline{T}^1$, there exist $(x_1,m_1),\dots,(x_t,m_t) \in X \times M(X)$ such that

$$(1)\quad (x,m)T(x_1,m_1)T\cdots T(x_t,m_t)T(x',m'),$$

where $T = \succ_F$ or T^1. Suppose, in (1), all T are equal to $\succ_F$. Then $(x,m) \succ_F (x',m')$ holds so that $(x,m)\overline{T}_D^1(x',m')$ holds.

Suppose, in (1), just one T is T^1. Let it be $(x_i,m_i)T^1(x_{i+1},m_{i+1})$.[8] Note that $x_i = x_{i+1}$. By $T \neq \succ_F$, there exist $(x_{i1},m_{i1}),\dots,(x_{ik},m_{ik}) \in X \times M(X)$ and $\overline{x}_i \in X$ such that

$$(\overline{x}_i,m_i) \succ^0 (x_{i1},m_{i1}) \succ^0 \cdots \succ^0 (x_{ik},m_{ik}) \succ^0 (\overline{x}_i,m_{i+1})$$

with at least one $\succ^0 = \succ_R$. By mFm_i and $m_{i+1}Fm'$,

$$(2)\quad (\overline{x}_i,m) \succ^0 (\overline{x}_i,m_i) \succ^0 (x_{i1},m_{i1}) \succ^0 \cdots \succ^0 (x_{ik},m_{ik}) \succ^0 (\overline{x}_i,m_{i+1}) \succ^0 (\overline{x}_i,m')$$

holds. Let $\left(x_1^d,m_1^d\right),\dots,\left(x_\ell^d,m_\ell^d\right)$ be the elements of $\{(x_{i1},m_{i1}),\dots,(x_{ik},m_{ik})\}\bigcap(X_D \times M_D)$. Since $\succ_F$ is transitive,

$$(3)\quad (\overline{x}_i,m) \succ^0 (\overline{x}_i,m_i) \succ^0 \left(x_1^d,m_1^d\right) \succ^0 \cdots \succ^0 \left(x_\ell^d,m_\ell^d\right) \succ^0 (\overline{x}_i,m_{i+1}) \succ^0 (\overline{x}_i,m')$$

follows from (2). If $\overline{x}_i \in X_D$, then let $\widehat{x} = \overline{x}_i$, and if $\overline{x}_i \notin X_D$, then let $\widehat{x} = x_1^d$. Then

$$(4)\quad (\widehat{x},m) \succ^0 (\widehat{x},m_i) \succ^0 \left(x_1^d,m_1^d\right) \succ^0 \cdots \succ^0 \left(x_\ell^d,m_\ell^d\right) \succ^0 (\widehat{x},m')$$

holds. Since, for all $(x^d,m^d),(x'^d,m'^d) \in X_D \times M_D$, $(x^d,m^d) \succ^0 (x'^d,m'^d)$ implies $(x^d,m^d) \succ_D^0 (x'^d,m'^d)$, then

$$(5)\quad (\widehat{x},m)\,T_D^1\,(\widehat{x},m')$$

[8] Below, we set $(x,m) = (x_0,m_0)$ and $(x',m') = (x_{t+1},m_{t+1})$.

holds. Thus, for all $x^d \in X_D$,

$$\left(x^d, m\right) T_D^1 \left(x^d, m'\right)$$

holds. Thus, by

$$(x, m)\, T_D^1\, (x, m')\, T_D^1 (x', m'),$$

$$(6) \quad (x, m)\, \overline{T}_D^1\, (x', m')$$

holds.

Suppose two T are T^1. Let them be $(x_i, m_i) T^1 (x_{i+1}, m_{i+1})$ and $(x_j, m_j) T^1 (x_{j+1}, m_{j+1})$, where $j > i$. Note that $x_i = x_{i+1}$ and $x_j = x_{j+1}$. By $T \neq \succ_F$, there exist $(x_{i1}, m_{i1}), \ldots, (x_{ik}, m_{ik}) \in X \times M(X)$ and $\overline{x}_i \in X$ such that

$$(\overline{x}_i, m_i) \succ^0 (x_{i1}, m_{i1}) \succ^0 \cdots \succ^0 (x_{ik}, m_{ik}) \succ^0 (\overline{x}_i, m_{i+1})$$

with at least one $\succ^0 = \succ_R$. Similarly, there exist $(x_{j1}, m_{j1}), \ldots, (x_{jk'}, m_{jk'}) \in X \times M(X)$ and $\overline{x}_j \in X$ such that

$$(\overline{x}_j, m_j) \succ^0 (x_{j1}, m_{j1}) \succ^0 \cdots \succ^0 (x_{jk'}, m_{jk'}) \succ^0 (\overline{x}_j, m_{j+1})$$

with at least one $\succ^0 = \succ_R$. Since mFm_i and $m_{j+1}Fm'$,

$$(7) \quad (\overline{x}_i, m) \succ^0 (\overline{x}_i, m_i) \succ^0 (x_{i1}, m_{i1}) \succ^0 \cdots \succ^0 (x_{ik}, m_{ik}) \succ^0 (\overline{x}_i, m_{i+1})$$

and

$$(8) \quad (\overline{x}_j, m_j) \succ^0 (x_{j1}, m_{j1}) \succ^0 \cdots \succ^0 (x_{jk'}, m_{jk'}) \succ^0 (\overline{x}_j, m_{j+1}) \succ^0 (\overline{x}_j, m')$$

holds. Let $\left(x_1^d, m_1^d\right), \ldots, \left(x_\ell^d, m_\ell^d\right)$ be the elements of $\{(x_{i1}, m_{i1}), \ldots, (x_{ik}, m_{ik})\} \bigcap (X_D \times M_D)$ and $(x_1^e, m_1^e), \ldots, (x_{\ell'}^e, m_{\ell'}^e)$ be the elements of $\left\{(x_{j1}, m_{j1}), \ldots, \left(x_{jk'}, m_{jk'}\right)\right\} \bigcap (X_D \times M_D)$. Since $\succ_F$ is transitive,

$$(9) \quad (\overline{x}_i, m) \succ^0 (\overline{x}_i, m_i) \succ^0 \left(x_1^d, m_1^d\right) \succ^0 \cdots \succ^0 \left(x_\ell^d, m_\ell^d\right) \succ^0 (\overline{x}_i, m_{i+1})$$

and

$$(10) \quad (\overline{x}_j, m_j) \succ^0 (x_1^e, m_1^e) \succ^0 \cdots \succ^0 (x_{\ell'}^e, m_{\ell'}^e) \succ^0 (\overline{x}_j, m_{j+1}) \succ^0 (\overline{x}_j, m')$$

follow from (7) and (8). If $\overline{x}_i \in X_D$, then let $\widehat{x} = \overline{x}_i$,and if $\overline{x}_i \notin X_D$, then let $\widehat{x} = x_1^d$. Similarly, if $\overline{x}_j \in X_D$, then let $\widetilde{x} = \overline{x}_j$, and if $\overline{x}_j \notin X_D$, then let $\widetilde{x} = x_1^e$. Then

$$(11) \quad (\widehat{x}, m) \succ^0 (\widehat{x}, m_i) \succ^0 \left(x_1^d, m_1^d\right) \succ^0 \cdots \succ^0 \left(x_\ell^d, m_\ell^d\right) \succ^0 (\widehat{x}, m_\ell^d)$$

and

$$(12) \quad (\widetilde{x}, m_j) \succ^0 (x_1^e, m_1^e) \succ^0 \cdots \succ^0 (x_{\ell'}^e, m_{\ell'}^e) \succ^0 \left(\widetilde{x}, m_{j+1}\right) \succ^0 (\widetilde{x}, m')$$

hold. Since, for all $(x^d, m^d), (x'^d, m'^d) \in X_D \times M_D$, $(x^d, m^d) \succ^0 (x'^d, m'^d)$ implies $(x^d, m^d) \succ^0_D (x'^d, m'^d)$, then, by $m^d_\ell F m_j$,

$$(13) \quad (\widehat{x}, m)\, T^1_D\, (\widehat{x}, m_j)$$

and

$$(14) \quad (\widetilde{x}, m_j)\, T^1_D\, (\widetilde{x}, m')$$

hold. Thus, for all $x^d \in X_D$,

$$\left(x^d, m\right) T^1_D (x^d, m_j) T^1_D \left(x^d, m'\right)$$

holds. Thus, by

$$(x, m)\, T^1_D\, (x, m')\, T^1_D (x', m'),$$

$$(15) \quad (x, m)\, \overline{T}^1_D\, (x', m')$$

holds.

If more than two T are T^1, almost the same argument leads to

$$(16) \quad (x, m)\, \overline{T}^1_D\, (x', m').$$

Since $\succ^1 = \succ^0 \bigcup \overline{T}^1$ and $\succ^1_D = \succ^0_D \bigcup \overline{T}^1_D$, by the same arguments as the above, $(x, m) \overline{\succ}^1 (x', m')$ implies $(x, m) \overline{\succ}^1_D (x', m')$

(ii) Suppose, for all $(x, m), (x', m') \in X_D \times M_D, (x, m)\overline{T}^{h-1}(x', m')$ implies $(x, m)\overline{T}^{h-1}_D(x', m')$ and $(x, m) \overline{\succ}^{h-1} (x', m')$ implies $(x, m) \overline{\succ}^{h-1}_D (x', m')$.

Suppose there exist $(x, m), (x', m') \in X_D \times M_D$ such that $(x, m)\overline{T}^h(x', m')$. By the definition of $\overline{T}^h$, there exist $(x_1, m_1), \ldots, (x_t, m_t) \in X \times M(X)$ such that

$$(17) \quad (x, m)T(x_1, m_1)T \cdots T(x_t, m_t)T(x', m'),$$

where $T = \succ_F$ or T^h. Suppose, in (17), all $T = \succ_F$. Then, by $(x, m) \succ_F (x', m')$, $(x, m)\overline{T}^h_D(x', m')$ holds. Suppose, in (17), just one T is T^h. Then the same argument as in (i) leads to

$$(18) \quad (x, m)\, \overline{T}^h_D\, (x', m').$$

Suppose, in (17), two T are T^h. Then the same argument as in (i) leads to

$$(19) \quad (x, m)\, \overline{T}^h_D\, (x', m').$$

Suppose, in (17), more than two T are T^h, almost the same argument leads to

$$(20) \quad (x, m)\, \overline{T}^h_D\, (x', m').$$

Since $\succ^h = \succ^{h-1} \bigcup \overline{T}^h$ and $\succ^h_D = \succ^{h-1}_D \bigcup \overline{T}^h_D$, by the same arguments as the above, $(x,m) \overline{\succ}^{h-1} (x',m')$ implies $(x,m) \overline{\succ}^{h-1}_D (x',m')$.

By the same arguments as for $\overline{\succ}^h$ and $\overline{T}^h$, clearly

$$(c) \quad (x,m) \overline{\succ}^{\#} (x',m') \quad \text{implies} \quad (x,m) \overline{\succ}^{\min}_D (x',m').$$

Q.E.D.

Lemma **5.3** . *Under Assumption 3.2, $\overline{\succ}^{\overline{h}}$ and $\overline{\succ}^{\#}$ are acyclic.*

Proof. Below, we prove that, for all nonnegative integer h, if there exist $(x,m), (x',m') \in X \times M(X)$ such that $(x,m)\overline{\succ}^h(x',m')$ and $(x',m')\overline{\succ}^h(x,m)$, then there exist $(x^d,m^d),(x'^d,m'^d) \in D^d$ such that $(x^d,m^d)\overline{\succ}^h_D(x'^d,m'^d)$ and $(x'^d,m'^d)\overline{\succ}^h_D(x^d,m^d)$. This is clearly sufficient for the proof of this lemma. That is if $\overline{\succ}^h$ is cyclic, then $\overline{\succ}^h_D$ is also cyclic and this contradicts Assumption 3.2. We prove this by induction.

(i) Suppose there exist $(x,m),(x',m') \in X \times M(X)$ such that $(x,m)\overline{\succ}^0(x',m')$ and $(x',m')\overline{\succ}^0(x,m)$. Then there exist $(x_1,m_1),\ldots,(x_k,m_k),(x_{k+1},m_{k+1}),\ldots,(x_{k+\ell},m_{k+\ell}) \in X \times M(X)$ such that

$$\begin{aligned}(1) \quad & (x,m) \succ^0 (x_1,m_1) \succ^0 \ldots \succ^0 (x_k,m_k) \succ^0 (x',m') \\ & \succ^0 (x_{k+1},m_{k+1}) \succ^0 \ldots \succ^0 (x_{k+\ell},m_{k+\ell}) \succ^0 (x,m).\end{aligned}$$

Case 1. Suppose, in (1), there is no $\succ_R$. (Recall that $\succ^0 = \succ_R \bigcup \succ_F$.) Then $(x,m) \succ_F (x',m') \succ_F (x,m)$ hold. This contradicts the definition of $\succ_F$.

Case 2. Suppose, in (1), there is just one $\succ_R$. Let it be $(x^d,m^d) \succ_R (\widehat{x},\widehat{m})$. On the other hand, by (1), $(\widehat{x},\widehat{m}) \succ_F (x^d,m^d)$. By the definition of $\succ_{RD}$, $(x^d,m^d) \succ_{RD} (x^d,m^d)$ holds, i.e., $\succ^0_D$ is cyclic. This contradicts the fact that $\succ^{\min}_D$ is total on D^d.

Case 3. Suppose, in (1), there are more than one $\succ_R$. Let the first one and the second one from the left be

$$(2) \quad \ldots \succ_F (x^d,m^d) \succ_R (\overline{x},\overline{m}) \succ_F \cdots \succ_F (x'^d,m'^d) \succ_R (\widehat{x},\widehat{m}) \succ_F \cdots.$$

By the definition of $\succ_{RD}$, (2) becomes

$$\cdots \succ_F (x^d,m^d) \succ_{RD} (x'^d,m'^d) \succ_R (\widehat{x},\widehat{m}) \succ_F \cdots.$$

Applying the same argument to all $\succ_R$ in (1), we obtain a cycle of $\succ_{RD}$. By $\succ^0_D = \succ_{RD} \bigcup \succ_{FD}$, $\succ^0_D$ is cyclic.

(ii) Suppose that there do not exist $(\overline{x},\overline{m}),(\overline{x}',\overline{m}') \in X \times M(X)$ such that $(\overline{x},\overline{m})\overline{\succ}^{h-1}(\overline{x}',\overline{m}')$ and $(\overline{x}',\overline{m}')\overline{\succ}^{h-1}(\overline{x},\overline{m})$ and that there exist $(x,m),(x',m') \in X \times M(X)$ such that $(x,m)\overline{\succ}^h(x',m')$

and $(x', m')\overline{\succ}^h(x, m)$. Then there exist $(x_1, m_1), \ldots, (x_k, m_k), (x_{k+1}, m_{k+1}), \ldots, (x_{k+\ell}, m_{k+\ell}) \in X \times M(X)$ such that

$$(3) \quad (x, m) \succ^h (x_1, m_1) \succ^h \cdots \succ^h (x_k, m_k) \succ^h (x', m') \succ^h (x_{k+1}, m_{k+1}) \succ^h \cdots \succ^h (x_{k+\ell}, m_{k+\ell}) \succ^h (x, m).$$

Case 1. Suppose, in (3), there is no $\succ_R$, i.e., all $\succ^h$ are $\overline{T}^h$. Then, by (3),

$$x \geq x_1 \geq \cdots \geq x_k \geq x' \geq x_{k+1} \geq \cdots \geq x_{k+\ell} \geq x$$

holds so that $x = x_1 = \cdots = x_k = x' = x_{k+1} = \cdots = x_{k+\ell}$ holds. Thus $(x, m)\overline{T}^h(x, m')$ and $(x, m')\overline{T}^h(x, m)$ holds. Since $(x, m) \succ_F (x, m')$ is not consistent with $(x, m') \succ_F (x, m)$, there exist $m_1^d, \ldots, m_{k'}^d, m_{k'+1}^d, \ldots, m_{k'+\ell'}^d \in M_D$ such that

$$(4) \; (x, m)\overline{T}^h \left(x, m_1^d\right) \overline{T}^h \left(x, m_2^d\right) \overline{T}^h \cdots \overline{T}^h \left(x, m_{k'}^d\right) \overline{T}^h (x, m') \overline{T}^h \left(x, m_{k'+1}^d\right) \overline{T}^h \cdots \overline{T}^h \left(x, m_{k'+\ell'}^d\right) \overline{T}^h (x, m).$$

Thus, by Lemma 5.2 and (4),

$$\left(x^d, m_1^d\right) \overline{T}_D^h \left(x^d, m_2^d\right) \overline{T}_D^h \cdots \overline{T}_D^h \left(x^d, m_{k'}^d\right) \overline{T}_D^h \left(x^d, m_{k'+1}^d\right) \overline{T}_D^h \cdots \overline{T}_D^h \left(x^d, m_{k'+\ell'}^d\right) \overline{T}_D^h \left(x^d, m_1^d\right)$$

holds, where $(x^d, m_1^d) \in D^d$. This contradicts Assumption 3.2.

Case 2. Suppose, in (3), there is just one $\succ_R$. Let it be $\left(x^d, m^d\right) \succ_R (\widehat{x}, \widehat{m})$. On the other hand, by (3), $(\widehat{x}, \widehat{m}) \succ_F \left(x^d, \widehat{m}\right) \overline{T}^h \left(x^d, m^d\right)$ holds. Thus, as the proof of Lemma 5.2, either

$$(a) \quad \left(x^d, \widehat{m}\right) \succ_F \left(x^d, m^d\right),$$

or

$$(b) \quad \text{there exist} \quad \widehat{m}_1^d, \ldots, \widehat{m}_k^d \in M_D \text{ such that}$$
$$\left(x^d, \widehat{m}\right) \succeq_F \left(x^d, \widehat{m}_1^d\right) T^h \cdots T^h \left(x^d, \widehat{m}_k^d\right) T^h \left(x^d, m^d\right)$$

holds. Thus, by Lemma 5.2 and the definition of $\succ_{RD}$,

$$\left(x^d, m^d\right) \succ_{RD} \left(x^d, m^d\right) \text{ if (a) holds}$$

and

$$\left(x^d, m^d\right) \succ_{RD} \left(x^d, \widehat{m}_1^d\right) \overline{T}_D^h \left(x^d, m^d\right) \text{ if (b) holds.}$$

This contradicts Assumption 3.2.

Case 3. Suppose, in (3), there are more than one $\succ_R$. Let the first one and the second one from the left of (3) be

$$(5) \quad \cdots \overline{T}^h \left(\widehat{x}^d, \widehat{m}^d\right) \succ_R (\widehat{x}, \widehat{m}) \overline{T}^h \cdots \overline{T}^h \left(\overline{x}^d, \overline{m}^d\right) \succ_R (\overline{x}, \overline{m}) \overline{T}^h \cdots.$$

In (5), clearly, $\widehat{x} \geq \overline{x}^d$ and $\left(\overline{x}^d, \widehat{m}\right) \overline{T}^h \left(\overline{x}^d, \overline{m}^d\right)$ hold.

Thus, as the proof of Lemma 5.2, either

$$(a) \quad \left(\overline{x}^d, \widehat{m}\right) \succ_F \left(\overline{x}^d, \overline{m}^d\right),$$

or

$$(b) \quad \text{there exist} \quad \widehat{m}_1^d, ..., \widehat{m}_k^d \in M_D \text{ such that}$$
$$\left(\overline{x}^d, \widehat{m}\right) \succeq_F \left(\overline{x}^d, \widehat{m}_1^d\right) T^h \cdots T^h \left(\overline{x}^d, \widehat{m}_k^d\right) T^h \left(\overline{x}^d, \overline{m}^d\right) \succ_R (\overline{x}, \overline{m})$$

holds. Thus, by Lemma 5.2 and the definition of $\succ_{RD}$,

$$\left(\widehat{x}^d, \widehat{m}^d\right) \succ_{RD} \left(\overline{x}^d, \overline{m}^d\right) \succ_R (\overline{x}, \overline{m}) \text{ if (a) holds}$$

and

$$\left(\widehat{x}^d, \widehat{m}^d\right) \succ_{RD} \left(\overline{x}^d, \widehat{m}_1^d\right) \overline{T}_D^h \left(\overline{x}^d, \overline{m}^d\right) \succ_R (\overline{x}, \overline{m}) \text{ if (b) holds.}$$

Applying the same argument to the other $\succ_R$ in (3), we obtain a contradiction.

Since $\overline{\succ}_D^{\min}$ is acyclic and total on D^d, the acyclicity of $\overline{\succ}^{\#}$ follows from the same argument as the above.

Q.E.D.

The Construction of μ

First, we construct μ.

LEMMA **5.4** . *These exists a weak* continuous function $\mu : M(X) \rightarrow R$ such that, for all $m, m' \in M(X)$,*

$$m\overline{F}^{\#}m' \qquad \textit{implies} \qquad \mu(m) > \mu(m').$$

Proof. (i) By the separability of $M(X)$, there exists a countable dense subset Φ of $M(X)$ such that $M_D \subset \Phi$. Since Φ is countable, it can be written as

$$\Phi = \{m(1), m(2), \ldots, m(s), \ldots\}.$$

(ii) Using Φ, we extend $F^{\#}$ to a continuous complete binary relation on $M(X)$. [9]

[9]For a set A, a binary relation $G \subset A \times A$ is said to be complete if $\forall a_1, a_2$ either $a_1 G a_2$ or $a_2 G a_1$ holds.

First, we introduce a new binary relation using $m(1)$. Let

$$L(m(1)) = \{m \in M(X) \mid m(1)\overline{F}^{\#}m\}.$$

and

$$U(m(1)) = \{m \in M(X) \mid m\overline{F}^{\#}m(1)\}.$$

Since $M(X)$ is separable, there exist a countable dense subset $DL(m(1))$ of $L(m(1))$. For each $m \in DL(m(1))$, take $B_{\varepsilon_m}(m)$, the ball with the center m and the radius ε_m, where

$$\varepsilon_m = \frac{1}{3} \inf_{m' \in U(m(1))} d^p(m, m').$$

Note that since $L(m(1)) \bigcap U(m(1)) = \emptyset$, then $\varepsilon_m > 0$ holds. Let $\tilde{L}(m(1)) = \bigcup_{m \in DL(m(1))} B_{\varepsilon_m}(m)$. Similarly, we take a countable dense subset $DL(m(1))$ of $U(m(1))$. For each $m \in DL(m(1))$, take $B_{\varepsilon_m}(m)$, where

$$\varepsilon_m = \frac{1}{3} \inf_{m' \in D(m(1))} d^p(m, m').$$

Let $\tilde{U}(m(1)) = \bigcup_{m \in DL(m(1))} B_{\varepsilon_m}(m)$. By the construction,

$$(1) \quad \tilde{L}(m(1)) \bigcup \tilde{U}(m(1)) = \emptyset$$

and

$$(2) \quad L(m(1)) \subset \tilde{L}(m(1)) \quad \text{and} \quad U(m(1)) \subset \tilde{U}(m(1))$$

hold.

Then we define a binary relation $\tilde{F}^1$ as follows:

$$m(1)\tilde{F}^1 m \quad \text{if} \quad m \in \tilde{L}(m(1))$$

and

$$m\tilde{F}^1 m(1) \quad \text{if} \quad m \in \tilde{U}(m(1)).$$

Let $\overline{G}^1$ be the transitive closure of $G^1 = \tilde{F}^1 \bigcup F^{\#}$. Below, we show that, $\overline{G}^1$ is acyclic, i.e.,

$$m\overline{G}^1\overline{m} \quad \text{implies} \quad \sim (\overline{m}\overline{G}^1 m).$$

Suppose the contrary. Then there exist $m_1, m_2, ..., m_k \in M(X)$ such that

$$(3) \quad m_1 G^1 m_2 G^1 \cdots G^1 m_{k-1} G^1 m_k G^1 m_1.$$

Since $\overline{F^{\#}}$ is acyclic, at least one of G^1 in (3) must be $\tilde{F}^1$. Thus, in (3), some $m_i = m(1)$. This contradicts (1).

Similarly, we extend $\overline{G}^1$ using $m(2)$. We repeat this procedure for $m(3), m(4), \ldots$.

(iii) Next, we define a complete binary relation on $M(X)$. For all $m, m' \in M(X)$, we define
(a) mF^*m' if $\exists \varepsilon > 0, \forall \overline{m} \in B_\varepsilon(m) \bigcap \Phi, \forall \overline{m}' \in B_\varepsilon(m') \bigcap \Phi, \exists t, \ \ \overline{m}\overline{G}^t\overline{m}'$,
(b) $m'F^*m$ if $\exists \varepsilon > 0, \forall \overline{m} \in B_\varepsilon(m) \bigcap \Phi, \forall \overline{m}' \in B_\varepsilon(m') \bigcap \Phi, \exists t, \ \ \overline{m}'\overline{G}^t\overline{m}$,
(c) $m \sim_F m'$ otherwise.

Define

$$mF_e^*m' \quad \text{if} \quad mF^*m' \quad \text{or} \quad m \sim_F m'.$$

Next, we show that

1. $m\overline{F}^{\#}m'$ implies mF^*m',
2. F_e^* is transitive, complete, and reflexive,
3. $\forall \overline{m} \in M(X), \{m | mF_e^*\overline{m}\}$ and $\{m | \overline{m}F_e^*m\}$ are closed.

First, we prove 1. If $m, m' \in \Phi$ holds, then 1 immediately follows from the definition of F^*. Otherwise , by the construction of $\overline{F}^{\#}$, (a) mFm' or (b) $\exists m^d \in M_D$, $m\overline{F}^{\#}m^d\overline{F}^{\#}m'$ holds. In case (a), by the definition of F, there exist $\bar{m} \in \Phi \bigcap B_\varepsilon(m)$ and $\bar{m}' \in \Phi \bigcap B_\varepsilon(m')$ such that $\bar{m}F\bar{m}'$, where $0 < \varepsilon < \frac{1}{5}d^p(m, m')$. Thus, for some $\delta > 0$, $\forall \overline{m} \in B_\delta(m) \bigcap \Phi, \forall \overline{m}' \in B_\delta(m') \bigcap \Phi, \exists t, \ \ \overline{m}\overline{G}^t\overline{m}'$. In case (b), 1 immediately follows from $m\overline{F}^{\#}m^d\overline{F}^{\#}m'$ and the definition of F^*.

Clearly, 2 holds.

Below, we prove 3. For $m^* \in \{m | \overline{m}F^*m\}$, there exists a real number $\varepsilon > 0$, such that $\forall \widehat{m} \in B_\varepsilon(\overline{m}) \bigcap \Phi, \forall \widehat{m}' \in B_\varepsilon(m^*) \bigcap \Phi, \exists t, \ \ \widehat{m}\overline{G}^t\widehat{m}'$. Thus, for $m_a \in B_\varepsilon(m^*)$, there exist a real number $\delta > 0$ such that $\forall \widehat{m} \in B_\delta(\overline{m}) \bigcap \Phi, \forall \widehat{m}' \in B_\delta(m_a) \bigcap \Phi, \exists t, \ \ \widehat{m}\overline{G}^t\widehat{m}'$. Thus $\{m | \overline{m}F^*m\}$ is an open set. Since F_e^* is complete $\{m | mF_e^*\overline{m}\}$ is the complement of $\{m | \overline{m}F^*m\}$ and thus it is closed . The same argument applies to $\{m | \overline{m}F_e^*m\}$.

By Debreu [1954] and 1,2, and 3, there exists a weak* continuous function $\mu : M(X) \to R$ such that

$$\begin{aligned} mF_e^*m' &\quad \text{if and only if} \quad \mu(m) \geq \mu(m') \\ mF^*m' &\quad \text{if and only if} \quad \mu(m) > \mu(m') \end{aligned}$$

Moreover, clearly

$$mF^{\#}m' \quad \text{implies} \quad \mu(m) > \mu(m')$$

holds. This concludes the proof. Q.E.D.

Next, $\forall m, m' \in M(X)$,

$$\text{if} \quad \mu(m) > \mu(m'), \quad \text{then we define} \quad \forall x \in X, (x,m) \succ^{\#1} (x,m')$$

and

$$\text{if} \quad \mu(m) \geq \mu(m'), \quad \text{then we define} \quad \forall x \in X, (x,m) \succeq^{\#1} (x,m').$$

Let $\succ^{\#2} = \succ^{\#1} \bigcup \succ^{\#}$ and $\succeq^{\#2} = \succeq^{\#1} \bigcup \succ^{\#}$. Let $\overline{\succeq}^{\#2}$ be the transitive closure of $\succeq^{\#2}$. Then we define

$$(x,m) \overline{\succ}^{\#2} (x',m') \quad \text{if} \quad \begin{array}{l} \text{there exist} \ \ (x_1,m_1),\ldots,(x_t,m_t) \in M(X) \\ \text{such that} \\ (x,m) \succeq^{\#2} (x_1,m_1) \succeq^{\#2} \ldots \succeq^{\#2} (x_t,m_t) \succeq^{\#2} (x',m'), \\ \text{where at least one} \ \ \succeq^{\#2} \ \ \text{is} \ \ \succ^{\#2}. \end{array}$$

We prove the following lemma.

Lemma 5.5 . $\forall (x,m), (x',m') \in X \times M(X)$, $\overline{\succ}^{\#2}$ *is acyclic.*

Proof. Suppose the contrary. Then there exist $(x,m), (x',m') \in X \times M(X)$, and

$$(x_1,m_1), \ldots, (x_s,m_s), (x_{s+1},m_{s+1}), \ldots, (x_{s+t},m_{s+t}) \in X \times M(X)$$

such that

(1) $(x,m) \succeq^{\#2} (x_1,m_1) \succeq^{\#2} \cdots \succeq^{\#2} (x_s,m_s) \succeq^{\#2} (x',m')$

(2) $(x',m') \succeq^{\#2} (x_{s+1},m_{s+1}) \succeq^{\#2} \cdots \succeq^{\#2} (x_{s+t},m_{s+t}) \succeq^{\#2} (x,m)$,

where, in (1), at least one $\succeq^{\#2}$ is $\succ^{\#2}$. There are three cases.

Case 1. Suppose, in (1) and (2), no $\succeq^{\#2}$ is $\succ^{*}_{D}$, where $\succ^{*}_{D}$ was defined in Assumption 3.1. Then, by (1) and (2), $x = x'$ holds. Thus, by (1), $\mu(m) > \mu(m')$ holds. On the other hand, by (2), $\mu(m') \geq \mu(m)$ holds. This is a contradiction.

Case 2. Suppose, either in (1) or in (2), just one $\succeq^{\#2}$ is $\succ^{*}_{D}$. Let it be

$$(3) \quad \left(x^d_a, m^d_a\right) \succ^{*}_{D} (x_a, m_a),$$

where $\left(x^d_a, m^d_a\right) \in D^d$. That is

$$(4) \quad (x,m) \succeq^{\#2} \cdots \succeq^{\#2} \left(x^d_a, m^d_a\right) \succ^{*}_{D} (x_a,m_a) \succeq^{\#2} \cdots \succeq^{\#2} (x,m).$$

Then

$$(5) \quad x_a \geq x_a^d$$

holds so that, by (3),

$$(6) \quad m_a^d F^{\#} m_a$$

holds. Thus $\mu\left(m_a^d\right) > \mu(m_a)$ holds. On the other hand, by (4),

$$(7) \quad \mu(m_a) \geq \mu\left(m_a^d\right)$$

holds. This is a contradiction.

Case 3. Suppose, in (1) and (2), more than one $\succeq^{\#2}$ are $\succ_D^*$. Let them be as follows:

$$\begin{aligned}(8) \quad (x, m) \succeq^{\#2} \cdots \succeq^{\#2} \left(x_1^d, m_1^d\right) \succ_D^* (x_1, m_1) \succeq^{\#2} \cdots \succeq^{\#2} \left(x_2^d, m_2^d\right) \succ_D^* (x_2, m_2) \\ \succeq^{\#2} \cdots \succeq^{\#2} \left(x_t^d, m_t^d\right) \succ_D^* (x_t, m_t) \succeq^{\#2} \cdots \succeq^{\#2} (x, m),\end{aligned}$$

where $\left(x_i^d, m_i^d\right), i = 1, \ldots, t$, belongs to D^d and no $\succeq^{\#2}$ in (8) is $\succ_D^*$. By the fact that $\succ_D^{\min}$ is total on D^d, either $\left(x_1^d, m_1^d\right) \succ_D^* \left(x_2^d, m_2^d\right)$ or $\left(x_2^d, m_2^d\right) \succ_D^* \left(x_1^d, m_1^d\right)$ holds. Suppose $\left(x_2^d, m_2^d\right) \succ_R \left(x_1^d, m_1^d\right)$ holds. Then $\left(x_2^d, m_2^d\right) \overline{\succ}^{\#2} (x_1, m_1)$ holds. On the other hand, $x_1 \geq x_2^d$ and $\mu(m_1) \geq \mu\left(m_2^d\right)$ hold. By $x_1 \geq x_2^d$ and $\left(x_2^d, m_2^d\right) \overline{\succ}^{\#2} (x_1, m_1)$, $\mu\left(m_2^d\right) > \mu(m_1)$ holds. This is a contradiction. Thus $\left(x_1^d, m_1^d\right) \succ_D^* \left(x_2^d, m_2^d\right)$ holds.

Similarly, $\left(x_i^d, m_i^d\right) \succ_D^* \left(x_{i+1}^d, m_{i+1}^d\right), i = 2, \ldots, t-1$, and $\left(x_t^d, m_t^d\right) \succ_D^* \left(x_1^d, m_1^d\right)$ hold. Thus they generate a cycle

$$\left(x_1^d, m_1^d\right) \succ_D^* \left(x_2^d, m_2^d\right) \succ_D^* \cdots \succ_D^* \left(x_t^d, m_t^d\right) \succ_D^* \left(x_1^d, m_1^d\right).$$

This contradicts the fact that $\succ_D^{\min}$ is acyclic on D^d.

Q.E.D.

The Construction of W

Finally, we construct W. First, we introduce a complete binary relation on $X \times \mu(M(X))$, where $\mu(M(X)) = \{a \in R | \exists m \in M(X), a = \mu(m)\}$.

For all $(x, a), (x', a') \in X \times \mu(M(X))$, we define

$$(x, a) \overline{\succ}^{\#3} (x', a') \quad \text{if} \quad \exists m, m' \in M(X), \mu(m) = a, \mu(m') = a', \text{ and } (x, m) \overline{\succ}^{\#2} (x', m')$$

and

$$(x,a)\succsim^{\#3}(x',a') \quad \text{if} \quad \exists m,m' \in M(X), \mu(m)=a, \mu(m')=a', \text{ and } (x,m)\succsim^{2}(x',m').$$

Clearly, $\succ^{\#3}$ is well-defined. Indeed, suppose, for $(x,a),(x',a') \in X \times \mu(M(X))$, there exist $m,m' \in M(X)$ such that $\mu(m)=a, \mu(m')=a'$, and $(x,m)\succ^{\#2}(x',m')$. Then, by the definition of $\succ^{\#2}$, for all $\overline{m},\overline{m}' \in M(X)$ satisfying $\mu(\overline{m})=a$ and $\mu(\overline{m}')=a', (x,\overline{m})\succ^{\#2}(x',\overline{m}')$ holds. Thus $\succ^{\#3}$ is well-defined. The same argument applies to $\succsim^{\#3}$.

Finally, we construct a continuous function $W: X \times \mu(M(X)) \to R$ such that

$$(x,a)\succ^{\#3}(x',a') \quad \text{implies} \quad W(x,a) > W(x',a')$$

and

$$(x,a)\succsim^{\#3}(x',a') \quad \text{implies} \quad W(x,a) \geq W(x',a').$$

Such a W, together with μ, is clearly a risk separable rationalization of $\{(x^n,m^n),B^n\}_{n=1}^N$.

In order to construct W, we first take a countable dense subset Φ' in $X \times \mu(M(X))$. Then the existence of W follows from the same argument as the proof of Lemma 5.4.

The Necessity of Assumption 3.2

Finally, we prove the necessity of Assumption 3.2.

Suppose that there exists a rationalization (W,μ) and that there is no total binary relation on D^d of which risk separable, time consistent extension is acyclic. Then we take the equivalence classes $A_1,\dots,A_L$, subsets of $X_D \times M_D$, such that $A_i \bigcap A_j = \emptyset$ for $i \neq j$, for all $(x,m),(x',m') \in A_i$, $W(x,\mu(m)) = W(x',\mu(m'))$, and, for all $(x,m),(x',m') \in (X_D \times M_D)\setminus\bigcup_{i=1}^L A_i$, $W(x,\mu(m)) \neq W(x',\mu(m'))$. Let $\succ_i$ be the acyclic total order on A_i. Let $\succ_D$ be the binary relation on $X_D \times M_D$ naturally defined by (W,μ), i.e., $(x,m) \succ_D (x',m')$ if and only if $W(x,\mu(m)) > W(x',\mu(m'))$. Then $\succ = \succ_D \bigcup(\bigcup_{i=1}^L \succ_i)$ must be cyclic. Let the cycle be

$$(1) \quad (x_1,m_1) \succ (x_2,m_2) \succ \cdots \succ (x_1,m_1).$$

In (1), if $\succ = \succ_i$ for some i, then we replace $\succ_i$ by $\sim$. Suppose all $\succ$ in (1) are replaced by $\sim$. Then this contradicts the fact that $\succ_i$ is acyclic on A_i.[10] Suppose there are some $\succ$ not replaced by $\sim$. Then this contradicts the fact that $\succ_D$ is derived from (W,μ). That is $W(x_1,\mu(m_1)) > \cdots > W(x_1,\mu(m_1))$ must hold.

[10] In this case, all (x,m) in (1) must be in the same equivalence class.

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