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Tables of Limiting Distributions Useful for Testing Unit Roots and Co-integration with Multiple Structural Breaks

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Abstract

We give tables of the limiting distributions of some statistics useful for testing the hypotheses of unit roots and cointegration with multiple structural break points. Since the limiting random variables are some complicated functionals of Brownian motions with drift functions, we have used a large number of simulations in order to obtain their empirical distribution functions. We give the tables of the distribution functions of some functionals of Brownian motions when there are several structural break points or change points in the drift functions and their exact locations are not necessarily known. We also consider a situation when the number of change points is not known, but it is less than a pre-specified number.

Key Words

Unit Roots, Cointegration, Functionals of Brownian Motions with Drift Functions, Multiple Structural Breaks, Unknown Number of Change Points

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1. Introduction

Recently, there has been a growing interest among econometricians and statisticians on the unit roots hypotheses and the cointegration hypotheses for describing economic time series. Many test procedures have been proposed by statisticians and econometricians in the past decade and they have been applied to many empirical studies on macro and financial time series. Among many test procedures, Kunitomo (1992) and Kunitomo (1995) have introduced a class of statistics for testing the hypotheses of unit roots and cointegration in the multivariate linear time series models with trend functions. They have considered the situation when there are some unit roots in the multivariate linear time series models and there are multiple structural break points in the data period, whose locations are known or unknown. Since the limiting distributions of the test statistics proposed by Kunitomo (1995) under the hypotheses of unit roots or cointegration are some quadratic functionals of the multi-dimensional Brownian motions with deterministic drift functions, their limiting distribution functions have not been known and not available in the literature.

In recent econometric studies there has been a growing suspect among econometricians and macro economists whether the possible existence of structural break points or change points in the trend functions could make significant effects on the existing testing procedures for the unit roots and cointegration hypotheses. One influential empirical example on the U.S. economy in time series econometrics of this line has been given by Perron (1989). The works by Kunitomo (1992) and Kunitomo (1995) can be regarded as the direct generalizations of the test statistics proposed by Perron (1989) in this respect. We should note that, however, Perron (1989) proposed to use the t-type test statistics when there is one trend break point and its location is known while Kunitomo (1995) has proposed to use the χ^2-type test statistics when there can be multiple breaks or change points and their locations are not necessarily known in multivariate time series. There have been some related empirical works done already in econometrics. For instance, Ohara (1994) has used the formulation of Kunitomo (1992) and examined the hypothesis of unit roots with two structural break points for some Japanese macro-data by using the t-type statistics. Soejima (1995) has used the unit roots and cointegration hypotheses with some non-linear trend functions and re-examined previous empirical studies on Japanese macro time series such as real GNP, money supply, and price indeces by using the χ^2-type statistics as we shall explain in Section 3.4 and Section 3.5.

In this report, we shall give the tables of the limiting distribution functions of the test statistics proposed by Kunitomo (1992) and Kunitomo (1995) in some details. The method of calculations adopted in this report is based on Monte Carlo simulations. Since the numbers of replications in our simulations are relatively large, we can rely on the numerical values of the tables in this report for practical purposes. The tables given in this report include the limiting distribution functions of statistics when there are unknown multiple structural breaks and the number of break points is less than a prespecified number and unknown. These cases have not been investigated in econometrics and statistics as far as we know. We hope the tables given in this report will be useful

for possible empirical studies.

2. Test Statistics and Representations of their Limiting Random Variables

Let $\boldsymbol{y}_t$ $(t = 1, \cdots, T)$ be a $G \times 1$ vector of time series satisfying

$$\boldsymbol{y}_t = \boldsymbol{\Gamma} \boldsymbol{z}_t^* + \boldsymbol{A}_1 \boldsymbol{y}_{t-1} + \cdots + \boldsymbol{A}_p \boldsymbol{y}_{t-p} + \boldsymbol{v}_t \ , \tag{2.1}$$

where $\boldsymbol{z}_t^{*\prime} = (\boldsymbol{z}_{1t}^{*\prime}, \boldsymbol{z}_{2t}^{*\prime})$ is a $1 \times K^*$ $(K^* = K_1^* + K_2^*)$ vector of strictly exogenous variables, $\boldsymbol{\Gamma} = (\boldsymbol{\Gamma}_1, \boldsymbol{\Gamma}_2)$ is a $G \times K^*$ coefficient matrix, $\boldsymbol{A}_1, \cdots, \boldsymbol{A}_p$ are $G \times G$ coefficient matrices, and $\boldsymbol{v}_t$ is a $G \times 1$ vector of disturbances with $E(\boldsymbol{v}_t | \mathcal{F}_{t-1}) = \boldsymbol{O}$ and $E(\boldsymbol{v}_t \boldsymbol{v}_t^{\prime} | \mathcal{F}_{t-1}) = \boldsymbol{\Omega}_t$. In this formulation $\mathcal{F}_{t-1}$ is the $\sigma-$field generated by $\boldsymbol{y}_s, \boldsymbol{v}_s$ $(s \leq t-1)$ and $\boldsymbol{z}_s^*$ $(s \leq t)$. It may be convenient to re-written (2.1) as

$$\Delta \boldsymbol{y}_t = \boldsymbol{\Gamma}_1 \boldsymbol{z}_{1t}^* + (\boldsymbol{\Gamma}_2, \boldsymbol{B}^*) \begin{pmatrix} \boldsymbol{z}_{2t}^* \\ \boldsymbol{y}_{t-1} \end{pmatrix} + \sum_{i=2}^{p} \boldsymbol{B}_i \Delta \boldsymbol{y}_{t-(i-1)} + \boldsymbol{v}_t \ , \tag{2.2}$$

where Δ is the difference operator $(\Delta \boldsymbol{y}_t = \boldsymbol{y}_t - \boldsymbol{y}_{t-1})$. The coefficient matrices $\boldsymbol{B}^*$ and $\boldsymbol{B}_i$ $(i = 2, \cdots, p)$ are successively defined from the original parameter matrices $\boldsymbol{A}_i$ $(i = 1, \cdots, p)$.

The cointegration hypothesis considered in Kunitomo (1995) is given by

$$H_2 : \ rank(\boldsymbol{\Gamma}_2, \boldsymbol{B}^*) = rank(\boldsymbol{B}^*) = r \ , \tag{2.3}$$

where $0 \leq r \leq G$. When $\boldsymbol{\Gamma}_2 = \boldsymbol{O}$, the hypothesis H_2 is the standard cointegration hypothesis originally formulated by Engle and Granger (1987). Further, when $r = 0$ and $G = 1$, it is equivalent to the unit root hypothesis for the univariate time series with a deterministic trend function.

Let λ_i^* $(i = 1, \cdots, G_0)$ be the G_0 $(G_0 = G - r)$ smaller characteristic roots $0 \leq \lambda_1^* \leq \cdots \leq \lambda_G^*$ of the determinantal equation

$$\left| \Delta \boldsymbol{Y}^{\prime} (\boldsymbol{P}_Z - \boldsymbol{P}_{Z_1}) \Delta \boldsymbol{Y} - \lambda^* \Delta \boldsymbol{Y}^{\prime} \bar{\boldsymbol{P}}_Z \Delta \boldsymbol{Y} \right| = 0 \ , \tag{2.4}$$

where $\Delta \boldsymbol{Y}$ is a $T \times G$ matrix whose t-th row vector is $\Delta \boldsymbol{y}_t^{\prime}$. The $T \times T$ matrix $\boldsymbol{P}_Z = \boldsymbol{Z}(\boldsymbol{Z}^{\prime}\boldsymbol{Z})^{-1}\boldsymbol{Z}^{\prime}$ denotes the projection operator onto the space spanned by the column vectors of $\boldsymbol{Z}$ whose t-th row vector is $\boldsymbol{z}_t^{\prime}$ and $\bar{\boldsymbol{P}}_Z = \boldsymbol{I}_T - \boldsymbol{P}_Z$ for any (full column) matrix $\boldsymbol{Z}$.

Let take a $1 \times K$ vector $\boldsymbol{z}_t^{\prime} = (\boldsymbol{z}_{1t}^{\prime}, \boldsymbol{z}_{2t}^{\prime})$ such that

$$\boldsymbol{z}_{1t}^{\prime} = (\boldsymbol{z}_{1t}^{*\prime}, \Delta \boldsymbol{y}_{t-1}^{\prime}, \cdots, \Delta \boldsymbol{y}_{t-(p-1)}^{\prime})$$

and

$$\boldsymbol{z}_{2t}^{\prime} = (\boldsymbol{z}_{2t}^{*\prime}, \boldsymbol{y}_{t-1}^{\prime}) \ .$$

Then the class of test statistic we shall consider is given by

$$RT_2 = T \times g(\lambda_1^*, \cdots, \lambda_{G_0}^*), \tag{2.5}$$

where $0 \leq \lambda_1^* \leq \cdots \leq \lambda_{G_0}^*$ are the G_0 smaller characteristic roots of (2.4), and the function $g(\cdot)$ satisfies (i) $G(0,\cdots,0) = 0$, (ii) $g(\lambda_1^*,\cdots,\lambda_{G_0}^*)$ being totally differentiable at $(\lambda_1^*,\cdots,\lambda_{G_0}^*) = (0,\cdots,0)$, and (iii)

$$\frac{\partial g}{\partial \lambda_i^*}|_{\lambda_1^*=\cdots=\lambda_{G_0}^*=0} = 1 \ (i = 1,\cdots,G_0) \ .$$

This class of statistics includes many test statistics as special cases including the likelihood ratio statistic, the Lagrange Multiplier statistic, and the Wald statistic. It will be called the trace-type statistic in this paper. The likelihood ratio (LR) statistic for testing H_2 against the alternative hypothesis

$$H_{2A} : rank(\boldsymbol{\Gamma}_2, \boldsymbol{B}^*) = G \tag{2.6}$$

is given by

$$LR_2 = T\sum_{i=1}^{G_0} \log(1+\lambda_i^*) \ , \tag{2.7}$$

where $G_0 = G - r$. The Lagrange Multiplier (LM) statistic and Wald statistic are given by

$$LM_2 = T\sum_{i=1}^{G_0} \frac{\lambda_i^*}{1+\lambda_i^*} \ , \tag{2.8}$$

and

$$W_2 = T\sum_{i=1}^{G_0} \lambda_i^* \ , \tag{2.9}$$

respectively. (See Kunitomo (1992) (1995), and Anderson and Kunitomo (1992) for the details of these forms of statistics for testing the rank hypothesis.)

The limiting distributions of LR_2, LM_2 and W_2 when $T \to +\infty$ (T is the number of observations) are the same under a set of fairly general (Lindeberge type) conditions and $\Delta \boldsymbol{z}_{2t}^* = \boldsymbol{z}_{1t}^*$ ($\boldsymbol{z}_t^*$ is $\mathcal{F}_0-$ measurable) which have been given in Theorem 5 of Kunitomo (1995). They have the same representation of

$$T_2^* = \mathrm{tr}\left(\tilde{\boldsymbol{N}}'\left[\tilde{\boldsymbol{M}}^{-1} - \begin{pmatrix} \boldsymbol{M}_{11}(1)^{-1} & O \\ O & O \end{pmatrix}\right]\tilde{\boldsymbol{N}}\right) \ , \tag{2.10}$$

where

$$\tilde{\boldsymbol{M}} = \int_0^1 \begin{bmatrix} \boldsymbol{m}(s) \\ \boldsymbol{B}(s) \end{bmatrix} \begin{bmatrix} \boldsymbol{m}(s) \\ \boldsymbol{B}(s) \end{bmatrix}' ds \ , \tag{2.11}$$

$$\tilde{\boldsymbol{N}} = \int_0^1 \begin{bmatrix} \boldsymbol{m}(s) \\ \boldsymbol{B}(s) \end{bmatrix} d\boldsymbol{B}(s)' \ , \tag{2.12}$$

and

$$\boldsymbol{M}_{11}(1) = \int_0^1 \boldsymbol{m}_1(s)\boldsymbol{m}_1(s)' ds \ . \tag{2.13}$$

In the above representation $\{\boldsymbol{B}(s)\}$ is a $G_0 \times 1$ vector of multi-dimensional standard Brownian motions and $\boldsymbol{m}(t)' = (\boldsymbol{m}_1(t)', \boldsymbol{m}_2(t)')$ is a $1 \times K^*$ $(K^* = K_1^* + K_2^*)$ vector of linearly independent functions of t satisfying

$$\lim_{T\to\infty} \boldsymbol{D}_T^{*'-1} \sum_{t=1}^{T} \boldsymbol{z}_t^* \boldsymbol{z}_t^{*\prime} \boldsymbol{D}_T^{*-1} = \int_0^1 \boldsymbol{m}(s)\boldsymbol{m}(s)' ds \,, \tag{2.14}$$

where $\boldsymbol{D}_T^*$ is a $K^* \times K^*$ normalization matrix ($\mathcal{F}_0-$ measurable), whose components are typically some polynomial functions of T.

We also consider the testing problem of H_2 against the alternative hypothesis

$$H_{2A}' : \; rank(\boldsymbol{\Gamma}_2, \boldsymbol{B}^*) = rank(\boldsymbol{B}^*) = r + 1 \,. \tag{2.15}$$

Then the likelihood ratio (LR) statistic is given by

$$\begin{aligned} LR_2 &= T \sum_{i=1}^{G-r} \log(1 + \lambda_i^*) - T \sum_{i=1}^{G-(r+1)} \log(1 + \lambda_i^*) \\ &= T \log(1 + \lambda_{G_0}^*) \,, \end{aligned} \tag{2.13}$$

where $G_0 = G - r$. Hence we can define the class of test statistic by

$$RT_2' = T \times g(\lambda_{G_0}^*), \tag{2.16}$$

where $\lambda_{G_0}^*$ is the $G_0 - th$ smallest characteristic root of (2.4) and the function $g(\cdot)$ satisfies the same conditions (i)$G(0) = 0$, (ii) $g(\lambda_{G_0}^*)$ being totally differentiable at $\lambda_{G_0}^* = 0$, and (iii)

$$\frac{\partial g}{\partial \lambda_{G_0}^*}|_{\lambda_{G_0}^*=0} = 1$$

as for (2.5). This type of statistics including the likelihood ratio statistic will be called as the max-type statistics in this paper below. Then we need the limiting distribution function of $T\times$ (the $G_0 - th$ smallest characteristic root of (2.4)). We denote the limiting random variable of this max-type statistic as

$$T_3^* = \max_{i=1,\cdots,G_0} \lambda(\tilde{\boldsymbol{N}}' \left[\tilde{\boldsymbol{M}}^{-1} - \begin{pmatrix} \boldsymbol{M}_{11}(1)^{-1} & O \\ O & O \end{pmatrix} \right] \tilde{\boldsymbol{N}}) \,, \tag{2.17}$$

where the right hand side means the maximum characteristic root of the random matrix in the parenthesis.

We shall give the numerical values of the distribution functions of the random variables represented by T_2^* in (2.10) and T_3^* in (2.17) for various special cases in the following tables, which may be useful for empirical studies.

3. Tables of Limiting Distributions

3.1 Simulation Procedure

We shall give many tables of the distribution functions calculated from a set of systematic simulations. Our simulations have been basically done by the following procedure. First, we divide the interval $[0,1]$ into 400 sub-intervals and we denote $n = 400$. Second, we generate a sequence of G_0-dimensional independent normal random variables $\{\boldsymbol{v}_j; j = 1, \cdots, n\}$, which follow $N_{G_0}(0, \frac{1}{n}\boldsymbol{I}_{G_0})$. Then we discretize the continuous functionals and approximate Brownian motions $\{\boldsymbol{B}(t), 0 \leq t \leq 1\}$ by $\sum_{s=1}^{j} \boldsymbol{v}_s$ $(j = [tn])$. The continuous drift functions $\{\boldsymbol{m}(t)\}$ have been directly calculated by $\boldsymbol{m}(j/n)$ $(j = 1, \cdots, n)$. Third, we repeat the above steps by generating different sets of normal random variables and calculate the empirical distributions of test statistics. The number of replications in our simulations is either 5,000 or 8,000. We have checked the accuracy of the resulting empirical distribution functions obtained by our methods. Generally, we can rely on the number of distribution functions up to the two decimal places in the following tables.

3.2 Table 1

Let q be the number of break points in the linear trend functions of $\boldsymbol{y}_t$. We assume that the locations of break points $0 < T_1(T) < \cdots < T_q(T) < T$ are functions of T and satisfy

$$\lim_{T \to +\infty} \frac{T_i(T)}{T} = \delta_i \tag{3.1}$$

and

$$0 < \delta_1 < \cdots < \delta_q < 1 \, . \tag{3.2}$$

In this case the exogenous variables we used are

$$\boldsymbol{z}_{1t}^{*\prime} = (1, DU_1(t), \cdots, DU_q(t)) \, ,$$

$$\boldsymbol{z}_t^{*\prime} = (1, DU_1(t), \cdots, DU_q(t), t, DT_1(t), \cdots, DT_q(t)) \, ,$$

where

$$DU_i(t) = \begin{cases} 0 & \text{if} \quad 0 < t \leq T_i \\ 1 & \text{if} \quad T_i < t \leq T \end{cases} \, , \tag{3.3}$$

and

$$DT_i(t) = \begin{cases} 0 & \text{if} \quad 0 < t \leq T_i \\ t - T_i & \text{if} \quad T_i < t \leq T \end{cases} \, . \tag{3.4}$$

We denote the trace-type statistic (2.5) as $RT_2(1; T_1, \cdots, T_q)$ and its limiting random variable (2.10) as $T_2^*(1; \delta_1, \cdots, \delta_q)$. The max-type statistic (2.16) is denoted as as $RT_2'(1; T_1, \cdots, T_q)$ and its limiting random variable in (2.17) is denoted as $T_3^*(1; \delta_1, \cdots, \delta_q)$. Then we give the distribution functions of $T_2^*(1; \delta_1, \cdots, \delta_q)$ (trace) and $T_3^*(1; \delta_1, \cdots, \delta_q)$ (max) for $q = 1$ as Table 1.1 and $q = 2$ as Table 1.2, respectively. The dimension of the Brownian motions is denoted by G_0, and we have tables for $G_0 = 1, \cdots, 5$ in all cases.

3.3 Table 2

When the precise periods of structural break points $0 < T_1 < \cdots < T_q < T$ are unknown, we cannot use Table 1. In this situation we can calculate the statistic

$$\sup_{0<T_1<\cdots<T_q<T} RT_2(1; T_1, \cdots, T_q) \,. \tag{3.5}$$

We note that the class of statistics RT_2 includes LR_2, LM_2, and W_2 as special cases. The trace-type statistic including these three statistics can be written as

$$RT_2(1; q) = \sup_{0<T_1<\cdots<T_q<T} RT_2(1; T_1, \cdots, T_q) \,, \tag{3.6}$$

and its limiting random variable is denoted by $T_4^*(1; q)$. Also the limiting random variable of the max-type statistic (2.16) of the form (2.17) in this case can be denoted by $T_5^*(1; q)$. The distribution functions of $T_4^*(1; q)$ and $T_5^*(1; q)$ for $q = 1, 2, 3$ are given as Table 2.1, Table 2.2, and Table 2.3, respectively.

3.4 Table 3

When there is a non-linear trend in certain particular segment of data peirod, we can use a mixture of linear and quadratic functions as the trend function. For this purpose, let $0 < T_1 < T_2 < T$. The strictly exogenous variables we can use are

$$\boldsymbol{z}_{1t}^{*\prime} = (1, DU_1(t), DT_1(t) - DT_2(t)) \,,$$
$$\boldsymbol{z}_t^{*\prime} = (1, DU_1(t), DT_1(t) - DT_2(t), DT_2(t), t, QT_1(t) - QT_2(t)) \,,$$

where

$$QT_i(t) = \begin{cases} 0 & (0 < t \leq T_i) \\ (t - T_i)^2 & (T_i < t \leq T) \end{cases} \,. \tag{3.7}$$

The resulting trace-type statistic is denoted by $RT_2(2; T_1, T_2)$ and the limiting random variable of this statistic is denoted by $T_2^*(2; \delta_1, \delta_2)$. The limiting distribution of the corresponding max-type statistic in this case is also denoted by $T_3^*(2; \delta_1, \delta_2)$. We give the distribution functions of $T_2^*(2; \delta_1, \delta_2)$ (trace) and $T_3^*(2; \delta_1, \delta_2)$ (max) as Table 3.1 and Table 3.2, respectively. Although the condition $\Delta \boldsymbol{z}_{2t}^* = \boldsymbol{z}_{1t}^*$ is not strictly satisfied in this case, we can show that the limiting random variables of statistics under the null-hypothesis (2.3) have the representations of T_2^* in (2.10) and T_3^* in (2.17) because the variables in $\boldsymbol{z}_{1t}^*$ are spanned by the variables in $\Delta \boldsymbol{z}_t^*$. (See the proof of Theorem 5 of Kunitomo(1995).) Soegima (1995) has used the curved trend function we have explained in this section and re-examined some previous empirical studies on macro economic variables in Japan.

3.5 Table 4

When the precise periods of structural break points $0 < T_1 < T_2 < T$ are unknown for $RT_2(2; T_1, T_2)$, we can use the statistics

$$\sup_{0<T_1<T_2<T} RT_2(2; T_1, T_2) \ . \tag{3.8}$$

The limiting random variables for the trace-type statistic and max-type statistic in this case are denoted by $T_4^*(2;2)$ (trace) and $T_5^*(2;2)$ (max), respectively. The distribution functions of these statistics are given as Table 4.1 and Table 4.2, respectively.

3.6 Table 5

When the number of structural break points is unknown, we can construct some statistics from a sequence of statistics with fixed q's which we have discussed in this section. For instance, we can construct a statistic for testing the unit roots hypothesis as well as the cointegration hypothesis with unknown number of linear change points from a sequence of statistics $\{RT_2(1; T_1, \cdots, T_q)\}$ (or $\{RT_2'(1; T_1, \cdots, T_q)\}$) we have discussed in Section 2. Since the degrees of freedom for the statistics $RT_2(1; T_1, \cdots, T_q)$ and $RT_2'(1; T_1, \cdots, T_q)$ are different, we need to adjust them by using

$$d_1(q) = G_0 \times (G_0 + 1 + q) \ .$$

Then the resulting trace-type statistic is given by

$$FRT_2(1; q_0) = \sup_{0\le q\le q_0} \sup_{0<T_1<\cdots<T_q<T} \frac{1}{d_1(q)} RT_2(1; T_1, \cdots, T_q) \ , \tag{3.9}$$

where the maximum integer number q_0 should be pre-specified. The limiting random variable of the adjusted (or F-type) trace statistic is denoted by $T_6^*(1; q_0)$. We tentatively give the limiting distribution functions of this trace-type statistic in Table 5.1 and Table 5.2 only for the case of $q_0 = 2$ and $q_0 = 3$.

Since the dimension for the max-type statistic (2.13) is one in a sense, we can also use

$$d_2(q) = G_0 + 1 + q$$

as the degrees of freedom instead of d_1 in (3.9). Then the limiting random variable of the resulting max-type statistic in the form of (2.16) is denoted by $T_7^*(1; q_0)$. We tentatively give the limiting distribution functions of this max-type statistics in Table 5.1 and Table 5.2 only for the case of $q_0 = 2$ and $q_0 = 3$.

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Table 1.1

$q = 1$: one linear kink case

$$z_t^* = \begin{bmatrix} 1 \\ DU_1(t) \\ t \\ DT_1(t) \end{bmatrix}, \quad z_{1t}^* = \begin{bmatrix} 1 \\ DU_1(t) \end{bmatrix}$$

G_0 = dimension of Brownian motions
δ_1 = location of changing point $(0 < \delta_1 < 1)$

(i) trace statistic

$G_0 = 1$

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	2.12	2.65	3.12	3.87	7.50	13.48	15.56	17.82	20.20	8.17	3.89
δ_1=0.2	2.68	3.18	3.80	4.56	8.55	14.83	16.96	18.80	21.17	9.21	4.13
δ_1=0.3	3.18	3.74	4.47	5.35	9.53	15.84	18.21	20.35	23.02	10.19	4.26
δ_1=0.4	3.76	4.45	5.11	5.94	10.00	16.36	18.67	20.77	23.55	10.70	4.24
δ_1=0.5	3.96	4.57	5.20	6.06	10.21	16.56	18.66	20.67	22.96	10.84	4.16
δ_1=0.6	3.72	4.31	5.03	5.86	10.11	16.30	18.39	20.65	22.91	10.71	4.17
δ_1=0.7	3.17	3.82	4.45	5.24	9.33	15.74	17.85	20.05	22.92	10.04	4.21
δ_1=0.8	2.59	3.20	3.83	4.58	8.55	14.79	16.92	19.30	21.93	9.23	4.16
δ_1=0.9	2.11	2.70	3.17	3.82	7.45	13.26	15.43	17.52	19.86	8.11	3.83

$G_0 = 2$

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	9.82	11.01	12.15	13.58	19.90	28.36	31.30	33.54	36.80	20.57	5.87
δ_1=0.2	10.98	12.25	13.49	15.13	21.80	30.38	33.53	36.13	39.40	22.41	6.13
δ_1=0.3	12.19	13.73	15.11	16.66	23.48	32.31	35.50	38.25	42.10	24.13	6.31
δ_1=0.4	13.09	14.57	15.86	17.45	24.35	33.74	36.56	39.35	42.94	25.09	6.39
δ_1=0.5	13.18	14.63	15.98	17.69	24.59	33.76	36.83	39.44	42.80	25.27	6.36
δ_1=0.6	12.80	14.43	15.73	17.26	24.24	33.57	36.40	39.12	42.67	24.91	6.38
δ_1=0.7	12.37	13.57	14.87	16.47	23.43	32.35	35.34	37.75	41.12	24.05	6.26
δ_1=0.8	11.05	12.42	13.61	15.11	21.85	30.63	33.75	36.39	39.68	22.51	6.19
δ_1=0.9	9.82	10.98	11.98	13.48	19.65	28.31	31.32	34.15	37.38	20.41	5.93

$G_0 = 3$

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	21.81	23.36	25.23	27.34	36.11	46.73	50.16	53.17	57.17	36.68	7.67
δ_1=0.2	24.09	25.92	27.82	30.10	39.27	50.52	54.18	57.55	61.28	39.88	8.05
δ_1=0.3	25.83	27.77	29.62	31.99	41.26	52.61	56.23	59.86	63.75	41.93	8.18
δ_1=0.4	26.79	28.77	30.48	32.72	42.36	54.05	57.75	61.20	64.76	42.97	8.30
δ_1=0.5	26.85	29.09	30.95	33.33	42.76	54.09	57.93	61.63	65.61	43.37	8.24
δ_1=0.6	26.68	28.58	30.42	32.85	42.38	54.06	57.34	60.62	64.16	43.00	8.27
δ_1=0.7	26.01	27.95	29.88	32.26	41.33	53.16	56.47	59.81	64.17	42.13	8.19
δ_1=0.8	24.12	26.13	28.03	30.21	39.44	50.68	54.21	57.12	61.26	40.00	8.02
δ_1=0.9	21.52	23.38	25.14	27.22	35.96	46.62	50.47	53.76	57.56	36.56	7.71

$G_0 = 4$

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	38.42	40.61	42.93	45.53	56.34	69.77	73.64	77.32	82.01	57.13	9.48
δ_1=0.2	40.83	43.78	45.87	49.06	60.60	73.88	77.92	81.74	85.78	61.05	9.71
δ_1=0.3	43.43	46.24	48.66	51.42	62.90	76.44	80.82	84.49	89.35	63.57	9.81
δ_1=0.4	44.69	47.68	50.12	52.85	64.11	77.90	82.28	86.01	90.70	64.84	9.91
δ_1=0.5	44.68	47.22	49.80	52.69	64.55	78.20	82.62	86.62	91.26	65.10	10.04
δ_1=0.6	44.07	47.30	49.50	52.37	64.34	78.19	82.49	86.26	90.77	64.92	10.08
δ_1=0.7	43.77	46.29	48.57	51.59	63.01	76.60	80.72	84.63	89.12	63.67	9.84
δ_1=0.8	41.51	44.07	46.48	49.46	60.40	73.77	78.28	81.95	87.10	61.18	9.64
δ_1=0.9	37.59	40.25	42.54	45.23	56.18	69.27	72.87	76.76	81.19	56.83	9.41

$G_0 = 5$

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	58.19	61.05	63.76	67.30	80.62	96.29	100.80	105.45	110.01	81.26	11.27
δ_1=0.2	62.40	65.87	68.88	72.18	85.54	101.14	106.06	110.15	115.11	86.28	11.38
δ_1=0.3	64.46	67.72	70.62	74.31	88.17	103.61	108.46	112.83	118.28	88.73	11.51
δ_1=0.4	65.87	68.64	71.69	75.64	89.42	105.44	110.25	114.63	119.73	90.08	11.69
δ_1=0.5	65.60	68.94	72.11	75.90	89.44	105.81	110.89	115.28	120.82	90.29	11.84
δ_1=0.6	66.22	69.47	72.40	75.75	89.39	105.23	110.40	115.16	121.35	90.22	11.64
δ_1=0.7	64.57	67.73	70.67	74.27	88.30	104.00	108.95	113.61	118.26	88.88	11.67
δ_1=0.8	62.52	65.85	68.98	72.34	85.69	101.28	105.87	110.15	115.68	86.25	11.30
δ_1=0.9	57.52	60.76	63.87	67.09	80.55	95.81	100.54	105.34	109.72	81.26	11.31

(ii) max statistic

$G_0 = 2$

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	6.20	6.98	7.84	8.95	13.76	20.52	23.00	25.19	27.64	14.34	4.63
δ_1=0.2	7.03	7.88	8.74	9.86	14.81	21.72	24.28	26.53	29.28	15.40	4.78
δ_1=0.3	7.84	8.72	9.64	10.77	15.65	22.69	25.24	27.62	30.64	16.31	4.84
δ_1=0.4	8.11	9.03	9.89	11.04	16.07	23.43	25.80	27.85	30.60	16.72	4.91
δ_1=0.5	8.08	8.99	9.99	11.05	16.18	23.26	25.82	28.08	30.76	16.76	4.87
δ_1=0.6	8.02	9.01	9.83	11.05	15.99	23.03	25.60	28.18	31.16	16.61	4.89
δ_1=0.7	7.78	8.66	9.47	10.59	15.66	22.71	25.03	27.25	30.06	16.29	4.81
δ_1=0.8	7.06	7.87	8.72	9.86	14.94	21.91	24.35	26.93	29.66	15.50	4.85
δ_1=0.9	6.12	6.98	7.81	8.83	13.56	20.42	22.81	25.12	27.99	14.19	4.67

$G_0 = 3$

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	10.50	11.52	12.53	13.81	19.34	26.92	29.43	31.66	34.75	19.97	5.20
δ_1=0.2	11.58	12.63	13.72	15.03	20.77	28.32	31.12	33.59	37.06	21.34	5.37
δ_1=0.3	12.01	13.14	14.22	15.54	21.43	29.00	31.72	33.96	36.66	21.94	5.33
δ_1=0.4	12.23	13.46	14.54	15.83	21.69	29.49	32.21	34.50	37.59	22.24	5.44
δ_1=0.5	12.25	13.63	14.69	16.01	21.67	29.21	31.66	34.09	37.16	22.25	5.27
δ_1=0.6	12.34	13.44	14.53	15.80	21.54	29.28	31.99	34.30	37.22	22.16	5.38
δ_1=0.7	12.19	13.36	14.48	15.81	21.44	29.13	31.77	34.14	37.51	22.08	5.32
δ_1=0.8	11.68	12.75	13.80	15.17	20.78	28.73	31.22	33.48	36.12	21.47	5.33
δ_1=0.9	10.38	11.53	12.49	13.84	19.25	26.81	29.55	31.97	34.87	19.90	5.22

$G_0 = 4$

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	15.31	16.49	17.49	18.95	25.10	33.32	36.00	38.26	41.23	25.68	5.65
δ_1=0.2	16.19	17.35	18.62	20.04	26.35	34.40	37.36	39.77	42.22	26.88	5.70
δ_1=0.3	16.67	17.97	19.18	20.67	26.83	34.97	37.85	40.55	44.11	27.45	5.76
δ_1=0.4	16.93	18.22	19.41	20.92	27.03	35.18	37.77	40.34	44.06	27.62	5.69
δ_1=0.5	16.84	18.18	19.42	20.81	27.12	35.31	38.09	40.50	43.66	27.68	5.72
δ_1=0.6	16.75	18.09	19.35	20.88	27.18	35.66	38.24	40.90	43.96	27.83	5.84
δ_1=0.7	16.66	17.97	19.20	20.71	26.92	34.97	37.63	40.05	43.43	27.46	5.71
δ_1=0.8	16.53	17.65	18.74	20.19	26.36	34.54	37.29	39.59	43.40	26.98	5.69
δ_1=0.9	14.89	16.13	17.35	18.73	24.99	32.87	35.75	38.34	41.59	25.56	5.70

$G_0 = 5$

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	19.75	21.16	22.44	23.87	30.62	39.20	41.83	44.53	47.60	31.19	6.03
δ_1=0.2	20.62	22.16	23.55	25.20	31.93	40.69	43.75	46.25	50.32	32.52	6.17
δ_1=0.3	21.36	22.53	23.85	25.51	32.21	40.81	43.50	45.92	49.35	32.80	6.03
δ_1=0.4	21.21	22.70	24.17	25.79	32.41	41.21	44.10	46.77	49.55	33.02	6.09
δ_1=0.5	21.28	22.76	24.09	25.76	32.35	41.06	43.63	46.42	49.15	32.98	6.05
δ_1=0.6	21.66	22.96	24.15	25.79	32.46	41.03	44.02	46.39	49.36	33.08	6.01
δ_1=0.7	21.25	22.58	23.89	25.52	32.29	40.87	43.75	46.42	50.30	32.90	6.13
δ_1=0.8	20.80	22.41	23.76	25.41	31.93	40.52	43.34	45.70	48.80	32.52	6.00
δ_1=0.9	19.77	20.97	22.30	23.90	30.64	39.06	42.12	44.81	48.11	31.19	6.08

Table 1.2

$q = 2$: two linear kinks case

$$z_t^* = \begin{bmatrix} 1 \\ DU_1(t) \\ DU_2(t) \\ t \\ DT_1(t) \\ DT_2(t) \end{bmatrix}, \quad z_{1t}^* = \begin{bmatrix} 1 \\ DU_1(t) \\ DU_2(t) \end{bmatrix}$$

G_0 = dimension of Brownian motions
$\delta_1,\ \delta_2$ = location of changing point $(0 < \delta_1, \delta_2 < 1)$

(i) trace statistic

$G_0 = 1$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	3.00	3.67	4.33	5.19	9.63	16.54	18.97	21.06	24.22	10.40	4.59

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	3.57	4.36	5.13	6.23	11.11	18.44	20.75	23.26	25.79	11.84	4.88
δ_1=0.2	3.80	4.44	5.19	6.21	11.18	18.63	21.11	23.36	26.25	11.90	4.95

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	4.42	5.12	6.06	7.15	12.30	19.51	22.29	24.73	28.15	12.96	5.02
δ_1=0.2	4.48	5.38	6.27	7.43	12.68	20.34	22.81	25.10	27.35	13.40	5.10
δ_1=0.3	4.50	5.27	6.06	7.15	11.99	19.33	21.79	24.10	26.83	12.79	4.90

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	5.10	5.76	6.60	7.66	12.77	20.10	22.50	25.34	28.04	13.44	4.97
δ_1=0.2	5.59	6.39	7.28	8.44	13.88	20.82	23.62	25.74	28.93	14.37	5.00
δ_1=0.3	5.59	6.44	7.35	8.46	13.80	21.08	23.38	25.76	28.50	14.38	5.03
δ_1=0.4	5.14	5.93	6.83	7.82	12.73	20.01	22.19	24.30	27.01	13.40	4.77

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	4.90	5.92	6.78	7.88	12.71	19.79	22.27	24.61	27.14	13.33	4.81
δ_1=0.2	6.10	6.88	7.65	8.90	14.20	21.23	23.64	25.82	27.91	14.69	4.86
δ_1=0.3	6.49	7.43	8.39	9.49	14.60	21.84	24.11	26.83	29.81	15.25	4.93
δ_1=0.4	6.01	6.75	7.60	8.77	14.12	21.39	24.20	26.33	29.40	14.74	5.04
δ_1=0.5	5.11	5.82	6.73	7.74	12.70	19.72	22.26	24.53	27.81	13.38	4.83

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	4.27	5.13	5.99	7.08	12.22	19.49	22.00	24.12	26.96	12.84	4.95
δ_1=0.2	5.62	6.40	7.31	8.58	13.80	20.90	23.46	26.11	28.99	14.41	4.97
δ_1=0.3	6.39	7.50	8.43	9.49	14.76	21.90	24.45	26.99	30.27	15.35	4.98
δ_1=0.4	6.64	7.48	8.37	9.51	14.59	21.84	24.33	26.34	29.20	15.21	4.88
δ_1=0.5	5.31	6.28	7.22	8.40	13.69	21.19	23.53	25.87	28.59	14.35	5.06
δ_1=0.6	4.41	5.21	6.10	7.19	12.18	19.43	21.71	24.00	26.90	12.84	4.85

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	3.54	4.24	5.17	6.24	11.19	18.50	20.88	23.28	25.74	11.86	4.87
δ_1=0.2	4.47	5.42	6.26	7.39	12.59	20.49	23.24	25.42	27.82	13.33	5.14
δ_1=0.3	5.37	6.31	7.20	8.40	13.74	21.26	23.71	25.94	28.68	14.41	5.07
δ_1=0.4	5.83	6.81	7.81	8.91	14.27	21.27	23.43	25.45	28.36	14.75	4.84
δ_1=0.5	5.50	6.42	7.29	8.42	13.81	21.52	24.19	26.42	29.06	14.52	5.20
δ_1=0.6	4.42	5.27	6.17	7.25	12.68	20.19	22.62	24.78	27.40	13.26	5.05
δ_1=0.7	3.65	4.34	5.13	6.26	11.16	18.31	20.94	22.91	25.94	11.85	4.87

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	3.05	3.65	4.26	5.22	9.70	16.41	19.13	21.19	24.09	10.39	4.56
δ_1=0.2	3.67	4.49	5.20	6.18	11.06	18.32	20.86	23.32	26.15	11.78	4.84
δ_1=0.3	4.41	5.24	6.09	7.16	12.23	19.54	22.16	24.75	27.87	12.94	5.01
δ_1=0.4	5.04	5.91	6.67	7.78	12.70	19.97	22.26	24.79	27.72	13.39	4.86
δ_1=0.5	5.05	5.84	6.53	7.61	12.79	19.88	22.46	24.70	27.87	13.38	4.92
δ_1=0.6	4.47	5.23	6.06	7.04	12.09	19.48	22.21	24.71	28.15	12.85	5.03
δ_1=0.7	3.76	4.50	5.24	6.24	11.06	18.21	20.71	22.96	25.92	11.77	4.83
δ_1=0.8	3.13	3.77	4.42	5.25	9.61	16.72	19.24	21.37	24.22	10.42	4.62

$G_0 = 2$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	12.25	14.04	15.16	17.07	24.53	34.42	37.29	40.51	44.27	25.25	6.81

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	14.46	15.91	17.43	19.42	27.13	37.12	40.52	43.88	47.96	27.89	7.04
δ_1=0.2	14.71	16.25	17.68	19.52	27.29	37.39	40.43	43.70	47.58	27.94	7.06

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	16.09	17.69	19.22	21.02	28.94	39.50	42.86	45.99	49.42	29.73	7.29
δ_1=0.2	16.67	18.12	19.74	21.59	29.87	39.94	43.54	46.78	50.08	30.52	7.27
δ_1=0.3	15.87	17.36	19.00	20.80	29.02	39.46	42.98	45.93	50.08	29.72	7.33

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	16.81	18.43	19.83	21.82	29.74	39.50	42.41	45.15	48.72	30.30	6.90
δ_1=0.2	18.15	19.91	21.56	23.36	31.79	42.14	45.47	48.55	53.16	32.37	7.42
δ_1=0.3	18.53	20.02	21.67	23.55	32.05	42.51	45.57	48.82	53.06	32.66	7.39
δ_1=0.4	16.59	18.32	19.73	21.67	29.88	39.94	43.14	46.51	50.06	30.47	7.22

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	16.17	18.17	19.77	21.85	30.05	40.47	43.50	46.52	51.06	30.68	7.34
δ_1=0.2	18.48	20.44	22.19	24.39	32.57	43.30	46.32	48.80	53.90	33.25	7.43
δ_1=0.3	19.68	21.39	23.07	25.10	33.51	44.13	47.46	50.35	54.11	34.09	7.46
δ_1=0.4	19.04	20.96	22.22	24.04	32.51	43.06	46.50	49.75	52.83	33.18	7.50
δ_1=0.5	16.74	18.64	20.12	21.93	29.96	39.64	43.03	46.28	49.98	30.61	7.08

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	15.93	17.13	18.83	20.98	29.08	39.30	42.50	45.41	49.01	29.68	7.18
δ_1=0.2	18.50	20.26	21.89	23.88	31.99	42.29	45.61	48.95	52.65	32.63	7.26
δ_1=0.3	19.97	21.62	23.21	25.10	33.34	43.87	47.30	50.28	54.02	34.05	7.41
δ_1=0.4	19.78	21.35	23.08	25.24	33.52	43.80	47.35	50.84	55.15	34.13	7.42
δ_1=0.5	18.14	19.98	21.67	23.69	31.95	42.68	46.08	48.65	52.13	32.65	7.43
δ_1=0.6	16.08	17.71	19.04	20.95	29.04	39.78	43.35	46.50	50.57	29.87	7.40

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	14.48	16.09	17.76	19.53	27.24	37.36	40.73	43.35	47.51	27.98	7.03
δ_1=0.2	16.75	18.34	19.76	21.72	29.97	40.35	44.00	47.17	51.29	30.68	7.42
δ_1=0.3	17.76	19.55	21.35	23.47	31.87	42.42	45.86	48.68	52.56	32.54	7.55
δ_1=0.4	18.94	20.63	22.16	24.18	32.81	42.71	45.73	48.67	52.12	33.22	7.28
δ_1=0.5	18.59	20.06	21.77	23.79	32.09	42.16	45.48	48.74	52.37	32.63	7.24
δ_1=0.6	16.43	18.78	20.18	22.01	29.97	40.62	43.40	46.13	49.20	30.68	7.14
δ_1=0.7	13.99	15.54	17.21	19.19	26.97	37.18	40.19	43.51	47.30	27.73	7.12

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	12.93	14.19	15.52	17.19	24.45	33.84	37.04	39.70	43.17	25.15	6.60
δ_1=0.2	14.47	16.01	17.50	19.29	26.98	37.03	40.52	43.53	47.14	27.74	7.10
δ_1=0.3	15.70	17.35	18.97	20.88	28.88	39.45	42.64	45.31	49.38	29.63	7.26
δ_1=0.4	16.66	18.58	19.87	21.92	29.80	39.82	43.20	46.36	49.90	30.45	7.11
δ_1=0.5	16.07	17.93	19.71	21.62	30.05	40.22	43.39	46.54	49.99	30.52	7.31
δ_1=0.6	15.72	17.41	19.14	20.96	28.67	38.98	42.17	45.22	49.07	29.44	7.13
δ_1=0.7	14.70	16.00	17.27	18.91	26.78	37.19	40.44	43.41	47.28	27.56	7.12
δ_1=0.8	12.33	13.95	15.39	17.25	24.66	34.41	37.77	40.72	44.35	25.39	6.86

$G_0 = 3$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	26.91	28.93	30.94	33.34	43.62	55.96	60.02	63.15	66.92	44.29	8.80

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	29.88	32.36	34.52	36.95	47.13	59.77	63.65	67.46	71.10	47.88	8.93
δ_1=0.2	30.20	32.38	34.57	37.19	47.76	60.39	64.32	68.29	72.57	48.29	9.04

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	32.46	34.60	36.78	39.19	49.63	62.81	66.64	70.00	74.81	50.43	9.13
δ_1=0.2	33.59	35.69	38.10	40.83	51.58	63.99	68.53	72.08	76.11	52.11	9.15
δ_1=0.3	32.51	34.75	36.82	39.34	49.64	62.32	66.80	70.32	75.21	50.47	9.03

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	32.95	35.28	37.38	40.09	50.70	63.38	68.11	72.03	75.95	51.38	9.18
δ_1=0.2	35.68	38.07	40.40	42.95	54.18	67.01	70.96	74.22	78.69	54.69	9.31
δ_1=0.3	35.98	38.31	40.35	42.93	53.67	67.01	71.32	74.98	80.21	54.49	9.49
δ_1=0.4	33.42	36.01	38.08	40.49	50.90	64.02	67.84	71.45	74.87	51.72	9.19

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	33.35	35.55	37.80	40.33	50.66	63.30	67.10	70.90	75.39	51.30	9.03
δ_1=0.2	36.29	38.58	40.96	43.58	54.62	68.02	72.07	75.74	80.34	55.36	9.57
δ_1=0.3	37.30	40.06	42.11	45.08	55.87	68.76	72.69	76.45	80.49	56.46	9.30
δ_1=0.4	36.34	38.56	40.76	43.49	54.51	67.01	71.40	75.77	79.97	55.09	9.37
δ_1=0.5	33.08	35.15	37.70	40.41	50.88	64.27	67.87	71.74	75.81	51.76	9.32

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	32.41	34.24	36.47	39.29	49.62	62.43	66.33	69.52	74.09	50.37	9.05
δ_1=0.2	36.00	38.40	40.56	43.32	53.95	66.43	70.90	74.87	79.55	54.56	9.16
δ_1=0.3	37.22	39.57	41.90	44.66	55.67	68.56	72.64	76.25	80.29	56.34	9.38
δ_1=0.4	37.23	39.47	41.78	44.88	55.75	68.91	72.86	76.73	80.99	56.44	9.42
δ_1=0.5	35.78	38.73	40.77	43.41	54.08	66.87	70.91	74.55	77.94	54.64	9.12
δ_1=0.6	31.67	34.15	36.32	39.21	49.85	62.53	66.42	69.66	74.96	50.53	9.12

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	30.15	32.62	34.51	37.10	47.44	60.42	64.13	67.39	72.44	48.19	9.03
δ_1=0.2	33.91	36.30	38.28	40.88	51.77	64.90	69.03	72.18	76.44	52.43	9.29
δ_1=0.3	35.35	37.84	40.12	42.90	53.61	66.52	70.56	74.45	78.58	54.31	9.27
δ_1=0.4	36.34	38.89	41.02	43.84	54.68	67.82	72.01	75.34	80.11	55.35	9.41
δ_1=0.5	35.79	37.97	39.93	42.81	53.95	66.90	71.01	75.30	79.19	54.51	9.42
δ_1=0.6	33.98	36.14	38.54	41.23	51.72	64.42	68.58	72.28	76.99	52.46	9.23
δ_1=0.7	30.41	32.64	34.68	37.19	47.70	60.44	64.41	67.97	72.49	48.39	9.15

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	27.14	29.12	31.02	33.57	43.51	55.83	59.40	63.46	67.53	44.21	8.71
δ_1=0.2	29.84	32.57	34.62	37.12	47.57	60.27	64.14	67.75	71.54	48.16	9.00
δ_1=0.3	32.53	34.62	36.78	39.41	49.82	62.56	66.78	70.31	74.59	50.55	9.08
δ_1=0.4	32.76	35.29	37.27	40.07	50.61	63.46	67.46	70.26	74.81	51.26	9.08
δ_1=0.5	33.09	35.29	37.52	39.95	50.60	63.45	67.83	72.13	76.36	51.34	9.33
δ_1=0.6	31.56	34.25	36.59	39.44	49.67	61.99	65.92	68.91	73.07	50.26	8.94
δ_1=0.7	30.67	32.63	34.37	37.29	47.40	60.45	64.49	68.14	72.57	48.18	9.15
δ_1=0.8	26.88	29.13	30.96	33.45	43.58	55.95	60.26	63.26	67.96	44.27	8.86

$G_0 = 4$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	46.10	48.86	51.56	54.74	66.91	81.00	85.88	89.59	94.99	67.50	10.48

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	50.97	53.83	56.50	59.32	72.00	86.97	91.40	95.17	100.09	72.59	10.68
δ_1=0.2	50.95	53.60	56.28	59.31	71.94	87.03	92.01	96.38	102.77	72.71	10.91

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	52.52	55.92	58.47	62.10	74.58	90.21	94.85	98.66	104.62	75.45	10.99
δ_1=0.2	54.63	57.42	60.26	63.98	77.19	92.59	97.14	100.80	105.53	77.78	11.09
δ_1=0.3	52.73	55.50	57.98	61.34	74.68	89.08	93.62	97.90	103.04	75.01	10.78

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	53.54	56.58	59.17	62.26	75.80	90.37	95.19	99.59	103.57	76.18	10.96
δ_1=0.2	57.11	59.66	62.72	66.23	79.27	94.85	99.35	103.56	108.47	80.07	11.07
δ_1=0.3	56.46	59.75	62.67	66.15	79.76	95.43	100.49	104.64	108.91	80.34	11.40
δ_1=0.4	53.08	56.50	59.34	62.46	75.20	90.32	95.17	99.73	105.49	75.99	10.98

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	53.57	56.55	58.67	62.18	75.31	90.00	94.28	98.23	102.59	75.79	10.78
δ_1=0.2	57.28	60.04	62.88	66.40	80.10	95.19	99.76	104.18	110.27	80.57	11.28
δ_1=0.3	58.90	61.91	64.89	68.20	81.40	96.27	100.77	105.14	110.19	81.96	11.09
δ_1=0.4	57.14	60.42	63.35	66.78	80.26	95.78	100.91	105.09	109.70	80.93	11.37
δ_1=0.5	53.68	56.50	59.26	62.58	75.67	90.39	94.77	98.86	104.75	76.33	10.96

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	53.09	55.83	58.70	61.92	74.43	89.18	93.57	97.91	102.94	75.14	10.76
δ_1=0.2	57.31	60.34	62.83	66.29	79.76	94.92	99.22	102.96	107.78	80.25	11.10
δ_1=0.3	58.92	61.96	64.76	68.12	81.80	97.01	101.45	106.09	111.20	82.26	11.26
δ_1=0.4	58.86	62.27	64.75	67.93	81.35	96.68	101.09	105.11	110.08	81.90	11.13
δ_1=0.5	56.66	59.76	62.75	66.23	79.32	94.49	99.17	103.89	108.78	79.96	11.13
δ_1=0.6	52.70	55.67	58.39	61.65	74.52	89.79	94.16	97.73	102.17	75.21	10.90

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	50.33	53.13	56.02	59.08	71.76	86.94	91.47	95.95	100.77	72.56	10.90
δ_1=0.2	55.43	58.49	60.94	64.06	77.00	92.32	96.83	100.83	105.98	77.74	10.97
δ_1=0.3	57.52	60.73	63.04	66.33	79.29	94.97	99.29	103.48	108.88	80.06	11.08
δ_1=0.4	57.29	60.70	63.42	66.92	80.33	95.00	99.41	104.26	109.02	80.75	11.06
δ_1=0.5	56.77	60.22	62.98	66.26	79.36	94.40	99.27	103.58	108.88	79.99	11.04
δ_1=0.6	54.72	57.48	60.28	64.20	77.13	91.69	96.02	100.26	106.04	77.62	10.88
δ_1=0.7	50.69	53.57	55.97	59.57	72.03	87.00	92.05	96.09	101.87	72.81	10.85

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	45.97	48.45	51.26	54.66	66.42	81.03	85.24	89.89	94.49	67.17	10.42
δ_1=0.2	50.16	53.38	56.14	59.37	71.93	86.32	90.77	94.87	98.83	72.43	10.59
δ_1=0.3	52.00	54.94	57.65	60.70	74.31	88.96	93.31	97.70	102.38	74.68	10.99
δ_1=0.4	52.80	56.31	59.22	62.46	75.51	90.65	94.89	99.16	103.83	76.07	10.99
δ_1=0.5	52.57	56.31	58.69	62.30	75.37	90.85	95.35	100.27	105.14	76.07	11.17
δ_1=0.6	52.32	55.52	58.04	61.41	74.46	89.45	93.95	98.24	103.68	75.00	10.98
δ_1=0.7	50.80	53.79	56.59	59.23	71.67	86.94	91.36	95.20	101.37	72.60	10.79
δ_1=0.8	45.70	49.27	51.76	54.93	67.11	81.12	85.68	89.68	94.15	67.63	10.35

$G_0 = 5$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	69.78	72.44	75.81	79.52	94.16	111.25	116.33	120.52	125.46	94.90	12.35

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	75.27	78.92	81.87	85.29	100.18	117.13	122.26	126.98	132.39	100.84	12.39
δ_1=0.2	74.75	78.42	81.71	85.26	99.84	116.59	121.55	126.21	132.49	100.55	12.31

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	76.37	80.21	83.91	87.83	102.95	120.20	125.87	129.84	135.03	103.63	12.71
δ_1=0.2	80.01	83.54	86.96	90.80	106.11	123.75	129.43	134.02	141.58	106.76	13.06
δ_1=0.3	76.56	80.63	83.99	87.89	102.78	120.18	125.53	129.55	134.53	103.48	12.59

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	77.46	81.93	84.98	88.93	104.04	121.10	126.66	132.08	137.85	104.75	12.74
δ_1=0.2	82.68	85.92	89.40	93.52	108.84	126.59	132.05	136.28	141.98	109.47	12.97
δ_1=0.3	82.18	85.89	89.51	93.27	108.80	125.67	131.52	135.87	142.49	109.29	12.80
δ_1=0.4	77.22	80.79	84.32	88.25	104.27	121.54	126.95	131.36	137.54	104.79	12.96

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	77.81	81.52	84.78	88.76	103.86	121.13	126.56	130.63	136.29	104.55	12.72
δ_1=0.2	82.74	86.41	89.36	93.33	109.17	126.66	131.81	136.14	141.35	109.76	12.87
δ_1=0.3	83.44	87.69	90.85	94.82	111.03	128.53	133.99	139.53	144.84	111.44	13.16
δ_1=0.4	82.27	86.37	89.90	93.86	109.20	126.84	132.25	136.69	142.38	109.89	12.91
δ_1=0.5	77.63	81.21	84.37	88.52	103.89	121.50	127.29	131.81	137.20	104.58	12.93

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	77.56	81.10	84.35	87.90	102.72	119.79	124.73	129.21	134.60	103.37	12.41
δ_1=0.2	82.31	85.63	88.85	92.97	108.33	126.76	132.45	136.84	141.40	109.25	13.01
δ_1=0.3	84.04	88.09	91.21	95.38	110.76	128.52	133.44	138.80	144.79	111.39	12.94
δ_1=0.4	83.77	87.97	91.28	95.37	111.52	128.77	133.58	138.45	143.49	111.87	12.99
δ_1=0.5	82.11	85.87	89.10	93.12	108.67	126.03	131.13	135.85	141.06	109.14	12.87
δ_1=0.6	78.07	81.11	84.36	88.02	102.82	120.51	126.19	131.03	137.46	103.75	12.70

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	74.85	78.05	81.07	84.87	100.06	116.96	122.25	126.88	132.83	100.65	12.55
δ_1=0.2	80.08	84.04	87.15	90.89	106.30	123.27	128.54	133.78	139.41	106.89	12.66
δ_1=0.3	82.49	85.90	89.20	93.51	108.71	126.52	131.89	136.96	142.35	109.38	12.99
δ_1=0.4	83.26	86.85	89.86	93.45	109.38	127.14	132.95	137.33	143.27	110.05	13.08
δ_1=0.5	81.87	85.55	89.53	93.34	108.67	126.35	131.67	135.84	142.80	109.38	12.90
δ_1=0.6	79.86	83.81	87.19	91.11	106.20	123.23	128.32	133.61	139.13	106.86	12.63
δ_1=0.7	73.95	77.66	81.22	85.26	100.11	117.14	122.64	127.17	132.08	100.73	12.56

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	68.73	72.64	75.95	79.66	94.11	110.31	115.30	119.38	125.62	94.65	12.05
δ_1=0.2	74.47	78.37	81.35	84.82	99.76	116.76	121.80	126.66	132.72	100.48	12.47
δ_1=0.3	77.64	80.21	83.87	88.09	103.10	120.17	124.85	129.28	134.87	103.67	12.55
δ_1=0.4	77.73	81.42	84.74	88.60	104.03	121.61	126.25	130.91	136.46	104.61	12.80
δ_1=0.5	77.37	81.66	85.12	88.83	103.54	121.12	126.40	131.66	136.72	104.43	12.71
δ_1=0.6	76.03	80.36	83.59	87.53	102.45	119.48	124.97	129.48	135.15	103.15	12.57
δ_1=0.7	73.78	78.22	80.94	84.96	100.09	116.89	122.01	126.91	132.56	100.62	12.45
δ_1=0.8	68.95	72.58	75.34	79.66	93.94	110.89	116.31	120.77	125.74	94.74	12.26

(ii) max statistic

$G_0 = 2$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	7.68	8.83	9.67	10.97	16.64	24.43	27.16	29.35	32.53	17.28	5.37

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	9.05	10.12	11.14	12.49	18.13	26.04	28.70	31.10	33.62	18.81	5.38
δ_1=0.2	9.03	10.13	11.25	12.62	18.30	26.36	28.83	30.77	33.59	18.95	5.37

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	9.91	10.93	12.05	13.25	18.92	27.06	29.92	32.09	35.03	19.70	5.50
δ_1=0.2	10.24	11.23	12.28	13.75	19.83	27.45	30.09	32.42	35.28	20.29	5.42
δ_1=0.3	9.66	10.79	11.83	13.20	19.09	26.98	29.77	32.13	35.13	19.69	5.47

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	10.29	11.32	12.28	13.73	19.13	26.60	29.02	30.95	34.65	19.78	5.17
δ_1=0.2	11.19	12.16	13.28	14.53	20.36	28.26	30.96	33.30	36.30	20.99	5.43
δ_1=0.3	11.48	12.35	13.35	14.67	20.68	28.59	31.07	34.03	36.84	21.26	5.53
δ_1=0.4	9.82	11.14	12.23	13.59	19.28	27.06	29.80	32.28	34.62	19.94	5.41

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	9.93	11.06	12.19	13.63	19.40	27.35	30.31	32.86	35.77	20.05	5.53
δ_1=0.2	11.28	12.48	13.50	14.98	20.85	28.57	31.21	33.51	36.72	21.43	5.45
δ_1=0.3	11.51	12.89	14.08	15.30	21.14	29.17	31.79	34.23	37.49	21.79	5.53
δ_1=0.4	11.36	12.48	13.43	14.89	20.77	28.54	31.42	34.00	36.78	21.36	5.47
δ_1=0.5	10.43	11.32	12.40	13.80	19.41	26.98	29.52	32.22	35.29	20.03	5.30

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	9.67	10.70	11.86	13.24	19.05	26.87	29.43	31.95	35.27	19.69	5.43
δ_1=0.2	11.26	12.41	13.57	14.78	20.67	28.28	30.95	33.49	36.02	21.21	5.34
δ_1=0.3	11.92	12.88	13.92	15.33	21.18	29.00	31.81	34.63	37.74	21.81	5.53
δ_1=0.4	11.81	12.93	14.01	15.39	21.25	29.02	31.75	34.39	37.69	21.84	5.49
δ_1=0.5	11.00	12.24	13.35	14.71	20.64	28.46	31.40	33.66	37.07	21.22	5.49
δ_1=0.6	9.89	11.03	11.96	13.31	19.06	27.08	29.70	32.62	35.52	19.77	5.54

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	8.94	10.12	11.27	12.59	18.26	26.24	28.86	31.29	34.62	18.96	5.45
δ_1=0.2	10.30	11.37	12.40	13.84	19.81	27.61	30.38	32.98	36.32	20.37	5.53
δ_1=0.3	10.89	12.13	13.23	14.55	20.55	28.57	31.15	33.67	36.86	21.18	5.55
δ_1=0.4	11.32	12.34	13.50	14.91	21.01	28.43	30.64	33.12	36.24	21.38	5.34
δ_1=0.5	11.06	12.31	13.37	14.78	20.63	28.49	30.87	33.38	36.01	21.19	5.41
δ_1=0.6	10.41	11.57	12.64	14.01	19.76	27.54	30.13	32.41	35.63	20.39	5.35
δ_1=0.7	8.66	9.77	11.05	12.37	18.17	26.00	28.54	30.95	34.07	18.75	5.40

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	7.96	8.86	9.85	11.06	16.65	24.31	26.83	29.08	31.91	17.22	5.17
δ_1=0.2	9.12	10.05	11.19	12.52	18.09	25.99	28.48	30.88	34.36	18.75	5.39
δ_1=0.3	9.71	10.91	11.95	13.22	18.96	27.00	29.49	32.08	35.17	19.66	5.47
δ_1=0.4	10.06	11.31	12.30	13.62	19.20	26.79	29.62	32.42	35.11	19.89	5.34
δ_1=0.5	9.73	10.87	12.10	13.59	19.32	27.47	29.98	32.60	35.54	20.01	5.53
δ_1=0.6	9.80	10.82	11.85	13.15	18.97	26.87	29.43	31.90	35.34	19.60	5.43
δ_1=0.7	9.05	9.99	11.09	12.25	17.93	26.03	28.56	31.08	33.97	18.62	5.42
δ_1=0.8	7.96	8.84	9.85	11.20	16.69	24.49	27.30	29.58	32.91	17.38	5.34

$G_0 = 3$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	12.65	13.82	15.11	16.60	22.97	31.45	34.20	36.54	39.34	23.57	5.86

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	14.20	15.38	16.56	17.96	24.32	32.62	35.13	38.33	41.14	24.96	5.83
δ_1=0.2	14.25	15.51	16.67	18.17	24.59	33.30	36.22	38.86	41.89	25.25	5.96

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	15.08	16.39	17.48	18.94	25.27	33.70	36.83	38.90	41.94	25.84	5.85
δ_1=0.2	15.63	16.74	18.12	19.59	25.95	34.03	37.03	39.96	43.24	26.51	5.83
δ_1=0.3	14.76	16.02	17.51	19.08	25.17	33.21	35.92	38.67	42.60	25.73	5.74

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	14.80	16.29	17.46	18.90	25.38	33.66	36.56	39.29	42.25	25.94	5.86
δ_1=0.2	16.20	17.43	18.61	20.05	26.74	35.02	37.62	40.06	42.64	27.22	5.79
δ_1=0.3	16.49	17.79	18.85	20.15	26.52	35.10	38.20	40.46	43.27	27.20	5.96
δ_1=0.4	15.31	16.58	17.77	19.19	25.42	33.86	36.54	39.05	41.67	26.05	5.76

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	15.42	16.68	17.81	19.19	25.25	33.49	36.14	38.65	42.50	25.90	5.70
δ_1=0.2	16.12	17.41	18.75	20.25	26.75	35.35	37.87	40.07	43.66	27.34	5.90
δ_1=0.3	16.57	17.81	19.23	20.73	27.01	35.40	38.05	40.58	43.77	27.61	5.81
δ_1=0.4	16.50	17.83	18.99	20.31	26.60	34.83	37.77	40.49	43.74	27.27	5.76
δ_1=0.5	15.11	16.39	17.79	19.25	25.48	33.89	36.67	39.39	42.75	26.11	5.83

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	14.94	16.22	17.36	18.82	25.20	33.16	35.68	37.96	40.88	25.67	5.65
δ_1=0.2	16.46	17.74	18.89	20.30	26.62	34.65	37.63	40.15	43.75	27.17	5.73
δ_1=0.3	16.23	17.74	18.89	20.58	27.02	35.13	37.81	40.20	42.93	27.50	5.78
δ_1=0.4	16.54	17.87	19.10	20.76	27.13	35.21	37.74	39.86	42.87	27.60	5.65
δ_1=0.5	16.37	17.60	18.82	20.29	26.58	34.88	37.67	39.97	42.68	27.22	5.78
δ_1=0.6	14.83	16.21	17.28	18.74	25.18	33.79	36.70	38.98	42.26	25.83	5.92

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	14.27	15.60	16.74	18.19	24.45	33.06	35.82	38.16	40.34	25.14	5.79
δ_1=0.2	15.80	17.06	18.08	19.55	26.06	34.70	37.68	40.55	43.62	26.73	5.98
δ_1=0.3	15.94	17.38	18.63	20.06	26.40	34.60	37.35	39.84	42.45	26.97	5.76
δ_1=0.4	16.39	17.72	18.84	20.33	26.72	35.15	38.20	40.67	43.74	27.35	5.91
δ_1=0.5	16.02	17.42	18.76	20.19	26.48	34.83	37.62	40.36	43.68	27.13	5.86
δ_1=0.6	15.85	17.05	18.29	19.75	26.19	34.18	36.94	39.52	43.10	26.72	5.80
δ_1=0.7	14.21	15.49	16.73	18.20	24.55	33.38	36.14	39.06	42.85	25.30	6.02

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	12.91	14.09	15.21	16.80	22.90	31.24	34.07	37.04	39.97	23.52	5.80
δ_1=0.2	14.42	15.52	16.78	18.29	24.56	32.97	35.55	38.38	41.68	25.20	5.83
δ_1=0.3	14.93	16.15	17.39	18.95	25.22	33.66	36.44	39.02	41.91	25.81	5.80
δ_1=0.4	14.86	16.29	17.49	18.99	25.32	33.71	36.36	38.58	41.48	25.91	5.80
δ_1=0.5	15.03	16.13	17.48	18.89	25.20	33.67	36.52	39.31	42.62	25.89	5.90
δ_1=0.6	14.77	16.09	17.34	18.77	25.08	33.37	36.24	39.08	42.70	25.72	5.86
δ_1=0.7	14.13	15.49	16.72	18.27	24.54	32.96	35.66	38.37	41.29	25.17	5.84
δ_1=0.8	13.00	14.12	15.27	16.72	23.00	31.25	34.00	36.76	40.21	23.58	5.79

$G_0 = 4$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	18.07	19.42	20.74	22.29	29.08	37.65	40.55	42.82	46.24	29.61	6.06

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	19.63	20.89	22.17	23.66	30.62	39.33	42.08	44.47	48.27	31.13	6.14
δ_1=0.2	19.12	20.82	22.16	23.83	30.44	39.47	42.74	45.37	49.17	31.14	6.27

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	20.06	21.52	22.74	24.44	31.10	39.93	42.94	45.59	49.79	31.72	6.22
δ_1=0.2	20.69	21.91	23.15	24.88	31.86	40.74	43.46	46.15	49.18	32.42	6.19
δ_1=0.3	19.89	21.35	22.53	24.10	30.89	39.53	42.46	44.67	48.15	31.45	6.05

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	19.97	21.37	22.58	24.21	31.06	39.71	42.60	45.21	48.50	31.64	6.15
δ_1=0.2	21.49	22.80	24.00	25.47	32.28	40.98	43.77	46.29	49.09	32.83	6.07
δ_1=0.3	21.02	22.46	23.85	25.44	32.34	41.41	44.14	46.39	49.82	32.93	6.27
δ_1=0.4	19.92	21.38	22.70	24.25	31.09	39.82	42.66	45.40	48.23	31.65	6.13

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	19.73	21.28	22.48	24.10	30.88	39.66	42.57	45.22	48.45	31.56	6.11
δ_1=0.2	20.83	22.49	23.82	25.40	32.18	40.80	43.74	46.29	50.08	32.74	6.13
δ_1=0.3	21.31	22.70	23.88	25.61	32.53	41.06	43.86	46.07	49.64	33.08	6.11
δ_1=0.4	20.91	22.43	23.86	25.40	32.27	41.13	44.08	46.69	50.02	32.92	6.24
δ_1=0.5	20.14	21.44	22.78	24.38	31.14	39.93	42.69	45.31	49.39	31.76	6.15

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	19.99	21.37	22.57	24.14	31.03	39.83	43.12	45.31	48.56	31.57	6.19
δ_1=0.2	21.28	22.43	23.96	25.55	32.26	41.01	43.82	46.31	49.56	32.90	6.15
δ_1=0.3	21.36	23.00	24.20	25.82	32.59	41.32	44.34	47.00	49.85	33.17	6.13
δ_1=0.4	21.40	22.90	24.12	25.66	32.43	41.16	43.86	46.55	50.02	32.96	6.07
δ_1=0.5	21.08	22.52	23.83	25.28	32.16	40.79	43.73	46.26	49.60	32.75	6.10
δ_1=0.6	19.71	21.15	22.52	24.22	31.04	39.88	42.86	45.32	48.32	31.64	6.22

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	19.35	20.68	22.07	23.61	30.28	39.44	42.58	45.13	47.78	31.08	6.26
δ_1=0.2	20.51	22.06	23.41	24.88	31.94	40.53	43.17	45.80	49.15	32.35	6.12
δ_1=0.3	21.24	22.70	24.00	25.47	32.13	41.06	43.83	46.55	50.01	32.78	6.16
δ_1=0.4	21.23	22.37	23.74	25.48	32.33	41.16	44.00	46.29	50.28	32.95	6.20
δ_1=0.5	21.02	22.50	23.90	25.55	32.06	40.68	43.48	46.18	49.53	32.74	6.06
δ_1=0.6	20.48	22.07	23.28	24.96	31.84	40.32	43.14	45.97	48.32	32.37	6.08
δ_1=0.7	19.49	20.99	22.20	23.80	30.55	39.17	42.12	44.84	47.67	31.14	6.08

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	17.79	19.39	20.52	22.13	28.83	37.53	40.30	42.85	46.52	29.42	6.11
δ_1=0.2	19.39	20.88	22.15	23.63	30.47	39.12	42.09	44.71	47.34	31.08	6.13
δ_1=0.3	19.67	21.12	22.34	23.97	30.75	39.47	42.32	45.26	47.98	31.35	6.12
δ_1=0.4	19.91	21.25	22.66	24.30	31.09	39.85	42.83	45.46	48.78	31.67	6.21
δ_1=0.5	19.44	21.16	22.44	24.22	31.08	40.10	43.21	45.84	48.65	31.72	6.29
δ_1=0.6	19.72	21.14	22.46	24.08	30.74	39.52	42.27	44.87	47.90	31.37	6.10
δ_1=0.7	19.54	21.02	22.28	23.81	30.54	39.34	42.64	44.99	48.43	31.19	6.20
δ_1=0.8	17.68	19.20	20.58	22.28	29.06	38.14	40.84	43.38	46.48	29.70	6.21

$G_0 = 5$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	22.91	24.63	25.96	27.71	34.93	44.00	47.28	49.99	52.92	35.46	6.48

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	24.72	25.96	27.35	29.06	36.11	45.59	48.71	51.61	54.94	36.84	6.57
δ_1=0.2	24.20	25.66	27.26	28.89	36.07	45.34	48.12	50.95	54.54	36.68	6.44

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	24.34	26.06	27.67	29.36	36.62	45.55	48.50	52.01	55.08	37.19	6.44
δ_1=0.2	25.34	26.86	28.35	29.99	37.11	46.68	49.67	52.49	55.81	37.82	6.55
δ_1=0.3	24.69	26.07	27.66	29.44	36.62	45.84	48.84	51.43	54.14	37.18	6.47

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	24.76	26.49	27.79	29.55	36.66	45.94	48.93	52.00	55.48	37.28	6.49
δ_1=0.2	25.93	27.35	28.78	30.45	37.81	47.03	49.81	52.76	56.00	38.35	6.53
δ_1=0.3	26.18	27.52	28.57	30.34	37.47	46.67	49.55	52.07	55.35	38.11	6.43
δ_1=0.4	24.95	26.23	27.60	29.41	36.71	45.84	48.80	51.61	55.11	37.31	6.50

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	24.73	26.36	27.83	29.60	36.75	45.85	48.78	51.66	55.27	37.36	6.48
δ_1=0.2	25.56	27.32	28.69	30.46	37.75	46.72	49.90	52.87	56.65	38.30	6.54
δ_1=0.3	26.08	27.67	28.96	30.73	38.08	47.12	50.03	53.04	56.55	38.60	6.45
δ_1=0.4	25.66	27.27	28.80	30.50	37.76	46.79	49.82	53.00	56.03	38.29	6.44
δ_1=0.5	24.90	26.45	27.63	29.37	36.57	45.95	48.98	51.69	55.69	37.26	6.58

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	24.76	26.16	27.71	29.40	36.41	45.51	48.77	51.49	54.87	37.08	6.42
δ_1=0.2	25.72	27.18	28.57	30.30	37.68	46.85	49.90	52.85	56.70	38.23	6.57
δ_1=0.3	26.41	27.76	29.17	30.86	37.99	47.26	49.97	52.48	55.90	38.54	6.44
δ_1=0.4	25.68	27.47	29.12	30.79	38.15	47.24	50.14	52.43	55.80	38.66	6.46
δ_1=0.5	25.38	27.14	28.91	30.59	37.71	47.05	50.07	52.56	56.28	38.33	6.54
δ_1=0.6	24.87	26.25	27.86	29.51	36.69	46.02	49.36	52.15	55.83	37.34	6.58

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	24.35	25.85	27.25	28.81	36.31	45.28	48.30	50.95	54.59	36.74	6.50
δ_1=0.2	25.68	27.06	28.35	30.22	37.35	46.53	49.47	51.86	54.98	37.96	6.42
δ_1=0.3	25.59	27.26	28.73	30.51	37.73	46.69	49.88	52.31	55.38	38.24	6.42
δ_1=0.4	26.05	27.44	28.78	30.48	37.59	47.03	50.26	53.38	56.60	38.33	6.61
δ_1=0.5	25.61	27.19	28.72	30.41	37.82	46.79	49.77	52.28	55.85	38.33	6.50
δ_1=0.6	25.28	26.89	28.31	30.28	37.48	46.69	49.78	52.62	55.74	38.03	6.48
δ_1=0.7	24.16	25.77	27.29	29.01	36.19	45.27	48.18	51.47	54.73	36.77	6.47

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	23.21	24.68	26.10	27.67	34.83	44.00	47.01	49.76	53.59	35.44	6.46
δ_1=0.2	24.15	25.55	26.96	28.85	35.99	45.50	48.53	51.02	53.86	36.65	6.53
δ_1=0.3	25.00	26.37	27.92	29.48	36.62	45.76	48.88	52.01	56.03	37.23	6.49
δ_1=0.4	25.05	26.33	27.71	29.65	36.74	46.37	49.20	51.76	54.95	37.44	6.54
δ_1=0.5	24.90	26.46	27.76	29.56	36.81	45.93	49.13	52.01	55.29	37.34	6.48
δ_1=0.6	24.48	26.03	27.65	29.40	36.68	45.19	48.37	51.63	55.80	37.11	6.43
δ_1=0.7	24.59	26.04	27.20	28.95	36.13	45.40	48.26	51.11	53.98	36.76	6.46
δ_1=0.8	22.98	24.46	25.78	27.73	34.85	44.38	47.20	50.03	54.04	35.51	6.59

Table 2.1

one kink case with unknown kink point

$$z_t^* = \begin{bmatrix} 1 \\ DU_1(t) \\ t \\ DT_1(t) \end{bmatrix}, \quad z_{1t}^* = \begin{bmatrix} 1 \\ DU_1(t) \end{bmatrix}$$

G_0 = dimension of Brownian motions

(i) trace statistic

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
G_0=1	9.37	10.10	10.77	11.79	16.01	21.93	24.19	25.72	28.39	16.47	4.08
G_0=2	21.54	22.91	23.97	25.28	31.83	40.07	42.62	45.43	48.91	32.38	5.85
G_0=3	37.11	38.79	40.43	42.52	51.47	62.27	65.30	68.57	73.24	52.03	7.72
G_0=4	56.23	58.26	60.58	63.56	74.31	86.82	91.78	95.02	98.60	74.99	9.32
G_0=5	78.94	81.47	84.54	87.81	100.11	114.63	119.26	123.82	127.81	100.76	10.63

(ii) max statistic

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
G_0=2	15.06	16.01	16.57	17.74	22.42	28.92	31.40	33.50	36.97	23.04	4.55
G_0=3	20.22	21.21	22.19	23.10	28.74	35.91	38.62	40.90	44.15	29.28	5.10
G_0=4	25.10	26.55	27.67	29.04	34.84	42.59	45.19	47.32	50.43	35.37	5.43
G_0=5	30.19	31.57	32.81	34.27	40.24	48.26	50.58	52.78	55.84	40.82	5.56

Table 2.2

two kinks case with unknown kink points

$$z_t^* = \begin{bmatrix} 1 \\ DU_1(t) \\ DU_2(t) \\ t \\ DT_1(t) \\ DT_2(t) \end{bmatrix}, \quad z_{1t}^* = \begin{bmatrix} 1 \\ DU_1(t) \\ DU_2(t) \end{bmatrix}$$

G_0 = dimension of Brownian motions

(i) trace statistic

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
G_0=1	13.63	14.72	15.63	16.64	21.56	28.65	31.24	33.72	37.10	22.20	4.88
G_0=2	30.23	31.54	33.21	34.81	42.36	52.00	54.78	57.84	61.11	43.00	6.73
G_0=3	49.48	51.84	54.09	56.58	66.58	77.69	81.73	85.13	89.23	66.89	8.34
G_0=4	72.89	75.84	78.45	81.74	93.51	107.98	112.16	115.83	121.33	94.20	10.36
G_0=5	100.57	103.48	106.53	109.69	123.96	139.74	144.69	148.93	154.71	124.36	11.65

(ii) max statistic

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
G_0=2	20.33	21.01	22.18	23.41	28.94	36.25	38.64	40.49	42.87	29.46	5.06
G_0=3	25.52	26.91	28.18	29.68	35.44	43.07	45.65	48.28	51.89	36.03	5.42
G_0=4	31.06	32.57	33.71	35.21	41.93	49.85	52.34	54.61	56.83	42.25	5.67
G_0=5	36.75	37.94	39.01	40.80	47.39	55.90	58.79	61.24	64.47	47.88	5.94

Table 2.3

three kinks case with unknown kink points

$$z_t^* = \begin{bmatrix} 1 \\ DU_1(t) \\ DU_2(t) \\ DU_3(t) \\ t \\ DT_1(t) \\ DT_2(t) \\ DT_3(t) \end{bmatrix}, \quad z_{1t}^* = \begin{bmatrix} 1 \\ DU_1(t) \\ DU_2(t) \\ DU_3(t) \end{bmatrix}$$

G_0 = dimension of Brownian motions

(i) trace statistic

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
G_0=1	17.88	19.16	20.27	21.79	27.60	35.79	38.78	41.39	45.16	28.30	5.66
G_0=2	38.76	40.44	42.43	44.37	53.14	63.46	67.16	70.28	73.28	53.61	7.56
G_0=3	62.86	65.05	67.42	70.34	80.70	94.12	98.69	101.44	106.10	81.63	9.39
G_0=4	90.42	93.30	97.17	100.72	113.33	128.94	132.82	135.77	141.63	114.18	11.04
G_0=5	121.91	124.74	128.58	132.32	147.79	164.83	170.14	176.61	185.46	148.48	12.84

(ii) max statistic

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
G_0=2	25.60	26.73	27.79	29.50	35.29	43.10	45.61	48.74	51.31	35.92	5.56
G_0=3	32.46	33.79	34.66	35.63	42.39	50.53	53.35	55.99	59.69	42.96	5.78
G_0=4	37.29	38.67	40.24	41.73	49.06	58.11	60.34	62.40	66.17	49.54	6.26
G_0=5	42.73	44.41	45.91	47.86	55.10	63.70	66.53	70.32	72.96	55.46	6.38

Table 3

linear–curved–linear kink case

$$z_t^* = \begin{bmatrix} 1 \\ DU_1(t) \\ DT_1(t) - DT_2(t) \\ DT_2(t) \\ t \\ QT_1(t) - QT_2(t) \end{bmatrix}, \quad z_{1t}^* = \begin{bmatrix} 1 \\ DU_1(t) \\ DT_1(t) - DT_2(t) \end{bmatrix}$$

G_0 = dimension of Brownian motions
$\delta_1,\ \delta_2$ = location of changing point $(0 < \delta_1, \delta_2 < 1)$

(i) trace statistic

$G_0 = 1$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	3.04	3.41	4.25	4.95	9.25	16.10	18.80	21.34	23.55	10.19	4.60

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	3.64	4.50	5.22	6.08	11.26	18.47	21.56	24.01	27.00	11.93	5.01
δ_1=0.2	3.47	4.16	5.08	5.98	11.14	17.62	20.61	22.65	24.92	11.60	4.74

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	4.36	5.02	5.50	6.86	12.11	19.55	22.63	25.15	27.37	12.81	5.06
δ_1=0.2	4.30	5.10	6.05	7.07	12.53	20.71	22.94	25.41	28.73	13.31	5.31
δ_1=0.3	4.52	5.34	6.14	7.14	12.04	19.25	21.76	24.27	26.99	12.76	4.94

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	4.84	5.65	6.47	7.65	13.17	21.04	23.83	25.68	28.43	13.89	5.30
δ_1=0.2	5.44	6.39	7.22	8.45	13.77	21.34	24.30	27.60	31.17	14.63	5.33
δ_1=0.3	5.37	6.11	7.14	8.32	13.72	21.05	23.36	25.46	27.74	14.29	4.96
δ_1=0.4	4.60	5.31	6.17	7.18	12.33	19.42	21.50	24.28	27.28	12.94	4.90

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	6.10	6.74	7.32	8.23	13.29	20.84	23.18	25.48	28.07	14.11	5.03
δ_1=0.2	6.31	7.09	8.18	9.40	14.71	23.05	25.37	26.97	28.78	15.54	5.30
δ_1=0.3	6.69	7.43	8.26	9.59	15.09	23.06	25.30	27.35	31.10	15.68	5.24
δ_1=0.4	5.58	6.64	8.06	9.24	14.25	21.69	24.02	26.55	30.93	14.89	5.12
δ_1=0.5	5.27	5.86	6.52	7.65	12.38	19.46	22.40	24.31	26.16	13.12	4.74

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	5.41	6.42	7.53	8.70	13.62	22.08	24.78	27.53	30.00	14.56	5.33
δ_1=0.2	6.87	7.47	8.42	9.66	15.19	22.74	25.93	28.21	30.10	15.75	5.27
δ_1=0.3	7.10	7.87	8.84	10.34	15.57	23.41	25.71	27.12	30.03	16.24	5.08
δ_1=0.4	6.47	7.32	8.48	9.87	15.04	22.94	25.94	27.91	30.93	15.80	5.24
δ_1=0.5	5.67	6.38	7.10	8.58	13.99	21.67	23.79	25.67	29.83	14.57	5.15
δ_1=0.6	4.43	5.11	5.84	7.19	12.24	19.84	22.13	24.84	28.15	12.90	5.02

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	5.14	5.83	6.92	7.85	13.17	21.16	23.95	26.81	29.36	13.93	5.34
δ_1=0.2	6.01	7.23	7.93	9.35	14.38	21.97	24.33	27.38	29.92	15.18	5.12
δ_1=0.3	7.18	8.14	9.05	10.29	15.71	22.53	25.42	28.30	30.61	16.21	4.99
δ_1=0.4	7.22	7.90	8.56	9.66	14.98	22.75	25.45	27.18	30.93	15.72	5.13
δ_1=0.5	5.21	6.63	7.56	8.79	14.35	21.98	24.49	26.78	28.75	14.90	5.14
δ_1=0.6	4.16	4.93	6.07	7.20	12.83	20.71	23.58	25.68	29.33	13.49	5.36
δ_1=0.7	4.14	4.63	5.49	6.44	11.21	18.31	20.76	22.81	26.53	11.88	4.82

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	4.40	5.10	6.00	6.95	11.95	19.36	21.56	24.42	27.11	12.66	4.98
δ_1=0.2	4.94	5.72	6.89	8.06	13.59	20.97	23.50	25.78	28.14	14.15	5.07
δ_1=0.3	6.11	7.38	8.23	9.50	14.82	22.19	24.81	27.26	30.60	15.42	5.19
δ_1=0.4	6.50	7.06	8.25	9.39	14.80	22.25	25.12	27.07	30.58	15.36	5.21
δ_1=0.5	5.16	6.08	6.95	7.94	13.65	22.10	24.47	27.27	29.79	14.42	5.53
δ_1=0.6	4.22	5.03	5.90	7.19	12.55	20.84	23.37	25.46	28.76	13.45	5.40
δ_1=0.7	3.29	4.10	4.91	5.79	11.14	18.29	20.78	23.19	26.04	11.73	4.91
δ_1=0.8	2.96	3.59	4.19	5.27	9.69	16.65	18.98	21.54	23.54	10.43	4.53

$G_0 = 2$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	12.35	14.05	15.27	16.81	24.80	34.76	37.96	41.26	43.69	25.35	6.97

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	14.16	15.44	16.89	18.85	26.96	37.52	41.20	43.63	47.39	27.68	7.28
δ_1=0.2	13.74	16.06	17.19	19.01	26.67	36.33	39.62	42.26	45.03	27.31	6.93

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	16.41	17.61	19.21	21.37	29.40	39.12	43.68	47.08	51.07	30.04	7.39
δ_1=0.2	16.15	17.83	19.93	21.50	29.54	40.06	43.25	46.29	50.87	30.35	7.35
δ_1=0.3	15.37	17.13	18.61	21.00	28.72	39.12	42.27	45.35	48.68	29.47	7.22

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	17.72	19.46	20.78	22.50	31.28	42.61	45.41	48.23	51.73	32.05	7.59
δ_1=0.2	18.04	20.66	22.47	24.27	32.78	43.82	47.38	50.48	53.94	33.41	7.67
δ_1=0.3	17.83	19.98	21.46	23.45	31.66	42.66	47.15	50.21	54.11	32.50	7.74
δ_1=0.4	15.96	18.45	19.56	21.09	29.41	39.66	42.91	45.22	48.28	30.06	7.09

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	18.84	19.99	21.43	23.44	31.90	41.91	45.44	49.05	52.95	32.51	7.35
δ_1=0.2	19.42	21.72	23.01	25.12	33.57	43.95	47.51	52.31	55.71	34.31	7.49
δ_1=0.3	21.00	22.17	23.52	25.67	33.99	45.12	47.80	51.67	55.69	34.78	7.67
δ_1=0.4	18.41	20.14	21.29	23.51	32.19	44.01	47.04	50.90	54.48	32.99	7.84
δ_1=0.5	15.65	17.35	18.38	20.91	29.52	39.29	42.65	46.36	48.68	29.95	7.12

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	18.06	19.14	21.07	22.95	31.88	42.03	45.47	50.44	54.89	32.27	7.78
δ_1=0.2	19.33	21.23	23.67	25.97	34.64	45.08	48.15	51.49	55.07	35.10	7.58
δ_1=0.3	21.27	22.73	24.33	26.48	35.36	46.22	49.67	52.98	56.64	36.04	7.68
δ_1=0.4	19.64	22.32	24.04	25.83	34.30	44.86	47.28	51.16	54.24	34.80	7.41
δ_1=0.5	18.86	20.44	22.23	23.94	32.15	42.47	45.47	48.37	53.03	32.73	7.36
δ_1=0.6	15.04	16.89	19.06	21.19	29.02	40.05	42.96	45.97	48.39	29.85	7.35

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	17.68	19.08	20.89	22.91	30.71	40.98	44.47	48.16	53.90	31.48	7.38
δ_1=0.2	19.99	22.31	23.53	25.69	34.32	44.81	48.43	52.30	55.54	34.79	7.63
δ_1=0.3	20.37	22.33	24.44	26.11	34.66	45.58	48.39	51.57	56.14	35.36	7.59
δ_1=0.4	19.07	21.91	24.19	26.19	34.87	45.88	48.89	52.55	56.52	35.50	7.79
δ_1=0.5	19.46	21.25	22.93	24.58	33.37	43.01	47.36	50.13	54.84	33.63	7.40
δ_1=0.6	16.60	18.71	20.42	22.19	30.31	41.40	44.31	46.85	50.46	31.11	7.46
δ_1=0.7	14.80	16.07	17.42	19.28	27.06	36.51	39.80	42.92	44.92	27.51	6.74

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	15.89	17.25	18.84	20.90	29.16	39.26	42.39	45.25	50.26	29.68	7.25
δ_1=0.2	18.11	20.02	21.38	23.35	31.63	42.04	45.88	49.29	52.29	32.33	7.32
δ_1=0.3	19.52	21.18	23.10	25.26	33.26	43.27	46.52	49.87	53.84	33.83	7.21
δ_1=0.4	19.51	21.79	23.03	25.20	33.07	44.35	47.47	50.21	54.25	34.00	7.39
δ_1=0.5	18.37	20.29	22.25	24.21	32.63	42.27	45.93	49.36	52.42	33.09	7.28
δ_1=0.6	16.62	18.72	20.60	22.03	30.25	40.36	42.80	46.02	51.05	30.80	7.16
δ_1=0.7	15.11	16.46	17.73	19.48	27.36	37.94	41.01	43.47	47.86	28.17	7.17
δ_1=0.8	12.64	13.71	15.27	17.31	24.74	34.63	37.80	40.21	43.99	25.37	6.83

$G_0 = 3$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	27.13	28.71	30.96	33.47	43.22	54.09	57.95	62.13	68.81	43.78	8.45

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	30.81	33.03	34.89	37.82	47.27	59.45	64.31	69.40	73.45	48.31	8.86
δ_1=0.2	30.11	33.15	34.65	37.19	46.99	59.44	62.71	64.95	69.81	47.80	8.72

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	32.28	33.90	35.79	39.34	50.17	63.74	68.87	72.52	77.49	51.14	9.76
δ_1=0.2	32.25	35.23	38.22	40.84	51.17	63.59	68.20	72.70	77.26	52.03	9.34
δ_1=0.3	31.48	33.27	36.25	39.21	49.70	61.90	66.66	70.54	73.73	50.35	9.08

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	33.77	36.19	38.65	41.18	53.10	66.18	69.82	72.82	77.82	53.42	9.60
δ_1=0.2	35.81	38.50	40.69	43.85	54.62	67.16	70.99	75.23	79.53	55.12	9.27
δ_1=0.3	34.73	37.09	39.34	42.21	54.06	66.98	70.25	74.37	81.40	54.56	9.65
δ_1=0.4	32.83	34.58	37.73	40.21	50.75	63.74	67.19	70.74	73.74	51.27	9.10

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	35.37	38.21	40.01	42.60	53.66	67.05	70.73	74.29	80.06	54.41	9.62
δ_1=0.2	36.72	40.79	42.90	45.13	56.54	70.12	74.74	78.48	81.45	57.39	9.71
δ_1=0.3	39.02	40.47	42.91	45.07	56.32	69.78	74.24	77.95	83.72	57.24	9.68
δ_1=0.4	35.49	38.63	40.62	43.80	54.96	67.90	71.83	75.94	81.36	55.29	9.56
δ_1=0.5	32.71	35.16	37.02	40.02	50.93	63.30	66.90	70.25	72.87	51.26	9.04

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	35.59	38.15	40.78	43.19	54.31	66.95	69.89	73.34	77.17	54.78	9.14
δ_1=0.2	38.19	40.68	43.65	46.75	57.71	69.73	73.96	78.02	82.06	58.07	9.23
δ_1=0.3	40.29	42.44	44.52	47.61	58.44	72.51	76.87	80.64	86.69	59.35	9.94
δ_1=0.4	38.10	40.63	42.72	46.25	57.34	69.88	74.30	78.33	82.53	57.74	9.47
δ_1=0.5	35.50	37.53	40.78	43.49	54.71	67.50	70.71	74.30	77.27	55.11	9.31
δ_1=0.6	31.25	34.07	36.59	39.15	49.72	62.40	66.03	70.11	73.80	50.06	9.04

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	34.79	37.14	39.55	42.00	52.23	65.87	69.68	72.83	76.20	53.06	9.21
δ_1=0.2	36.53	39.72	42.56	45.37	57.11	69.97	73.86	77.24	80.52	57.51	9.63
δ_1=0.3	39.85	41.90	44.40	47.44	58.04	71.45	75.47	79.80	84.15	58.71	9.49
δ_1=0.4	39.09	41.70	43.93	46.73	58.23	71.66	76.18	79.75	83.55	58.75	9.74
δ_1=0.5	37.86	39.90	42.57	44.90	55.20	67.63	72.19	74.78	78.57	55.81	8.97
δ_1=0.6	34.96	36.66	39.04	41.30	52.00	64.26	68.05	71.33	74.98	52.39	8.82
δ_1=0.7	29.90	32.08	33.92	36.54	47.06	59.62	63.96	67.63	72.08	47.69	9.12

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	32.59	35.50	36.64	39.81	50.64	62.84	66.57	70.38	74.51	50.99	9.09
δ_1=0.2	36.10	38.73	40.95	43.52	53.99	67.13	70.77	74.41	80.98	54.77	9.25
δ_1=0.3	36.00	38.85	41.02	44.20	55.13	68.66	72.74	76.56	82.73	55.86	9.76
δ_1=0.4	36.70	40.04	41.87	44.40	56.04	68.92	73.44	76.51	80.04	56.30	9.50
δ_1=0.5	36.69	38.42	40.79	43.19	54.93	68.19	72.84	78.27	84.94	55.81	9.93
δ_1=0.6	32.77	35.40	37.86	40.70	52.20	64.56	67.96	73.42	77.58	52.60	9.36
δ_1=0.7	30.55	32.36	34.42	37.12	48.09	61.43	66.62	70.45	75.52	48.95	9.59
δ_1=0.8	27.88	29.49	31.90	34.26	43.47	56.43	59.74	63.23	69.14	44.54	8.73

$G_0 = 4$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	45.36	48.78	51.81	54.71	66.21	80.53	85.08	88.66	93.74	67.07	10.19

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	49.64	52.90	55.90	58.75	71.45	86.66	91.68	97.10	100.91	72.26	11.02
δ_1=0.2	50.05	52.17	54.75	58.18	70.42	83.80	89.43	94.57	97.93	71.09	10.24

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	53.04	55.47	58.31	62.74	75.24	90.60	94.96	101.44	106.97	76.28	11.35
δ_1=0.2	55.79	58.11	61.44	63.98	77.78	91.58	96.59	101.45	104.64	77.81	10.80
δ_1=0.3	53.33	56.23	58.63	62.28	74.77	88.92	93.70	98.10	103.95	75.26	10.59

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	55.60	58.79	61.57	65.21	77.81	93.72	97.84	100.32	106.90	78.40	10.94
δ_1=0.2	59.18	61.27	64.60	68.21	81.75	96.77	100.96	105.13	110.95	82.16	11.36
δ_1=0.3	55.80	59.81	62.25	65.63	79.58	95.31	99.83	104.40	109.86	80.14	11.51
δ_1=0.4	53.13	56.29	58.67	61.84	75.18	89.99	94.51	97.19	102.54	75.80	10.96

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	56.82	59.84	62.50	65.73	78.26	93.46	98.30	102.87	106.09	79.15	10.89
δ_1=0.2	59.52	61.93	66.19	69.98	83.83	99.17	104.30	108.05	112.49	84.14	11.58
δ_1=0.3	59.73	63.22	65.78	70.02	83.90	99.19	103.53	108.09	113.03	84.21	11.25
δ_1=0.4	56.81	60.26	63.75	67.13	79.65	94.75	100.09	104.95	111.58	80.66	11.22
δ_1=0.5	52.41	55.92	59.88	62.34	74.80	89.21	93.55	99.47	104.27	75.31	10.76

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	56.21	59.14	62.24	65.55	79.35	95.24	99.69	103.74	106.82	80.09	11.48
δ_1=0.2	59.88	63.85	66.41	70.63	84.30	99.88	104.38	107.84	111.95	85.01	11.46
δ_1=0.3	60.79	64.96	67.20	70.83	85.58	101.68	106.50	111.65	115.82	86.01	11.93
δ_1=0.4	59.66	63.36	65.31	69.37	84.51	99.68	106.89	110.84	115.18	84.87	12.13
δ_1=0.5	57.75	59.83	63.08	66.55	79.98	94.14	99.30	104.52	110.82	80.50	11.30
δ_1=0.6	50.56	53.80	56.02	60.66	73.70	89.54	93.13	97.92	104.26	74.55	11.23

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	55.88	59.35	61.85	64.56	78.48	95.08	99.05	103.86	108.42	79.41	11.31
δ_1=0.2	60.63	63.74	66.23	69.88	82.37	98.13	103.10	106.29	112.72	83.42	11.06
δ_1=0.3	60.74	64.29	68.12	71.08	84.73	100.33	105.57	109.85	114.78	85.19	11.44
δ_1=0.4	60.08	63.92	67.06	70.48	83.97	99.47	104.21	109.50	115.27	84.73	11.40
δ_1=0.5	58.78	62.01	64.29	68.45	82.53	99.03	104.41	109.64	112.61	83.16	12.00
δ_1=0.6	57.19	59.89	62.37	65.52	78.68	93.33	98.42	101.89	107.42	79.13	11.00
δ_1=0.7	49.96	52.72	55.38	58.96	71.32	86.83	90.45	93.89	99.99	72.11	10.65

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	52.30	55.94	58.16	61.67	74.67	91.21	94.88	99.15	104.01	75.54	11.20
δ_1=0.2	57.58	60.89	63.51	66.46	79.76	94.99	99.57	104.12	110.18	80.36	10.98
δ_1=0.3	59.95	62.75	65.01	68.18	81.75	96.44	101.48	105.73	112.54	82.31	11.31
δ_1=0.4	57.55	62.20	65.48	68.77	81.93	97.11	102.53	104.95	110.83	82.53	11.19
δ_1=0.5	57.99	60.61	62.99	66.39	80.36	96.42	100.49	105.76	108.89	81.07	11.37
δ_1=0.6	55.28	58.21	61.21	65.01	78.07	92.53	96.88	101.92	108.22	78.45	11.00
δ_1=0.7	50.75	54.14	57.51	60.86	73.64	88.55	92.69	95.81	101.85	74.15	10.74
δ_1=0.8	46.64	48.93	51.85	54.62	67.12	82.12	86.78	90.70	97.00	67.95	10.78

$G_0 = 5$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	68.34	71.22	74.91	78.83	94.02	111.39	115.65	120.03	127.02	94.66	12.60

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	74.28	78.07	81.71	85.54	99.99	118.36	124.50	128.94	133.97	101.10	12.89
δ_1=0.2	74.96	78.67	81.02	83.98	99.22	117.37	122.94	127.06	132.21	100.04	12.66

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	76.59	80.82	84.72	89.55	105.19	123.86	128.34	132.20	135.33	105.62	13.21
δ_1=0.2	80.62	83.14	85.79	89.87	106.61	124.82	129.18	134.81	140.07	106.65	13.14
δ_1=0.3	75.01	78.94	82.46	86.32	101.77	120.71	125.39	129.30	138.37	102.82	13.15

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	80.88	85.51	88.44	92.76	108.36	125.51	130.90	136.27	142.04	109.03	13.06
δ_1=0.2	87.04	89.44	92.33	95.99	110.93	128.72	133.91	138.90	143.98	111.81	12.78
δ_1=0.3	81.76	85.61	90.15	93.57	109.43	127.19	133.42	138.66	146.28	110.21	13.52
δ_1=0.4	77.50	81.68	85.57	89.68	103.66	121.82	127.37	132.05	137.04	104.79	12.71

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	78.60	84.97	89.10	93.10	108.55	126.43	132.02	137.10	142.30	109.48	13.08
δ_1=0.2	87.53	90.08	93.89	98.40	112.92	131.77	137.69	142.04	147.52	114.03	13.06
δ_1=0.3	86.04	91.00	93.40	97.08	114.02	130.52	135.81	141.56	146.33	114.01	13.05
δ_1=0.4	83.77	87.86	90.98	94.77	110.42	126.98	132.50	138.36	146.13	110.80	12.92
δ_1=0.5	78.24	82.41	84.50	89.16	103.22	121.42	126.86	132.03	137.71	104.48	12.87

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	81.21	85.78	89.56	93.73	109.51	127.32	132.72	137.81	143.75	110.00	13.22
δ_1=0.2	86.32	90.16	93.57	98.06	115.11	133.22	138.43	143.70	150.82	115.11	13.75
δ_1=0.3	88.05	91.69	95.63	100.08	115.73	133.42	139.43	143.83	148.44	116.39	13.12
δ_1=0.4	82.78	88.44	93.97	98.01	113.19	132.81	138.63	142.87	148.27	114.35	13.70
δ_1=0.5	83.58	87.08	90.02	93.67	109.57	126.21	132.16	138.16	141.74	110.06	12.75
δ_1=0.6	76.52	79.66	83.05	87.33	102.76	120.24	125.06	127.94	134.84	103.37	12.72

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	80.84	84.09	87.65	91.92	107.31	125.14	130.75	136.18	141.08	108.20	13.17
δ_1=0.2	86.39	89.87	94.23	97.96	114.73	131.66	137.77	142.58	148.25	114.86	13.39
δ_1=0.3	86.11	91.05	94.89	99.23	114.67	132.26	138.82	143.62	151.05	115.28	13.28
δ_1=0.4	87.37	90.63	93.93	99.31	114.99	133.31	138.41	145.33	151.79	115.66	13.49
δ_1=0.5	85.00	88.30	91.85	95.94	112.32	129.88	135.63	141.84	147.36	112.87	13.50
δ_1=0.6	79.55	83.12	86.64	90.58	106.90	123.68	129.33	135.27	143.59	107.24	13.04
δ_1=0.7	71.94	75.94	81.05	84.75	98.98	117.33	122.41	126.01	131.93	100.13	12.91

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	77.22	81.71	84.15	88.15	103.81	121.33	128.28	133.09	139.66	104.65	13.28
δ_1=0.2	82.01	87.38	89.84	93.14	109.38	126.99	132.80	136.87	143.33	109.76	13.10
δ_1=0.3	85.35	88.33	91.29	95.79	112.04	130.28	136.62	142.44	150.77	112.77	13.72
δ_1=0.4	84.52	87.13	91.49	95.61	111.42	129.46	135.81	141.17	143.78	112.30	13.25
δ_1=0.5	83.42	86.33	89.83	93.28	108.89	129.05	134.75	138.52	144.32	110.30	13.63
δ_1=0.6	80.45	84.63	88.65	92.60	107.47	125.15	130.05	134.76	144.47	108.07	12.83
δ_1=0.7	76.91	79.89	83.27	86.22	102.22	121.57	126.98	131.41	137.86	103.20	13.31
δ_1=0.8	69.64	73.74	76.24	80.16	95.39	112.09	117.69	122.05	128.89	95.94	12.57

(ii) max statistic

$G_0 = 2$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	7.80	8.94	9.89	11.00	16.70	24.73	27.21	29.43	33.57	17.46	5.48

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	9.20	10.11	11.10	12.28	18.15	25.99	28.92	31.06	36.91	18.83	5.66
δ_1=0.2	8.59	9.73	10.93	12.17	17.93	25.84	28.74	30.68	33.23	18.47	5.40

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	9.76	11.10	12.27	13.89	19.51	27.54	30.84	33.56	36.88	20.15	5.63
δ_1=0.2	9.80	11.26	12.46	13.77	19.77	27.79	30.75	32.66	36.25	20.34	5.52
δ_1=0.3	9.69	10.76	11.95	13.44	19.20	27.14	29.95	32.21	35.27	19.75	5.45

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	10.66	11.63	12.80	14.51	20.43	28.98	31.83	33.35	36.44	21.19	5.64
δ_1=0.2	11.19	12.38	13.91	15.30	21.32	29.19	32.18	34.29	36.85	21.92	5.59
δ_1=0.3	11.25	12.14	13.55	14.71	20.58	28.98	31.66	34.09	37.90	21.31	5.69
δ_1=0.4	9.93	10.97	12.11	13.33	19.18	26.77	28.96	31.44	35.59	19.74	5.30

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	10.99	12.31	13.49	14.73	20.53	28.74	30.93	33.05	36.28	21.22	5.56
δ_1=0.2	11.77	12.90	14.09	15.40	21.67	29.07	32.02	35.75	38.65	22.21	5.58
δ_1=0.3	12.30	13.02	14.19	15.52	21.74	29.56	32.63	35.47	37.09	22.33	5.58
δ_1=0.4	11.51	12.48	13.31	14.75	20.68	29.56	32.11	34.50	37.65	21.50	5.77
δ_1=0.5	9.40	10.44	11.51	13.10	19.14	26.35	29.39	31.97	34.20	19.62	5.35

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	11.18	11.78	12.91	14.12	20.35	28.85	32.07	33.98	37.35	21.05	5.84
δ_1=0.2	11.34	12.45	14.03	16.10	21.93	30.00	32.76	34.57	36.84	22.48	5.59
δ_1=0.3	12.81	13.87	14.76	16.35	22.16	30.37	33.74	36.23	39.96	22.90	5.71
δ_1=0.4	12.09	13.29	14.35	15.85	21.69	29.61	31.87	34.14	36.92	22.40	5.42
δ_1=0.5	11.48	12.41	13.47	14.85	20.62	28.56	30.82	32.44	36.39	21.26	5.42
δ_1=0.6	9.56	11.02	11.87	13.26	19.32	27.36	30.28	32.67	35.78	19.87	5.59

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	10.34	12.04	13.00	14.56	19.69	28.11	30.79	34.19	38.03	20.70	5.62
δ_1=0.2	11.88	13.60	14.43	15.88	21.85	30.17	31.95	34.62	38.23	22.47	5.55
δ_1=0.3	12.30	13.51	14.73	15.95	21.81	29.55	32.31	34.89	38.20	22.45	5.45
δ_1=0.4	11.64	13.28	14.69	16.19	22.10	30.55	33.12	35.30	38.18	22.77	5.66
δ_1=0.5	11.71	12.91	14.12	15.29	21.23	29.16	31.95	35.38	37.33	21.91	5.52
δ_1=0.6	10.40	11.69	12.57	14.04	20.08	28.51	31.21	34.00	37.38	20.77	5.76
δ_1=0.7	9.24	10.29	11.23	12.43	17.89	25.36	27.68	29.80	31.91	18.58	5.06

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	9.11	10.62	11.80	13.01	19.14	27.28	29.79	31.55	33.63	19.69	5.58
δ_1=0.2	11.13	12.23	13.17	14.91	20.67	28.31	30.61	33.16	35.40	21.16	5.30
δ_1=0.3	11.64	12.96	14.22	15.33	21.34	28.84	31.60	34.18	37.96	21.84	5.44
δ_1=0.4	12.02	13.34	14.33	15.60	21.22	28.95	32.41	34.67	37.20	21.88	5.41
δ_1=0.5	11.43	12.52	14.02	15.20	20.93	29.49	31.82	33.75	36.59	21.62	5.53
δ_1=0.6	10.21	11.34	12.63	14.17	20.08	27.95	30.37	33.21	36.29	20.62	5.54
δ_1=0.7	8.90	10.22	11.25	12.48	18.48	26.90	28.62	31.16	34.55	19.13	5.53
δ_1=0.8	8.13	9.22	9.95	11.04	17.10	24.57	27.49	29.08	33.62	17.51	5.37

$G_0 = 3$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	12.71	14.57	15.58	16.66	22.98	30.75	33.33	36.14	39.50	23.52	5.64

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	14.82	16.07	17.19	18.56	24.60	32.95	35.60	37.93	39.54	25.26	5.61
δ_1=0.2	14.40	15.84	17.08	18.57	24.39	32.72	36.13	38.04	41.53	25.23	5.78

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	15.04	16.51	17.49	18.96	25.84	34.70	37.87	40.22	43.62	26.52	6.15
δ_1=0.2	15.28	16.52	18.09	19.55	25.94	34.25	37.24	40.86	44.27	26.61	6.01
δ_1=0.3	14.91	16.04	17.25	18.77	25.19	33.77	36.37	39.79	42.83	25.88	5.90

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	15.82	16.75	18.14	19.37	26.26	34.56	37.83	40.22	43.45	26.87	5.99
δ_1=0.2	16.37	17.54	18.93	20.38	27.27	35.54	38.58	40.93	42.88	27.64	5.87
δ_1=0.3	15.88	17.47	18.64	19.95	27.04	34.89	37.26	39.76	43.28	27.32	5.95
δ_1=0.4	14.68	16.52	17.52	18.97	25.19	33.52	36.84	39.13	42.40	25.89	5.84

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	16.02	17.09	18.71	20.19	26.41	34.68	37.48	39.72	42.18	27.10	5.82
δ_1=0.2	16.42	18.18	19.48	20.97	27.70	36.10	39.00	42.14	46.42	28.27	6.12
δ_1=0.3	16.44	17.89	19.68	21.03	27.47	36.31	39.00	41.34	43.86	28.08	5.92
δ_1=0.4	16.59	18.13	19.38	20.74	26.76	35.07	37.38	39.97	43.44	27.46	5.75
δ_1=0.5	14.97	15.99	17.18	18.85	25.43	33.77	36.37	38.72	41.59	25.90	5.76

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	16.73	17.56	18.77	20.49	27.05	35.28	38.09	40.26	42.33	27.47	5.80
δ_1=0.2	17.32	18.58	20.36	21.66	27.77	36.21	38.41	41.37	45.76	28.49	5.76
δ_1=0.3	17.86	19.34	20.31	21.61	28.21	37.36	40.17	42.47	44.63	28.99	6.07
δ_1=0.4	17.04	18.16	19.75	21.08	27.75	36.12	38.79	42.28	45.84	28.37	5.98
δ_1=0.5	15.96	17.66	19.08	20.48	26.89	35.18	37.75	39.81	41.47	27.53	5.76
δ_1=0.6	14.31	15.73	17.11	18.91	24.92	33.18	35.55	38.99	41.20	25.56	5.73

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	15.89	16.86	18.29	19.72	26.13	34.91	37.81	40.08	42.68	26.81	5.91
δ_1=0.2	16.74	17.50	19.36	21.32	27.92	36.31	39.48	42.98	45.14	28.50	6.14
δ_1=0.3	17.34	18.77	20.05	21.41	27.97	36.51	39.51	41.82	43.70	28.66	5.87
δ_1=0.4	17.38	18.86	19.86	21.45	27.91	36.86	39.22	41.22	44.83	28.64	5.94
δ_1=0.5	16.91	18.05	19.70	20.94	26.93	35.49	38.67	40.95	43.13	27.67	5.76
δ_1=0.6	16.44	17.32	18.31	19.70	25.94	33.80	36.73	38.90	41.76	26.55	5.52
δ_1=0.7	13.95	15.35	16.30	17.77	24.41	32.63	35.64	38.08	40.47	24.96	5.86

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	15.01	16.23	17.40	18.88	25.53	34.35	37.18	39.37	42.81	26.23	6.04
δ_1=0.2	17.00	18.02	19.27	20.60	26.98	34.95	37.85	40.32	44.48	27.45	5.61
δ_1=0.3	16.02	17.66	18.74	20.44	27.19	35.73	38.74	41.20	44.22	27.66	6.07
δ_1=0.4	17.05	18.06	19.34	21.02	27.26	36.33	38.63	42.90	44.81	27.92	6.10
δ_1=0.5	16.62	17.76	18.56	20.28	27.38	36.18	39.26	42.49	44.40	27.98	6.25
δ_1=0.6	15.27	16.59	18.05	19.87	26.33	34.72	37.46	40.06	43.17	26.92	5.96
δ_1=0.7	14.33	15.57	16.75	18.23	24.90	33.27	36.00	38.69	42.45	25.51	5.96
δ_1=0.8	12.89	14.09	15.47	16.69	22.80	31.85	34.66	38.04	41.67	23.68	5.97

$G_0 = 4$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	17.83	19.26	20.50	22.48	28.98	37.32	40.58	44.28	47.32	29.60	6.04

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	19.04	20.54	21.85	23.88	30.80	39.41	42.83	45.44	47.88	31.32	6.30
δ_1=0.2	18.81	20.21	21.42	23.03	29.86	38.72	41.77	44.11	46.54	30.52	6.03

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	19.90	21.49	22.83	24.42	31.72	41.07	44.20	46.06	49.30	32.26	6.43
δ_1=0.2	21.62	22.60	24.09	25.41	32.26	41.40	44.13	45.71	49.18	32.84	6.18
δ_1=0.3	20.62	21.79	22.98	24.07	31.38	39.42	42.18	45.49	47.75	31.67	5.96

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	20.69	22.61	23.97	25.39	31.84	40.44	43.12	47.03	50.53	32.62	6.20
δ_1=0.2	21.04	22.79	24.20	25.86	33.05	41.51	44.27	46.83	50.94	33.49	6.19
δ_1=0.3	20.39	22.33	23.73	25.31	32.67	42.39	45.16	47.69	51.42	33.30	6.52
δ_1=0.4	20.34	21.51	22.99	24.49	30.96	40.24	43.10	45.74	48.40	31.81	6.19

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	21.32	22.65	23.88	25.24	31.89	40.80	43.47	46.70	49.56	32.61	6.06
δ_1=0.2	22.59	23.76	24.97	26.34	33.45	42.58	45.69	48.12	50.85	34.10	6.28
δ_1=0.3	22.23	23.34	24.52	26.51	33.77	42.45	44.97	46.73	52.10	34.10	6.18
δ_1=0.4	21.97	23.32	24.62	25.86	32.42	41.38	43.69	46.04	48.78	33.03	6.07
δ_1=0.5	19.93	21.16	22.74	24.27	30.78	39.42	42.15	44.41	48.10	31.29	5.96

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	21.07	22.48	24.07	25.49	32.78	41.55	44.29	47.02	49.59	33.19	6.27
δ_1=0.2	21.82	23.42	24.64	26.44	33.54	42.61	46.23	48.40	50.18	34.13	6.23
δ_1=0.3	22.17	23.95	25.33	26.66	33.45	42.73	46.85	49.49	52.71	34.36	6.43
δ_1=0.4	22.37	23.21	24.69	26.28	33.72	42.93	45.10	49.40	51.50	34.32	6.53
δ_1=0.5	21.30	22.17	23.68	25.64	32.54	41.45	43.69	45.90	48.72	33.01	6.07
δ_1=0.6	18.90	20.63	22.20	23.42	30.76	40.10	43.25	45.32	47.52	31.29	6.45

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	21.19	22.49	23.67	25.40	32.35	41.62	44.42	46.12	50.64	33.05	6.36
δ_1=0.2	22.21	23.74	25.01	26.68	33.53	42.66	45.64	48.75	52.89	34.15	6.34
δ_1=0.3	21.83	23.02	24.66	26.52	33.53	42.96	45.97	48.05	52.21	34.19	6.45
δ_1=0.4	22.04	23.57	24.95	26.53	33.77	41.70	44.49	48.13	50.69	34.17	6.31
δ_1=0.5	21.76	23.52	24.75	26.57	33.39	42.19	45.03	47.97	52.10	34.05	6.39
δ_1=0.6	21.81	22.88	23.99	25.48	32.42	41.25	44.62	48.04	52.49	33.25	6.45
δ_1=0.7	19.45	20.80	21.99	23.59	30.27	39.13	42.21	44.40	46.47	30.95	6.10

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	20.04	21.67	22.87	24.53	31.45	40.66	43.15	46.62	50.64	32.14	6.38
δ_1=0.2	21.44	22.81	23.85	25.30	32.62	41.54	44.89	47.30	51.37	33.13	6.24
δ_1=0.3	21.74	23.06	24.31	25.90	32.88	41.57	44.02	46.08	48.49	33.26	6.02
δ_1=0.4	21.48	23.11	24.42	26.06	33.02	41.00	44.08	46.67	50.07	33.39	6.01
δ_1=0.5	21.49	22.70	23.98	25.74	32.56	41.48	44.67	46.57	49.98	33.13	6.28
δ_1=0.6	20.31	22.17	23.36	25.05	32.16	41.75	44.11	46.38	48.96	32.80	6.29
δ_1=0.7	20.60	21.51	22.85	24.54	31.23	40.56	43.40	45.99	48.15	32.00	6.19
δ_1=0.8	18.29	19.34	20.34	22.36	29.24	38.48	41.17	44.20	47.21	29.91	6.34

$G_0 = 5$

δ_2=0.2

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	23.25	24.48	26.13	27.68	35.01	43.91	46.56	49.70	54.29	35.59	6.49

δ_2=0.3

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	24.85	26.06	27.35	29.01	36.63	46.17	50.11	52.29	56.39	37.25	6.78
δ_1=0.2	24.60	26.26	27.22	28.73	36.26	45.54	48.24	50.46	55.52	36.70	6.49

δ_2=0.4

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	24.48	25.95	27.93	29.70	37.52	46.78	49.44	52.32	54.80	37.96	6.70
δ_1=0.2	24.67	26.32	28.45	30.21	37.67	47.15	49.51	51.70	54.54	38.15	6.51
δ_1=0.3	24.47	26.26	27.77	29.17	36.57	46.06	49.73	51.94	55.93	37.21	6.67

δ_2=0.5

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	25.70	27.57	28.75	30.52	38.04	48.00	50.68	53.45	57.50	38.72	6.81
δ_1=0.2	27.00	28.08	30.05	31.44	38.65	48.29	51.09	53.71	56.12	39.38	6.47
δ_1=0.3	25.45	27.10	29.04	30.61	38.06	47.48	50.75	53.23	58.14	38.71	6.68
δ_1=0.4	24.74	26.53	28.01	29.80	36.87	46.47	49.58	51.68	53.86	37.60	6.59

δ_2=0.6

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	25.78	27.25	28.86	30.56	38.06	47.31	49.55	52.29	55.97	38.48	6.50
δ_1=0.2	25.94	27.84	29.50	31.49	39.09	47.83	50.65	53.34	56.68	39.39	6.52
δ_1=0.3	26.09	28.11	29.81	31.71	39.29	48.29	50.57	54.56	58.69	39.71	6.66
δ_1=0.4	26.32	27.71	29.39	30.96	38.50	47.44	50.57	53.72	56.70	38.94	6.60
δ_1=0.5	25.13	26.94	27.83	29.58	36.75	45.52	49.18	51.76	55.90	37.32	6.44

δ_2=0.7

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	26.34	27.35	29.10	30.85	38.23	47.10	49.92	53.90	57.27	38.68	6.50
δ_1=0.2	26.97	28.86	29.91	31.16	39.02	48.37	51.89	54.38	60.32	39.56	6.86
δ_1=0.3	27.17	29.05	30.33	31.78	39.38	48.69	50.83	53.92	57.69	39.81	6.51
δ_1=0.4	26.79	28.02	29.57	31.31	38.82	48.69	51.32	54.75	56.91	39.49	6.79
δ_1=0.5	26.67	28.23	29.61	31.12	38.18	47.55	51.69	54.71	57.41	38.91	6.65
δ_1=0.6	25.09	26.49	27.86	29.59	36.91	46.02	48.64	50.95	54.93	37.21	6.36

δ_2=0.8

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	25.81	27.15	28.66	30.37	38.03	47.48	50.86	53.71	56.98	38.60	6.78
δ_1=0.2	26.76	28.30	30.54	31.81	39.12	48.73	51.80	54.55	57.65	39.96	6.73
δ_1=0.3	27.24	28.26	29.75	31.30	39.39	48.54	51.06	53.62	55.55	39.75	6.53
δ_1=0.4	27.63	29.05	30.12	32.10	39.24	48.39	51.40	53.28	56.64	39.82	6.36
δ_1=0.5	26.12	28.41	29.63	31.40	38.86	48.17	51.13	53.78	56.35	39.43	6.59
δ_1=0.6	25.31	26.84	28.22	30.33	37.67	46.96	49.77	52.59	56.82	38.14	6.64
δ_1=0.7	23.89	25.56	27.01	28.77	36.29	45.49	48.11	49.87	52.41	36.66	6.45

δ_2=0.9

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
δ_1=0.1	25.10	26.39	27.49	29.31	37.32	46.65	50.27	52.47	55.13	37.86	6.67
δ_1=0.2	25.59	27.38	28.93	30.80	38.36	47.64	51.20	54.01	57.11	38.99	6.83
δ_1=0.3	26.65	27.51	28.79	30.94	38.93	48.67	51.46	53.42	56.52	39.36	6.80
δ_1=0.4	26.61	27.93	29.18	31.06	38.28	47.75	50.08	52.92	58.82	38.94	6.64
δ_1=0.5	25.97	27.29	28.93	30.60	38.45	47.70	50.74	53.92	59.94	38.95	6.89
δ_1=0.6	26.18	27.79	29.22	30.63	37.82	47.37	50.48	53.26	56.84	38.54	6.58
δ_1=0.7	24.92	26.52	28.23	29.86	37.17	46.60	49.94	53.75	57.67	37.73	6.76
δ_1=0.8	23.48	25.05	26.33	28.07	35.25	44.52	47.78	50.82	54.11	35.93	6.66

Table 4

linear–curved–linear kink case with unknown kink points

$$z_t^* = \begin{bmatrix} 1 \\ DU_1(t) \\ DT_1(t) - DT_2(t) \\ DT_2(t) \\ t \\ QT_1(t) - QT_2(t) \end{bmatrix}, \quad z_{1t}^* = \begin{bmatrix} 1 \\ DU_1(t) \\ DT_1(t) - DT_2(t) \end{bmatrix}$$

G_0 = dimension of Brownian motions

(i) trace statistic

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
G_0=1	14.47	15.62	16.51	17.72	22.77	29.96	32.53	34.68	37.57	23.41	4.98
G_0=2	31.74	33.04	34.60	36.93	44.56	54.46	57.33	60.30	64.66	45.12	6.93
G_0=3	51.91	54.09	56.52	58.85	68.90	81.11	85.30	88.30	93.60	69.52	8.72
G_0=4	76.09	78.74	81.94	84.36	96.88	111.65	116.00	119.12	124.90	97.53	10.56
G_0=5	104.60	107.32	110.04	114.03	128.04	144.58	149.46	153.39	159.46	128.87	11.88

(ii) max statistic

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
G_0=2	21.04	22.37	23.41	24.86	30.32	38.01	40.74	42.61	46.11	30.96	5.25
G_0=3	26.61	28.17	29.35	30.84	36.83	45.00	47.22	49.44	51.87	37.46	5.52
G_0=4	32.39	33.72	34.97	36.46	42.93	51.21	54.21	56.74	61.00	43.52	5.92
G_0=5	38.02	39.47	40.81	42.33	49.15	57.63	60.90	62.89	65.76	49.68	6.06

Table 5.1

$q_0 = 2$: linear kink points
number of kink points ≤ 2
G_0 = dimension of Brownian motions

(i) trace statistic

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
G_0=1	3.63	3.83	3.97	4.24	5.53	7.52	8.01	8.73	9.47	5.74	1.29
G_0=2	2.96	3.15	3.31	3.50	4.24	5.26	5.63	5.79	5.93	4.32	0.68
G_0=3	2.70	2.93	3.01	3.15	3.64	4.29	4.44	4.68	5.13	3.69	0.47
G_0=4	2.68	2.76	2.83	2.92	3.32	3.86	3.99	4.15	4.29	3.36	0.36
G_0=5	2.48	2.58	2.67	2.73	3.11	3.51	3.66	3.78	3.86	3.12	0.31

(ii) max statistic

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
G_0=2	4.04	4.18	4.44	4.66	5.85	7.48	7.99	8.39	8.62	5.99	1.07
G_0=3	4.36	4.65	4.75	4.98	5.96	7.23	7.63	8.11	8.75	6.06	0.92
G_0=4	4.44	4.68	4.85	5.11	6.02	7.14	7.47	7.83	8.36	6.07	0.83
G_0=5	4.49	4.75	4.93	5.13	6.00	7.02	7.50	7.95	8.48	6.07	0.83

Table 5.2

$q_0 = 3$: linear kink points
number of kink points ≤ 3
G_0 = dimension of Brownian motions

(i) trace statistic

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
G_0=1	3.59	3.88	4.16	4.51	5.80	7.50	8.21	9.17	10.10	5.95	1.29
G_0=2	3.30	3.43	3.53	3.76	4.51	5.42	5.70	5.98	6.36	4.58	0.67
G_0=3	3.04	3.16	3.28	3.42	3.93	4.54	4.83	4.93	5.12	3.96	0.46
G_0=4	2.81	2.89	3.02	3.09	3.51	3.98	4.09	4.25	4.45	3.53	0.35
G_0=5	2.67	2.79	2.87	2.97	3.28	3.65	3.73	3.82	4.04	3.29	0.27

(ii) max statistic

	1%	2.5%	5%	10%	50%	90%	95%	97.5%	99%	mean	SD
G_0=2	4.39	4.54	4.84	5.13	6.15	7.82	8.50	8.83	9.64	6.37	1.10
G_0=3	4.73	4.93	5.12	5.38	6.37	7.64	8.07	8.43	8.65	6.43	0.89
G_0=4	4.93	4.99	5.14	5.36	6.14	7.29	7.75	7.91	8.45	6.27	0.77
G_0=5	4.90	5.03	5.19	5.40	6.19	7.18	7.55	7.86	8.51	6.24	0.75