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# Estimation of the Frequency of Unbounded Spectral Densities

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**Abstract**

We consider estimation of the frequency at which an unbounded spectral density diverges and derive its asymptotic properties. Next we apply these results to estimation of parametric models whose spectral densities have this property. A Gegenbauer autoregressive moving-average model, which is a generalization of a fractional autoregressive moving-average model, is an example.

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# 1 Introduction.

Estimation and testing of a periodical structure in time series have been discussed for a long time. Especially estimation and testing of frequencies of trigonometric components in a regression model have been considered by Fisher [5], Whittle [25], Grenander and Rosenblatt [9], Hannan [10], [12], Walker [23],Chen [1] [2], Lin and Kedem [16] and many others. While Damsleth and Spjøtvoll[3], Quinn [17], Wang [24], Kavalieris and Hannan [15] considered estimation of the number of these components.

The regression model with deterministic trigonometric functions has been useful to analyze many time series in the natural science. However it is unable to deal with some data whose periodicities and amplitudes are not exact and are likely to change rather than remain constant, though exhibiting a periodical structure.

On the other hand Gray et al. [8] proposed a new model called a Gegenbauer autoregressive moving-average(GARMA) model to analyze time series with persistent periodical behavior. This model is a stationary process and has an absolutely continuous spectral distribution function but its spectral density diverges at some frequency $\lambda$ in $[0, \pi]$.

Hence its sample function exhibits a strong periodical behavior but its periodicities and amplitudes can change over time unlike those of a regression model with deterministic trigonometric functions so that a GARMA model can be an alternative model for analyzing data having properties described above.

And a GARMA model includes a fractional autoregressive moving-average (FARMA) model whose spectral density diverges at the origin $\lambda = 0$ as a special case. A FARMA model is proposed by Granger and Joyeux [7] and Hosking [13] and is one of long-memory time series models to which much attention has been paid both recently.

In this paper first we discuss estimation of the frequency of unbounded spectral densities. We estimate this frequency by the value which maximizes the periodogram in the same way as that of a deterministic trigonometric component in some of the papers cited above and derive its asymptotic properties. Next we apply it to estimation of parametric time series models including a GARMA model.

Section 2 contains our model, notation, basic assumptions and the presentation of the main thorem and its proof. In Section 3, we apply the theorem in Section 2 to estimate parameters of time series models. Lemmas and propositions which are necessary to prove the theorems in Sections 2 and 3 are compiled in Section 4.

## 2 Model and theorem.

First we state assumptions and introduce notation. Let $\{X(t)\}$ be a Gaussian stationary process with mean 0 and spectral density $f(\lambda)$ and covariance function $r(h) = Cov(X(t), X(t+h)) = \int_{\Pi} e^{it\lambda} f(\lambda) d\lambda$ for $h = 0, \pm 1, \pm 2, \ldots$, where $\Pi = [-\pi, \pi]$. And $f(\lambda)$ is assumed to have the form

$$f(\lambda) = \frac{g(\lambda)}{|\lambda - \lambda_0|^{2d}}, \quad \lambda \in [0, \pi], \tag{1}$$

where $0 \leq \lambda_0 \leq \pi$ and $0 < d < 1/2$.

Further the following assumptions are imposed on $f(\lambda)$.

**Assumption A.**

(A1) $f(\lambda) = f(-\lambda), \quad \lambda \in [-\pi, 0)$.

(A2) $g(\lambda)$ is positive in $[0, \pi]$ and continuously differentiable in $(0, \lambda_0)$ and $(\lambda_0, \pi)$ and right(left) continuously differentiable at $\lambda = 0(\pi)$.

(A3) $g(\lambda)$ satisfies

$$|g'(\lambda)/g(\lambda)| = O(1/|\lambda - \lambda_0|).$$

Then we note that $f(\lambda)$ diverges to infinity as $\lambda \to \lambda_0$.

Here we introduce some notation. First let $I_T(\lambda)$ be the periodogram

$$I_T(\lambda) = \frac{1}{T}|\sum_{t=1}^{T} X(t)\exp(it\lambda)|^2.$$

Next set

$$\Lambda_\epsilon = [0, \lambda_0 - \epsilon] \cup [\lambda_0 + \epsilon, \pi],$$

for $\epsilon > 0$. And define the maximum of the normalized periodogram in $\Lambda_\epsilon$ by

$$M_{T,\epsilon} = \max_{\lambda \in \Lambda_\epsilon} \frac{I_T(\lambda)}{2\pi f(\lambda)}.$$

Finally $C, C_1, C_2$ and so on stand for general constants being independent of $T$ but are not always the same constants in each context.

Now we shall consider estimation of $\lambda_0$. Let $\hat{\lambda}_T$ be the value of $\lambda$ in $[0, \pi]$ which maximizes $I_T(\lambda)$. We use $\hat{\lambda}_T$ to estimate $\lambda_0$. Then we have the following result.

**Theorem 2.1** *Under Assumption A, for any $\alpha \in (0,1)$,*

$$T^\alpha(\hat{\lambda}_T - \lambda_0) \xrightarrow{p} 0 \text{ as } T \to \infty.$$

**Proof.** We can assume $\lambda_0 = 0$ without loss of generality. For any $\epsilon > 0$, put

$$\epsilon(T) = \epsilon / T^\alpha.$$

Then we have

$$\begin{aligned} & P[T^\alpha|\hat{\lambda}_T| > \epsilon] = P[|\hat{\lambda}_T| > \epsilon(T)] \qquad (2) \\ \leq\ & P[I_T(0) < \max_{\Lambda_{\epsilon(T)}} I_T(\lambda)] \\ \leq\ & P[I_T(0) < M_{T,\epsilon(T)} \max_{\Lambda_{\epsilon(T)}} 2\pi f(\lambda)] \\ \leq\ & P[I_T(0) < C M_{T,\epsilon(T)} \epsilon(T)^{-2d}] \\ =\ & P[I_T(0) < C M_{T,\epsilon(T)} \epsilon(T)^{-2d},\ \overline{\lim}_{T\to\infty} M_{T,\epsilon(T)}/\log T \leq 12], \end{aligned}$$

where the third ineqaulity follows from (1) and Assumption A and the last equality follows from Proposition 4.1 in Section 4. By Egoroff's Theorem, for any $\eta > 0$ and any sufficiently large $T$, the last term of (2) is bounded by

$$\begin{aligned} & P[I_T(0) < C_1 M_{T,\epsilon(T)}\epsilon(T)^{-2d},\ M_{T,\epsilon(T)}/\log T \le C_2] + \eta \\ \le\ & P[I_T(0)/T^{2d} < C \log T/(T\epsilon(T))^{2d}] + \eta, \end{aligned}$$

where $C_2$ satisfies $C_2 > 12$. The assumption implies that $\log T/(T\epsilon(T))^{2d}$ converges to 0 as $T \to \infty$. Then it follows from Proposition 4.2 in Section 4 and Pólya's theorem[Serfling [19], page 18] that

$$P[I_T(0)/T^{2d} < C \log T/(T\epsilon(T))^{2d}] \to 0 \text{ as } T \to \infty.$$

Hence the proof is completed.

**Remark 2.1**

(i) The Gaussian property of $\{X(t)\}$ is essential only for evaluating the moment generating function of the periodogram of the proof of Propositon 4.1. Propositon 4.2 still holds without the Gaussian property if the central limit theorems can be proved for $\sum X(t) \sin \lambda_0 t$ and $\sum X(t) \cos \lambda_0 t$[see for example Theorem 3.1 of Yajima [27]].

(ii) For a deterministic trigonometric component in a regression model , $T(\hat{\lambda_T} - \lambda_0)$ converges to 0 almost surely as $T \to \infty$ and the limting distribution of $T^{3/2}(\hat{\lambda_T} - \lambda_0)$ is a normal distribution if $\lambda_0 \neq 0, \pi$(see Hannan [12]).

Hence the speed of convergence is slower for unbouded spectral densities. On the hand the limiting distribution has never been derived as yet but is conjectured to be nonnormal by the following reason.

We have, expanding $I_T'(\lambda)$ in the first two terms of its Taylor series, about $\lambda_0$

$$0 = I_T'(\hat{\lambda}_T) = I_T'(\lambda_0) + (\hat{\lambda}_T - \lambda_0) I_T''(\tilde{\lambda}_T), \quad |\tilde{\lambda}_T - \lambda_0| \le |\hat{\lambda}_T - \lambda_0|.$$

Hence

$$T(\hat{\lambda_T} - \lambda_0) = \frac{-I_T'(\lambda_0)/T^{1+2d}}{I_T''(\tilde{\lambda}_T)/T^{2+2d}}. \tag{3}$$

Now let

$$S_{T,i} = \sum_{t=1}^{T} X(t)t^i \sin \lambda_0 t, \quad i = 0, 1, 2,$$

$$C_{T,i} = \sum_{t=1}^{T} X(t)t^i \cos \lambda_0 t, \quad i = 0, 1, 2.$$

Then the derivatives of the periodogram are expressed by

$$I'_T(\lambda_0) = 2T^{-1}[S_{T,0}C_{T,1} - C_{T,0}S_{T,1}],$$

$$I''_T(\lambda_0) = 2T^{-1}[S_{T,1}^2 + C_{T,1}^2 - S_{T,0}S_{T,2} - C_{T,0}C_{T,2}].$$

Then we can show in the same way as Proposition 4.2 that the limiting distribution of

$$T^{-d}\left(C_{T,0}/T^{1/2}, C_{T,1}/T^{3/2}, C_{T,2}/T^{5/2}, S_{T,0}/T^{1/2}, S_{T,1}/T^{3/2}, S_{T,2}/T^{5/2}\right)$$

is a 6-dimensional multivariate normal distribution. Hence if it could be proved that

$$[I''_T(\tilde{\lambda}_T) - I''_T(\lambda_0)]/T^{2+2d} \xrightarrow{p} \quad as\ T \to \infty,$$

it would follow from (3) that the limiting distribution is nonnormal.

# 3 Application.

As an application of Theorem 2.1, we consider estimation of parametric time series models. Let $\{X(t)\}$ be a Gaussian stationary process with mean 0. And its spectral density $f(\lambda; \lambda_0, \theta)$ is assumed to have the form

$$f(\lambda; \lambda_0, \theta) = \frac{g(\lambda; \lambda_0, \theta)}{|\lambda - \lambda_0|^{2d}}, \quad \lambda \in [0, \pi], \tag{4}$$

where

$$\theta = (\theta_1, \theta_2, \ldots, \theta_k)' \ (\in \Theta \subset \boldsymbol{R}^k),$$

is a vector of parameters and $d$ is a component of $\theta$. We assume that the parameter space $\Theta$ is a compact set.

In addition to Assumption A, we impose the following assumptions on $f(\lambda; \lambda_0, \theta)$ and $g(\lambda; \lambda_0, \theta)$.

**Assumption B.**

(B1) If $f(\lambda; \lambda_0, \theta_1) = (\lambda; \lambda_0, \theta_2)$ almost surely with respect to Lebesgue measure on $[0, \pi]$, then $\theta_1 = \theta_2$ for any fixed $\lambda_0$.

(B2) $g(\lambda; \lambda_0, \theta)$ is a positive and continuous function of $(\lambda, \lambda_0, \theta)$ in $[0, \pi]^2 \times \Theta$.

(B3) $g(\lambda; \lambda_0, \theta)$ has the first partial derivative $\partial g(\lambda; \lambda_0, \theta)/\partial \theta_i (i = 1, 2, \ldots, k)$ and the second partial derivative $\partial^2 g(\lambda; \lambda_0, \theta)/\partial \theta_i \partial \theta_j (i, j = 1, 2, \ldots, k)$ with respect to $\theta$. And $\partial g(\lambda; \lambda_0, \theta)/\partial \theta_i$ and $\partial^2 g(\lambda; \lambda_0, \theta)/\partial \theta_i \theta_j$ are continuous functions of $(\lambda, \lambda_0, \theta)$ in $[0, \pi]^2 \times \Theta$.

(B4) For any suffciently small $\epsilon > 0$,

$$|\frac{\partial^2 f^{-1}(\lambda; \lambda_0, \theta)}{\partial \lambda \partial \theta_i}| \leq C|\lambda - \lambda_0|^{2d-1-\epsilon}.$$

(B5) For any sufficiently small $\epsilon > 0$, $g(\lambda; \lambda_0, \theta)$ and $\partial g(\lambda; \lambda_0, \theta)/\partial \theta_i (i = 1, 2, \ldots, k)$ satisfy

$$|g(\lambda; \lambda_0, \theta) - g(\lambda; \lambda_0^*, \theta)| \leq C|\lambda_0 - \lambda_0^*|^d$$
$$|\frac{\partial g(\lambda; \lambda_0, \theta)}{\partial \theta_i} - \frac{\partial g(\lambda; \lambda_0^*, \theta)}{\partial \theta_i}| \leq C|\lambda_0 - \lambda_0^*|^{d-\epsilon},$$

uniformly in $\lambda$ and $\theta$ on $[0, \pi] \times \Theta$.

Let $\theta_0$ be the true vector of parameters. Now consider the estimation of $\theta_0$ when $\lambda_0$ is unknown. Then the exact ML procedure for $\theta_0$ and $\lambda_0$ is very tedious. Hence we propose a simpler procedure. Set

$$U_T(\lambda_0, \theta) = \int_{\Pi} [\log f(\lambda; \lambda_0, \theta) + \frac{I_T(\lambda)}{2\pi f(\lambda; \lambda_0, \theta)}] d\lambda.$$

$U_T(\lambda_0, \theta)$ is an approximate function for the exact log likelifood function multiplied by -1. Now we substitute the estimator $\hat{\lambda}_T$ of Section 2 for $\lambda_0$ of $U_T(\lambda_0, \theta)$. And let $\hat{\theta}_T$ be the value of $\theta$ which minimizes $U_T(\hat{\lambda}_T, \theta)$. Then we have the following result.

**Theorem 3.1**

$$\hat{\theta}_T \xrightarrow{p} \theta_0, \quad (T \to \infty).$$

**Proof.** First we shall prove that

$$p - \lim_{T\to\infty} [U_T(\lambda_0, \theta) - U_T(\hat{\lambda}_T, \theta)] = 0, \tag{5}$$

for any $\theta$ in $\Theta$. We have

$$\begin{aligned} & U_T(\lambda_0, \theta) - U_T(\hat{\lambda}_T, \theta) \\ = & \int_\Pi \left(\log f(\lambda; \lambda_0, \theta) - \log f(\lambda; \hat{\lambda}_T, \theta)\right) d\lambda \\ & + \int_\Pi \left(f^{-1}(\lambda; \lambda_0, \theta) - f^{-1}(\lambda; \hat{\lambda}_T, \theta)\right) \frac{I_T(\lambda)}{2\pi} d\lambda. \end{aligned} \tag{6}$$

Assumption (B2) assures that $f^{-1}(\lambda; \lambda_0, \theta)$ is an uniformly continuous function of $(\lambda, \lambda_0, \theta)$ in $\Pi \times [0, \pi] \times \Theta$. Hnece it is easily shown that the second term on the right hand side of (6) converges to 0 in probability as $T \to \infty$. While from (4),

$$\begin{aligned} & \log f(\lambda; \lambda_0, \theta) \\ = & \log g(\lambda; \lambda_0, \theta) - 2d \log |\lambda - \lambda_0|. \end{aligned} \tag{7}$$

From Assumption (B2), the first term of (7) is an uniformly continuous function of $(\lambda, \lambda_0, \theta)$ in $\Pi \times [0, \pi] \times \Theta$. And

$$\begin{aligned} & \int_0^\pi \log |\lambda - \lambda_0| d\lambda \\ = & \lambda_0 \log \lambda_0 + (\pi - \lambda_0) \log(\pi - \lambda_0) - \pi, \end{aligned} \tag{8}$$

which is a continous function of $\lambda_0$. Hence the first term on the right hand side of (6) converges to 0 in probability as $T \to \infty$. Then the proof of (5) is completed.

Hereafter we shall prove the assertion by following the same procedure as in Lemma 2 and Theorem 1 of Walker [22]. Let $\theta$ be any other point of $\Theta$.

First it follows from Lemma 4.3 in Section 4 and (5) that

$$\begin{aligned} & p - \lim_{T\to\infty} [U_T(\hat{\lambda}_T, \theta_0) - U_T(\hat{\lambda}_T, \theta)] \\ = & \int_\Pi (\log f(\lambda; \lambda_0, \theta_0) + 1)\, d\lambda \\ & - \int_\Pi \left( \log f(\lambda; \lambda_0, \theta) + \frac{f(\lambda; \lambda_0, \theta_0)}{f(\lambda; \lambda_0, \theta)} \right) d\lambda \\ = & -\mu(\theta_0, \theta), \quad \text{(say)}. \end{aligned}$$

From Lemma 4.4 in Section 4, $\mu(\theta_0, \theta) > 0$. Let $K(\theta_0, \theta)$ be any positive constant less than $\mu(\theta_0, \theta)$. Then

$$\lim_{T\to\infty} P\{[U_T(\hat{\lambda}_T, \theta_0) - U_T(\hat{\lambda}_T, \theta)] < -K(\theta_0, \theta)\} = 1. \tag{9}$$

Next we show that there exist a sequence of random variables $\{H_{\delta,T}(\theta_1)\}$ and a function $H_\delta(\theta_1)$ which satisfy

$$|U_T(\hat{\lambda}_T, \theta_2) - U_T(\hat{\lambda}_T, \theta_1)| < H_{\delta,T}(\theta_1), \tag{10}$$

for any $\theta_1, \theta_2 (\in \Theta)$ such that $|\theta_2 - \theta_1| < \delta$ where

$$p - \lim_{T\to\infty} H_{\delta,T}(\theta_1) = H_\delta(\theta_1), \tag{11}$$

for any $\delta > 0$ and

$$\lim_{\delta\to 0} H_\delta(\theta_1) = 0. \tag{12}$$

We have

$$\begin{aligned} & U_T(\hat{\lambda}_T, \theta_2) - U_T(\hat{\lambda}_T, \theta_1) \\ = & -2(d_2 - d_1) \int_\Pi \log|\lambda - \hat{\lambda}_T| d\lambda \\ & + \int_\Pi \left( \log g(\lambda; \hat{\lambda}_T, \theta_2) - \log g(\lambda; \hat{\lambda}_T, \theta_1) \right) d\lambda \\ & + \int_\Pi \left( f^{-1}(\lambda; \hat{\lambda}_T, \theta_2) - f^{-1}(\lambda; \hat{\lambda}_T, \theta_1) \right) \frac{I_T(\lambda)}{2\pi} d\lambda. \end{aligned} \tag{13}$$

Hence if we put

$$H_{\delta,T}(\theta_1) = 2\delta \left| \int_\Pi \log|\lambda - \hat{\lambda}_T| d\lambda \right| + 2\pi M_{\delta,1}(\theta_1) + M_{\delta,2}(\theta_1) \hat{r}_T(0),$$

and

$$H_\delta(\theta_1) = 2\delta|\int_\Pi \log|\lambda - \lambda_0|d\lambda| + 2\pi M_{\delta,1}(\theta_1) + M_{\delta,2}(\theta_1)r(0),$$

where

$$M_{\delta,1}(\theta_1) = \sup_{\lambda,\omega,|\theta_2-\theta_1|<\delta} |\log g(\lambda;\omega,\theta_2) - \log g(\lambda;\omega,\theta_1)|,$$
$$M_{\delta,2}(\theta_1) = \sup_{\lambda,\omega,|\theta_2-\theta_1|<\delta} |f^{-1}(\lambda;\omega,\theta_2) - f^{-1}(\lambda;\omega,\theta_1)|,$$
$$\hat{r}_T(0) = \sum_{t=1}^{T} X(t)^2/T,$$

then it is easily shown by Assumption (B2) and (13) that $H_{\delta,T}(\theta_1)$ and $H_\delta(\theta_1)$ satisfy (10),(11) and (12).

Then the result follows from (9),(10),(11) and (12) by the same argument as in Lemma 2 and Theorem 1 of [22].

**Remark 3.1**

Walker [22] assumed in his original proof that

$$\lim_{\delta\to 0} E(H_{\delta,T}(\theta_1)) = 0,$$

uniformly in $T$ and

$$\lim_{T\to\infty} Var(H_{\delta,T}(\theta_1)) = 0,$$

for each $\delta$. However as is seen from his proofs of Lemma 2 and Theorem 1, the same result still holds if $H_{\delta,T}(\theta_1)$ and $H_\delta(\theta_1)$ satisfy (11) and (12).

**Theorem 3.2** *Let $\theta_0$ be the inner point of $\Theta$. If $1/4 < d_0$, then*

$$\sqrt{T}(\hat{\theta}_T - \theta_0) \xrightarrow{\mathcal{D}} N(\mathbf{o}, 4\pi W^{-1}(\theta_0)),$$

*where*

$$W(\theta) = \int_\Pi \frac{\partial \log f(\lambda;\lambda_0,\theta)}{\partial\theta}\frac{\partial \log f(\lambda;\lambda_0,\theta)}{\partial\theta'}d\lambda.$$

*and $\partial\log f(\lambda;\lambda_0,\theta)/\partial\theta$ is the $k$-dimensional random vector of the first partial derivatives of $\log f(\lambda;\lambda_0,\theta)$ evaluated at $\theta$.*

**Proof.** Let $U_T^{(1)}(\lambda_0, \theta)$ be the $k$-dimensional random vector of the first partial derivatives and $U_T^{(2)}(\lambda_0, \theta)$ be the $k \times k$ random matrix of the second partial derivatives of $U_T(\lambda_0, \theta)$ evaluated at $\theta$ respectively. $U_T^{(1)}(\hat{\lambda}_T, \theta)$ and $U_T^{(2)}(\hat{\lambda}_T, \theta)$ are defined similarly.

And let $\partial f^{-1}(\lambda; \lambda_0, \theta)/\partial\theta$ be the $k$-dimensional random vector of the first partial derivatives and $\partial^2 f^{-1}(\lambda; \lambda_0, \theta)/\partial\theta\partial\theta'$ be the $k \times k$ matrix of the second partial derivatives of $f^{-1}(\lambda; \lambda_0, \theta)$ evaluated at $\theta$ respectively. And $\partial^2 \log f(\lambda; \lambda_0, \theta)/\partial\theta\partial\theta'$ is defined similarly.

Then we have

$$\begin{aligned} 0 &= U_T^{(1)}(\hat{\lambda}_T, \hat{\theta}_T) \\ &= U_T^{(1)}(\hat{\lambda}_T, \theta_0) + U_T^{(2)}(\hat{\lambda}_T, \theta_T^*)(\hat{\theta}_T - \theta_0), \end{aligned}$$

where $\theta_T^* = \theta_0 + \tau_T(\hat{\theta}_T - \theta_0)$ and $\tau_T$ is a $k \times k$ random matrix. Strictly $\tau_T$ is dependent on each component of $U_T^{(2)}(\hat{\lambda}_T, \theta_T^*)$. But we do not express it explicitly for notational simplicity. Hence

$$\begin{aligned} &(\hat{\theta}_T - \theta_0) \\ &= -\left[U_T^{(2)}(\hat{\lambda}_T, \theta_T^*)\right]^{-1} U_T^{(1)}(\hat{\lambda}_T, \theta_0) \end{aligned} \tag{14}$$

First we shall evaluate the first term of (14). It follows from Assumptions (B2) and (B3) by the same argument as in (5) that

$$p - \lim_{T\to\infty} \left[U_T^{(2)}(\hat{\lambda}_T, \theta_T^*) - U_T^{(2)}(\lambda_0, \theta_0)\right] = 0. \tag{15}$$

While it is shown by the same argument as in Lemma 4.3

$$\begin{aligned} &\lim_{T\to\infty} U_T^{(2)}(\lambda_0, \theta_0) \\ &= \int_\Pi \left( \frac{\partial^2 \log f(\lambda; \lambda_0, \theta_0)}{\partial\theta\partial\theta'} + \frac{\partial^2 f^{-1}(\lambda; \lambda_0, \theta_0)}{\partial\theta\partial\theta'} f(\lambda; \lambda_0, \theta_0) \right) d\lambda. \end{aligned} \tag{16}$$

From Lemma 4.4,

$$\int_\Pi \left( \frac{\partial \log f(\lambda; \lambda_0, \theta_0)}{\partial\theta} + \frac{\partial f^{-1}(\lambda, \lambda_0, \theta_0)}{\partial\theta} f(\lambda; \lambda_0, \theta_0) \right) d\lambda = 0, \tag{17}$$

and

$$\int_{\Pi}\left(\frac{\partial^2 \log f(\lambda;\lambda_0,\theta_0)}{\partial\theta\partial\theta'}+\frac{\partial^2 f^{-1}(\lambda,\lambda_0,\theta_0)}{\partial\theta\partial\theta'}f(\lambda;\lambda_0,\theta_0)\right)d\lambda$$
$$+\int_{\Pi}\left(\frac{\partial f^{-1}(\lambda;\lambda_0,\theta_0)}{\partial\theta}\frac{\partial f(\lambda;\lambda_0,\theta_0)}{\partial\theta'}\right)d\lambda=0. \tag{18}$$

Hence from (18), the right hand side term of (16) is equal to $W(\theta_0)$.

Next we shall evaluate the second term of (14). First we shall show that

$$p-\lim_{T\to\infty} T^{1/2}\left[U_T^{(1)}(\hat{\lambda}_T,\theta_0)-U_T^{(1)}(\lambda_0,\theta_0)\right]=0 \tag{19}$$

We have

$$T^{1/2}\left[U_T^{(1)}(\hat{\lambda}_T,\theta_0)-U_T^{(1)}(\lambda_0,\theta_0)\right] \tag{20}$$
$$= T^{1/2}\int_{\Pi}\left(\frac{\partial \log f(\lambda;\hat{\lambda}_T,\theta_0)}{\partial\theta}-\frac{\partial \log f(\lambda;\lambda_0,\theta_0)}{\partial\theta}\right)d\lambda$$
$$+T^{1/2}\int_{\Pi}\left(\frac{\partial f^{-1}(\lambda;\hat{\lambda}_T,\theta_0)}{\partial\theta}-\frac{\partial f^{-1}(\lambda;\lambda_0,\theta_0)}{\partial\theta}\right)\frac{I_T(\lambda)}{2\pi}d\lambda.$$

From Assumptions (B2),(B3), and (B5), the first term on the right hand side of (20) is bounded by

$$CT^{1/2}\left(\left|\int_{\Pi}(\log|\lambda-\hat{\lambda}_T|-\log|\lambda-\lambda_0|)d\lambda\right|+|\hat{\lambda}_T-\lambda_0|^{2d-\epsilon}\right),$$

with any sufficiently small $\epsilon>0$, which is shown to converge to 0 in probabilty as $T\to\infty$ by Theorem 2.1 and (8).

The second term on the right hand side of (20) is bounded by

$$CT^{1/2}|\hat{\lambda}_T-\lambda_0|^{2d-\epsilon}\hat{r}_T(0),$$

which is also shown to converge to 0 as $T\to\infty$ by Theorem 2.1. Hence the proof of (19) is completed. Then it follows from Lemma 4.5 in Section 4 that

$$T^{1/2}U_T^{(1)}(\hat{\lambda}_T,\theta_0)\xrightarrow{\mathcal{D}} N(\mathbf{o},4\pi W(\theta_0)). \tag{21}$$

Finally we have the assertion by noting (14),(15),(16),(18) and (21).

**Example 3.1**

We give two examples which satisfy Assumptions A and B. The first one is a GARMA model proposed by Gray et al. [8]. The spectral density of a GARMA$(p, q, u, \tilde{d})$ model is expressed by

$$f(\lambda; \lambda_0, \theta) = \frac{\sigma^2 |\beta(e^{i\lambda})|^2}{2\pi |\alpha(e^{i\lambda})|^2 p(e^{i\lambda})},$$

where

$$\begin{aligned} \alpha(z) &= 1 - \alpha_1 z - \ldots - \alpha_p z^p, \\ \beta(z) &= 1 - \beta_1 z - \ldots - \beta_q z^q, \end{aligned}$$

and all of the roots of $\alpha(z) = \beta(z) = 0$ exist outside the unit circle and there are no common roots and

$$p(z) = |1 - 2uz + z^2|^{2\tilde{d}},$$

with $|u| \leq 1$ and $\lambda_0 = \cos^{-1} u$ and $0 < \tilde{d} < 1/4$ if $\lambda_0 = 0, \pi$ and $0 < \tilde{d} < 1/2$ if $\lambda_0 \neq 0, \pi$.

If we put

$$\begin{aligned} g(\lambda; \lambda_0, \theta) &= \frac{\sigma^2 |\beta(e^{i\lambda})|^2 |\lambda - \lambda_0|^{2d}}{2\pi |\alpha(e^{i\lambda})|^2 p(e^{i\lambda})} \\ &= \frac{\sigma^2 |\beta(e^{i\lambda})|^2 |\lambda - \lambda_0|^{2d}}{2\pi |\alpha(e^{i\lambda})| (2^2 \sin(\lambda + \lambda_0)/2) \sin(\lambda - \lambda_0)/2)^{2\tilde{d}}}, \end{aligned}$$

and

$$\theta = (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q, d, \sigma^2)',$$

where $d = 2\tilde{d}$ if $\lambda_0 = 0, \pi$ and $d = \tilde{d}$ if $\lambda \neq 0, \pi$, then we can show by an elementary calculation that $f(\lambda; \lambda_0, \theta)$ and $g(\lambda; \lambda_0, \theta)$ saitsfy Assumptions A and B.

The second one is a model for a "signal" observed with "noise". Dunsmuir [4] and Hosoya and Taniguchi [14] considered estimation of an autoregressive signal with white noise.

Here We consider a GARMA-type signal with ARMA(p,q) noise. And the spectral density is expressed by

$$f(\lambda;\lambda_0,\theta)=\frac{\sigma_s^2}{2\pi|\lambda-\lambda_0|^{2d}}+\frac{\sigma_n^2|\beta(e^{i\lambda})|^2}{2\pi|\alpha(e^{i\lambda})|^2}.$$

If we put

$$g(\lambda;\lambda_0,\theta)=\frac{\sigma_s^2}{2\pi}\left(1+\frac{\sigma_n^2|\beta(e^{i\lambda})|^2|\lambda-\lambda_0|^{2d}}{\sigma_s^2|\alpha(e^{i\lambda})|^2}\right),$$

and

$$\theta=(\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q,d,\sigma_s^2,\sigma_n^2)',$$

then $f(\lambda;\lambda_0,\theta)$ and $g(\lambda;\lambda_0,\theta)$ satsify Assumptions A and B.

# 4 Lemmas and propositions.

**Lemma 4.1** *Let $\{\epsilon(T)\}$ be any sequence of positive constants such that* $\lim_{T\to\infty}\epsilon(T)=0$.

*Then for any $\lambda\in\Lambda_{\epsilon(T)}$ and any $\delta>0$,*

$$\frac{E[I_T(\lambda)]}{2\pi f(\lambda)}-1=O\left(\frac{1+|\log(T\epsilon(T))|}{T\epsilon(T)}+\frac{1}{T\epsilon(T)^{1+\delta}}\right),$$

*and the convergence is uniform on $\Lambda_{\epsilon(T)}$.*

**Proof.** First we have

$$|E[I_T(\lambda)]-2\pi f(\lambda)|\leq\int_\Pi|f(\omega)-f(\lambda)|K_T(\omega-\lambda)d\omega, \tag{22}$$

where $K_T(\omega)$ is proportional to Fejér's kernel,

$$K_T(\omega)=\frac{1}{T}|\sum_{t=1}^{T}\exp(it\omega)|^2.$$

Now we shall prove the result in a similar way as in Robinson [18]. We only consider the case that $\lambda\in[\lambda_0+\epsilon(T),\pi]$ since the assertion for the case that $\lambda\in[0,\lambda_0-\epsilon(T)]$ can be proved similarly.

We partition the integral on the right hand side of (22) into

$$
\begin{aligned}
&\int_{\Pi} |f(\omega) - f(\lambda)| K_T(\omega - \lambda) d\omega \\
= \; &\left[ \int_{\lambda-\epsilon(T)/2}^{\lambda+\epsilon(T)/2} + \int_{\lambda+\epsilon(T)/2}^{\lambda+\epsilon} + \int_{\lambda-\epsilon}^{\lambda-\epsilon(T)/2} + \int_{|\omega-\lambda|>\epsilon} \right] \qquad (23) \\
&|f(\omega) - f(\lambda)| K_T(\omega - \lambda) d\omega,
\end{aligned}
$$

where $\epsilon$ is a sufficiently small fixed constant.

Now we shall evaluate each integral of (23). It follows from Assumption A that

$$
\begin{aligned}
|f'(\lambda)| &\leq \frac{2dg(\lambda)}{|\lambda - \lambda_0|^{2d+1}} + \frac{|g'(\lambda)|}{|\lambda - \lambda_0|^{2d}} \qquad (24) \\
&\leq C|\lambda - \lambda_0|^{-2d-1}
\end{aligned}
$$

And define $D_T(\omega)$ by

$$
D_T(\omega) = \sum_{t=1}^{T} \exp(it\omega).
$$

Then it follows from (1) and (24) that

$$
\begin{aligned}
&\int_{\lambda-\epsilon(T)/2}^{\lambda+\epsilon(T)/2} |f(\omega) - f(\lambda)| K_T(\omega - \lambda) d\omega \qquad (25) \\
\leq \; &\max_{\lambda-\epsilon(T)/2 \leq \omega \leq \lambda+\epsilon(T)/2} |f(\omega)'| \int_{\lambda-\epsilon(T)/2}^{\lambda+\epsilon(T)/2} |\omega - \lambda| K_T(\omega - \lambda) d\omega \\
\leq \; &C|\lambda - \lambda_0 - \epsilon(T)/2|^{-2d-1} \int_{\lambda-\epsilon(T)/2}^{\lambda+\epsilon(T)/2} \frac{|D_T(\omega - \lambda)|}{T} d\omega \\
\leq \; &C(T\epsilon(T))^{-1} |\lambda - \lambda_0|^{-2d} \left( \int_0^{1/T} T d\omega + \int_{1/T}^{\epsilon(T)/2} \frac{1}{\omega} d\omega \right) \\
\leq \; &Cf(\lambda)(1 + |\log(T\epsilon(T))|)/(T\epsilon(T)).
\end{aligned}
$$

Next

$$
\begin{aligned}
&\int_{\lambda+\epsilon(T)/2}^{\lambda+\epsilon} |f(\omega) - f(\lambda)| K_T(\omega - \lambda) d\omega \qquad (26) \\
\leq \; &\left( \max_{\lambda+\epsilon(T)/2 \leq \omega \leq \lambda+\epsilon} f(\omega) + f(\lambda) \right) \int_{\lambda+\epsilon(T)/2}^{\infty} \frac{1}{T|\omega - \lambda|^2} d\omega \\
\leq \; &Cf(\lambda)/(T\epsilon(T)).
\end{aligned}
$$

Next

$$
\int_{\lambda-\epsilon}^{\lambda-\epsilon(T)/2} |f(\omega) - f(\lambda)| K_T(\omega - \lambda) d\omega \qquad (27)
$$

$$\begin{aligned} &\le \int_{\lambda-\epsilon}^{\lambda-\epsilon(T)/2} f(\omega)K_T(\omega-\lambda)d\omega \\ &\quad +f(\lambda)\int_{\lambda-\epsilon}^{\lambda-\epsilon(T)/2} K_T(\omega-\lambda)d\omega. \end{aligned}$$

For the second term on the right hand side of (27), we have

$$\begin{aligned} & f(\lambda)\int_{\lambda-\epsilon}^{\lambda-\epsilon(T)/2} K_T(\omega-\lambda)d\omega \\ &= f(\lambda)\int_{\epsilon(T)/2}^{\epsilon} K_T(\omega)d\omega \\ &\le Cf(\lambda)/(T\epsilon(T)) \end{aligned} \tag{28}$$

Next we evaluate the first term on the right hand side of (27). First consider the case that $\lambda_0 > 0$. We have

$$\begin{aligned} & \int_{\lambda-\epsilon}^{\lambda-\epsilon(T)/2} f(\omega)K_T(\omega-\lambda)d\omega \\ &\le CT^{-1}\int_{\lambda-\epsilon}^{\lambda-\epsilon(T)/2} |\omega-\lambda_0|^{-2d}|\omega-\lambda|^{-2}d\omega \\ &= CT^{-1}\int_{\epsilon(T)/2}^{\epsilon} |\lambda-\lambda_0-\omega|^{-2d}\omega^{-2}d\omega. \end{aligned} \tag{29}$$

If $\lambda-\lambda_0 > \epsilon$, for any $\delta > 0$,

$$\begin{aligned} & \int_{\epsilon(T)/2}^{\epsilon} |\lambda-\lambda_0-\omega|^{-2d}\omega^{-2}d\omega \\ &\le C\epsilon(T)^{-1-\delta}\int_0^{\lambda-\lambda_0} |\lambda-\lambda_0-\omega|^{-2d}\omega^{\delta-1}d\omega \\ &\le C\epsilon(T)^{-1-\delta}|\lambda-\lambda_0|^{-2d+\delta} \\ &\le C\epsilon(T)^{-1-\delta}f(\lambda). \end{aligned} \tag{30}$$

On the other hand, if $\lambda-\lambda_0 \le \epsilon$, for any $\delta > 0$,

$$\begin{aligned} & \int_{\epsilon(T)/2}^{\epsilon} |\lambda-\lambda_0-\omega|^{-2d}\omega^{-2}d\omega \\ &= \int_{\epsilon(T)/2}^{\lambda-\lambda_0} (\lambda-\lambda_0-\omega)^{-2d}\omega^{-2}d\omega \\ &\quad +\int_{\lambda-\lambda_0}^{\epsilon} (\omega-\lambda+\lambda_0)^{-2d}\omega^{-2}d\omega \\ &\le C\epsilon(T)^{-1-\delta}\int_0^{\lambda-\lambda_0} (\lambda-\lambda_0-\omega)^{-2d}\omega^{\delta-1}d\omega \\ &\quad +C\int_{\lambda-\lambda_0}^{\infty} (\omega-\lambda+\lambda_0)^{-2d}\omega^{-2}d\omega \end{aligned} \tag{31}$$

$$\leq C[\epsilon(T)^{-1-\delta}(\lambda-\lambda_0)^{-2d+\delta}+(\lambda-\lambda_0)^{-2d-1}]$$
$$\leq C\epsilon(T)^{-1-\delta}f(\lambda).$$

Now consider the case that $\lambda_0 = 0$. If $\lambda > \epsilon$, for any $\delta > 0$,

$$\begin{aligned} &\int_{\lambda-\epsilon}^{\lambda-\epsilon(T)/2} f(\omega)K_T(\omega-\lambda)d\omega \qquad (32)\\ \leq\ & CT^{-1}\int_{\epsilon(T)/2}^{\epsilon}(\lambda-\omega)^{-2d}\omega^{-2}d\omega\\ \leq\ & CT^{-1}\epsilon(T)^{-1-\delta}\int_0^{\lambda}(\lambda-\omega)^{-2d}\omega^{\delta-1}d\omega\\ \leq\ & CT^{-1}\epsilon(T)^{-1-\delta}\lambda^{-2d+\delta}\\ \leq\ & CT^{-1}\epsilon(T)^{-1-\delta}f(\lambda). \end{aligned}$$

On the other hand if $\lambda \leq \epsilon$, for any $\delta > 0$,

$$\begin{aligned} &\int_{\lambda-\epsilon}^{\lambda-\epsilon(T)/2} f(\omega)K(\omega-\lambda)d\omega \qquad (33)\\ \leq\ & CT^{-1}\int_0^{\lambda-\epsilon(T)/2}(\lambda-\omega)^{-2}\omega^{-2d}d\omega\\ & +CT^{-1}\int_{\lambda-\epsilon}^{0}(\lambda-\omega)^{-2}(-\omega)^{-2d}d\omega\\ \leq\ & CT^{-1}\epsilon(T)^{-1-\delta}\int_0^{\lambda}(\lambda-\omega)^{\delta-1}\omega^{-2d}d\omega\\ & +CT^{-1}\int_{\lambda}^{\infty}(\omega-\lambda)^{-2d}\omega^{-2}d\omega\\ \leq\ & CT^{-1}[\epsilon(T)^{-1-\delta}\lambda^{-2d+\delta}+\lambda^{-1-2d}]\\ \leq\ & CT^{-1}\epsilon(T)^{-1-\delta}f(\lambda). \end{aligned}$$

From (27),(28),(29),(30),(31),(32),and (33), we see

$$\int_{\lambda-\epsilon}^{\lambda-\epsilon(T)/2}|f(\omega)-f(\lambda)|K_T(\omega-\lambda)d\omega \leq CT^{-1}\epsilon(T)^{-1-\delta}f(\lambda). \qquad (34)$$

Finally we have

$$\begin{aligned} &\int_{|\omega-\lambda|>\epsilon}|f(\omega)-f(\lambda)|K_T(\omega-\lambda)d\lambda \qquad (35)\\ \leq\ & \frac{C}{T\epsilon^2}\left(\int_{|\omega-\lambda|>\epsilon}f(\omega)d\omega+f(\lambda)\right)\\ \leq\ & \frac{C}{T\epsilon^2}f(\lambda). \end{aligned}$$

Then the result follows from (23),(25),(26),(34) and (35).

**Lemma 4.2** *Let $\epsilon$ be any sufficiently small positive constant and $\theta$ be any constant with $0 < \theta < 1$. Then there exist a constant $K$ and an interval $A_T = [a_T, b_T]$ almost surely which satisfy the following properties (i) $\sim$ (iv);*

*(i) $A_T \subset \Lambda_\epsilon$,*

*(ii) For any $\lambda \in A_T$,*

$$\theta M_{T,\epsilon} \leq I_T(\lambda)/(2\pi f(\lambda)),$$

*(iii) The length of $A_T$, $b_T - a_T$, satisfies,*

$$\frac{K(1-\theta)}{T\epsilon^{-2d} + \epsilon^{-1}} \leq b_T - a_T,$$

*(iv) $A_T$ is dependent on $T, \epsilon, \theta$ and a sample path, but $K$ is independent of all of them.*

**Proof.** We shall prove the assertion by following the procedure developed by Lemma 3.1 of Turkman and Walker [21]. Let $\lambda_1$ be the value of $\lambda$ which satisfies

$$M_{T,\epsilon} = \frac{I_T(\lambda_1)}{2\pi f(\lambda_1)}.$$

We cam assume that $\lambda_1 \in [0, \lambda_0 - \epsilon]$ without loss of generality. Consider the case that $0 < \lambda_1 < \lambda_0 - \epsilon$ since the assertion for the case that $\lambda_1 = \lambda_0 - \epsilon$ and $\lambda_1 = 0$ can be shown similarly.

First assume that there exists $\lambda_2$ in $[\lambda_1, \lambda_0 - \epsilon]$ such that

$$\theta M_{T,\epsilon} = \frac{I_T(\lambda_2)}{2\pi f(\lambda_2)}. \tag{36}$$

Hereafter let $\lambda_2$ be the first point to the right of $\lambda_1$ which satisfies (36). Then

$$\begin{aligned}(1-\theta)M_{T,\epsilon} &= \frac{I_T(\lambda_1)}{2\pi f(\lambda_1)} - \frac{I_T(\lambda_2)}{2\pi f(\lambda_2)} \qquad (37)\\ &\leq (\lambda_2 - \lambda_1) \max_{[\lambda_1,\lambda_2]} |\frac{d}{d\lambda}\frac{I_T(\lambda)}{2\pi f(\lambda)}| \\ &\leq (\lambda_2 - \lambda_1)\left(\max_{[\lambda_1,\lambda_2]} |\frac{I_T'(\lambda)}{2\pi f(\lambda)}| + \max_{[\lambda_1,\lambda_2]} \frac{I_T(\lambda)|f(\lambda)'|}{2\pi f(\lambda)^2}\right).\end{aligned}$$

It follows from (1), Assumption A and Theorem 3.1 of [21] that

$$\begin{aligned}&\max_{[\lambda_1,\lambda_2]} \left|\frac{I_T'(\lambda)}{2\pi f(\lambda)}\right| \\ \leq\ & CTM_{T,\epsilon}\frac{\max_{[\lambda_1,\lambda_2]} f(\lambda)}{\min_{[\lambda_1,\lambda_2]} f(\lambda)} \\ \leq\ & CT\epsilon^{-2d}M_{T,\epsilon}.\end{aligned} \tag{38}$$

While from Assumption A, (1) and (24),

$$\max_{[\lambda_1,\lambda_2]} \frac{I_T(\lambda)|f(\lambda)'|}{2\pi f(\lambda)^2} \leq C\epsilon^{-1}M_{T,\epsilon}. \tag{39}$$

Then from (37),(38) and (39),

$$(1-\theta)M_{T,\epsilon} \leq C(\lambda_2-\lambda_1)(T\epsilon^{-2d}+\epsilon^{-1})M_{T,\epsilon}.$$

Hence there exists a constant $K$ such that

$$\frac{K(1-\theta)}{T\epsilon^{-2d}+\epsilon^{-1}} \leq \lambda_2-\lambda_1.$$

And we can put $a_T=\lambda_1$ and $b_T=\lambda_2$.

Next assume that

$$\theta M_{T,\epsilon} < \frac{I_T(\lambda)}{2\pi f(\lambda)},$$

for any $\lambda \in [\lambda_1, \lambda_0-\epsilon]$. If there exists $\lambda_2 \in [0,\lambda_1]$ which satisfies (36), by letting $\lambda_2$ be the first point to the left of $\lambda_1$, it is shown similarly that $a_T=\lambda_2$ and $b_T=\lambda_0-\epsilon$ have the desired property. Otherwise we can put $a_T=0$ and $b_T=\lambda_0-\epsilon$.

**Proposition 4.1** *Let $\alpha$ be any constant such that $0<\alpha<1$ and $\{\epsilon(T)\}$ be any sequence of positive constants such that $\lim_{T\to\infty}\epsilon(T)=0$ and $\lim_{T\to\infty}T^\alpha\epsilon(T)>0$. And let $\beta$ be any constant with $\beta>2+2d\alpha$ and $\delta$ be any constant with $\delta<1/4$.*

*Then*

$$\limsup_{T\to\infty}\,(\delta M_{T,\epsilon(T)}-\beta\log T)=-\infty, \quad a.s.$$

**Proof.** We shall prove the assertion by following the procedure of Lemma 3.2 of [21].

We can assume that $\lambda_0 = 0$ without loss of generality. The assumption implies $\epsilon(T)^{-1} = o(T\epsilon(T)^{-2d})$. Then from Lemma 4.2, for any $\theta$ with $0 < \theta < 1$, there exist a constant $K$ and an interval $A_T = [a_T, b_T](\subset \Lambda_{\epsilon(T)})$ such that

$$\frac{K(1-\theta)}{T\epsilon(T)^{-2d}} \leq b_T - a_T,$$

and

$$\theta M_{T,\epsilon(T)} \leq \frac{I_T(\lambda)}{2\pi f(\lambda)},$$

for any $\lambda \in A_T$.

Let $\gamma$ be a positive constant, being specified later. Then

$$\begin{aligned}
&\frac{K(1-\theta)}{T\epsilon(T)^{-2d}} \exp(\theta\gamma M_{T,\epsilon(T)}) \\
\leq\; & \int_{A_T} \exp(\theta\gamma M_{T,\epsilon(T)}) d\lambda \\
\leq\; & \int_{A_T} \exp(\gamma I_T(\lambda)/2\pi f(\lambda)) d\lambda \\
\leq\; & \int_{\Lambda_{\epsilon(T)}} \exp(\gamma I_T(\lambda)/2\pi f(\lambda)) d\lambda.
\end{aligned}$$

Hence

$$E[\exp(\theta\gamma M_{T,\epsilon(T)})] \leq \frac{T\epsilon(T)^{-2d}}{K(1-\theta)} \int_{\Lambda_{\epsilon(T)}} E\exp(\gamma I_T(\lambda)/2\pi f(\lambda)) d\lambda. \tag{40}$$

Now we evaluate the right hand side term of (40). Let

$$\begin{aligned}
C_T(\lambda) &= T^{-1/2} \sum_1^T X(t) \cos \lambda t / \sqrt{2\pi f(\lambda)}, \\
S_T(\lambda) &= T^{-1/2} \sum_1^T X(t) \sin \lambda t / \sqrt{2\pi f(\lambda)}.
\end{aligned}$$

Then

$$\begin{aligned}
&\exp(\gamma I_T(\lambda)/2\pi f(\lambda)) \\
=\; & \exp[\gamma(C_T(\lambda)^2 + S_T(\lambda)^2)] \\
\leq\; & \frac{1}{2}[\exp(2\gamma C_T(\lambda)^2) + \exp(2\gamma S_T(\lambda)^2)].
\end{aligned}$$

Now define

$$\sigma^2_{T,C,\lambda} = Var(C_T(\lambda)),$$
$$\sigma^2_{T,S,\lambda} = Var(S_T(\lambda)).$$

Then if we choose $\delta$ such that $1+\delta < 1/\alpha$, then it follows from Lemma 4.1 that

$$\begin{aligned} \sigma^2_{T,C,\lambda} &\leq E\left(\frac{I_T(\lambda)}{2\pi f(\lambda)}\right) = 1 + o(1), \\ \sigma^2_{T,S,\lambda} &\leq E\left(\frac{I_T(\lambda)}{2\pi f(\lambda)}\right) = 1 + o(1), \end{aligned}$$

uniformly in $\lambda$ on $\Lambda_{\epsilon(T)}$.

While if $4\gamma\sigma^2_{T,C,\lambda} < 1$ and $4\gamma\sigma^2_{T,S,\lambda} < 1$,

$$\begin{aligned} E[\exp(2\gamma C_T(\lambda)^2)] &= (1 - 4\gamma\sigma^2_{T,C,\lambda})^{-1/2}, \\ E[\exp(2\gamma S_T(\lambda)^2)] &= (1 - 4\gamma\sigma^2_{T,S,\lambda})^{-1/2}. \end{aligned}$$

Hence if $T$ is sufficiently large and $\gamma < 1/4$,

$$\begin{aligned} E[\exp(2\gamma C_T(\lambda)^2)] &= O(1), \\ E[\exp(2\gamma S_T(\lambda)^2)] &= O(1), \end{aligned}$$

and consequently

$$E\exp(\gamma I_T(\lambda)/2\pi f(\lambda)) = O(1),$$

uniformly in $\lambda$ on $\Lambda_{\epsilon(T)}$.

Then it follows from (40) that

$$E[\exp(\theta\gamma M_{T,\epsilon(T)})] \leq CT\epsilon(T)^{-2d} = O(T^{1+2d\alpha}).$$

Hence

$$\begin{aligned} &\sum_{T=1}^{\infty} E[\exp(\theta\gamma M_{T,\epsilon(T)} - \beta\log T)] \\ \leq\ & C\sum_{t=1}^{\infty} T^{1+2d\alpha-\beta} < \infty. \end{aligned}$$

Hence

$$\exp(\theta\gamma M_{T,\epsilon(T)} - \beta \log T) \rightarrow \ 0 \ a.s.,$$

as $T \rightarrow \infty$.

Finally we have the assertion if we put $\delta = \theta\gamma$.

**Proposition 4.2** *As* $T \rightarrow \infty$,

$$\frac{I_T(\lambda_0)}{T^{2d}} \xrightarrow{\mathcal{D}} \begin{cases} h(d)\chi^2(1) & , \quad \lambda_0 = 0, \pi, \\ h(d)\chi^2(2)/2 & , \quad \lambda_0 \neq 0, \pi, \end{cases}$$

*where* $\xrightarrow{\mathcal{D}}$ *implies convergence in distribution and*

$$h(d) = \frac{2\pi g(\lambda_0)\Gamma(1-2d)}{(1+2d)\Gamma(1+d)\Gamma(1-d)}.$$

**Proof.** $\{X(t)\}$ is a Gaussian stationary process. Hence if $\lambda_0 = 0, \pi$, the result follows immediately from Theorem 2.2 of Yajima [26].

Next consider the case that $\lambda_0 \neq 0, \pi$. We have only to show

$$\lim_{T\rightarrow\infty} Var(\sum_{t=1}^{T} X(t) \cos \lambda_0 t)/T^{1+2d} = h(d)/2, \tag{41}$$

$$\lim_{T\rightarrow\infty} Var(\sum_{t=1}^{T} X(t) \sin \lambda_0 t)/T^{1+2d} = h(d)/2, \tag{42}$$

$$\lim_{T\rightarrow\infty} Cov(\sum_{t=1}^{T} X(t) \cos \lambda_0 t, \sum_{t=1}^{T} X(t) \sin \lambda_0 t)/T^{1+2d} = 0. \tag{43}$$

First we show (41). We have

$$\begin{aligned} & Var(\sum_{t=1}^{T} X(t) \cos \lambda_0 t) \\ = \ & \frac{1}{4}\int_{\Pi} \left(\sum_{t=1}^{T}(\exp(i(\lambda+\lambda_0)t) + \exp(i(\lambda-\lambda_0)t))\right) \\ & \times \left(\sum_{s=1}^{T}(\exp(-i(\lambda+\lambda_0)s) + \exp(-i(\lambda-\lambda_0)s))\right) f(\lambda)d\lambda. \end{aligned} \tag{44}$$

Now we evaluate each term of (44). First we can show in the same way as Theorem 2.1 of Yajima [27] that for any $c > 0$,

$$\lim_{T\rightarrow\infty} \int_{\Pi} |\sum_{t=1}^{T} \exp(i(\lambda+\lambda_0)t)|^2 f(\lambda)d\lambda/T^{1+2d} \tag{45}$$

$$
\begin{aligned}
&= \lim_{T\to\infty} \int_{|\lambda+\lambda_0|\le c} |\sum_{t=1}^{T} \exp(i(\lambda+\lambda_0)t)|^2 f(\lambda) d\lambda / T^{1+2d} \\
&= \lim_{T\to\infty} \int_{|\lambda|\le c} |\sum_{t=1}^{T} \exp(i\lambda t)|^2 f(\lambda-\lambda_0) d\lambda / T^{1+2d} \\
&= g(\lambda_0) \lim_{T\to\infty} \int_{|\lambda|\le c} |\sum_{t=1}^{T} \exp(i\lambda t)|^2 |\lambda|^{-2d} d\lambda / T^{1+2d} \\
&= h(d),
\end{aligned}
$$

where the last equality follows from Theorem 2.2 of [26].

Similarly

$$
\lim_{T\to\infty} \int_{\Pi} |\sum \exp(i(\lambda-\lambda_0)t)|^2 f(\lambda) d\lambda / T^{1+2d} = h(d). \tag{46}
$$

Next consider

$$
\begin{aligned}
&\int_{\Pi} (\sum_{t=1}^{T} \exp(i(\lambda+\lambda_0)t))(\sum_{s=1}^{T} \exp(-i(\lambda-\lambda_0)s)) f(\lambda) d\lambda \\
&= \left[\int_{|\lambda+\lambda_0|\le c} + \int_{|\lambda-\lambda_0|\le c} + \int_{|\lambda+\lambda_0|>c, |\lambda-\lambda_0|>c}\right] \\
&\quad (\sum_{t=1}^{T} \exp(i(\lambda+\lambda_0)t))(\sum_{s=1}^{T} \exp(-i(\lambda-\lambda_0)s)) f(\lambda) d\lambda.
\end{aligned} \tag{47}
$$

Then by Schwarz' inequality,

$$
\begin{aligned}
&| \int_{|\lambda+\lambda_0|\le c} (\sum_{t=1}^{T} \exp(i(\lambda+\lambda_0)t))(\sum_{s=1}^{T} \exp(-i(\lambda-\lambda_0)s)) \\
&\quad \times f(\lambda) d\lambda / T^{1+2d}| \\
&\le [\int_{|\lambda+\lambda_0|\le c} |\sum_{t=1}^{T} \exp(i(\lambda+\lambda_0)t)|^2 f(\lambda) d\lambda / T^{1+2d}]^{1/2} \\
&\quad \times [\int_{|\lambda+\lambda_0|\le c} |\sum_{t=1}^{T} (\exp(-i(\lambda-\lambda_0)t)|^2 f(\lambda) d\lambda / T^{1+2d}]^{1/2}.
\end{aligned} \tag{48}
$$

The first term on the right hand side of (48) is bounded and the second one converges to 0 as $T \to \infty$. The other integrals of (47) can be evaluated similarly. Hence

$$
\lim_{T\to\infty} \int_{\Pi} (\sum_{t=1}^{T} \exp(i(\lambda+\lambda_0)t))(\sum_{s=1}^{T} \exp(-i(\lambda-\lambda_0)s)) f(\lambda) d\lambda / T^{1+2d} = 0. \tag{49}
$$

Similarly

$$\lim_{T\to\infty}\int_{\Pi}(\sum_{t=1}^{T}\exp(i(\lambda-\lambda_0)t))(\sum_{s=1}^{T}\exp(-i(\lambda+\lambda_0)s))f(\lambda)d\lambda/T^{1+2d} \tag{50}$$
$$=0.$$

Then We have (41) from (44),(45),(46),(49) and (50). (42) is shown similarly.

Finally (43) is shown in the same way as (49). Then the proof is completed.

Now we prepare some lemmas on $U_T(\lambda_0,\theta)$ in order to prove thoerems stated in Section 3.

**Lemma 4.3** *Under Assumptions A and B,*

$$\begin{aligned}&\lim_{T\to\infty} U_T(\lambda_0,\theta)\\ =&\int_{\Pi}\left(\log f(\lambda;\lambda_0,\theta)+\frac{f(\lambda;\lambda_0,\theta_0)}{f(\lambda;\lambda_0,\theta)}\right)d\lambda, \quad a.s.,\end{aligned}$$

*and the convergence is uniform on* $\Theta$.

This lemma is obtained by the same argument as in Lemma 1 of Hannan [11].

**Lemma 4.4** *Under Assumptions A and B,*

$$\begin{aligned}&\int_{\Pi}\left(\log f(\lambda;\lambda_0,\theta)+\frac{f(\lambda;\lambda_0,\theta_0)}{f(\lambda;\lambda_0,\theta)}\right)d\lambda\\ \geq&\int_{\Pi}(\log f(\lambda;\lambda_0,\theta_0)+1)d\lambda,\end{aligned}$$

*and the equality holds if and only if* $\theta=\theta_0$.

This lemma follows immediatley from Theorem 1 of Taniguchi [20] and Assumption (B1).

**Lemma 4.5** *Under Assumptions A and B, as* $T\to\infty$,

$$T^{1/2}U_T^{(1)}(\lambda_0,\theta_0)\xrightarrow{\mathcal{D}} N(\mathbf{o},4\pi W(\theta_0)).$$

**Proof.** From Assumptions (B2) and (B3),

$$
\begin{aligned}
& U_T^{(1)}(\lambda_0,\theta_0) \\
= & \int_\Pi \left( \frac{\partial \log f(\lambda;\lambda_0,\theta_0)}{\partial\theta} + \frac{\partial f^{-1}(\lambda;\lambda_0,\theta_0)}{\partial\theta}\frac{I_T(\lambda)}{2\pi} \right) d\lambda.
\end{aligned}
$$

Next it follows from (17) that

$$U_T^{(1)}(\lambda_0,\theta_0) = \int_\Pi \frac{\partial f^{-1}(\lambda;\lambda_0,\theta_0)}{\partial\theta}\left(\frac{I_T(\lambda)}{2\pi} - f(\lambda;\lambda_0,\theta_0)\right) d\lambda.$$

For $\lambda_0 = 0$, the result has already been proved by Giratis and Surgailis [6]. We shall show by following their procedure that the result still holds for the case that $\lambda_0 \neq 0$.

First we evaluate the bias term. Let

$$m_T = T^{1/2}\int_\Pi \frac{\partial f^{-1}(\lambda;\lambda_0,\theta_0)}{\partial\theta}\left(E\frac{I_T(\lambda)}{2\pi} - f(\lambda;\lambda_0,\theta)\right) d\lambda.$$

Hereafter we put $f(\lambda) = f(\lambda;\lambda_0,\theta_0)$ and $h(\lambda) = \partial f^{-1}(\lambda;\lambda_0,\theta_0)/\partial\theta$ for notational simplicity. Then

$$m_T = T^{-1/2}\frac{1}{2\pi}\int_{\Pi^2} |D_T(\omega+\lambda)|^2 f(\omega)(h(\lambda)-h(\omega))d\omega d\lambda.$$

Hence

$$
\begin{aligned}
|m_T| &\leq CT^{-1/2}\int_{\Pi^2} |D_T(\omega+\lambda)|^2 f(\omega)|h(\lambda)-h(\omega)|d\omega d\lambda \qquad (51)\\
&\leq CT^{-1/2}\int_{[0,\pi]^2} (|D_T(\omega+\lambda)|^2 + |D_T(\omega-\lambda)|^2) f(\omega)|h(\omega)-h(\lambda)|d\omega d\lambda.
\end{aligned}
$$

Let $\psi_T(\lambda)$ be a periodic function with period $2\pi$ defined by

$$\psi_T(\lambda) = \frac{1}{1+T|\lambda|}, \qquad \lambda \in \Pi.$$

Then

$$|D_T(\lambda)| \leq CT\psi_T(\lambda). \qquad (52)$$

By (52), the last term of (51) is boundend by

$$
\begin{aligned}
& CT^{-1/2}\int_{[0,\pi]^2} T^2\psi_T(\omega-\lambda)^2 f(\omega)|h(\omega)-h(\lambda)|d\omega d\lambda \\
= & CT^{-1/2}\int_{[-\lambda_0,\pi-\lambda_0]^2} T^2\psi_T(\omega-\lambda)^2 f(\omega+\lambda_0)|h(\omega+\lambda_0)-h(\lambda+\lambda_0)|d\omega d\lambda \\
\leq & CT^{-1/2}\left[\int_{[0,\pi-\lambda_0]^2} + \int_{[-\lambda_0,0]\times[0,\pi-\lambda_0]} + \int_{[0,\pi-\lambda_0]\times[-\lambda_0,0]} + \int_{[-\lambda_0,0]^2}\right] \qquad (53)\\
& T^2\psi_T(\omega-\lambda)^2 f(\omega+\lambda_0)|h(\omega+\lambda_0)-h(\lambda+\lambda_0)|d\omega d\lambda.
\end{aligned}
$$

Now we evaluate each integral of (53). Consider the first integral. For $\omega < \lambda$, by Assumption (B.4)

$$
\begin{aligned}
& |h(\omega+\lambda_0) - h(\lambda+\lambda_0)| \\
\leq & \max_{\omega+\lambda_0 \leq \tau \leq \lambda+\lambda_0} |\partial h(\tau)/\partial\tau||\omega-\lambda| \\
\leq & C|\omega|^{2d-1-\epsilon}|\omega-\lambda|,
\end{aligned}
$$

for any sufficiently small $\epsilon > 0$. Hence

$$
\begin{aligned}
& f(\omega+\lambda_0)|h(\omega+\lambda_0) - h(\lambda+\lambda_0)| \\
\leq & C|\omega|^{-2d}|\omega|^{(2d-1-\epsilon)(1-\delta)}|\omega-\lambda|^{1-\delta} \\
= & C|\omega|^{-1+\delta'}|\omega-\lambda|^{1-\delta},
\end{aligned} \tag{54}
$$

where $0 < \delta < 1, \delta' = \delta(1+\epsilon-2d) - \epsilon$. We can assume that $0 < \delta' < 1$. Similary for $\lambda < \omega$

$$
\begin{aligned}
& f(\omega+\lambda_0)|h(\omega+\lambda_0) - h(\lambda+\lambda_0)| \\
\leq & C|\omega|^{-2d}|\lambda|^{(2d-1-\epsilon)(1-\delta)}|\omega-\lambda|^{1-\delta} \\
= & C|\lambda|^{-1+\delta'}|\omega-\lambda|^{1-\delta}.
\end{aligned} \tag{55}
$$

From (54) and (55),

$$
\begin{aligned}
& CT^{-1/2}\int_{[0,\pi-\lambda_0]^2} T^2\psi_T(\omega-\lambda)^2 f(\omega+\lambda_0) \\
& \times|h(\omega+\lambda_0) - h(\lambda+\lambda_0)|d\omega d\lambda \\
\leq & CT^{-1/2}\int_{\Pi^2} T^2|\omega|^{-1+\delta'}|\omega-\lambda|^{1-\delta}(1+T|\omega-\lambda|)^{-2}d\omega d\lambda.
\end{aligned} \tag{56}
$$

Then it is shown by the same argument in Lemma 4 of [6] that (56) converges to 0 as $T \to \infty$.

Next consider the second integral of (53). We have

$$
\begin{aligned}
& CT^{-1/2}\int_{[-\lambda_0,0]\times[0,\pi-\lambda_0]} T^2\psi_T(\omega-\lambda)^2 f(\omega+\lambda_0)|h(\omega+\lambda_0) - h(\lambda+\lambda_0)|d\omega d\lambda \\
= & CT^{-1/2}\int_{[0,\lambda_0]\times[0,\pi-\lambda_0]} T^2\psi_T(-\omega-\lambda)^2 f(-\omega+\lambda_0)|h(-\omega+\lambda_0) - h(\lambda+\lambda_0)|d\omega d\lambda \\
\leq & CT^{-1/2}\int_{[0,\lambda_0]\times[0,\pi-\lambda_0]} T^2\psi_T(-\omega-\lambda)^2 f(-\omega+\lambda_0)|h(-\omega+\lambda_0)|d\omega d\lambda \\
& +CT^{-1/2}\int_{[0,\lambda_0]\times[0,\pi-\lambda_0]} T^2\psi_T(-\omega-\lambda)^2 f(-\omega+\lambda_0)|h(\lambda+\lambda_0)|d\omega d\lambda.
\end{aligned} \tag{57}
$$

From Assumptions (B2), (B3 ) and (4), the first term of (57) is bounded by

$$\begin{aligned} & CT^{-1/2}\int_{[0,\pi]^2} T^2[1+T(\omega+\lambda)]^{-2}|\omega|^{-2d}|\omega|^{2d-\epsilon}d\omega d\lambda \\ = \; & CT^{-1/2+\epsilon}\int_{[0,\pi T]^2} u^{-\epsilon}(1+u+v)^{-2}dudv, \end{aligned} \tag{58}$$

for any sufficiently small $\epsilon > 0$. Then (58) converges to 0 as $T \to \infty$ since the integral is bounded with respect to $T$.

Similarly the second term of (57) is bounded by

$$\begin{aligned} & CT^{-1/2}\int_{[0,\pi]^2} T^2[1+T(\omega+\lambda)]^{-2}|\omega|^{-2d}|\lambda|^{2d-\epsilon}d\omega d\lambda \\ = \; & CT^{-1/2+\epsilon}\int_{[0,\pi T]^2} u^{-2d}v^{2d-\epsilon}(1+u+v)^{-2}dudv. \end{aligned} \tag{59}$$

Also (59) converges to 0 as $T \to \infty$. Hence the second integral of (53) converges to 0 as $T \to \infty$.

Similarly the third and fourth integrals of (53) converge to 0 as $T \to \infty$. Then we have

$$\lim_{T\to\infty} |m_T| = 0. \tag{60}$$

Next we shall show

$$T^{1/2}\int_{-\pi}^{\pi} h(\lambda)\left(\frac{I_T(\lambda)}{2\pi} - E\frac{I_T(\lambda)}{2\pi}\right)d\lambda \xrightarrow{\mathcal{D}} N(\mathbf{o}, 4\pi W(\theta_0)). \tag{61}$$

Let $b(t) = \int_\Pi e^{it\lambda}h(\lambda)d\lambda$. And let $R_T$ and $B_T$ be the $T \times T$ Toepliz matrices with $(t,s)$ elements, $r(t-s)$ and $b(t-s)$ respectively. Then it suffices to show that

$$\lim_{T\to\infty} Tr(R_T B_T)^2/T = (2\pi)^3\int_\Pi (f(\lambda)h(\lambda))^2\, d\lambda. \tag{62}$$

Since if (62) holds, the assertion (61) for the case that $\lambda_0 \neq 0$ also follows from Theorem 2 of [6]. We show (62) by following the proecedure of Lemmas 5 and 6 of [6].

First for any measurable set $A(\in \Pi^4)$, let

$$\mu_T(A) = T^{-1}\int_A D_T(-x_1+x_3)D_T(x_2-x_3)\overline{D_T(-x_1+x_4)D_T(x_2-x_4)}dx_1dx_2dx_3dx_4,$$

be a signed measure on $\Pi^4$. And put

$$F(x_1, x_2, x_3, x_4) = f(x_1)f(x_2)|h(x_3)||h(x_4)|,$$

and

$$\begin{aligned} S_{K,j} &= \{\boldsymbol{x} \in \Pi^4 | f(x_j) \geq K)\}, \quad (j = 1, 2), \\ S_{K,j} &= \{\boldsymbol{x} \in \Pi^4 || h(x_j)| \geq K)\}, \quad (j = 3, 4), \end{aligned}$$

for $K > 0$ and $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$. And put

$$I_j = \int_{\Pi^4} F(x_1, x_2, x_3, x_4)\mathbf{1}(S_{K,j})d|\mu_T(\boldsymbol{x})|, \quad (j = 1, 2, 3, 4).$$

Then by noting the proof of Lemma 5 of [6], it suffices to show that

$$\lim_{K\to\infty} \limsup_{T\to\infty} I_j = 0, \quad (j = 1, 2, 3, 4), \tag{63}$$

in order to prove (62). By symmetry, we have only to prove (63) for $j = 1$.

First by transforming variables ,if necessary, we can change the domain of the integral of $I_1$ from $\Pi^4$ to $[0, \pi]^4$ and ,hence, the definition of $S_{K,1}$ to

$$S_{K,1} = \{\boldsymbol{x} \in [0, \pi]^4 | f(x_1) \geq K)\}.$$

Then since $f(x_1)$ is bounded outside of $x_1 = \lambda_0$, $S_{K,1} \subset \{\boldsymbol{x} \in [0, \pi]^4 || x_1 - \lambda_0| < \epsilon\}$ for some $\epsilon = \epsilon(K)$ with the property that $\lim_{K\to\infty} \epsilon(K) = 0$. Next we have

$$\begin{aligned} \{\boldsymbol{x} \in [0, \pi]^4 || x_1 - \lambda_0| < \epsilon\} &\subset \cup_{i=3}^4 \{\boldsymbol{x} \in [0, \pi]^4 || x_1 - \lambda_0| < |x_i - \lambda_0|/2\} \\ &\quad \cup \{\boldsymbol{x} \in [0, \pi]^4 | \epsilon > |x_1 - \lambda_0| \geq \max_{i=3,4} |x_i - \lambda_0|/2\} \\ &= B_3 \cup B_4 \cup W, \quad \text{(say)}. \end{aligned}$$

Then (63) follows from that

$$\lim_{T\to\infty} \int_{B_i} F d|\mu_T(\boldsymbol{x})| = 0, \quad (i = 3, 4), \tag{64}$$

and

$$\lim_{\epsilon\to 0} \limsup_{T\to\infty} \int_W F d|\mu_T(\boldsymbol{x})| = 0. \tag{65}$$

Now we shall prove (64). Let $\alpha = 2d$ and $\beta = -\alpha + \delta$ with any sufficiently small $\delta > 0$ Then by noting (B2) and (B3), for any $v$ with $0 < v < 1$, it is shown in a similar way as in (5.9) of [6] that

$$\begin{aligned}
&\int_{B_3} F d|\mu_T(\boldsymbol{x})| \\
\leq\ & CT^{4v-1} \int_{B_3} \prod_{i=1}^{2} \prod_{j=3}^{4} |x_i - \lambda_0|^{-\alpha} |x_j - \lambda_0|^{-\beta} |x_i - x_j|^{v-1} d^4\boldsymbol{x} \\
\leq\ & CT^{4v-1} \int_{[0,\pi]^4} |x_1 - \lambda_0|^{-\alpha} |x_2 - \lambda_0|^{-\alpha} |x_3 - \lambda_0|^{v-1-\beta} \\
& \times |x_4 - x_0|^{-\beta} |x_1 - x_4|^{v-1} |x_2 - x_3|^{v-1} |x_2 - x_4|^{v-1} d^4\boldsymbol{x} \\
=\ & CT^{4v-1} \int_{[-\lambda_0, \pi-\lambda_0]^4} |x_1|^{-\alpha} |x_2|^{-\alpha} |x_3|^{v-1-\beta} \\
& \times |x_4|^{-\beta} |x_1 - x_4|^{v-1} |x_2 - x_3|^{v-1} |x_2 - x_4|^{v-1} d^4\boldsymbol{x} \\
\leq\ & CT^{4v-1} \int_{\Pi^4} |x_1|^{-\alpha} |x_2|^{-\alpha} |x_3|^{v-1-\beta} \\
& \times |x_4|^{-\beta} |x_1 - x_4|^{v-1} |x_2 - x_3|^{v-1} |x_2 - x_4|^{v-1} d^4\boldsymbol{x}. \qquad (66)
\end{aligned}$$

Then it is shown by the same argument as in Lemma 6 of [6] that (66) converges to 0 as $T \to \infty$. Hence the proof of (64) is completed. (65) is shown in a similar way as in Lemma 6 of [6]. Now we obtain (62). Finally the assertion follows from (60) and (61).

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