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Weights of $\bar{\chi}^2$ distribution for smooth or piecewise smooth cone alternatives

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Abstract

We study the problem of testing a simple null hypothesis on multivariate normal mean vector against smooth or piecewise smooth cone alternatives. We show that the mixture weights of the $\bar{\chi}^2$ distribution of the likelihood ratio test can be characterized as mixed volumes of the cone and its dual. The weights can be calculated by integration involving the second fundamental form on the boundary of the cone. We illustrate our technique by spherical cone and cone of non-negative definite matrices.

Key words: multivariate one-sided alternative, mixed volume, second fundamental form, internal angle, external angle, Gauss-Bonnet theorem, Shapiro's conjecture.

1 Introduction

We first state our problem and then give outline of the paper. In Section 1.2 we prepare basic material from convex analysis.

1.1 The problem

We consider the problem of testing a simple null hypothesis on multivariate normal mean vector against a convex cone alternative in the following canonical form. Let $Z \in R^p$ be distributed according to the p-dimensional multivariate normal distribution $N_p(\mu, I)$. Let K be a closed convex cone with non-empty interior in R^p. Our testing problem is

$$H_0 : \mu = 0 \ \text{ vs. } \ H_1 : \mu \in K. \tag{1}$$

The main objective of this paper is to study the null distribution of the likelihood ratio statistic for K with smooth or piecewise smooth boundary using techniques of convex analysis and differential geometry.

In addition to (1) consider a complementary testing problem

$$H_1 : \mu \in K \quad \text{vs.} \quad H_2 : \mu \in R^p. \tag{2}$$

In describing the complementary testing problem we need to use the dual cone K^* of K:

$$K^* = \{y : \langle y, x\rangle \le 0,\ \forall x \in K\},$$

where $\langle\ ,\ \rangle$ denotes the inner product.

For $x \in R^p$ let x_K denote the orthogonal projection of x onto K and x_{K^*} denote the orthogonal projection of x onto K^*. Then the likelihood ratio test of (1) is equivalent to rejecting H_0 when

$$\bar{\chi}^2_{01} = \|Z_K\|^2 \tag{3}$$

is large and the likelihood ratio test of (2) is equivalent to rejecting H_1 when

$$\bar{\chi}^2_{12} = \|Z_{K^*}\|^2 \tag{4}$$

is large. We consider joint distribution of $\bar{\chi}^2_{01}$ and $\bar{\chi}^2_{12}$ under H_0.

The statistics $\bar{\chi}^2_{01}$ and $\bar{\chi}^2_{12}$ in (3) and (4) are called *chi-bar-square statistics*, and known to have a finite mixture of the chi-square distributions when H_0 is true. In this paper we call the mixing probabilities as the *weights*. Generally, it is not easy to derive the explicit expression of the weights. Here we list some examples of K whose weights are known explicitly or can be easily calculated numerically.

$$\begin{aligned}
K_1 &= \{\mu \,|\, \mu_1 \le \cdots \le \mu_p\} \\
K_2 &= \{\mu \,|\, \mu_1 \le \mu_j,\ \ j = 2, \ldots, p\} \\
K_3 &= \Big\{\mu \,|\, \frac{\mu_1 + \cdots + \mu_j}{j} \le \frac{\mu_{j+1} + \cdots + \mu_p}{p-j},\quad j = 1, \ldots, p-1\Big\} \\
K_4 &= \Big\{\mu \,|\, \frac{\mu_1}{\|\mu\|} \ge \cos\psi\Big\} \\
K_5 &= \{M : p \times p \text{ symmetric} \,|\, M \text{ is non-negative definite}\}.
\end{aligned}$$

The cones K_1 and K_2 are defined by the partial orders referred as *simple order* and *simple tree order*, respectively. The corresponding weights can be given by the recurrence formulas (Section 2.4 of Robertson et al. (1988)). In particular, the weights for K_1 are known to be expressed in terms of the Stirling number of the first kind (Theorem 2.4 of Robertson et al. (1988)). The cone K_3 is the dual cone of $-K_1$, and the corresponding weights are derived directly from the weights of K_1 due to this duality.

The three cones K_1, K_2 and K_3 are polyhedral, i.e., the cones defined by a finite number of linear constraints. On the other hand, K_4 and K_5 are non-polyhedral. K_4 is spherical cone which is smooth in the sense of Section 2.2 with no singularities except for the origin. K_5 is a piecewise smooth cone in the sense of Section 2.3. In section 3 we show that the singularities of K_5 exhibit a beautiful recurrence structure. The weights for K_4 and K_5 were given by Pincus (1975) and Kuriki (1993), respectively.

For the polyhedral cone, the geometrical meaning of the weights is clear, since the weights can be expressed in terms of the internal and external angles. Compared with the polyhedral cone, the meaning of the weights for non-polyhedral cones is not clear. In this paper we clarify the geometrical meaning of the weights in the case that the boundary of the cone is smooth or piecewise smooth.

In Section 2 we prove our basic theorem relating the weights to the mixed volumes of K and its dual K^*. For smooth or piecewise smooth cones we express the mixed volumes as integrals involving the second fundamental form on the boundary of the cone. In section 3 we apply our technique to the case of multivariate one-sided alternative on covariance matrices and clarify geometrical meaning of the problem.

1.2 Preparation from convex analysis

Here we summarize basic results from convex analysis. These results are taken from Webster (1994). Let $U = U_p$ be the closed unit ball and K be a convex set in R^p. For $\lambda \geq 0$, λ*-neighborhood of* K or *outer parallel set of* K *at distance* λ is defined as

$$(K)_\lambda = K + \lambda U,$$

where the addition is the vector sum. The *Hausdorff distance* between two non-empty compact convex sets K_1, K_2 is defined by

$$\rho(K_1, K_2) = \inf\{\lambda \geq 0 : K_1 \subset (K_2)_\lambda \text{ and } K_2 \subset (K_1)_\lambda\}.$$

Endowed with the Hausdorff distance, the set of compact convex sets becomes a complete metric space (Theorem 9 of Gruber (1993)).

A *polytope* is the convex hull of finite number of points. Any compact convex set can be approximated by polytopes.

Lemma 1.1 *(Theorem 3.1.6 of Webster (1994)) Let K be a non-empty compact convex set in R^p and let $\epsilon > 0$. Then there exist polytopes P, Q in R^p such that $P \subset K \subset Q$ and $\rho(K, P) \leq \epsilon$, $\rho(K, Q) \leq \epsilon$.*

We deal with convex cones which are not bounded. However uniform convergence on any bounded region is sufficient for us because we are concerned with the standard normal probabilities of the cones. Let K be a convex cone and denote $K_{(\lambda)} = K \cap \lambda U$. In view of the fact that polytopes are bounded polyhedral sets (Theorem 3.2.5 of Webster (1994)) the next lemma follows easily from Lemma 1.1.

Lemma 1.2 *Let K be a closed convex cone in R^p and let $\lambda \geq 0$, $\epsilon \geq 0$. Then there exist polyhedral cones P, Q in R^p such that $P \subset K \subset Q$ and $\rho(K_{(\lambda)}, P_{(\lambda)}) \leq \epsilon$, $\rho(K_{(\lambda)}, Q_{(\lambda)}) \leq \epsilon$.*

Now we introduce the notion of mixed volumes. Let K_1, K_2 be two compact convex sets in R^p and let $v_p(\cdot)$ denote the volume in R^p. For $\nu, \lambda \geq 0$ consider the volume $v_p(\nu K_1 + \lambda K_2)$. Then the following lemma holds.

Lemma 1.3 *(Theorem 6.4.3 of Webster (1994)) $v_p(\nu K_1 + \lambda K_2)$ is a homogeneous polynomial of degree p in ν and λ, i.e.,*

$$\begin{aligned} v_p(\nu K_1 + \lambda K_2) &= \nu^p v_{p,0}(K_1, K_2) + p\nu^{p-1}\lambda v_{p,1}(K_1, K_2) + \cdots + \lambda^p v_{0,p}(K_1, K_2) \\ &= \sum_{i=0}^{p} \binom{p}{i} \nu^{p-i}\lambda^i v_{p-i,i}(K_1, K_2), \end{aligned}$$

where $v_{p,0}(K_1, K_2) = v_p(K_1)$ *and* $v_{0,p}(K_1, K_2) = v_p(K_2)$.

$v_{p-i,i}(K_1, K_2)$, $i = 0, \ldots, p$, are called *mixed volumes* of K_1 and K_2 . In the particular case $\nu = 1$ and $K_2 = U$, i.e., when we are considering the outer parallel set of K_1 , $v_{p-i,i}(K_1, U)$ is called *quermassintegral* of K_1 and $\binom{p}{i} v_{i,p-i}(K_1, U)/\omega_{p-i}$ is called *intrinsic volume* of K_1 , where

$$\omega_q = \frac{\pi^{q/2}}{\Gamma(\frac{q}{2}+1)} \tag{5}$$

is the volume of the unit ball U_q in R^q . It is also known that mixed volumes are continuous in K_1, K_2 with respect to Hausdorff metric (Theorem 6.4.7 of Webster (1994)).

2 Weights of $\bar{\chi}^2$ distribution as mixed volumes

In this section we first prove our basic theorem which states that the weights of $\bar{\chi}^2$ distribution are the mixed volumes of the convex cone K and its dual cone K^* . Then we apply the basic theorem to the case of smooth convex cone using the fact that mixed volumes can be evaluated as integrals involving the second fundamental form on the boundary of K . Our result for the case of R^3 is very easily stated and connection to the classical Gauss-Bonnet theorem will be discussed. We illustrate our result for the smooth cone with the case of spherical cone. Finally we discuss the case of "piecewise smooth" cone. Full treatment of piecewise smooth cone is needed to discuss multivariate one-sided alternatives for covariance matrices in Section 3.

2.1 Basic theorem

Here we prove our basic theorem stating that the weights of $\bar{\chi}^2$ distributions are mixed volumes. Since the concept of mixed volumes applies equally to polyhedral as well as smooth cones, our Theorem 2.1 covers both cases.

Theorem 2.1 *Consider the testing problems (1) and (2). Let* $K_{(1)} = K \cap U$ *and* $K^*_{(1)} = K^* \cap U$ *and let* $v_{p-i,i}(K_{(1)}, K^*_{(1)}), i = 0, \ldots, p,$ *be the mixed volumes of* $K_{(1)}$ *and* $K^*_{(1)}$. *Then under* H_0

$$P(\bar{\chi}^2_{01} \le a, \bar{\chi}^2_{12} \le b) = \sum_{i=0}^{p} \binom{p}{i} \frac{v_{p-i,i}(K_{(1)}, K^*_{(1)})}{\omega_i \omega_{p-i}} G_{p-i}(a) G_i(b), \tag{6}$$

where ω_q *is the volume of the unit ball in* R^q *given in (5) and* $G_q(t)$ *is the cumulative distribution function of chi-square distribution with* q *degrees of freedom.*

Proof. Let $P_n, n = 1, 2, \ldots,$ be sequence of polyhedral cones converging to K in the sense of Lemma 1.2. For a given point $x \in R^p$ let x_{P_n} denote the orthogonal projection onto P_n . Then it is easy to show that x_{P_n} converges to x_K . At the same time the dual cone P_n^* converges to K^* and the projection $x_{P_n^*}$ converges to x_{K^*} . Since pointwise convergence implies convergence in law we have

$$\begin{aligned} P(\bar{\chi}^2_{01} \le a, \bar{\chi}^2_{12} \le b) &= P(\|Z_K\|^2 \le a, \|Z_K^*\|^2 \le b) \\ &= \lim_{n\to\infty} P(\|Z_{P_n}\|^2 \le a, \|Z_{P_n^*}\|^2 \le b). \end{aligned} \tag{7}$$

In view of the continuity of the mixed volumes, (7) shows that it is enough to prove our theorem for polyhedral cones.

From now on let K be a polyhedral cone. In this case the weights of $\bar{\chi}^2$ distribution is well understood in terms of the internal and external angles. Therefore we only need to verify that these angles can be expressed in terms of mixed volumes. Let F be a (closed) face of K and let $\beta(0,F)$ and $\gamma(F,K)$ be the internal angle and the external angle at F. See Appendix for more precise definition. Then it is well known that the joint distribution of $\bar{\chi}^2_{01}$ and $\bar{\chi}^2_{12}$ is a mixture of independent χ^2 distributions.

$$P(\bar{\chi}^2_{01} \le a, \bar{\chi}^2_{12} \le b) = \sum_{i=0}^{p} w_{p-i} G_{p-i}(a) G_i(b),$$

The mixture weight is expressed as

$$w_d = \sum_{\substack{F\in\mathcal{F}(K)\\ \dim F = d}} \beta(0,F)\gamma(F,K).$$

where $\mathcal{F}(K)$ is the set of faces of K and $\dim F$ is the dimension of the linear subspace spanned by F (Wynn (1975), Section 2.3 of Robertson et al. (1988)).

Let F^* be the face of K^* dual to the face F of K. Then $\dim F^* = p - \dim F$ and F is orthogonal to F^*. Consider the orthogonal sum $F \oplus F^*$. For different faces F_1, F_2, interiors of the sets $F_1 \oplus F_1^*$, $F_2 \oplus F_2^*$ are disjoint and R^p is covered by these sets

$$R^p = \bigcup_{F\in\mathcal{F}(K)} F \oplus F^*$$

(Lemma 3 of McMullen (1975), Wynn (1975)). Then

$$\begin{aligned}
\nu K_{(1)} + \lambda K^*_{(1)} &= (\nu K_{(1)} + \lambda K^*_{(1)}) \cap \Big(\bigcup_{F\in\mathcal{F}(K)} F \oplus F^* \Big) \\
&= \bigcup_{F\in\mathcal{F}(K)} (F \oplus F^*) \cap (\nu K_{(1)} + \lambda K^*_{(1)}) \\
&= \bigcup_{F\in\mathcal{F}(K)} (F \cap \nu U) \oplus (F^* \cap \lambda U).
\end{aligned}$$

Therefore

$$v_p(\nu K_{(1)} + \lambda K^*_{(1)}) = \sum_{F\in\mathcal{F}(K)} v_p((F\cap \nu U) \oplus (F^* \cap \lambda U)).$$

Because of the orthogonality

$$\begin{aligned}
v_p((F\cap \nu U) \oplus (F^* \cap \lambda U)) &= v_d(F \cap \nu U) \times v_{p-d}(F^* \cap \lambda U) \\
&= \nu^d \omega_d \beta(0,F) \times \lambda^{p-d} \omega_{p-d} \gamma(F,K),
\end{aligned}$$

where $d = \dim F$. Therefore

$$v_p(\nu K_{(1)} + \lambda K^*_{(1)}) = \sum_{d=0}^{p} \sum_{\dim F = d} \nu^d \lambda^{p-d} \omega_d \omega_{p-d} \beta(0,F)\gamma(F,K)$$

and
$$\binom{p}{d} v_{d,p-d}(K_{(1)}, K^*_{(1)}) = \omega_d \omega_{p-d} \sum_{\dim F = d} \beta(0,F)\gamma(F,K) = \omega_d \omega_{p-d} \times w_d,$$
or
$$\omega_d = \binom{p}{d} \frac{v_{p-d,d}(K_{(1)}, K^*_{(1)})}{\omega_d \omega_{p-d}}.$$
Therefore (6) holds for polyhedral cones. This proves the theorem for general cones as well by the argument given at the beginning of the proof. ∎

Remark 2.1 *The argument of approximating a non-polyhedral cone with sequence of polyhedral cones is also found in Theorem 3.1 of Shapiro (1985).*

To characterize the set $\nu K_{(1)} + \lambda K^*_{(1)}$ the following lemma is useful.

Lemma 2.1 *Let* K *be a closed convex cone in* R^p *and* K^* *be its dual. Then for* $\nu, \lambda \geq 0$,
$$\nu K_{(1)} + \lambda K^*_{(1)} = (K + \lambda U) \cap (K^* + \nu U).$$

Proof. Note that $\nu K_{(1)} = \nu(K \cap U) = K \cap (\nu U)$ and $\lambda K^*_{(1)} = K^* \cap (\lambda U)$. Now suppose that $x \in K \cap \nu U$ and $y \in K^* \cap \lambda U$. Then $x \in K$, $y \in \lambda U$ and $x + y \in K + \lambda U$. At the same time $x \in \nu U$, $y \in K^*$ and $x + y \in K^* + \nu U$. Therefore $x + y \in (K + \lambda U) \cap (K^* + \nu U)$. This implies
$$(K \cap \nu U) + (K^* \cap \lambda U) \subset (K + \lambda U) \cap (K^* + \nu U).$$

To prove the converse let $z \in (K + \lambda U) \cap (K^* + \nu U)$. Since $z \in K^* + \nu U$ there exist x and y such that $z = x + y$ and $x \in K^*, ||y|| \leq \nu$. Write $z = z_K + z_{K^*}$ and $y = y_K + y_{K^*}$. Then
$$\begin{aligned} ||z_K||^2 &= ||z - z_{K^*}||^2 \leq ||z - x - y_{K^*}||^2 = ||y_K||^2 \\ &= ||y||^2 - ||y_{K^*}||^2 \leq ||y||^2 \leq \nu^2. \end{aligned}$$

Therefore $z_K \in K \cap \nu U$. Similarly $z_{K^*} \in K^* \cap \lambda U$. Hence $z = z_K + z_{K^*} \in (K \cap \nu U) + (K^* \cap \lambda U)$ and this implies
$$(K + \lambda U) \cap (K^* + \nu U) \subset (K \cap \nu U) + (K^* \cap \lambda U).$$
∎

In evaluating mixed volumes, the p-dimensional volumes $v_p(K_{(1)}) = v_{p,0}(K_{(1)}, K^*_{(1)})$ and $v_{0,p}(K_{(1)}, K^*_{(1)}) = v_p(K^*_{(1)})$ have to be evaluated individually. Other mixed volumes turn out to be easier to evaluate. Consider
$$(\nu K_{(1)} + \lambda K^*_{(1)}) \cap (\nu K_{(1)})^C \cap (\lambda K^*_{(1)})^C \tag{8}$$
where A^C is the complement of A. By Lemma 2.1, $x \notin K, \notin K^*$ belongs to the set of (8) if and only if $||x - x_K|| \leq \lambda$ and $||x - x_{K^*}|| \leq \nu$, i.e., x is not more than λ distant

from the boundary surface ∂K of K and $\|x_K\| \le \nu$. Therefore the evaluation of mixed volumes is reduced to the evaluation of quermassintegrals, or more precisely, the volume of "local parallel sets" defined below in (9). In the case of polyhedral cones, the evaluation reduces to the evaluation of lower dimensional internal and external angles. On the other hand in the case of smooth cone the evaluation reduces to integral of principal curvatures on the boundary surface ∂K.

2.2 The case of smooth cone

One of the main objectives of this research is to characterize the weights of $\bar{\chi}^2$ distributions for cones with smooth boundaries. Although the characterization by the mixed volumes in Theorem 2.1 applies to smooth cones, the definition of mixed volumes is not necessarily easy to work with for computational purposes. Here we can use the result that the volume of local parallel set of a smooth cone K can be expressed as an integral of principal curvatures on ∂K. See section III.13.5 of Santaló (1976), Section 2.5 of Schneider (1993a), or Schneider (1993b). We summarize the result below.

Let K be a closed convex set with smooth boundary ∂K. For a relatively open subset S of ∂K the *local parallel set of* S *at distance* λ is defined as

$$A_\lambda(K,S) = \{x \mid x_K \in S \text{ and } 0 < \|x - x_K\| \le \lambda\}. \tag{9}$$

Let $H = H(s)$ be the positive semidefinite matrix of the second fundamental form at $s \in \partial K$ with respect to an orthonormal basis. The *principal curvatures* $\kappa_1, \ldots, \kappa_{p-1}$ are the eigenvalues of H. Denote the j-th trace of H, i.e., the j-th elementary symmetric function of the eigenvalues of H, by

$$\begin{aligned} \mathrm{tr}_j H = \mathrm{tr}_j H(s) &= \sum_{1 \le i_1 < \ldots < i_j \le p-1} \kappa_{i_1} \cdots \kappa_{i_j}, \quad j = 1, \ldots, p-1, \qquad (10) \\ \mathrm{tr}_0 H &\equiv 1, \end{aligned}$$

and let ds denote the ($p-1$ dimensional) volume element of ∂K. Then we have the following lemma.

Lemma 2.2 *(Formula (2.5.31) of Schneider (1993a))*

$$v_p(A_\lambda(K,S)) = \sum_{j=1}^{p} \lambda^j \frac{1}{j} \int_S \mathrm{tr}_{j-1} H(s) ds. \tag{11}$$

We now apply Lemma 2.2 to our problem. Let K be a closed convex cone with smooth boundary and $\mathrm{tr}_j H(s)$ be defined by (10) on the boundary ∂K. Consider the base set

$$S = \{s \mid s \in \partial K \text{ and } 0 < \|s\| < \nu\},$$

then $A_\lambda(K,S)$ equals to the set of (8) except for the boundary, i.e.,

$$\mathrm{int} A_\lambda(K,S) = \mathrm{int}\Big((\nu K_{(1)} + \lambda K^*_{(1)}) \cap (\nu K_{(1)})^C \cap (\lambda K^*_{(1)})^C\Big).$$

Note that for each $s \in \partial K$, ∂K contains a half line starting at the origin in the direction of s. Therefore the principal curvature for the direction s is 0 and $\mathrm{tr}_{p-1}H(s) = 0$. Other principal directions lie in the tangent space $T_s(\partial K \cap \partial(lU))$, where $l = \|s\|$. Furthermore because of the cone structure the integration on ∂K can be reduced to the product of integration on $\partial K \cap \partial U$ and the 1-dimensional integration with respect to l.

Theorem 2.2 *Let K be a closed convex cone with smooth boundary. Then the mixed volumes $v_{p-i,i}(K_{(1)}, K^*_{(1)})$, $1 \le i \le p-1$, in (6) of Theorem 2.1 is expressed as*

$$\binom{p}{i} v_{p-i,i}(K_{(1)}, K^*_{(1)}) = \frac{1}{i(p-i)} \int_{\partial K \cap \partial U} \mathrm{tr}_{i-1} H(u) du,$$

where du denotes the ($p-2$ dimensional) volume element of $\partial K \cap \partial U$.

Proof. Let $l = \|s\|$ for $s \in \partial K$. The half line in the direction of s and $T_s(\partial K \cap \partial(lU))$ are orthogonal and the volume element of $\partial K \cap \partial(lU)$ is $l^{p-2}du$. Therefore

$$ds = dl \times (l^{p-2} du).$$

The principal curvatures are inversely proportional to l, i.e., $\kappa_i(s) = \kappa_i(u)/l$, where $u = s/l$. Therefore

$$\mathrm{tr}_j H(s) = l^{-j} \mathrm{tr}_j H(u), \qquad l = \|s\|,\ u = s/l.$$

Then

$$\int_S \mathrm{tr}_{j-1} H(s) ds = \int_0^\nu \frac{l^{p-2}}{l^{j-1}} dl \int_{\partial K \cap \partial U} \mathrm{tr}_{j-1} H(u) du = \frac{\nu^{p-j}}{p-j} \int_{\partial K \cap \partial U} \mathrm{tr}_{j-1} H(u) du.$$

By (11)

$$v_p(A_\lambda(K, S)) = \sum_{j=1}^{p-1} \frac{\lambda^j \nu^{p-j}}{j(p-j)} \int_{\partial K \cap \partial U} \mathrm{tr}_{j-1} H(u) du.$$

Therefore

$$\binom{p}{j} v_{p-j,j}(K_{(1)}, K^*_{(1)}) = \frac{1}{j(p-j)} \int_{\partial K \cap \partial U} \mathrm{tr}_{j-1} H(u) du$$

and this proves the theorem. ∎

Remark 2.2 *Theorem 2.2 is stated in terms of K. However because of the duality of K and K^*, equivalent statement can be made in terms of K^*.*

Remark 2.3 *(The case of R^3 and the classical Gauss-Bonnet theorem)*
For $p = 3$ the mixed volumes take particularly simple form. Furthermore Shapiro's conjecture (Shapiro (1987)) reduces to the classical Gauss-Bonnet theorem. Let

$$P(\bar{\chi}^2_{01} \le a, \bar{\chi}^2_{12} \le b) = w_3 G_3(a) + w_2 G_2(a) G_1(b) + w_1 G_1(a) G_2(b) + w_0 G_3(b).$$

Then clearly

$$w_3 = \frac{\text{total area of } K \cap \partial U}{4\pi}, \qquad w_0 = \frac{\text{total area of } K^* \cap \partial U}{4\pi},$$

where 4π *is the total surface area of the unit sphere* ∂U *in* R^3 *. By Theorem 2.2*

$$\begin{aligned} w_2 &= \frac{1}{2\omega_1\omega_2}\int_{\partial K\cap\partial U} \mathrm{tr}_0 H(u)du = \frac{1}{4\pi}\int_{\partial K\cap\partial U} 1du \\ &= \frac{\text{total length of the curve } \partial K \cap \partial U}{4\pi} \end{aligned}$$

and considering K^*

$$w_1 = \frac{\text{total length of the curve } \partial K^* \cap \partial U}{4\pi}.$$

On the other hand by Theorem 2.2

$$w_1 = \frac{1}{4\pi}\int_{\partial K\cap\partial U} \kappa(u)du,$$

where $\kappa(u) = \mathrm{tr}_1 H(u)$ *is the geodesic curvature* $\kappa(u)$ *of the curve* $\partial K \cap \partial U$ *on* ∂U *. Since the Gaussian curvature is 1 on* ∂U *, the the classical Gauss-Bonnet theorem states*

$$2\pi = \int_{\partial K\cap\partial U} \kappa(u)du + (\text{total area of } K \cap \partial U).$$

Therefore

$$\frac{1}{2} = w_1 + w_3,$$

which is a particular case of Shapiro's conjecture.

Remark 2.4 *Shapiro's conjecture is known to hold for polyhedral cones. Because of the continuity of mixed volumes, Shapiro's conjecture holds for smooth or piecewise smooth cones as well.*

Example 2.1 *The case of spherical cone (Pincus (1975), Akkerboom (1990))*

$$K = \{\mu = (\mu_1, \ldots, \mu_p) \mid \frac{\mu_1}{||\mu||} \geq \cos\psi\}, \qquad 0 < \psi < \frac{\pi}{2}.$$

This is the spherical cone K_4 mentioned in Section 1.1. Using our geometric approach the weights of $\bar{\chi}^2$ distribution can be obtained in a straightforward manner. Let

$$F(x) = F(x_1, \ldots, x_p) = x_1^2 \sin^2\psi - (x_2^2 + \cdots + x_p^2)\cos^2\psi. \tag{12}$$

Then the boundary ∂K of K can be written as

$$\partial K = \{x \mid F(x) = 0,\ x_1 \geq 0\}.$$

By our Theorem 2.2 we consider a point $s \in \partial K,\ \|s\| = 1$. Because of spherical symmetry with respect to $x_2, \ldots, x_p$ we take $s^0 = (\cos\psi, \sin\psi, 0, \ldots, 0)$ as a representative point. The values of $\mathrm{tr}_j H(u)$ are the same for all $u \in \partial K \cap \partial U$. The outward unit normal vector at s^0 is easily seen to be

$$N_p = (-\sin\psi, \cos\psi, 0, \ldots, 0).$$

Consider the rotation of coordinates $(x_1, \ldots, x_p) \mapsto (u_1, \ldots, u_p)$

$$\begin{aligned} u_1 &= -\sin\psi\ x_1 + \cos\psi\ x_2, \\ u_2 &= \cos\psi\ x_1 + \sin\psi\ x_2, \\ u_i &= x_i, \quad i = 3, \ldots, p. \end{aligned}$$

Note that u_2 is the coordinate for the direction of s^0 . Substituting the inverse rotation $x_1 = -\sin\psi\ u_1 + \cos\psi\ u_2$, $x_2 = \cos\psi\ u_1 + \sin\psi\ u_2$ into (12), ∂K can be written as

$$\begin{aligned} F &= x_1^2 \sin^2\psi - x_2^2 \cos^2\psi - (x_3^2 + \cdots + x_p^2)\cos^2\psi \\ &= -u_1^2 \cos 2\psi - u_1 u_2 \sin 2\psi - (u_3^2 + \cdots + u_p^2)\cos^2\psi \\ &= 0. \end{aligned} \tag{13}$$

The particular point s^0 expressed in the new coordinates is

$$u^0 = (u_1^0, \ldots, u_p^0) = (0, 1, 0, \ldots, 0).$$

Now we want to calculate the elements of the second fundamental form

$$-\frac{\partial^2 u_1}{\partial u_i \partial u_j}, \quad i, j \geq 2. \tag{14}$$

Recall that s^0 itself is the principal direction with zero principal curvature and u_2 is the coordinate for this direction. Therefore actually we only need to calculate (14) for $i, j = 3, \ldots, p$. (Or one can easily verify that derivatives with respect to u_2 are indeed 0.) Now regard (13) as an equation determining u_1 in terms of $u_2, \ldots, u_p$. Taking partial derivative of (13) with respect to $u_i,\ i \geq 3$, we have

$$0 = \frac{\partial F}{\partial u_i} = -2\frac{\partial u_1}{\partial u_i} u_1 \cos 2\psi - \frac{\partial u_1}{\partial u_i} u_2 \sin 2\psi - 2u_i \cos^2\psi.$$

Differentiating this once more we obtain

$$0 = -2\frac{\partial^2 u_1}{\partial u_i \partial u_j} u_1 \cos 2\psi - \frac{\partial^2 u_1}{\partial u_i \partial u_j} u_2 \sin 2\psi - 2\delta_{ij} \cos^2\psi,$$

where δ_{ij} is the Kronecker's delta. Evaluating this at $u^0 = (0, 1, 0, \ldots, 0)$ we obtain

$$H(u^0) = \mathrm{diag}\Big(0, \underbrace{\frac{1}{\tan\psi}, \ldots, \frac{1}{\tan\psi}}_{p-2}\Big).$$

Therefore

$$\mathrm{tr}_j H(u^0) = \binom{p-2}{j} \frac{1}{\tan^j \psi}.$$

As mentioned earlier this value is the same for all u , i.e., $\mathrm{tr}_j H(u^0) = \mathrm{tr}_j H(u)$, $\forall u \in \partial K \cap \partial U$. Furthermore

$$\partial K \cap \partial U = \{x \mid x_1 = \cos\psi, x_2^2 + \cdots + x_p^2 = 1 - \cos^2\psi = \sin^2\psi\}.$$

Therefore the ($p-2$ dimensional) total volume of $\partial K \cap \partial U$ equals to the total surface volume of sphere of radius $\sin\psi$ in R^{p-1} , i.e.,

$$v_{p-2}(\partial K \cap \partial U) = v_{p-2}(\partial(\sin\psi U_{p-1})) = (p-1)\sin^{p-2}\psi\ \omega_{p-1}.$$

Combining the above results the weight for $\bar{\chi}^2$ distribution is

$$\begin{aligned} \binom{p}{i} v_{p-i,i}(K_{(1)}, K^*_{(1)}) &= \frac{1}{i(p-i)} \binom{p-2}{i-1} \frac{1}{\tan^{i-1}\psi} \times (p-1)\sin^{p-2}\psi\ \omega_{p-1} \\ &= \frac{(p-1)!}{i!(p-i)!} \omega_{p-1} \sin^{p-i-1}\psi\ \cos^{i-1}\psi. \end{aligned} \tag{15}$$

Further manipulation of (15) shows that

$$\binom{p}{i} \frac{v_{p-i,i}(K_{(1)}, K^*_{(1)})}{\omega_i \omega_{p-i}} = \frac{1}{2} \binom{p-2}{i-1} \frac{B(\frac{p-i}{2}, \frac{i}{2})}{B(\frac{1}{2}, \frac{p-1}{2})} \sin^{p-i-1}\psi\ \cos^{i-1}\psi,$$

which coincides with the result given by Pincus (1975).

2.3 The case of piecewise smooth cone

Here we consider an intermediate case between the polyhedral cone and everywhere smooth cone, namely a cone K whose boundary ∂K consists of both smooth surfaces and edges. This case is rather complicated but we need a full treatment of these cones to be able to discuss one important natural example of testing multivariate one-sided alternative for covariance matrices in Section 3. We could not find a ready reference of the needed theory (Theorem 2.3 below) in standard books on convex analysis. Therefore we give a sketch of the proof of Theorem 2.3 in Appendix.

To fix ideas let us consider a generalization of Example 2.1.

Example 2.2 *Let K be defined as*

$$K = \left\{ \mu \in R^p \mid \frac{\mu_1}{\|\mu\|} \geq \cos\psi_1 \text{ and } \frac{\mu_2}{\|\mu\|} \geq \cos\psi_2 \right\},$$

where

$$\cos^2\psi_1 + \cos^2\psi_2 < 1, \quad 0 < \psi_i < \frac{\pi}{2}, \ i = 1, 2, \quad p \geq 3.$$

In this example $K = K_1 \cap K_2$ where

$$K_i = \left\{ \mu \mid \frac{\mu_i}{\|\mu\|} \geq \cos\psi_i \right\}, \qquad i = 1, 2$$

are cones of Example 2.1. Note that ∂K *is no longer smooth at* $\partial K_1 \cap \partial K_2$ *. At a point* s *of the common boundary* $\partial K_1 \cap \partial K_2$ *, the outward unit normal vector is no longer unique and contribution to the mixed volume from* $s \in \partial K_1 \cap \partial K_2$ *can not be expressed as an integral with respect to the volume element of the* $p-1$ *dimensional surface of* ∂K *.*

Let K be a convex set. For each point s on the boundary ∂K of K, the *normal cone* $N(K,s)$ is defined as

$$N(K,s) = \{y \mid \langle y, z-s\rangle \le 0,\ \forall z \in K\} \tag{16}$$

(see Section 2.2 of Schneider (1993a)). Define

$$D_m(\partial K) = \{s \in \partial K \mid \dim N(K,s) = m\}, \quad m = 1,\dots,p.$$

Then

$$\partial K = D_1(\partial K) \cup \ldots \cup D_p(\partial K).$$

In Example 2.2, $D_2(\partial K) = \partial K_1 \cap \partial K_2$ and $D_1(\partial K)$ consists of 2 relatively open connected components $\mathrm{relint}(\partial K_1 \cap \partial K)$, $\mathrm{relint}(\partial K_2 \cap \partial K)$. Other D_i 's are empty. With Example 2.2 in mind, we make the following assumption on convex set K and we call such K *piecewise smooth.*

Assumption 2.1 $D_m(\partial K)$ *is a smooth* $p-m$ *dimensional manifold consisting of finite number of relatively open connected components. Furthermore* $N(K,s)$ *is continuous in* s *on* $D_m(\partial K)$ *in the sense of Lemma 1.2.*

Let $s \in D_m(\partial K)$. In a neighborhood of s we take an orthonormal system of vectors $e_1,\dots,e_{p-m},N_{p-m+1},\dots,N_p$ where $e_1,\dots,e_{p-m}$ constitute an orthonormal basis for the tangent space $T_s(D_m(\partial K))$ and $N_{p-m+1},\dots,N_p$ constitute an orthonormal basis for the orthogonal complement $T_s(D_m(\partial K))^\perp$ of $T_s(D_m(\partial K))$. Clearly $N(K,s) \subset T_s(D_m(\partial K))^\perp$.

Let

$$H_{ij}^\alpha, \quad i,j = 1,\dots,p-m,\ \alpha = p-m+1,\dots,p$$

be the element of the second fundamental tensor with respect to the chosen coordinate system. For a unit vector v in $T_s(D_m(\partial K))^\perp$

$$v = \sum_{\alpha=p-m+1}^{p} v_\alpha N_\alpha, \qquad \|v\| = 1,$$

define

$$H_{ij}(s,v) = \sum_{\alpha=p-m+1}^{p} v_\alpha H_{ij}^\alpha.$$

Furthermore let

$$\mathrm{tr}_j H(s,v) = \sum_{1\le i_1<\ldots<i_j\le p-m} \kappa_{i_1}(s,v)\cdots\kappa_{i_j}(s,v), \qquad j = 1,\dots,p-m,$$

where $\kappa_1(s,v),\ldots,\kappa_{p-m}(s,v)$ are eigenvalues of the $(p-m)\times(p-m)$ matrix $H_{ij}(s,v)$, i.e., the principal curvatures against a particular normal direction v at s.

We now generalize Lemma 2.2 to the case of piecewise smooth convex set. We use the same notation as in Lemma 2.2

Theorem 2.3 *Let K be a piecewise smooth closed convex set satisfying Assumption 2.1. Let ds_{p-m} denote the ($p-m$ dimensional) volume element of $D_m(\partial K)$ and let dv_{m-1} denote the $m-1$ dimensional volume element of the surface ∂U_m. Then*

$$v_p(A_\lambda(K,S)) = \sum_{m=1}^{p}\sum_{j=m}^{p}\lambda^j\frac{1}{j}\int_{S\cap D_m(\partial K)}\Big[\int_{N(K,s_{p-m})\cap\partial U}\mathrm{tr}_{j-m}H(s_{p-m},v_{m-1})dv_{m-1}\Big]ds_{p-m}. \tag{17}$$

For a sketch of the proof see Appendix. From Theorem 2.3 we obtain the corresponding result for our problem.

Theorem 2.4 *Let K be a closed convex cone satisfying Assumption 2.1. Let du_{p-m-1} denote the ($p-m-1$ dimensional) volume element of $D_m(\partial K)\cap\partial U$, $m=1,\ldots,p-1$. Then the mixed volumes $v_{p-i,i}(K_{(1)},K^*_{(1)})$, $1\le i\le p-1$, in (6) of Theorem 2.1 is expressed as*

$$\binom{p}{i}v_{p-i,i}(K_{(1)},K^*_{(1)}) = \frac{1}{i(p-i)} \times\sum_{m=1}^{i}\int_{D_m(\partial K)\cap\partial U}\Big[\int_{N(K,u_{p-m-1})\cap\partial U}\mathrm{tr}_{i-m}H(u_{p-m-1},v_{m-1})dv_{m-1}\Big]du_{p-m-1}. \tag{18}$$

Proof. It is easy to show that

$$N(K,s)=N(K,u),\qquad l=\|s\|,\ u=s/l.$$

As in the proof of Theorem 2.2

$$\mathrm{tr}_{j-m}H(s,v)=\mathrm{tr}_{j-m}H(u,v)/l^{j-m},$$

Therefore in (17)

$$\int_{N(K,s)\cap\partial U}\mathrm{tr}_{j-m}H(s_{p-m},v_{m-1})dv_{m-1} = \frac{1}{l^{j-m}}\int_{N(K,u_{p-m-1})\cap\partial U}\mathrm{tr}_{j-m}H(u_{p-m-1},v_{m-1})dv_{m-1}.$$

Moreover

$$ds_{p-m}=dl\times(l^{p-m-1}du_{p-m-1}).$$

Therefore for $S = \{s \mid s \in \partial K \text{ and } 0 < \|s\| < \nu\}$

$$\int_{S\cap D_m(\partial K)} \Big[\int_{N(K,s)\cap\partial U} \mathrm{tr}_{j-m} H(s_{p-m}, v_{m-1}) dv_{m-1}\Big] ds_{p-m}$$
$$= \int_0^\nu l^{p-j-1} dl \int_{D_m(\partial K)\cap\partial U} \Big[\int_{N(K,u_{p-m-1})\cap\partial U} \mathrm{tr}_{j-m} H(u_{p-m-1}, v_{m-1}) dv_{m-1}\Big] du_{p-m-1}$$
$$= \frac{\nu^{p-j}}{p-j} \int_{D_m(\partial K)\cap\partial U} \Big[\int_{N(K,u_{p-m-1})\cap\partial U} \mathrm{tr}_{j-m} H(u_{p-m-1}, v_{m-1}) dv_{m-1}\Big] du_{p-m-1}.$$

It follows that

$$\binom{p}{j} v_{p-j,j} = \frac{1}{j(p-j)} \times \sum_{m=1}^{j} \int_{D_m(\partial K)\cap\partial U} \Big[\int_{N(K,u_{p-m-1})\cap\partial U} \mathrm{tr}_{j-m} H(u_{p-m-1}, v_{m-1}) dv_{m-1}\Big] du_{p-m-1}$$

and this proves the theorem. ∎

Example 2.2 *(continued)*

Using Theorem 2.4 we evaluate the weights of $\bar{\chi}^2$ distribution. First we consider $D_1(\partial K) = \mathrm{relint}(\partial K_1 \cap \partial K) \cup \mathrm{relint}(\partial K_2 \cap \partial K)$. Note that $\mathrm{relint}(\partial K_1 \cap \partial K) = \partial K_1 \cap \mathrm{int} K_2$. Therefore

$$\mathrm{relint}(\partial K_1 \cap \partial K) \cap \partial U = \{x \mid x_1 = \cos\psi_1,\ x_2 > \cos\psi_2,\ \|x\| = 1\}.$$

Now consider the following ratio of volumes

$$\frac{v_{p-2}(\{(x_2,\dots,x_p) \mid x_2 > \cos\psi_2, x_2^2 + \cdots + x_p^2 = \sin^2\psi_1\})}{v_{p-2}(\{(x_2,\dots,x_p) \mid x_2^2 + \cdots + x_p^2 = \sin^2\psi_1\})}.$$

This is obviously equal to the following incomplete beta function

$$\beta_1 = \frac{1}{2} \int_{\cos^2\psi_2/\sin^2\psi_1}^{1} u^{-\frac{1}{2}} (1-u)^{\frac{p-4}{2}} du. \tag{19}$$

The contribution to the weights from $\partial K_1 \cap \partial K \cap \partial U$ is just as (15) multiplied by β_1 with $\psi = \psi_1$. Similarly the contribution from $\partial K_2 \cap \partial K \cap U$ is (15) multiplied by β_2 with $\psi = \psi_2$, where

$$\beta_2 = \frac{1}{2} \int_{\cos^2\psi_1/\sin^2\psi_2}^{1} u^{-\frac{1}{2}} (1-u)^{\frac{p-4}{2}} du. \tag{20}$$

It remains to evaluate the contribution from $\partial K_1 \cap \partial K_2$. Consider a representative point

$$s^0 = (\cos\psi_1, \cos\psi_2, \tau, 0, \dots, 0),$$

where

$$\tau = \sqrt{1 - \cos^2\psi_1 - \cos^2\psi_2}. \tag{21}$$

The outward unit normal vector to K_1 at s^0 is

$$n_1 = \Big(-\sin\psi_1, \frac{\cos\psi_2}{\tan\psi_1}, \frac{\cos\psi_2}{\tan\psi_1}\tau, 0, \dots, 0\Big).$$

Similarly the outward unit normal vector to K_2 at s^0 is

$$\Big(\frac{\cos\psi_1}{\tan\psi_2}, -\sin\psi_2, \frac{\cos\psi_1}{\tan\psi_2}\tau, 0, \ldots, 0\Big).$$

The normal cone $N(K, s^0)$ is the positive combination of these two vectors

$$N(K, s^0) = an_1 + bn_2, \quad a, b \geq 0.$$

The inner product of these two vectors is

$$\langle n_1, n_2\rangle = -\frac{1}{\tan\psi_1 \tan\psi_2}.$$

Let

$$N_{p-1} = n_1, \quad N_p = \Big(0, -\frac{\tau}{\sin\psi_1}, \frac{\cos\psi_2}{\sin\psi_1}, 0, \ldots, 0\Big).$$

Then N_{p-1}, N_p form an orthonormal basis of $T_{s^0}(D_2(\partial K))^\perp$. Now consider the rotation of coordinates based on N_{p-1}, N_p and s^0 :

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -\sin\psi_1 & \frac{\cos\psi_2}{\tan\psi_1} & \frac{\cos\psi_2}{\tan\psi_1}\tau \\ 0 & -\frac{\tau}{\sin\psi_1} & \frac{\cos\psi_2}{\sin\psi_1} \\ \cos\psi_1 & \cos\psi_2 & \tau \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
$$= \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

and $u_i = x_i, i = 4, \ldots, p$ with the inverse transformation

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{21} & g_{31} \\ g_{12} & g_{22} & g_{32} \\ g_{13} & g_{23} & g_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

s^0 in the new coordinates is

$$u^0 = (0, 0, 1, 0, \ldots, 0).$$

Now consider (12) for K_1 and K_2 :

$$0 = F_1 = x_1^2 \sin\psi_1^2 - (x_2^2 + x_3^2)\cos^2\psi_1 - (u_4^2 + \cdots + u_p^2)\cos^2\psi_1, \tag{22}$$
$$0 = F_2 = x_2^2 \sin\psi_2^2 - (x_1^2 + x_3^2)\cos^2\psi_2 - (u_4^2 + \cdots + u_p^2)\cos^2\psi_2. \tag{23}$$

In (22) and (23) x_1, x_2, x_3 are functions of u_1, u_2, u_3 , i.e., $x_i = \sum_{j=1}^3 g_{ji}u_j$. We regard (22) and (23) as a system of equations for determining u_1, u_2 in terms of $u_3, \ldots, u_p$. Furthermore as in Example 2.1 we can ignore differentiation with respect to u_3 and we differentiate (22) and (23) with respect to $u_4, \ldots, u_p$. At u^0

$$0 = \frac{\partial u_1}{\partial u_i}\Big|_{u^0} = \frac{\partial u_2}{\partial u_i}\Big|_{u^0}, \quad i \geq 4.$$

Therefore

$$0 = \frac{\partial x_j}{\partial u_i}\Big|_{u^0} = 0, \quad i \geq 4, \; j = 1, 2, 3.$$

Using this it can be easily shown that $0 = \partial^2 F_1/(\partial u_i \partial u_j)$, $i, j \geq 4$, evaluated at u^0 reduces to

$$0 = -2\frac{\partial^2 u_1}{\partial u_i \partial u_j} \cos\psi_1 \sin\psi_1 - 2\delta_{ij}\cos^2\psi_1 \tag{24}$$

and $0 = \partial^2 F_2/(\partial u_i \partial u_j)$ evaluated at u^0 reduces to

$$0 = 2\frac{\partial^2 u_1}{\partial u_i \partial u_j}\frac{\cos^2\psi_2}{\tan\psi_1} - 2\frac{\partial^2 u_2}{\partial u_i \partial u_j}\frac{\tau\cos\psi_2}{\sin\psi_1} - 2\delta_{ij}\cos^2\psi_2. \tag{25}$$

Solving (24) and (25) we obtain

$$\begin{aligned} -\frac{\partial^2 u_1}{\partial u_i^2} &= \frac{1}{\tan\psi_1}, \\ -\frac{\partial^2 u_2}{\partial u_i^2} &= \frac{\cos^2\psi_2}{\sin^2\psi_1}. \end{aligned}$$

All the other second order derivatives evaluated at u^0 are 0.

Let

$$\theta_0 = \arccos\Big(-\frac{1}{\tan\psi_1 \tan\psi_2}\Big), \qquad \frac{\pi}{2} < \theta_0 < \pi.$$

Then $v \in N(k, s^0)$, $\|v\| = 1$ can be written as

$$v = \cos\theta N_{p-1} + \sin\theta N_p, \qquad 0 < \theta < \theta_0.$$

Therefore

$$H(s^0, v) = \text{diag}(0, \underbrace{h(\theta, \psi_1, \psi_2), \ldots, h(\theta, \psi_1, \psi_2)}_{p-3}),$$

where

$$h(\theta, \psi_1, \psi_2) = \cos\theta\frac{1}{\tan\psi_1} + \sin\theta\frac{\cos^2\psi_2}{\sin^2\psi_1}$$

and we obtain

$$\text{tr}_j H(s^0, v) = \binom{p-3}{j} h(\theta, \psi_1, \psi_2)^j.$$

Therefore

$$\int_{N(K,s^0)\cap\partial U} \text{tr}_j H(x^0, v_1) dv_1 = \binom{p-3}{j} \int_0^{\theta_0} h(\theta, \psi_1, \psi_2)^j d\theta. \tag{26}$$

The value of (26) is the same for all $s \in \partial K_1 \cap \partial K_2 \cap \partial U$ and

$$v_{p-3}(\partial K_1 \cap \partial K_2 \cap \partial U) = (p-2)\tau^{p-3}\omega_{p-2}.$$

Therefore the contribution from $\partial K_1 \cap \partial K_2$ to the mixed volume $\binom{p}{i} v_{p-i,i}(K_{(1)}, K^*_{(1)})$ is obtained as

$$\binom{p-3}{i-2}\frac{1}{i(p-i)}\int_0^{\theta_0} h(\theta, \psi_1, \psi_2)^{i-2} d\theta \times (p-2)\tau^{p-3}\omega_{p-2}.$$

Summarizing the above calculations the mixed volume is

$$\binom{p}{i} v_{p-i,i}(K_{(1)}, K^*_{(1)}) = \frac{(p-1)!}{i!(p-i)!}\omega_{p-1}(\beta_1 \sin^{p-i-1}\psi_1 \cos^{i-1}\psi_1 + \beta_2 \sin^{p-i-1}\psi_2 \cos^{i-1}\psi_2) + \frac{(i-1)(p-2)!}{i!(p-i)!}\tau^{p-3}\omega_{p-2}\int_0^{\theta_0} h(\theta,\psi_1,\psi_2)^{i-2}d\theta.$$

where τ is defined in (21) and β_1, β_2 are defined in (19),(20). Note that the last term vanishes for $i = 1$.

3 The cone of non-negative definite matrices

In this section, we treat the cone of non-negative definite matrices, which is a typical example of the piecewise smooth cone. We reveal the "recurrence structure" of the singularities of the cone of non-negative definite matrices which plays an essential role in the derivation of the mixed volumes and the weights of $\bar{\chi}^2$ distribution.

3.1 Testing problem and $\bar{\chi}^2$ statistic

Let $A = (a_{ij})$ be a $p \times p$ symmetric random matrix whose components are independently distributed as $a_{ii} \sim N(\mu_{ii}, 1)$ and $\sqrt{2}\, a_{ij} \sim N(\sqrt{2}\, \mu_{ij}, 1)$ $(i < j)$. The joint density of A is

$$\frac{1}{2^{p/2}\pi^{p(p+1)/4}} \exp\Big\{ -\frac{1}{2}\mathrm{tr}(A-M)^2 \Big\},$$

where $M = (\mu_{ij})$ is the mean matrix.

Let K be the cone formed by the $p \times p$ non-negative matrices, i.e.,

$$K = \{W : p \times p \,|\, W \geq O\},$$

where $\geq$ denotes the *Löwner order*. The likelihood ratio tests we consider here are

$$H_0 : M = O \quad \text{vs.} \quad H_1 : M \in K, \tag{27}$$

and

$$H_1 : M \in K \quad \text{vs.} \quad H_2 : M \in \mathcal{S}_p, \tag{28}$$

where $\mathcal{S}_p$ is the set of $p \times p$ symmetric matrices.

The test statistics are shown to be

$$\bar{\chi}^2_{01} = \sum_{l_i > 0} l_i^{\,2} \qquad \text{and} \qquad \bar{\chi}^2_{12} = \sum_{l_i < 0} l_i^{\,2}, \tag{29}$$

where $l_1 > \cdots > l_p$ are the eigenvalues of the random matrix A . In this case, the marginal distributions of $\bar{\chi}^2_{01}$ and $\bar{\chi}^2_{12}$ under H_0 are the same because of the facts that the distribution of $-A$ is equivalent to A and that K is self-dual, i.e., $K^* = -K$. In our previous paper, Kuriki (1993) gave an expression for the weights $\{w_d\}$ as well as a method to evaluate them numerically. In this section, we will see that the weights derived by Theorem 2.4 give the same results as Kuriki (1993).

The testing problems (27) and (28), and the corresponding distributions of $\bar{\chi}^2_{01}$ and $\bar{\chi}^2_{12}$ in (29) arise as the limit of the likelihood ratio tests for the multivariate variance components in multivariate one-way random effects model with equal replications when the block size goes to infinity. See Kuriki (1993) and the references therein.

3.2 The second fundamental form

We identify $\mathcal{S}_p$ with $R^{p(p+1)/2}$ by the map

$$W = (w_{ij}) \in \mathcal{S}_p \leftrightarrow (w_{11}, \ldots, w_{pp}, \sqrt{2}w_{12}, \ldots, \sqrt{2}w_{p-1,p}) \in R^{p(p+1)/2}$$

and the corresponding inner product

$$\operatorname{tr} W_1 W_2 = \sum_i w_{1ii}w_{2ii} + \sum_{i<j}(\sqrt{2}w_{1ij})(\sqrt{2}w_{2ij}) \tag{30}$$

for $W_1 = (w_{1ij})$, $W_2 = (w_{2ij}) \in \mathcal{S}_p$. Define

$$\mathcal{S}_{r,p} = \{W \in \mathcal{S}_p \,|\, \operatorname{rank} W = r\},$$

and

$$\begin{aligned} \mathcal{S}^+_{r,p} &= \mathcal{S}_{r,p} \cap K \\ &= \{W \in \mathcal{S}_p \,|\, W \geq O, \ \operatorname{rank} W = r\}. \end{aligned}$$

$\mathcal{S}^+_{p-1,p}$ is the smooth surface of the boundary ∂K of K . $\mathcal{S}^+_{r,p}$, $r = 1, \ldots, p-2$, form singularities of ∂K . This can be shown by identifying the normal cone at any fixed point $W_0 \in \mathcal{S}^+_{r,p}$.

The spectral decomposition of W_0 is $W_0 = H_{10}\Lambda_0 H_{10}'$, where $\Lambda_0 = \operatorname{diag}(l_{10}, \ldots, l_{r0})$ with $l_{10} \geq \cdots \geq l_{r0} > 0$ and H_{10} is a $p \times r$ matrix such that $H_{10}'H_{10} = I_r$. Let H_{20} be a $p \times (p-r)$ matrix such that $H_0 = (H_{10}, H_{20})$ is $p \times p$ orthogonal.

Lemma 3.1 *The normal cone (16) of K at $W_0 \in \mathcal{S}^+_{r,p}$ is*

$$\begin{aligned} N(K, W_0) &= \{-H_{20}Y_{22}H_{20}' \,|\, Y_{22} \geq O\} \\ &= \Big\{ -H_0 Y H_0' \,\Big|\, Y = \begin{pmatrix} O & O \\ O & Y_{22} \end{pmatrix}, \ Y_{22} \geq O \Big\} \end{aligned}$$

with the dimension

$$\dim N(K, W_0) = (p-r)(p-r+1)/2.$$

Proof. Put

$$M(W_0) = \Big\{ -H_0 Y H_0' \,\Big|\, Y = \begin{pmatrix} O & O \\ O & Y_{22} \end{pmatrix}, \ Y_{22} \geq O \Big\}.$$

From the definition of

$$N(K, W_0) = \Big\{ -H_0 Y H_0' \,\Big|\, \operatorname{tr} \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{12}' & Y_{22} \end{pmatrix} \begin{pmatrix} Z_{11} - \Lambda_0 & Z_{12} \\ Z_{12}' & Z_{22} \end{pmatrix} \geq 0, \ \forall Z \in K \Big\},$$

it holds obviously that

$$N(K,W_0) \supset M(W_0).$$

The proof of the converse is as follows. Fix a point in $\mathcal{S}_p$ as

$$-H_0VH_0' = -\begin{pmatrix} H_{10} & H_{20} \end{pmatrix}\begin{pmatrix} V_{11} & V_{12} \\ V_{12}' & V_{22} \end{pmatrix}\begin{pmatrix} H_{10}' \\ H_{20}' \end{pmatrix} \in \mathcal{S}_p$$

such that

$$-H_0VH_0 \notin M(W_0). \tag{31}$$

Case 1) If V_{22} is not non-negative definite, there exists $-\lambda < 0$, a negative eigenvalue of V_{22}, and the corresponding eigenvector v. Putting

$$Z = \begin{pmatrix} \Lambda_0 & O \\ O & vv' \end{pmatrix} \in K,$$

we see that

$$\operatorname{tr}\Big\{V\Big(Z - \begin{pmatrix} \Lambda_0 & O \\ O & O \end{pmatrix}\Big)\Big\} = -\lambda v'v < 0.$$

Case 2) If $V_{11} \neq O$, we can choose $\varepsilon > 0$ such that

$$Z = \begin{pmatrix} \Lambda_0 - \varepsilon V_{11} & O \\ O & O \end{pmatrix} \in K$$

and

$$\operatorname{tr}\Big\{V\Big(Z - \begin{pmatrix} \Lambda_0 & O \\ O & O \end{pmatrix}\Big)\Big\} = -\varepsilon \operatorname{tr} V_{11}^2 < 0.$$

Case 3) If $V_{11} = O$ and $V_{12} \neq O$, we can choose a sufficiently small number $\varepsilon > 0$ such that

$$Z = \begin{pmatrix} \Lambda_0 + I_r & -\varepsilon V_{12} \\ -\varepsilon V_{12}' & \varepsilon^2 V_{12}' V_{12} \end{pmatrix} \in K$$

and

$$\operatorname{tr}\Big\{V\Big(Z - \begin{pmatrix} \Lambda_0 & O \\ O & O \end{pmatrix}\Big)\Big\} = -2\varepsilon \operatorname{tr} V_{12}V_{12}' + \varepsilon^2 \operatorname{tr} V_{12}V_{22}V_{12}' < 0.$$

The three cases 1-3 above cover (31) and we obtain

$$N(K,W_0) \subset M(W_0).$$

This completes the proof. ∎

Remark 3.1 *We see that the normal cone at each point of singularity of ∂K is a lower dimensional replica of the original cone K.*

Now we proceed to derive the second fundamental form at W_0. In order to do this we introduce a local coordinate system $X = (x_{ij}) = \begin{pmatrix} X_{11} & X_{12} \\ X_{12}' & X_{22} \end{pmatrix}$ of $\mathcal{S}_p$ in the neighborhood of W_0 as

$$\begin{aligned} \mathcal{S}_p \ni W &= W_0 + H_0XH_0' \\ &= \begin{pmatrix} H_{10} & H_{20} \end{pmatrix}\begin{pmatrix} \Lambda_0 + X_{11} & X_{12} \\ X_{12}' & X_{22} \end{pmatrix}\begin{pmatrix} H_{10}' \\ H_{20}' \end{pmatrix}. \end{aligned}$$

We note here that for a $p \times p$ orthogonal matrix H, the transform $W \mapsto HWH'$ is orthogonal and preserves the inner product (30), because $\mathrm{tr}\,(HW_1H')(HW_2H') = \mathrm{tr}\,W_1W_2$. So, the new coordinate system X, i.e., $(x_{11},\ldots,x_{pp},\sqrt{2}x_{12},\ldots,\sqrt{2}x_{p-1,p})$, is also orthonormal.

Here we can take $\partial/\partial x_{ii}$ $(r+1 \le i \le p)$, $\partial/\partial(\sqrt{2}x_{ij})$ $(r+1 \le i < j \le p)$ as an orthonormal basis of $N(K,W_0)$, and therefore, $\partial/\partial x_{ii}$ $(1 \le i \le r)$, $\partial/\partial(\sqrt{2}x_{ij})$ $(1 \le i \le r,\ i<j \le p)$ as an orthonormal basis of $N(K,W_0)^{\perp} = T_{W_0}(\mathcal{S}^{+}_{r,p})$.

In the neighborhood of W_0, $W \in \mathcal{S}^{+}_{r,p}$ is equivalent to

$$X_{22} = X_{12}'(\Lambda_0 + X_{11})^{-1}X_{12},$$

because $\Lambda_0 + X_{11}$ is positive definite in the neighborhood of W_0. Fix $\tilde{W} = -H_{20}YH_{20}' \in N(K,W_0)$. Then, the second fundamental form with respect to the normal direction $\tilde{W}$ becomes

$$H(W_0,\tilde{W}) = \left.\frac{\partial^2\,\mathrm{tr}\,(YX_{22})}{\partial((x_{ii})_{1\le i\le r},(\sqrt{2}x_{ij})_{1\le i\le r,\ i<j\le p})^2}\right|_{W_0}. \qquad (32)$$

The $(k-r,l-r)$-th element of X_{22} is

$$\begin{aligned} x_{kl} &= (X_{12}'(\Lambda_0+X_{11})^{-1}X_{12})_{kl} \\ &= (\,x_{1k}\ \cdots\ x_{rk}\,)(\Lambda_0+X_{11})^{-1}\begin{pmatrix} x_{1l} \\ \vdots \\ x_{rl}\end{pmatrix}, \qquad r+1 \le k \le l \le p. \qquad (33)\end{aligned}$$

Differentiating (33) twice with respect to $(x_{ii})_{1\le i\le r}$, $(\sqrt{2}x_{ij})_{1\le i\le r,\ i<j\le p}$, and putting $X_{11} = O$ and $X_{12} = O$, we see that the non-vanishing terms of (32) are only

$$\left.\frac{\partial^2 x_{kl}}{\partial(\sqrt{2}x_{ik})\partial(\sqrt{2}x_{jl})}\right|_{W_0} = \frac{\delta_{ij}}{l_{i0}}\cdot\frac{1+\delta_{kl}}{2},$$

$1 \le i \le j \le r$, $r+1 \le k \le l \le p$. So

$$\left.\frac{\partial^2\,\mathrm{tr}\,(YX_{22})}{\partial(\sqrt{2}x_{ik})\partial(\sqrt{2}x_{jl})}\right|_{W_0} = \frac{\delta_{ij}}{l_{i0}}\cdot y_{kl}$$

with $Y = (y_{kl})$, and other contributions are zero. Now we have established the following lemma.

Lemma 3.2 *The non-vanishing part of the second fundamental form at* $W_0 = H_{10}\Lambda_0H_{10}' \in \mathcal{S}^{+}_{r,p}$ *with respect to the direction* $\tilde{W} = -H_{20}YH_{20}' \in N(K,W_0)$ *is*

$$H(W_0,\tilde{W}) = \left(\frac{\delta_{ij}}{l_{i0}}\cdot y_{kl}\right) = \Lambda_0^{-1}\otimes Y.$$

Here $H_0 = (H_{10},H_{20})$ *is* $p\times p$ *orthogonal, and* $\otimes$ *denotes the Kronecker product.*

Let $\tilde{\Lambda} = \mathrm{diag}(\tilde{l}_1,\ldots,\tilde{l}_{p-r})$ be the eigenvalues of Y. Concerning the m-th trace

$$\mathrm{tr}_m H = \mathrm{tr}_m(\Lambda_0^{-1}\otimes Y) = \mathrm{tr}_m(\Lambda_0^{-1}\otimes\tilde{\Lambda}),$$

the following lemma holds.

Lemma 3.3 *For* $\Lambda = \mathrm{diag}(l_i)_{1\le i\le r}$ *and* $\tilde{\Lambda} = \mathrm{diag}(\tilde{l}_i)_{1\le i\le p-r}$

$$\det(\Lambda)^{p-r}\mathrm{tr}_m({\Lambda_0}^{-1}\otimes\tilde{\Lambda}) = \sum_{(q,\bar{q})} \frac{\det({l_i}^{q_j})_{1\le i,j\le r}}{\prod_{1\le i<j\le r}(l_i - l_j)} \cdot \frac{\det(\tilde{l}_i^{\bar{q}_j})_{1\le i,j\le p-r}}{\prod_{1\le i<j\le p-r}(\tilde{l}_i - \tilde{l}_j)},$$

where the summation $\sum_{(q,\bar{q})}$ *is over the set of integers*

$$(q_1,\ldots,q_r,\bar{q}_1,\ldots,\bar{q}_{p-r}) \in Q_{r,p}(-m + r(p-r) + r(r-1)/2)$$

with

$$Q_{r,p}(n) = \{(q_1,\ldots,q_r,\bar{q}_1,\ldots,\bar{q}_{p-r}) \in \pi_p \,|\, q_1 > \cdots > q_r,\ \bar{q}_1 > \cdots > \bar{q}_{p-r},\ \sum_{j=1}^{r} q_j = n\}$$

and π_p *denotes the set of all permutations of* $\{p-1, p-2, \ldots, 0\}$.

Proof. Define the generating function by

$$\Phi(x) = \sum_{m=0}^{r(p-r)} (-1)^m x^{r(p-r)-m}\det(\Lambda)^{p-r}\mathrm{tr}_m({\Lambda_0}^{-1}\otimes\tilde{\Lambda}).$$

Then

$$\begin{aligned}
\Phi(x) &= \det(\Lambda)^{p-r}\det(xI_r\otimes I_{p-r} - {\Lambda_0}^{-1}\otimes\tilde{\Lambda}) \\
&= (\prod_{i=1}^{r} {l_i}^{p-r}) \cdot \prod_{i=1}^{r}\prod_{j=1}^{p-r}\left(x - \frac{\tilde{l}_j}{l_i}\right) \\
&= \prod_{i=1}^{r}\prod_{j=1}^{p-r}(l_i x - \tilde{l}_j) \\
&= \det\begin{pmatrix} (xl_1)^{p-1} & \cdots & xl_1 & 1 \\ \vdots & & \vdots & \vdots \\ (xl_r)^{p-1} & \cdots & xl_r & 1 \\ \tilde{l}_1^{p-1} & \cdots & \tilde{l}_1 & 1 \\ \vdots & & \vdots & \vdots \\ \tilde{l}_{p-r}^{p-1} & \cdots & \tilde{l}_{p-r} & 1 \end{pmatrix} \Big/ \prod_{1\le i<j\le r}(xl_i - xl_j) \prod_{1\le i<j\le p-r}(\tilde{l}_i - \tilde{l}_j). \quad (34)
\end{aligned}$$

By the Laplace expansion of the determinant in (34), we have

$$\begin{aligned}
\Phi(x) &= \sum_{(q,\bar{q})} (-1)^{\frac{1}{2}r(r+1)+\sum_{j=1}^{r}(p-q_j)} \frac{\det(x{l_i}^{q_j})_{1\le i,j\le r}}{\prod_{1\le i<j\le r}(xl_i - xl_j)} \cdot \frac{\det(\tilde{l}_i^{\bar{q}_j})_{1\le i,j\le p-r}}{\prod_{1\le i<j\le p-r}(\tilde{l}_i - \tilde{l}_j)} \\
&= \sum_{(q,\bar{q})} (-1)^{\frac{1}{2}r(r+1)+\sum_{j=1}^{r}(p-q_j)} x^{\sum_{j=1}^{r} q_j - \frac{1}{2}r(r+1)} \\
&\qquad \times \frac{\det({l_i}^{q_j})_{1\le i,j\le r}}{\prod_{1\le i<j\le r}(l_i - l_j)} \cdot \frac{\det(\tilde{l}_i^{\bar{q}_j})_{1\le i,j\le p-r}}{\prod_{1\le i<j\le p-r}(\tilde{l}_i - \tilde{l}_j)}.
\end{aligned}$$

Comparing the coefficient of the term $(-1)^m x^{r(p-r)-m}$, we prove the lemma. ∎

Remark 3.2 *The polynomial* $\det(l_i^{q_j})/\prod(l_i - l_j)$ *is the Schur function, which is symmetric and homogeneous in* l_i *(Macdonald (1995)).*

Remark 3.3 *Lemma 3.3 allows us to separate the integration with respect to* l_i *'s and the integration with respect to* $\tilde{l}_i$ *'s. It also reflects the selfduality of* K *.*

3.3 Volume element in $\mathcal{S}_{r,p}$

To evaluate the mixed volumes in virtue of Theorems 2.3 or 2.4, we have to know the concrete forms of the volume elements of $\mathcal{S}_{r,p}$ or $\mathcal{S}_{r,p} \cap \partial U$.

Before proceeding to derive the volume elements, we prepare several facts on Stiefel manifold. Let $\mathcal{V}_{r,p} = \{H_1 : p \times r \,|\, H_1'H_1 = I_r\}$ be the Stiefel manifold. Regarded as a subset of R^{pr} , the volume element of $\mathcal{V}_{r,p}$ is given as follows. Let H_2 be $p \times (p-r)$ such that $H = (H_1, H_2) = (h_1, \ldots, h_r, h_{r+1}, \ldots, h_p)$ is $p \times p$ orthogonal. Then the volume element of $\mathcal{V}_{r,p}$ is

$$dH_1 = \bigwedge_{i=1}^{r} \bigwedge_{j=i+1}^{p} h_j' dh_i. \tag{35}$$

The total ($r(r-1)/2 + r(p-r)$ dimensional) volume of $\mathcal{V}_{r,p}$ is

$$v_{r(r-1)/2+r(p-r)}(\mathcal{V}_{r,p}) = \frac{2^r \pi^{pr/2}}{\Gamma_r(p/2)},$$

where

$$\Gamma_r(p/2) = \pi^{r(r-1)/4} \Gamma\Big(\frac{p}{2}\Big) \Gamma\Big(\frac{p-1}{2}\Big) \cdots \Gamma\Big(\frac{p-r+1}{2}\Big)$$

(Muirhead (1982), pages 62–70).

Let

$$W = (w_{ij}) = H_1 \Lambda H_1' \in \mathcal{S}_{r,p},$$

where $\Lambda = \mathrm{diag}(l_i)_{1 \le i \le r}$, $l_1 \ge \cdots \ge l_r$, and $H_1 \in \mathcal{V}_{r,p}$. Then, the volume element of $\mathcal{S}_{r,p}$ can be written as follows.

Lemma 3.4 *The volume element of* $\mathcal{S}_{r,p}$ *is*

$$dW_{r,p} = 2^{r(r-1)/4 + r(p-r)/2} \prod_{1 \le i < j \le r} (l_i - l_j) \prod_{i=1}^{r} l_i^{p-r} \prod_{i=1}^{r} dl_i \, dH_1,$$

where dH_1 *is the volume element of* $\mathcal{V}_{r,p}$ *defined in (35).*

Proof. Proof is similar to the derivation of the second fundamental form in Section 3.2. Fix an arbitrary point $W_0 \in \mathcal{S}_{r,p}$ and write $W_0 = H_{10} L_0 H_{10}'$, $L_0 = \mathrm{diag}(l_{10}, \ldots, l_{r0})$, $l_{10} \ge \cdots \ge l_{r0}$. We want to obtain the volume element at W_0 . Fix some H_{20} such that $H_0 = (H_{10}, H_{20}) = (h_1, \ldots, h_r, h_{r+1}, \ldots, h_p)$ is $p \times p$ orthogonal and take the elements of $\tilde{W} = H_0' W H_0$, $W \in \mathcal{S}_p$, as a local coordinate system.

Now we consider the elements of $d\tilde{W} = d(H_0{}'WH_0)$ where W (and hence $H_0{}'WH_0$) is restricted in $\mathcal{S}_{r,p}$. Write $dH = (dH_1, dH_2)$ and $dL = \mathrm{diag}(dl_1, \ldots, dl_r, 0, \ldots, 0)$. Then

$$\begin{aligned} d\tilde{W} &= d(H_0{}'WH_0) \\ &= H_0{}'dWH_0 \\ &= H_0{}'(dH\,\mathrm{diag}(l_{10}, \ldots, l_{r0}, 0, \ldots, 0)H_0{}' + H_0 dLH_0' \\ &\qquad + H_0\,\mathrm{diag}(l_{10}, \ldots, l_{r0}, 0, \ldots, 0)dH')H_0 \\ &= H_0{}'dH_1L_0 + dL + L_0\,dH_1{}'H_0. \end{aligned}$$

It is seen that the $(p-r) \times (p-r)$ lower-right block consists of 0's, i.e., $d\tilde{w}_{ij} = 0$ $(r+1 \le i,\, j)$. Therefore under the chosen local coordinate system, we can take $\partial/\partial\tilde{w}_{ij}$ $(1 \le i \le r\,,\; i < j \le p)$ as basis for the tangent space $T_{W_0}(\mathcal{S}_{r,p})$. Taking the exterior product, the volume element of $\mathcal{S}_{r,p}$ at W_0 is evaluated as

$$dW = \bigwedge_{i=1}^{r} d\tilde{w}_{ii} \bigwedge_{i=1}^{r} \bigwedge_{j=i+1}^{p} d(\sqrt{2}\tilde{w}_{ij}).$$

Now as on page 105 of Muirhead (1982)

$$\begin{aligned} &\tilde{w}_{ii} = dl_i, && 1 \le i \le r, \\ &\tilde{w}_{ij} = (l_i - l_j)h_j{}'dh_i, && 1 \le i < j \le r, \\ &\tilde{w}_{ij} = l_i h_j{}'dh_i, && 1 \le i \le r, \quad r+1 \le j \le p. \end{aligned}$$

Therefore

$$dW = 2^{r(r-1)/4 + r(p-r)/2} \prod_{1\le i<j\le r} (l_i - l_j) \prod_{i=1}^{r} l_i{}^{p-r} \prod_{i=1}^{r} dl_i \bigwedge_{i=1}^{r} \bigwedge_{j=i+1}^{p} h_j{}'dh_i$$

and this proves the lemma. ∎

Corollary 3.1 *The volume element of* $\mathcal{S}_{r,p} \cap \partial U$ *is*

$$dU_{r,p} = 2^{r(r-1)/4 + r(p-r)/2} \prod_{1\le i<j\le r} (l_i - l_j) \prod_{i=1}^{r} l_i{}^{p-r} d\mu_r(l)\, dH_1,$$

where $d\mu_r(l)$ *is the volume element of the surface of the unit ball*

$$\{(l_1, \ldots, l_r) \,|\, l_1{}^2 + \cdots + l_r{}^2 = 1\}.$$

Remark 3.4 *As mentioned in Muirhead (1982), we have to be careful that the sign of each* h_i *is not uniquely determined. If we integrate with respect to* dH_1 *over the whole* $\mathcal{V}_{r,p}$ *, then the same* W *is counted* 2^r *times. Therefore when integrating with respect to* $dW_{r,p}$ *, we have to divide by* 2^r *, e.g.*

$$\int_{\mathcal{S}_{r,p}^{+}} g(W) dW_{r,p} = \frac{1}{2^r} \times 2^{r(r-1)/4 + r(p-r)/2} \int_{\mathcal{L}_r^{+} \times \mathcal{V}_{r,p}} g(H_1 L H_1') \prod_{1\le i<j\le r} (l_i - l_j) \prod_{i=1}^{r} l_i{}^{p-r} \prod_{i=1}^{r} dl_i\, dH_1$$

with

$$\mathcal{L}_r^{+} = \{(l_1, \ldots, l_r) \,|\, l_1 > \cdots > l_r > 0\}.$$

3.4 Mixed volumes and weights of $\bar{\chi}^2$ statistic

Now we can evaluate the weights of $\bar{\chi}^2_{01}$ and $\bar{\chi}^2_{12}$ in (29). In the case of our problem, the double integral in (18) reduces to

$$I_{r,p}(i) = \int_{\mathcal{S}^+_{r,p}\cap\partial U} \Big[\int_{\mathcal{S}^+_{p-r,p-r}\cap\partial U} \mathrm{tr}_{i-(p-r)(p-r+1)/2}\, H(\Lambda^{-1}, \tilde{\Lambda})\, dU_{p-r,p-r} \Big] dU_{r,p}, \qquad (36)$$

where $H(\Lambda^{-1}, \tilde{\Lambda}) = \Lambda^{-1} \otimes \tilde{\Lambda}$. Note that

$$\mathcal{S}^+_{r,p} \cap \partial U = \partial\mathcal{L}^+_r \times \mathcal{V}_{r,p}$$

with

$$\partial\mathcal{L}^+_r = \{(l_1, \ldots, l_r) \mid l_1 > \cdots > l_r > 0,\ {l_1}^2 + \cdots + {l_r}^2 = 1\}.$$

From Lemma 3.3 and Remark 3.4, the integral (36) is separated into two parts as

$$I_{r,p}(i) = c_p \sum_{(q,\bar{q})} \int_{\partial\mathcal{L}^+_r} \det({l_k}^{q_j})_{1\le k,j\le r} d\mu_r(l) \cdot \int_{\partial\mathcal{L}^+_{p-r}} \det(\tilde{l}_k^{\bar{q}_j})_{1\le k,j\le p-r} d\mu_{p-r}(\tilde{l}),$$

where the summation $\sum_{(q,\bar{q})}$ is over

$$(q_1, \ldots, q_r, \bar{q}_1, \ldots, \bar{q}_{p-r}) \in Q_{r,p}(-i - r + p(p+1)/2),$$

and the constant is

$$\begin{aligned} c_p &= \frac{1}{2^p} 2^{p(p-1)/2}\, v_{r(r-1)/2+r(p-r)}(\mathcal{V}_{r,p})\, v_{(p-r)(p-r+1)/2}(\mathcal{V}_{p-r,p-r}) \\ &= \frac{2^{p(p-1)/4} \pi^{p(p+1)/4}}{\prod_{k=1}^p \Gamma(k/2)}. \end{aligned} \qquad (37)$$

Note that (37) does not depend on r .

Then, the mixed volume in (18) is

$$v_{p(p+1)/2-i,i} = \frac{(i-1)!\{p(p+1)/2 - i - 1\}!}{\{p(p+1)/2\}!} \sum_r I_{r,p}(i),$$

where the summation $\sum_r$ is over

$$r \in R_p(i) = \{r \mid 0 \le i - (p-r)(p-r+1)/2 \le r(p-r)\},$$

since $\mathrm{tr}_{m'} H(\Lambda^{-1}, \tilde{\Lambda}) = 0$ for $m' > r(p-r)$. Then, from Theorem 2.1, we obtain the weights of $\bar{\chi}^2_{01}$ and $\bar{\chi}^2_{12}$ as

$$\begin{aligned} & w_{p(p+1)/2-i} \\ &= \binom{p(p+1)/2}{i} \frac{v_{p(p+1)/2-i,i}}{\omega_i\, \omega_{p(p+1)/2-i}} \\ &= \frac{1}{i\{p(p+1)/2 - i\}} \Gamma\Big(\frac{i}{2}+1\Big) \Gamma\Big(\frac{p(p+1)/2 - i}{2} + 1\Big) \frac{2^{p(p-1)/4}}{\prod_{k=1}^p (k/2)} \\ &\quad \times \sum_r \sum_{(q,\bar{q})} \int_{\partial\mathcal{L}^+_r} \det({l_k}^{q_j})_{1\le k,j\le r} d\mu_r(l) \cdot \int_{\partial\mathcal{L}^+_{p-r}} \det(\tilde{l}_k^{\bar{q}_j})_{1\le k,j\le p-r} d\mu_{p-r}(\tilde{l}), \end{aligned} \qquad (38)$$

where the summations $\sum_r$ and $\sum_{(q,\bar{q})}$ are over $r \in R_p(i)$ and

$$(q_1, \ldots, q_r, \bar{q}_1, \ldots, \bar{q}_{p-r}) \in Q_{r,p}(-i - r + p(p+1)/2),$$

respectively.

Corresponding formula by Kuriki (1993) is

$$\begin{aligned} w_{p(p+1)/2-i} &= d_p \sum_r \sum_{(q,\bar{q})} \int_{\mathcal{L}_r^+} e^{-\frac{1}{2}({l_1}^2+\cdots+{l_r}^2)} \det({l_k}^{q_j})_{1\le k,j\le r} \prod_{k=1}^{r} dl_k \\ &\quad \cdot \int_{\mathcal{L}_{p-r}^+} e^{-\frac{1}{2}(\tilde{l}_1^2+\cdots+\tilde{l}_{p-r}^2)} \det(\tilde{l}_k^{\bar{q}_j})_{1\le k,j\le p-r} \prod_{k=1}^{p-r} d\tilde{l}_k \end{aligned} \tag{39}$$

where

$$d_p = \frac{1}{2^{p/2}\prod_{k=1}^{p}\Gamma(k/2)},$$

and the ranges of the summations $\sum_r$ and $\sum_{(q,\bar{q})}$ are the same as in (38). Letting ${l_1}^2 + \cdots + {l_r}^2 = R^2$ and $\tilde{l}_1^2 + \cdots + \tilde{l}_{p-r}^2 = \tilde{R}^2$, we have $\prod_{k=1}^{r} dl_k = R^{r-1}dR\, d\mu_r(l)$ and $\prod_{k=1}^{p-r} d\tilde{l}_k = \tilde{R}^{p-r-1}d\tilde{R}\, d\mu_{p-r}(\tilde{l})$. By integrating with respect to R and $\tilde{R}$ using

$$\int_0^\infty R^\alpha e^{-\frac{1}{2}R^2} dR = 2^{\frac{\alpha-1}{2}}\Gamma\Big(\frac{\alpha+1}{2}\Big),$$

we see that (39) coincides with (38).

4 Appendix

Internal angle and external angle

Let F be a face of a closed polyhedral convex cone K in R^p. The internal angle $\beta(0,F)$ of K at F is defined as

$$\beta(0,F) = \frac{v_d(U \cap F)}{\omega_d},$$

where v_d is restricted to the linear subspace $L(F)$ spanned by F. Let $C(F,K)$ be the smallest cone containing K and the linear subspace $L(F)$ spanned by F and let $F^* = C(F,K)^*$. F^* can also be written as

$$F^* = \{y : y \in K^* \text{ and } \langle x, y\rangle = 0,\ \forall x \in F\}.$$

Therefore F^* is the face of K^* dual to F of K. The external angle $\gamma(F,K)$ of K at F is defined as

$$\gamma(F,K) = \frac{v_{p-d}(U \cap F^*)}{\omega_{p-d}} = \beta(0, F^*),$$

where v_{p-d} is restricted to the linear subspace spanned by F^*. See McMullen (1975) and Section 2.4 of Schneider (1993a) for more detail.

Sketch of the Proof of Theorem 2.3

Let $s \in D_m(\partial K)$ and consider an infinitesimal spherical neighborhood $B(s) \subset D_m(\partial K)$ of s. The essential step of the proof is evaluating the infinitesimal contribution of $B(s)$ to $v_p(A_\lambda(K,S))$, i.e.,

$$v_p[\cup_{s' \in B(s)}(N(K,s') \cap \lambda U)]. \tag{40}$$

The rest of the proof is just integration similar to the proofs of Theorem of 2.2 or Theorem 2.4. Therefore we only discuss evaluation of (40).

Let Δ be the radius of $B(s)$. We we only need to evaluate terms of order $O(v_{p-m}(B(s)))$ $= O(\Delta^{p-m})$ in (40).

Now take a point P in the relative interior of $s + N(K,s)$ and let $y = P - s$. Consider $y + D_m(\partial K)$ which is $D_m(\partial K)$ translated so that it passes through P. Let $s' \in B(s)$. Because $\dim N(K,s) + \dim D_m(\partial K) = p$ and $N(K,s)$ is continuous in s, we see that $y + D_m(\partial K)$ and $s' + T_{s'}(D_m(\partial K))^\perp$ meets at one point $Q(y,s')$ in the relative interior of $s' + N(K,s')$. (Since we are considering infinitesimal neighborhood, we could have taken $y + T_s(D_m(\partial K))$ instead of $y + D_m(\partial K))$.) Now define

$$B(s,y) = \cup_{s' \in B(s)} Q(y,s').$$

$B(s,y)$ is orthogonal to $N(k,t)$ and the infinitesimal contribution of $B(s)$ to (40) can be evaluated as

$$v_p[\cup_{s' \in B(s)}(N(K,s') \cap \lambda U)] = \int_{N(K,s)\cap \lambda U} v_{p-m}(B(s,y))dy,$$

where dy is the standard volume element of R^m.

Now we explicitly evaluate $B(s,y)$ by introducing convenient coordinates in a neighborhood of s in R^p. Write $r = p - m$. By translating the origin to s, we can assume $s = 0$ without loss of generality. Furthermore by appropriate rotation we can assume without loss of generality that the first r coordinate vectors

$$e_1 = (1,0,\ldots,0),\ e_2 = (0,1,0,\ldots,0),\ \ldots,\ e_r = (\underbrace{0,\ldots,0}_{r-1},1,0,\ldots,0),$$

constitute an orthonormal basis of $T_s(D_m(\partial K)) = T_0(D_m(\partial K))$ and last m coordinate vectors $N_{r+1},\ldots,N_p$ constitute an orthonormal basis of $T_0(D_m(\partial K))^\perp$. Furthermore we can assume that $P = y + s = y$ is in the direction of N_{r+1} and the coordinates of y are just $(\underbrace{0,\ldots,0}_{r},l,0,\ldots,0)$ where $l = \|y\|$.

Under the above coordinate system, by the implicit function theorem we can write

$$s_{r+1} = s_{r+1}(s_1,\ldots,s_r),\ \ldots,\ s_p = s_p(s_1,\ldots,s_r)$$

for $s \in B(0)$. We see that under the above coordinate system individual elements are already of order $O(\Delta)$ and we can ignore higher order terms.

Now for fixed $s_1^0,\ldots,s_r^0$ and infinitesimal scalar t consider the point

$$s(t) = (ts_1^0,\ldots,ts_r^0, s_{r+1}(ts_1^0,\ldots,ts_r^0),\ldots,s_p(ts_1^0,\ldots,ts_r^0))$$

in $B(0)$. The tangent space $T_{s(t)}(D_m(\partial K))$ is spanned by

$$\Big(1,0,\dots,0,\frac{\partial s_{r+1}}{\partial s_1}(s(t)),\dots\frac{\partial s_p}{\partial s_1}(s(t))\Big),$$
$$\vdots$$
$$\Big(0\dots,0,1,\frac{\partial s_{r+1}}{\partial s_r}(s(t)),\dots\frac{\partial s_p}{\partial s_r}(s(t))\Big).$$

Therefore $T_{s(t)}(D_m(\partial K))^{\perp}$ consists of vectors z such that

$$z_1+z_{r+1}\frac{\partial s_{r+1}}{\partial s_1}(s(t))+\dots+z_p\frac{\partial s_p}{\partial s_1}(s(t))=0,$$
$$\vdots$$
$$z_r+z_{r+1}\frac{\partial s_{r+1}}{\partial s_r}(s(t))+\dots+z_p\frac{\partial s_p}{\partial s_r}(s(t))=0.$$

On the other hand the tangent space

$$T_s(D_m(\partial K))=T_0(D_m(\partial K))=\mathrm{Span}(e_1,\dots,e_r)$$

translated to go thorough $P=y$ is

$$\{(s_1,\dots,s_r,l,0,\dots,0)\},\quad l=\|y\|.$$

Now $(s_1,\dots,s_r,l,0,\dots,0)$ meets $s(t)+T_{s(t)}(D_m(\partial K))^{\perp}$ iff

$$(s_1-ts_1^0)+\frac{\partial s_{r+1}}{\partial s_1}(s(t))(l-s_{r+1}(t))+\frac{\partial s_{r+2}}{\partial s_1}(s(t))(-s_{r+2}(t))$$
$$+\dots+\frac{\partial s_p}{\partial s_1}(s(t))(-s_p(t))=0$$
$$\vdots$$
$$(s_r-ts_r^0)+\frac{\partial s_{r+1}}{\partial s_r}(s(t))(l-s_{r+1}(t))+\frac{\partial s_{r+2}}{\partial s_r}(s(t))(-s_{r+2}(t))$$
$$+\dots+\frac{\partial s_p}{\partial s_r}(s(t))(-s_p(t))=0.$$

Noting that

$$\frac{\partial s_\alpha}{\partial s_i}(s(t))=t\sum_{j=1}^{r}\frac{\partial^2 s_\alpha}{\partial s_i\partial s_j}\Big|_0 s_j^0+o(t),\quad i=1,\dots,r,\ \alpha=r+1,\dots,p,$$

and

$$s_\alpha(t)=o(t),\quad \alpha=r+1,\dots,p$$

we obtain the coordinates of $Q(y,s(t))$ as

$$s_i=t\Big(s_i^0-l\sum_{j=1}^{r}\frac{\partial^2 s_{r+1}}{\partial s_i\partial s_j}\Big|_0 s_j^0\Big)+o(t),\quad i=1,\dots,r.$$

On the other hand the elements of the second fundamental tensor are just

$$H_{ij}^{\alpha} = -\frac{\partial^2 s_\alpha}{\partial s_i \partial s_j}.$$

Hence

$$s_i = t(s_i^0 + l\sum_{j=1}^{r} H_{ij}^{r+1} s_j^0) + o(t), \quad i = 1, \dots, r.$$

So far we have assumed that y is in the direction of N_{r+1}. If $y = lv$, $v = \sum_{\alpha=r+1}^{p} v_\alpha N_\alpha$, $\|v\| = 1$, then the same argument shows that coordinates of $Q(y, s(t))$ are

$$s_i = t(s_i^0 + l\sum_{j=1}^{r} H_{ij}(0, v) s_j^0) + o(t).$$

Therefore

$$\begin{aligned} v_r(B(s,y)) &= |\det(I_r + lH(s,v))|\, v_r(B(s)) + o(v_r(B(s))) \\ &= (1 + l\kappa_1(s,v)) \cdots (1 + l\kappa_r(s,v)) v_r(B(s)) + o(v_r(B(s))) \\ &= (1 + l\mathrm{tr}_1 H(s,v) + \cdots + l^r \mathrm{tr}_r H(s,v)) v_r(B(s)) + o(v_r(B(s))). \end{aligned}$$

The rest of the proof is integration similar to the proof of Theorem 2.2 or Theorem 2.4 and omitted.

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