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Abstract

We propose a new methodology for the valuation problems of financial contingent claims when the underlying asset prices follow the general class of continuous Itô processes. Our method can be applicable to a wide range of the valuation problems including the complicated contingent claims associated with the term structure of interest rates. We illustrate our method by giving two examples: the valuation problems of Swaptions and Average (Asian) Options for Interest Rates. Our method gives some explicit formulae for solutions, which are numerically accurate enough for practical purposes in most cases. The continuous stochastic processes for spot interest rates and forward interest rates are not necessarily Markovian or diffusion processes in the usual sense; nevertheless our approach can be rigorously justified by the Malliavin-Watanabe Calculus in stochastic analysis.

Key Words

Derivatives, Term Structure of Interest Rates, Asymptotic Expansion, Small Disturbance Asymptotics, Malliavin-Watanabe Calculus

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1 Introduction

In the past decade various contingent claims have been introduced and actively traded in financial markets. In particular, the various types of interest rates based contingent claims have appeared and attracted much attention not only of financial economists but also financial practitioners in financial markets. This paper presents a new methodology which is applicable to the valuation problems of financial contingent claims including such as various options, swaps, and other derivative securities when the underlying asset prices follow the general class of continuous Itô processes. Especially, our method in this paper is suitable for the pricing problems of interest rates based derivatives when the underlying implicit forward rates follow the general class of continuous Itô processes.

In the valuation problems of financial contingent claims, it has been known that we can rarely obtain the explicit formulae on solutions when the underlying assets follow the general class of continuous Itô processes. This is particularly evident for the contingent claims based on the term structure of interest rates because their payoff functions are usually complicated functionals of the underlying asset prices and the term structure of interest rates must satisfy strong restrictions by the fundamental economic considerations. In order to cope with these problems, two methods, called the partial differential equation (PDE) approach and the Monte Carlo (MC) approach have been widely known among financial economists and used for practical valuation problems. (See Hull (1993) for the details of these methods.) The asymptotic expansion approach we are proposing in this paper is different from these conventional methods. As we shall show later in this paper, our method has several advantageous aspects over the existing methods.

The asymptotic expansion approach in this paper is based on the key empirical observation on many asset prices including interest rates, that is, the observed and estimated volatilities for financial asset prices may vary over time, but they are not very large in comparison with the observed levels of asset prices. This aspect of phenomena has been even true for the stock prices whose volatilities are relatively large compared with other financial prices. It was this key observation why Kunitomo and Takahashi (1992) have developed the asymptotic theory called *Small Disturbance Asymptotics* for solving the valuation problem of average (or Asian) options for the foreign exchange rates when the volatility parameter goes to zero. They have proposed to use the limiting distribution in this asymptotic theory as the first order approximation to the exact distribution of the payoff functions of average options when the underlying asset prices follow the geometric Brownian motion. Although the approximations they proposed have given relatively accurate numerical values in many cases, they are not completely satisfactory in some cases for practical purposes. For the same setting of the valuation problems in Kunitomo and Takahashi (1992), Yoshida (1992) has obtained some further results on average options when the undrelying asset prices follow the geometric Brownian motion by using the asymptotic expansion technique originally developed for some applications in statistics.

The main purpose of the present paper is to show that the asymptotic expansion method in the small disturbance asymptotics can be effectively applicable for the various valuation problems of contingent claims in financial economics when the underlying asset prices follow the general class of continuous Itô processes. In particular, we shall show that the asymptotic expansion approach is very simple, but gives an unified method to the valuation problems of interest rates based contingent claims. However, we shall point out that some economic considerations on the theoretical restrictions on the structure of stochastic processes should be indispensable when we apply the asymptotic expansion method to the valuation problems of financial contingent claims. In the term structure problems, for instance, we need the strong conditions on the form of drift functions because of the no-arbitrage theory, which has been standard in financial economics. It implies that the continuous stochastic processes for spot interest rates and forward rates are not necessarily Markovian or diffusion processes in the usual sense.

Also as we shall explain in Appendix, our method is not an ad-hoc approximation method because it can be rigorously justified by the Malliavin-Watanabe theory in stochastic analysis. However, we should mention that the spot and forward interest rates are not necessarily Markovian and the existing asymptotic expansion methods by Watanabe (1987) and Yoshida (1992) in stochastic analysis and statistics have been developed for the case of time homogeneous Markovian processes. Hence we need to extend the existing results on the validity of the asymptotic expansion approach to certain extents. In this respect the asymptotic expansion approach developed would be interesting for researchers in stochastic analysis as well as financial economics.

Furthermore, as we shall illustrate in Section 4, the resulting formulae we shall derive for the complicated contingent claims are numerically accurate in many practical situations. Thus the asymptotic expansion approach would be not only theoretically interesting, but also quite useful for researchers in financial economics.

In Section 2, we formulate the valuation problem of the contingent claims based on the term structure of interest rates. In Section 3, we shall explain the asymptotic expansion approach for this problem and give some theoretical results. Then, in Section 4, we shall show some numerical results on Average Options for the interest rates as an illustrative example. Section 5 will summarize our results and give concluding comments. Some mathematical details including useful formulae and discussions of the validity of our method via the Malliavin-Watanabe theory will be gathered in Section 6.

2 The Valuation Problem of Interest Rates Based Contingent Claims

We consider a continuous time economy with a trading interval $[0, \bar{T}]$, where $\bar{T} < +\infty$ and it is complete in the proper economic sense. Let $P(t,T)$ denote the price of the discount bond at t with the maturity date T $(0 \leq t \leq T \leq \bar{T} < +\infty)$. We use the notational convention that $P(T,T) = 1$ at the maturity date $t = T$ for normalization. Let also $P(t,T)$ be continuously differentiable with respect to T and $P(t,T) > 0$ for $0 \leq t \leq T \leq \bar{T}$. Then the instantaneous forward rate at s for the future date t $(0 \leq s \leq t \leq T)$ is defined by

$$f(s,t) = -\frac{\partial \log P(s,t)}{\partial t} \; . \tag{2.1}$$

In the term structure model of interest rates we assume that a family of forward rate processes $\{f(s,t)\}$ for $0 \leq s \leq t \leq T$ follow the stochastic integral equation:

$$\begin{aligned} f(s,t) = f(0,t) \; &+ \; \int_0^s \left[\sum_{i=1}^n \sigma_i^*(f(v,t),v,t) \int_v^t \sigma_i^*(f(v,y),v,y)dy \right] dv \\ &+ \; \int_0^s \sum_{i=1}^n \sigma_i^*(f(v,t),v,t)dB_i(v) \end{aligned} \tag{2.2}$$

where $f(0,t)$ are non-random initial forward rates, $\{B_i(v), i = 1, \cdots, n\}$ are n independent Brownian motions, and $\{\sigma_i^*(f(v,t),v,t), i = 1, \cdots, n\}$ are the volatility functions. We assume that the initial forward rates are observable and fixed.

In the above formulation there is a strong form of restrictions on the drift function on $\{f(s,t)\}$. This is because we shall use the arbitrage-free valuation method of financial contingent claims based on the equivalent martingale measure, which has been standard in financial economics. The restrictions in (2.2) we are imposing in this formulation have been derived by Heath, Jarrow, and Morton (1992). Let $f(s,t)$ be continuous at $s = t$ for $0 \leq s \leq t \leq T$. Then the spot interest rate at t can be defined by

$$r(t) = f(t,t). \tag{2.3}$$

We now consider the contingent claims based on the term structure of interest rates. There have been many interest rates based contingent claims developed and traded in financial markets. Most of those contingent claims can be regarded as functionals of bond prices with different maturities. Let $\{c_j, j = 1, \cdots, m\}$ be a sequence of non-negative coupon payments and $\{T_j, j = 1, \cdots, m\}$ be a sequence of payment periods satisfying the condition $0 \leq t \leq T_1 \leq \cdots \leq T_m \leq \bar{T}$ [1].

[1] To be more precise, the weight at the maturity date c_m should be interpreted as $1 + c_m'$, where c_m' is the coupon rate at T_m. This is because the principal of bond is redempted at the maturity date T_m.

Then the price of the coupon bond with coupon payments $\{c_j, j = 1, \cdots, m\}$ at t should be given by [2]

$$P_{m,\{T_j\},\{c_j\}}(t) = \sum_{j=1}^{m} c_j P(t, T_j), \tag{2.4}$$

where $\{P(t, T_j), j = 1, \cdots, m\}$ are the prices of zero-coupon bonds with different maturities. For illustrations we give two examples of interest rates based contingent claims, which are important for practice in financial markets.

Example 1 : The payoff functions of options on the coupon bond with coupon payments $\{c_j, j = 1, \cdots, m\}$ at $\{T_j, j = 1, \cdots, m\}$ and the swaptions at the expiry date T $(0 < T \leq T_m)$ can be written as

$$V^{(1)}(T) = \left[P_{m,\{T_j\},\{c_j\}}(T) - K\right]^+, \tag{2.5}$$

and

$$V^{(2)}(T) = \left[K - P_{m,\{T_j\},\{c_j\}}(T)\right]^+, \tag{2.6}$$

where K is a fixed strike price and the max function is defined by $[X]^+ = \max(X, 0)$. $V^{(1)}(T)$ and $V^{(2)}(T)$ are the payoffs of the call options and put options on the coupon bond, respectively.

Example 2 : The yield of a zero coupon bond at t with the time to maturity of τ $(0 < t < t + \tau < T_m)$ years is given by

$$L^{\tau}(t) = \left[\frac{1}{P(t, t+\tau)} - 1\right]\frac{1}{\tau}. \tag{2.7}$$

Then the payoffs of the options on average interest rates are given by

$$V^{(3)}(T) = \left[\frac{1}{T}\int_0^T L^{\tau}(t)dt - K\right]^+ \tag{2.8}$$

and

$$V^{(4)}(T) = \left[K - \frac{1}{T}\int_0^T L^{\tau}(t)dt\right]^+, \tag{2.9}$$

where K is a fixed strike price. $V^{(3)}(T)$ and $V^{(4)}(T)$ are the payoffs of the call options and put options of average options on interest rates, respectively.

[2] If the following equality is not satisfied, there is an arbitrage opportunity in the economy. We shall use the no-arbitrage condition in the economy, which has been standard in recent financial economics. Here we also have implicitly assumed that there does neither exist any default risk associated with bonds nor any transaction costs.

The valuation problem of a contingent claim can be simply defined as to find its "fair" value at financial markets. Let $V(T)$ be the payoff of a contingent claim at the terminal period T. Then the standard martingale theory in financial economics predicts that the fair price of $V(T)$ at time t $(0 \leq t < T)$ should be given by

$$V_t(T) = \mathbf{E}_t\left[e^{-\int_t^T r(s)ds} V(T)\right], \tag{2.10}$$

where $\mathbf{E}_t[\cdot]$ stands for the conditional expectation operator given the information available at t. When we do not impose the drift restrictions given by (2.2) for the implicit forward rates processes, (2.10) should be regarded as the expectation operation with respect to the equivalent martingale measure for the true forward rates processes and we can obtain the same results reported in this paper. Since this complicates our notations as well as explanations, we have directly imposed the restrictions given by (2.2) [3] from the begining of our discussions for later developments.

3 The Asymptotic Expansion Approach

There are two difficulties in the valuation problems of interest rates based contingent claims. First, the payoff functions are usually non-linear functions of functionals of coupon bonds with different maturities. More importantly, second, the coupon bond prices are also complicated functionals of the instantaneous forward rate processes. Therefore except some special cases we cannot obtain the explicit formulae for the solutions in the valuations problems of interest rates based contingent claims.

In order to develop a new asymptotic expansion approach, we first re-formulate (2.2) and we assume that a family of the instantaneous forward rate processes obey the stochastic integral equation:

(3.1)

$$\begin{aligned} f^{(\varepsilon)}(s,t) = f(0,t) &+ \varepsilon^2 \int_0^s \left[\sum_{i=1}^n \sigma_i(f^{(\varepsilon)}(v,t),v,t) \int_v^t \sigma_i(f^{(\varepsilon)}(v,y),v,y)dy\right] dv \\ &+ \varepsilon \int_0^s \sum_{i=1}^n \sigma_i(f^{(\varepsilon)}(v,t),v,t)dB_i(v), \end{aligned}$$

where $0 < \varepsilon \leq 1$ and $0 \leq s \leq t \leq T \leq \bar{T}$. The volatility function $\sigma_i(f^{(\varepsilon)}(s,t),s,t)$ depends not only on s and t, but also on $f^{(\varepsilon)}(s,t)$ in the general case. Let $f^{(\varepsilon)}(s,t)$ be continuous at $s = t$ for $0 \leq s \leq t \leq T \leq \bar{T}$. Then the instantaneous interest

[3] This issue has been systematically investigated by Heath, Jarrow, and Morton (1992), for instance. There have been other approaches to modelling the term structure of interest rates as discussed in Hull (1993).

rate process can be defined by

$$r^{(\varepsilon)}(t) = f^{(\varepsilon)}(t,t)\ . \tag{3.2}$$

We note that these equations on $\{r^{(\varepsilon)}(t)\}$ and $\{f^{(\varepsilon)}(s,t)\}$ can be obtained simply by substituting $\varepsilon\sigma_i(f^{(\varepsilon)}(v,t),v,t)$ for $\sigma_i^*(f(v,t),v,t)$ in (2.2).

The asymptotic expansion approach we are proposing in this paper consists of the following three steps. First, since we do not know the distribution of a smooth functional of the future forward rate processes:

$$U_T^{(\varepsilon)} = U(\{f^{(\varepsilon)}(s,t)\}), \tag{3.3}$$

we consider its stochastic expansion around the deterministic process

$$U_T^{(0)} = U(\{f^{(0)}(s,t)\}) \tag{3.4}$$

when the volatility parameter ε goes to zero. Second, the formal asymptotic expansion can be taken along the polynomial order of the volatility coefficients $\varepsilon^k(k = 1,2,\cdots)$. Then we truncate the resulting stochastic expansion and take the expectation in (2.10) given the information available at time t. In order to implement this procedure, we first need to obtain the stochastic expansion of the stochastic processes $\{f^{(\varepsilon)}(s,t)\}$ and $\{r^{(\varepsilon)}(t)\}$. We shall make the following assumptions:

Assumption I : The volatility functions $\{\sigma_i(f^{(\varepsilon)}(s,t),s,t)\}$ are non-negative, bounded, Lipschitz continuous, and smooth in its first argument, and all derivatives are bounded uniformly in ε, where $f^{(\varepsilon)}(s,t)$ are properly defined in $(\varepsilon,s,t,f^{(\varepsilon)}(s,t)) \in (0,1]\times\{0 < s \le t \le T\}\times R^1$. The initial forward rates $f(0,t)$ are also Lipschitz continuous with respect to t.

Assumption II : For any $0 < t \le T$,

$$\Sigma_t = \int_0^t \sum_{i=1}^n \sigma_i^{(0)}(v,t)^2 dv > 0, \tag{3.5}$$

where

$$\sigma_i^{(0)}(v,t) = \sigma_i(f^{(\varepsilon)}(v,t),v,t)|_{\varepsilon=0}. \tag{3.6}$$

The conditions we have made in Assumption I can exclude the possiblity of explosions for the solution of (3.1) [4]. They are quite strong and could be relaxed considerably, which may be interesting from the view of stochastic analysis. For practical purposes, however, we can often use the truncation arguments as an

[4]For example, Morton (1989) has shown that there does not exist any meaningful solution when the volatility function is proportional to the forward rate process.

example given by Heath, Jarrow, and Morton (1992). Assumption II ensures the key condition of non-degeneracy of the Malliavin-covariance in our problem, which is essential for the validity of the asymptotic expansion approach as we shall see in the following derivations. Under these assumptions we can get the stochastic expansions of the forward rates and spot interest rates processes. The outline of their derivations and their mathematical validity are given in Section 6.

Theorem 3.1 : *Under Assumption I, the stochastic expansion of the instantaneous forward rate* $\{f^{(\varepsilon)}(s,t)\}$ *is given by*

$$f^{(\varepsilon)}(s,t) = f(0,t) + \varepsilon A(s,t) + \varepsilon^2 B(s,t) + o_p(\varepsilon^2) \tag{3.7}$$

as $\varepsilon \to 0$. *In particular, the spot rate process can be expanded as*

$$r^{(\varepsilon)}(t) = f(0,t) + \varepsilon A(t,t) + \varepsilon^2 B(t,t) + o_p(\varepsilon^2). \tag{3.8}$$

The coefficients $A(s,t)$ *and* $B(s,t)$ *in (3.7) and (3.8) are defined by*

$$A(s,t) = \int_0^s \sum_{i=1}^n \sigma_i^{(0)}(v,t)dB_i(v), \tag{3.9}$$

$$B(s,t) = \int_0^s b^{(0)}(v,t)dv + \int_0^s \sum_{i=0}^n A(v,t)\partial\sigma_i^{(0)}(v,t)dB_i(v), \tag{3.10}$$

where

$$b^{(0)}(v,t) = b(f^{(\varepsilon)}(v,t),v,t)|_{\varepsilon=0}, \tag{3.11}$$

$$\partial\sigma_i^{(0)}(v,t) = \frac{\partial\sigma_i(f^{\varepsilon}(v,t),v,t)}{\partial f^{(\varepsilon)}(v,t)}|_{\varepsilon=0}, \tag{3.12}$$

and

$$b(f^{(\varepsilon)}(v,t),v,t) = \sum_{i=1}^n \sigma_i(f^{(\varepsilon)}(v,t),v,t)\int_v^t \sigma_i(f^{(\varepsilon)}(v,y),v,y)dy\ . \tag{3.13}$$

In the above representations, the first terms of (3.7) and (3.8) are deterministic functions. The second term $A(s,t)$ in (3.7) follows the normal distribution with zero mean and the variance Σ_t, which corresponds to the limit of the Malliavin-covariance in the theory of Malliavin-Watanabe calculus when $\varepsilon \to 0$. The stochastic expansion method around the normal distribution has been standard in the statistical asymptotic theories.

The next step in the asymptotic expansion approach is to obtain the stochastic expansions of the bond price processes and the discount factor. For this purpose, we utilize the relation between the bond price processes and the implicit forward rate processes:

$$P^{(\varepsilon)}(t,T) = exp\left[-\int_t^T f^{(\varepsilon)}(t,u)du\right] . \tag{3.14}$$

Using (3.7), we immediately have a stochastic expansion of the bond price process $\{P^{(\varepsilon)}(t,T)\}$ as

$$P^{(\varepsilon)}(t,T) = \frac{P(0,T)}{P(0,t)} \exp\left[-\varepsilon\int_t^T A(t,u)du - \varepsilon^2\int_t^T B(t,u)du + o_p(\varepsilon^2)\right] , \tag{3.15}$$

where $P(0,T)$ and $P(0,t)$ are the observable initial discount bond prices. Because the coupon bond prices $\{P^{(\varepsilon)}_{m,\{T_j\},\{c_j\}}\}$ are linear combinations of the prices of zero-coupon bonds, it has a stochastic expansion as

(3.16)

$$\begin{aligned} P_{m,\{T_j\},\{c_j\}}(t) &= \sum_{j=1}^m c_j\frac{P(0,T_j)}{P(0,t)} \exp\left[-\varepsilon\int_t^{T_j} A(t,u)du - \varepsilon^2\int_t^{T_j} B(t,u)du + o_p(\varepsilon^2)\right] \\ &= \sum_{j=1}^m c_j\frac{P(0,T_j)}{P(0,t)}\left[1-\varepsilon\int_t^{T_j} A(t,u)du - \varepsilon^2\int_t^{T_j} B(t,u)du\right. \\ &\quad \left. +\varepsilon^2\frac{1}{2}\left(\int_t^{T_j} A(t,u)du\right)^2 + o_p(\varepsilon^2)\right] . \end{aligned}$$

By using a Fubini-type result [5], we can write

$$\begin{aligned} \int_t^T A(t,u)du &= \int_t^T\left[\int_0^t\sum_{i=1}^n \sigma_i^{(0)}(v,u)dB_i(v)\right]du \\ &= \int_0^t \boldsymbol{\sigma}_{tT}^{(0)}(v)d\boldsymbol{B}(v), \end{aligned} \tag{3.17}$$

where $\boldsymbol{B}(v) = (B_i(v))$ is an $n\times 1$ vector of standard (i.e. mutually independent) Brownian motions and $\boldsymbol{\sigma}^{(0)}_{t,T}$ is a $1\times n$ vector

$$\boldsymbol{\sigma}_{tT}^{(0)}(v) = \left[\int_t^T \sigma_i^{(0)}(v,u)du\right] . \tag{3.18}$$

Since (3.17) is a linear combinations of $\{B_i(v)\}$ with deterministic coefficients, it follows a normal distribution. Also we have

(3.19)

$$\int_t^T B(t,u)du = k_1(t,T) + \int_0^t\left[\int_t^T\left[\int_0^s \boldsymbol{\sigma}^{(0)}(v,u)d\boldsymbol{B}(v)\right]\partial\boldsymbol{\sigma}^{(0)}(s,u)d\boldsymbol{B}(s)\right]du,$$

[5]We can use *Lemma* 4.1 of Ikeda and Watanabe (1989) under Assumption I as a generalized Fubini-type theorem.

where $\boldsymbol{\sigma}^{(0)}(v,u) = (\sigma_i^{(0)}(v,u))$ and $\partial\boldsymbol{\sigma}^{(0)}(s,u) = (\partial\sigma_i^{(0)}(s,u))$ are $1 \times n$ vectors of deterministic functions, and

$$k_1(t,T) = \int_0^t \left[\int_t^T b^{(0)}(v,u)du\right] dv. \tag{3.20}$$

Hence we notice that (3.19) is a quadratic functional of n standard Brownian motions. Similarly, by making use of (3.8), a stochastic expansion of the discount factor process is given by

(3.21)

$$\begin{aligned} e^{-\int_0^T r^{(\varepsilon)}(s)ds} &= P(0,T)exp\left[-\varepsilon\int_0^T A(s,s)ds - \varepsilon^2\int_0^T B(s,s)ds + o_p(\varepsilon^2)\right] \\ &= P(0,T)\left[1 - \varepsilon\int_0^T A(s,s)ds - \varepsilon^2\int_0^T B(s,s)ds \right. \\ &\quad \left. +\varepsilon^2\frac{1}{2}\left(\int_0^T A(s,s)ds\right)^2\right] + o_p(\varepsilon^2)\,. \end{aligned}$$

The second term of the discount factor process can be expressed as

$$\begin{aligned} \int_0^T A(t,t)dt &= \int_0^T \int_v^T \boldsymbol{\sigma}_i^{(0)}(v,t)dtdB_i(v), \\ &= \int_0^T \boldsymbol{\sigma}_T^{(0)}(v)d\boldsymbol{B}(v), \end{aligned} \tag{3.22}$$

where $\boldsymbol{\sigma}_T^{(0)}$ is a $1 \times n$ vector

$$\boldsymbol{\sigma}_T^{(0)} = \left[\int_v^T \sigma_i^{(0)}(v,t)dt\right]\,. \tag{3.23}$$

Since (3.22) is also a linear combinations of $\{B_i(v)\}$ with deterministic coefficients, the second term of (3.21) follows a normal distribution. The third term of (3.21) can be expressed as

$$\int_0^T B(t,t)dt = k_2(T) + \int_0^T \left[\int_0^t \left[\int_0^s \boldsymbol{\sigma}^{(0)}(v,t)dB(v)\right] \partial\boldsymbol{\sigma}^{(0)}(s,t)dB(s)\right] dt, \tag{3.24}$$

where

$$k_2(T) = \int_0^T \left[\int_v^T b^{(0)}(v,t)dt\right] dv. \tag{3.25}$$

The third step in our approach is to obain the asymptotic expansion of the discounted functional of the payoff function at the expiring date. We shall illustrate this procedure by using two examples we have mentioned to in Section 2.

By using (3.16) and (3.21), the asymptotic expansion of the discounted coupon bond price minus the strike price is given by

$$
\begin{aligned}
g^{(\varepsilon)} &= e^{-\int_0^T r^{(\varepsilon)}(s)ds}\left[P_{m,\{T_j\},\{c_j\}}(T)-K\right] \\
&= g_0+\varepsilon g_1+\varepsilon^2 g_2+o_p(\varepsilon^2),
\end{aligned} \tag{3.26}
$$

where the coefficients g_i $(i=1,2,3)$ in (3.26) are given by

$$
g_0=\sum_{j=1}^{m} c_j P(0,T_j)-KP(0,T), \tag{3.27}
$$

$$
g_1=\int_0^T \boldsymbol{\sigma}^*_{g_1}(v)dB(v), \tag{3.28}
$$

$$
\boldsymbol{\sigma}^*_{g_1}(v)=-g_0\boldsymbol{\sigma}^{(0)}_T(v)-\sum_{i=1}^{m} c_j P(0,T_j)\boldsymbol{\sigma}^{(0)}_{T,T_j}(v) \tag{3.29}
$$

and

(3.30)

$$
\begin{aligned}
g_2 &= \tfrac{1}{2}g_0\{\textstyle\int_0^T A(s,s)ds\}^2+\{\int_0^T A(s,s)ds\}\sum_{j=1}^{m} c_j P(0,T_j)\{\int_T^{T_j} A(T,u)du\} \\
&+ \tfrac{1}{2}\sum_{j=1}^{m} c_j P(0,T_j)\{\int_T^{T_j} A(T,u)du\}^2-g_0\int_0^T B(s,s)ds \\
&- \sum_{j=1}^{m} c_j P(0,T_j)\{\int_T^{T_j} B(T,u)du\}.
\end{aligned}
$$

What we really need is not to derive the stochastic expansion of the random variable $g^{(\varepsilon)}$, but to obtain the asymptotic expansion of its density function. For this purpose, we consider the characteristic function of the normalized random variable :

$$
X_T^{(\varepsilon)}=\frac{1}{\varepsilon}(g^{(\varepsilon)}-g_0). \tag{3.31}
$$

Then the characteristic function of $X_T^{(\varepsilon)}$ can be formally expanded as

$$
\begin{aligned}
\varphi_X(t) &= \mathbf{E}\left[e^{itX_T^{(\varepsilon)}}\right] \\
&= \mathbf{E}\left[e^{itg_1}(1+\varepsilon it\mathbf{E}[g_2|g_1])\right]+o(\varepsilon),
\end{aligned} \tag{3.32}
$$

where $\mathbf{E}[g_2|g_1]$ is the conditional expectation operator. By using the inversion formula in Section 6 (i.e. *Lemma* 6.2), we have the following result. The validity of this procedure will be discussed in Section 6.3.

Theorem 3.2 : *Under Assumptions I and II, the density function of* $X_T^{(\varepsilon)}$ *as* $\varepsilon \longrightarrow 0$ *can be expressed as*

$$f_X(x) = \phi_\Sigma(x) + \varepsilon \left[\frac{c}{\Sigma}x^3 + (\frac{f}{\Sigma} - 2c)x\right] \phi_\Sigma(x) + O(\varepsilon^2), \tag{3.33}$$

where $\phi_\Sigma(x)$ *stands for the normal density function with zero mean and variance*

$$\Sigma = \int_0^T \boldsymbol{\sigma}^*_{g_1}(t)\boldsymbol{\sigma}^{*'}_{g_1}(t)dt , \tag{3.34}$$

provided that $\Sigma > 0$. *The coefficients* c *and* f *in (3.33) are determined by the integral equation*

$$\mathbf{E}\left[g_2|g_1 = x\right] = cx^2 + f. \tag{3.35}$$

The asymptotic variance Σ is the limit of the Malliavin-covariance when $\varepsilon \longrightarrow 0$ for the call options of the coupon bond and the swaptions whose payoff function is given by (2.5). The explicit formulae of coefficients in (3.33) are quite complicated in this problem. By using *Lemma* 6.1 in Section 6, we can show that c and f for the call options of the coupon bond and swaptions in Example 1 are given by

(3.36)

$$\begin{aligned}
c &= \frac{1}{2}\frac{g_0}{\Sigma^2}\left[\int_0^T \boldsymbol{\sigma}_T^{(0)}(v)\boldsymbol{\sigma}^*_{g_1}(v)'dv\right]^2 \\
&+ \frac{1}{\Sigma^2}\left[\int_0^T \boldsymbol{\sigma}_T^{(0)}(v)\boldsymbol{\sigma}^*_{g_1}(v)'dv\right]\sum_{j=1}^m c_jP(0,T_j)\left[\int_0^T \boldsymbol{\sigma}_{T,T_j}^{(0)}(v)\boldsymbol{\sigma}^*_{g_1}(v)'dv\right] \\
&+ \frac{1}{2}\frac{1}{\Sigma^2}\sum_{j=1}^m c_jP(0,T_j)\left[\int_0^T \boldsymbol{\sigma}_{T,T_j}^{(0)}(v)\boldsymbol{\sigma}^*_{g_1}(v)'dv\right]^2 \\
&- \frac{g_0}{\Sigma^2}\left[\int_0^T\left[\int_0^t \boldsymbol{\sigma}^*_{g_1}(s)\partial\boldsymbol{\sigma}^{(0)}(s,t)'(\int_0^s \boldsymbol{\sigma}^{(0)}(v,t)\boldsymbol{\sigma}^*_{g_1}(v)'dv)ds\right]dt\right] \\
&- \frac{1}{\Sigma^2}\sum_{j=1}^m c_jP(0,T_j)\left[\int_T^{T_j}\left[\int_0^T \boldsymbol{\sigma}^*_{g_1}(s)\partial\boldsymbol{\sigma}^{(0)}(s,u)'(\int_0^s \boldsymbol{\sigma}^{(0)}(v,u)\boldsymbol{\sigma}^*_{g_1}(v)'dv)ds\right]du\right]
\end{aligned}$$

and

(3.37)

$$
\begin{aligned}
f &= -g_0 k_2(T) - \sum_{j=1}^{m} c_j P(0,T_j) k_1(T,T_j) \\
&- \frac{1}{2}\frac{g_0}{\Sigma}\left[\int_0^T \boldsymbol{\sigma}_T^{(0)}(v)\boldsymbol{\sigma}_{g_1}^*(v)'dv\right]^2 + \frac{g_0}{2}\left[\int_0^T \boldsymbol{\sigma}_T^{(0)}(v)\boldsymbol{\sigma}_T^{(0)}(v)'dv\right] \\
&+ \sum_{j=1}^{m} c_j P(0,T_j)\int_0^T \boldsymbol{\sigma}_T^{(0)}(v)\boldsymbol{\sigma}_{T,T_j}^{(0)}(v)'dv \\
&- \frac{1}{\Sigma}\left[\int_0^T \boldsymbol{\sigma}_T^{(0)}(v)\boldsymbol{\sigma}_{g_1}^*(v)'dv\right]\sum_{j=1}^{m} c_j P(0,T_j)\left[\int_0^T \boldsymbol{\sigma}_{T,T_j}^{(0)}(v)\boldsymbol{\sigma}_{g_1}^*(v)'dv\right] \\
&- \frac{1}{2}\frac{1}{\Sigma}\sum_{j=1}^{m} c_j P(0,T_j)\left[\int_0^T \boldsymbol{\sigma}_{T,T_j}^{(0)}(v)\boldsymbol{\sigma}_{g_1}^*(v)'dv\right]^2 \\
&+ \frac{1}{2}\sum_{j=1}^{m} c_j P(0,T_j)\left[\int_0^T \boldsymbol{\sigma}_{T,T_j}^{(0)}(v)\boldsymbol{\sigma}_{T,T_j}^{(0)}(v)'dv\right] + \frac{1}{2}\sum_{j=1}^{m} c_j P(0,T_j)\left[\int_0^T \boldsymbol{\sigma}_{g_1}^*(v)\boldsymbol{\sigma}_{T,T_j}^{0}(v)'dv\right] \\
&+ \frac{g_0}{\Sigma}\left[\int_0^T\left[\int_0^t \boldsymbol{\sigma}_{g_1}^*(s)\partial\boldsymbol{\sigma}^{(0)}(s,t)'(\int_0^s \boldsymbol{\sigma}^{(0)}(v,t)\boldsymbol{\sigma}_{g_1}^*(v)'dv)ds\right]dt\right] \\
&+ \frac{1}{\Sigma}\sum_{j=1}^{m} c_j P(0,T_j)\left[\int_T^{T_j}\left[\int_0^T \boldsymbol{\sigma}_{g_1}^*(s)\partial\boldsymbol{\sigma}^{(0)}(s,u)'(\int_0^s \boldsymbol{\sigma}^{(0)}(v,u)\boldsymbol{\sigma}_{g_1}^*(v)'dv)ds\right]du\right] .
\end{aligned}
$$

Also we can treat Example 2 in Section 2 by the same metod. After some tedious calculations for the call options of the average interest rates whose payoff function is given by (2.8), we have a stchastic expansion as

$$
\begin{aligned}
g^{(\varepsilon)} &= e^{-\int_0^T r^{(\varepsilon)}(s)ds}(\tfrac{1}{T\tau})\left[\int_0^T \tfrac{1}{P(t,t+\tau)}dt - k\right] \\
&= g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + o_p(\varepsilon^2),
\end{aligned}
\tag{3.38}
$$

where $k = (1 + K\tau)T$. In this stochastic expansion of the random variable $g^{(\varepsilon)}$, the coefficients g_i $(i = 1, 2, 3)$ are given by

$$
g_0 = \frac{P(0,T)}{T\tau}\left[\int_0^T \frac{P(0,t)}{P(0,t+\tau)}dt - k\right],
\tag{3.39}
$$

$$
g_1 = \int_0^T \boldsymbol{\sigma}_{g_1}^*(v)dB(v),
\tag{3.40}
$$

and

(3.41)

$$
\begin{aligned}
g_2 &= \frac{1}{2}\frac{P(0,T)}{T\tau}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \boldsymbol{\sigma}_{t,t+\tau}^{(0)}(v)d\boldsymbol{B}(v)\right]^2 dt \\
&- \frac{P(0,T)}{T\tau}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \boldsymbol{\sigma}_{t,t+\tau}^{(0)}(v)d\boldsymbol{B}(v)\right]\left[\int_0^T \boldsymbol{\sigma}_T^{(0)}(v)d\boldsymbol{B}(v)\right] dt \\
&+ \frac{1}{2}g_0\left[\int_0^t \boldsymbol{\sigma}_T^{(0)}(v)d\boldsymbol{B}(v)\right]^2 dt \\
&+ \frac{P(0,T)}{T\tau}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_t^{t+\tau} B(t,u)du\right] dt - g_0\left[\int_0^T B(t,t)dt\right],
\end{aligned}
$$

where we have used the notations

(3.42)

$$
\boldsymbol{\sigma}_{g_1}^*(v) = \frac{P(0,T)}{T\tau}\int_v^T \frac{P(0,t)}{P(0,t+\tau)}dt\boldsymbol{\sigma}_{t,t+\tau}^{(0)}(v) - \frac{P(0,T)}{T\tau}\left[\int_0^T \frac{P(0,t)}{P(0,t+\tau)}dt - k\right]\boldsymbol{\sigma}_T^{(0)}(v),
$$

$$
\boldsymbol{\sigma}_{t,t+\tau}^{(0)}(v) = \left[\int_t^{t+\tau} \boldsymbol{\sigma}_i^0(v,u)du\right]_i,
$$

and

$$
\boldsymbol{\sigma}_T^{(0)}(v) = \left[\int_v^T \boldsymbol{\sigma}_i^{(0)}(v,u)du\right]_i .
$$

Then the asymptotic variance Σ in (3.33) is given by the formula of (3.34) for Example 2, where we use (3.42) instead of (3.29). By a tedious but straightforward calculation in present case, we have

(3.43)

$$
\begin{aligned}
c &= \frac{1}{2}\frac{1}{\Sigma^2}\frac{P(0,T)}{T\tau}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \boldsymbol{\sigma}_{t,t+\tau}^{(0)}(v)\boldsymbol{\sigma}_{g_1}^*(v)'dv\right]^2 dt \\
&- \frac{P(0,T)}{T\tau}\frac{1}{\Sigma^2}\left[\int_0^T \frac{P(0,t)}{P(0,t+\tau)}(\int_0^t \boldsymbol{\sigma}_{t,t+\tau}^{(0)}(v)\boldsymbol{\sigma}_{g_1}^*(v)'dv)dt\right]\left[\int_0^T \boldsymbol{\sigma}_T^{(0)}(v)\boldsymbol{\sigma}_{g_1}^*(v)'dv\right] \\
&+ \frac{1}{2}\frac{1}{\Sigma^2}g_0\left[\int_0^t \boldsymbol{\sigma}_T^{(0)}(v)\boldsymbol{\sigma}_{g_1}^*(v)'dv\right]^2 \\
&+ \frac{P(0,T)}{T\tau\Sigma^2}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_t^{t+\tau}\left[\int_0^t \boldsymbol{\sigma}_{g_1}^*(s)\partial\boldsymbol{\sigma}^{(0)}(s,u)'(\int_0^s \boldsymbol{\sigma}^*(v,u)\boldsymbol{\sigma}_{g_1}^*(v)'dv)ds\right]du\right]dt \\
&- \frac{g_0}{\Sigma^2}\int_0^T\left[\int_0^t \boldsymbol{\sigma}_{g_1}^*(s)\partial\boldsymbol{\sigma}^{(0)}(s,t)'(\int_0^s \boldsymbol{\sigma}^{(0)}(v,t)\boldsymbol{\sigma}_{g_1}^*(v)'dv)ds\right]dt,
\end{aligned}
$$

and

(3.44)

$$
\begin{aligned}
f \;=\; & -\frac{1}{2}\frac{P(0,T)}{T\tau}\frac{1}{\Sigma}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \boldsymbol{\sigma}^{(0)}_{t,t+\tau}(v)\boldsymbol{\sigma}^{*}_{g_1}(v)'dv\right]^2 dt \\
& + \frac{1}{2}\frac{P(0,T)}{T\tau}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \boldsymbol{\sigma}^{(0)}_{t,t+\tau}(v)\boldsymbol{\sigma}^{(0)}_{t,t+\tau}(v)'dv\right] dt \\
& + \frac{P(0,T)}{T\tau}\frac{1}{\Sigma}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \boldsymbol{\sigma}^{(0)}_{t,t+\tau}(v)\boldsymbol{\sigma}^{(0)}_{g_1}(v)'dv\right] dt\left[\int_0^T \boldsymbol{\sigma}^{(0)}_{T}(v)\boldsymbol{\sigma}^{*}_{g_1}(v)'dv\right] \\
& - \frac{P(0,T)}{T\tau}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\left[\int_0^t \boldsymbol{\sigma}^{(0)}_{t,t+\tau}(v)\boldsymbol{\sigma}^{(0)}_{T}(v)'dv\right] dt \\
& - \frac{1}{2}\frac{1}{\Sigma}g_0\left[\int_0^t \boldsymbol{\sigma}^{(0)}_{T}(v)\boldsymbol{\sigma}^{*}_{g_1}(v)'dv\right]^2 \\
& + \frac{1}{2}g_0\left[\int_0^t \boldsymbol{\sigma}^{(0)}_{T}(v)\boldsymbol{\sigma}^{(0)}_{T}(v)'dv\right] \\
& + \frac{P(0,T)}{T\tau}\left[\int_0^T k_1(t,t+\tau)\frac{P(0,t)}{P(0,t+\tau)}dt\right] - g_0 k_1(T) \\
& - \frac{P(0,T)}{T\tau\Sigma}\int_0^T \frac{P(0,t)}{P(0,t+\tau)}\int_t^{t+\tau}\left[\int_0^t\left[\boldsymbol{\sigma}^{*}_{g_1}(s)\partial\boldsymbol{\sigma}^{(0)}(s,u)'\left(\int_0^s \boldsymbol{\sigma}^{(0)}(v,u)\boldsymbol{\sigma}^{*}_{g_1}(v)'dv\right)ds\right]du\right]dt \\
& + \frac{g_0}{\Sigma}\int_0^T\left[\int_0^t \boldsymbol{\sigma}^{*}_{g_1}(s)\partial\boldsymbol{\sigma}^{(0)}(s,t)'\left(\int_0^s \boldsymbol{\sigma}^{(0)}(v,t)\boldsymbol{\sigma}^{*}_{g_1}(v)'dv\right)ds\right]dt.
\end{aligned}
$$

The last step in our method is to derive the asymptotic expansion of the conditional expectations of the discounted terminal payoff based on the asymptotic expansion of the exact density function we have obtained in *Theorem* 3.2. For this purpose, we re-write the payoff function in (2.5) and (2.7) as

$$V(T) = \varepsilon\left[y^{(\varepsilon)} + X_T^{(\varepsilon)}\right]^+ , \tag{3.45}$$

where

$$y^{(\varepsilon)} = \frac{1}{\varepsilon}g_0 \, . \tag{3.46}$$

In order to evaluate the terminal payoff function of contingent claims at the initial period, we need the additional assumption:

Assumption III : There exists a constant y such that

$$y^{(\varepsilon)} = y + O(\varepsilon^2). \tag{3.47}$$

The above condition means that we are considering the situation where the strike price is near g_0 [6]. Hence we have omitted ε of K_ε and will use the notation

[6] This means that we are considering the valuation of contingent claim when the strike price is near its present value implied by the initial forward rates observed in financial markets.

K as before. The condition given by (3.47) can be relaxed to a certain extent, but then there would be some more complications in the following analyses. In Example 1 and Example 2, we should take

$$y^{(\varepsilon)} = \frac{1}{\varepsilon}\left[\sum_{j=1}^{m} c_j P(0,T_j) - KP(0,T)\right], \tag{3.48}$$

and

$$y^{(\varepsilon)} = \frac{1}{\varepsilon}\frac{P(0,T)}{T_T}\left[\int_0^T \frac{P(0,t)}{P(0,t+\tau)}dt - k\right], \tag{3.49}$$

respectively.

Theorem 3.3 : *Under Assumptions I,II, and III, the asymptotic expansions of $V_0(T)$ are given by*

$$\begin{aligned} V_0(T) &= \varepsilon \int_{-y}^{+\infty} (y+x)\phi_\Sigma(x)dx \\ &+ \varepsilon^2 \int_{-y}^{+\infty} (cx^2 + f)\phi_\Sigma(x)dx + o(\varepsilon^2), \end{aligned} \tag{3.50}$$

provided that $\Sigma > 0$. The coefficients c and f are given as the same as Theorem 3.2.

We note that all terms in the right hand side of (3.50) are some known functions of the distribution function and the density function of $N(0,\Sigma)$. For instance, we have the relation

$$\int_{-y}^{\infty} x^2\phi_\Sigma(x)dx = \Sigma\Phi(\Sigma^{-1/2}y) - y\Sigma\phi_\Sigma(y), \tag{3.51}$$

where $\Phi(\cdot)$ is the distribution function of the standard normal distribution. This and similar formulae are useful for the numerical implementation.

4 Numerical Examples

In this section, we will present some numerical results to illustrate the method introduced in Section 3. For this purpose, we use the pricing problem of average options of interest rates in the term structure model explained in Section 2.

For the simplicity of exposition, we assume that the instantaneous forward rates processes $\{f(s,t)\}$ follow the stochastic differential equation:

$$df(s,t) = \sigma^2(t-s)ds + \sigma dB(s). \tag{4.1}$$

For the expository purpose we assume that the volatility function $\sigma(s,t)$ is constant and we impose the condition on the drift term by the no-arbitrage condition.

This model corresponds to a continuous analogue of the discrete model by Ho and Lee (1986), which has been used in some studies. In this case it is apparent that the stochastic process of the zero coupon bond follows the geometric Brownian motion. Although the forward rates process is very simple, it has not been possible to obtain the explicit formula for the average interest rate options.

Tables 1-3 show the numerical values of call options on average interest rates for the case when the volatility function for the instantaneous forward rates is constant over time. The time to maturity of the underlying interest rates is one year and hence the average is taken over interest rates whose maturity are one year ($\tau = 1$). For simplicity, the present term structrure at $t = 0$ is assumed to be flat of 5% per year and the volatility parameter σ is assumed to be 150 basis point per year ($\sigma = 0.015$), which may be a reasonable level for practical purpose. We use the approximations based on the asymptotic expansions in Section 3 by setting $\sigma = \varepsilon$. We have given the results for the out-of-the money case ($K = 5.5\%$ for Table 1 and $K = 6\%$ for Tables 2 and 3), at-the-money case ($K = 5\%$ for Table 1-3), and in-the-money case ($K = 4.5\%$ for Table 1 and $K = 4\%$ for Tables 2 and 3). For the comparative purposes, the numerical values by Monte Carlo simulations and by the finite difference method in the PDE approach have been given. The number of replication in our Monte Calro simulations is 500,000 and we expect that the numerical values by Monte Carlo method are very accurate. The values of the finite difference method in tables are based on solving the PDE numerically for the average options of interest rates processes under the assumption that they follow (4.1). This method has been developed by Takahashi (1995).

Since the number of replications of our Monte Carlo simulations is large, we expect that they give the bench mark values for the average options on the interest rates. From these tables we can find that the differences in the option values by the asymptotic expansion approach and the Monte Carlo approach are very small and less than 1 percent of the underlying price levels in most cases. Also the differences in the option values by the asymptotic expansion approach and the finite difference approach are not very large. By these numerical comparisons the values of our approximations in our tables are reliable in two digits at least. Thus we can tentatively conclude that the approximation formulae we have obtained in Section 3 are very accurate and useful for practical purposes.

5 Concluding Remarks

This paper proposes a new methodology for the valuation problems of financial contingent claims when the underlying asset prices follow the general class of continuous Itô processes. Our method called the asymptotic expansions approach can be applicable to a wide range of valuation problems including complicated contingent claims associated with the term structure of interest rates. We have illustrated our methdology by deriving some useful formulae for the swaptions and average (or Asian) options for interest rates. Also we have given an evidence that the resulting formulae are numerically accurate enough for practical applications. Since the asymptotic expansion approach can be justified rigorously by the Malliavin-Watanabe calculus in stochastic analysis, it is not an ad-hoc method to give numerical approximations. The asymptotic expansions explained in Section 3 can be made up to any order of precision $O(\varepsilon^k)(k = 1, 2, \cdots)$ in principle.

There are several advantageous aspects in our method over the PDE method and the Monte Carlo method, which have been extensively used in practical applications. First, our method is applicable in an unified manner to the pricing problems of various types of functionals of asset prices in the economy governed by the general class of continuous Itô processes. This problem has been known to be difficult by using existing methods. Second, our method is computationally efficient in comarison with other methods since it is very fast to obtain the numerical results by PC. Third, the distributions of the underlying assets and their functionals at any date can be evaluated by our method. This aspect is quite useful in various kinds of simulations. For instance, the pricing formulae derived by our method can be used as control variates to improve the efficiency of Monte Carlo simulations and PDE method. The PDE method, on the other hand, requires a tough task in its implementation especially when the underlying assets follow multi-factor processes including the term structure model of interest rates. Also the Monte Carlo simulations are often quite time consuming in this case. Takahashi (1995) has discussed some extensions of our method for the pricing problem of derivatives in more complicated multi-countries and multi-factors situations.

Finally, we should mention to the fact that the asymptotic expansion approach in this paper can give a powerful and useful tool not only to the valuation problem of contingent claims associated with the term structure of interest rates, but also to other problems in financial economics. Our method usually gives some explicit formulae which may shed some new light on the solutions of problems when the underlying asset prices follow a general class of continuous Itô processes. Hence we do not need to use simple stochastic processes among the class of diffusion or Markovian processes in the usual sense only because the resulting solutions are manageable. We suspect that there have been some works in financial economics, which have used simple but unreasonable stochastic processes mainly because the resulting analyses are mathematically convenient.

6 Mathematical Appendix

In this appendix, we gather some mathematical details which we have omitted in the previous sections. We also discuss the validity of our method by the use of the Malliavin-Watanabe theory in stochastic analysis.

6.1 Two Useful Lemmas

We first give some formulae on the conditional expectation operations as Lemma 6.1, which is a slight generalization of Lemma 5.7 of Yoshida (1992). The proof is a direct result of calculation by making use of the Gaussianity of continuous processes.

Lemma 6.1 : *Let $\boldsymbol{B}(t)$ be an $n \times 1$ vector of independent Brownian motions and $\boldsymbol{x}$ be a k dimensional vector. Let also $\boldsymbol{q}_1(t)$ be an $R^1 \mapsto R^{k\times n}$ non-stochastic function and*

$$\boldsymbol{\Sigma} = \int_0^T \boldsymbol{q}_1(t)\boldsymbol{q}_1'(t)dt \tag{6.1}$$

is a positive definite matrix. (i) Suppose $\boldsymbol{q}_2(u)$ and $\boldsymbol{q}_3(u)$ be $R^1 \mapsto R^{m\times n}$ non-stochastic functions. Then for $0 < s \leq t \leq T$

$$\begin{aligned} &\mathbf{E}\left[\int_0^t \left[\int_0^s \boldsymbol{q}_2(u)d\boldsymbol{B}(u)\right]' \boldsymbol{q}_3(s)d\boldsymbol{B}(s) | \int_0^T \boldsymbol{q}_1(u)d\boldsymbol{B}(u) = \boldsymbol{x}\right] \\ &= \ trace \int_0^t \boldsymbol{q}_1(s)\boldsymbol{q}_3(s)'(\int_0^s \boldsymbol{q}_2(u)\boldsymbol{q}_1(u)'du)ds\boldsymbol{\Sigma}^{-1}\left[\boldsymbol{x}\boldsymbol{x}' - \boldsymbol{\Sigma}\right]\boldsymbol{\Sigma}^{-1} . \end{aligned} \tag{6.2}$$

(ii) Suppose $\boldsymbol{q}_2(u)$ and $\boldsymbol{q}_3(u)$ be $R^1 \mapsto R^n$ non-stochastic functions. Then for $0 < s \leq t \leq T$

$$\begin{aligned} &\mathbf{E}\left[\left[\int_0^s \boldsymbol{q}_2(u)d\boldsymbol{B}(u)\right]\left[\int_0^t \boldsymbol{q}_3(v)d\boldsymbol{B}(v)\right] | \int_0^T \boldsymbol{q}_1(u)d\boldsymbol{B}(u) = \boldsymbol{x}\right] \\ &= \int_0^s \boldsymbol{q}_2(u)\boldsymbol{q}_3(u)'du + \left[\int_0^s \boldsymbol{q}_2(u)\boldsymbol{q}_1(u)'du\right]\boldsymbol{\Sigma}^{-1}\left[\boldsymbol{x}\boldsymbol{x}' - \boldsymbol{\Sigma}\right]\boldsymbol{\Sigma}^{-1}\left[\int_0^t \boldsymbol{q}_1(v)\boldsymbol{q}_3(v)'dv\right] . \end{aligned} \tag{6.3}$$

The second lemma is on the inversion formulae of the characteristic functions of some random variables. The proof is also a direct result of calculation, which has been given in Fujikoshi et.al. (1982), for instance.

Lemma 6.2 : *Suppose that $\boldsymbol{x}$ follows an n-dimensional normal distribution with mean $\mathbf{o}$ and variance-covariance matrix $\boldsymbol{\Sigma}$. The density function of $\boldsymbol{x}$ is denoted by $\phi_\Sigma(\cdot)$. Then for any polynomial functions $g(\cdot)$ and $h(\cdot)$,*

$$\mathcal{F}^{-1}\left[h(-i\boldsymbol{t})\mathbf{E}\left[g(\boldsymbol{x})e^{i\boldsymbol{t}'\boldsymbol{x}}\right]\right]_{<\boldsymbol{\xi}>} = h\left[\frac{\partial}{\partial\boldsymbol{\xi}}\right] g(\boldsymbol{\xi})\phi_\Sigma(\boldsymbol{\xi}), \tag{6.4}$$

where

$$\mathcal{F}^{-1}\left[h(-i\boldsymbol{t})\mathbf{E}\left[g(\boldsymbol{x})e^{i\boldsymbol{t}'\boldsymbol{x}}\right]\right]_{<\boldsymbol{\xi}>} = (\frac{1}{2\pi})^n \int_{R^n} e^{-i\boldsymbol{t}'\boldsymbol{\xi}} h(-i\boldsymbol{t})\mathbf{E}\left[g(\boldsymbol{x})e^{i\boldsymbol{t}'\boldsymbol{x}}\right] d\boldsymbol{t}, \tag{6.5}$$

and the expectation operation $\mathbf{E}[\cdot]$ is taken over $\boldsymbol{x} \in \boldsymbol{R}^n$, and $\mathcal{F}^{-1}[\cdot]_{<\boldsymbol{\xi}>}$ denotes $\mathcal{F}^{-1}[\cdot]$ being evaluated at $\boldsymbol{\xi}$.

6.2 A Sketch of Derivations of Asymptotic Expansions

We give a brief sketch of our derivations used in Section 3. The following derivations are formal and the validity of our method will be discussed in the next subsection. From (3.1), the deterministic process of $\{f^{(\varepsilon)}(s,t)\}$ follows when $\varepsilon \to 0$ is given by

$$f^{(0)}(s,t) = \lim_{\varepsilon \to 0} f^{(\varepsilon)}(s,t) = f(0,t) \,. \tag{6.6}$$

Then we define the random variables $A(s,t)$ and $B(s,t)$ by

$$A(s,t) = \frac{\partial f^{(\varepsilon)}(s,t)}{\partial \varepsilon}|_{\varepsilon=0}, \tag{6.7}$$

and

$$B(s,t) = \frac{1}{2}\frac{\partial^2 f^{(\varepsilon)}(s,t)}{\partial^2 \varepsilon}|_{\varepsilon=0} \,. \tag{6.8}$$

By a direct calculation of differentiation, we have

$$\begin{aligned} A(s,t) &= \int_0^s \left[2\varepsilon b(f^{(\varepsilon)}(v,t),v,t) + \varepsilon^2 \frac{\partial b(f^{(\varepsilon)}(v,t),v,t)}{\partial \varepsilon}\right]_{\varepsilon=0} dv \\ &+ \int_0^s \sum_{i=1}^n \left[\sigma_i(f^{(\varepsilon)}(v,t),v,t) + \varepsilon \frac{\partial \sigma_i(f^{\varepsilon}(v,t),v,t)}{\partial \varepsilon}\right]_{\varepsilon=0} dB_i(v) \\ &= \int_0^s \sum_{i=1}^n \sigma_i^{(0)}(v,t) dB_i(v) \,. \end{aligned} \tag{6.9}$$

Similarly, we have

$$\begin{aligned} B(s,t) &= \int_0^s \left[b(f^{(\varepsilon)}(v,t),v,t) + 2\varepsilon \frac{\partial b(f^{(\varepsilon)}(v,t),v,t)}{\partial \varepsilon} + \frac{\varepsilon^2}{2}\frac{\partial^2 b(f^{(\varepsilon)}(v,t),v,t)}{\partial \partial^2 \varepsilon}\right]_{\varepsilon=0} dv \\ &+ \int_0^s \sum_{i=1}^n \left[\frac{\partial \sigma_i(f^{(\varepsilon)}(v,t),v,t)}{\partial \varepsilon} + \frac{1}{2}\varepsilon \frac{\partial^2 \sigma_i(f^{(\varepsilon)}(v,t),v,t)}{\partial^2 \varepsilon}\right]_{\varepsilon=0} dB_i(v) \\ &= \int_0^s b^{(0)}(v,t) dv + \int_0^s \sum_{i=1}^n \partial \sigma_i^{(0)}(v,t) A(v,t) dB_i(v) \,. \end{aligned} \tag{6.10}$$

Hence we have obtained the stochastic differential equations which $\{A(s,t)\}$ and $\{B(s,t)\}$ must satisfy.

Next, we substitute $\{f(t,u)\}$ in (3.14) and use the fact that $\{P^{(\varepsilon)}(s,t)\}$ are non-stochastic functions at $s=0$, which lead to (3.15) and (3.16). The stochastic expansion of the discounted factor can be obtained by using $\{r^{(\varepsilon)}(t)\}$ instead of $\{f^{(\varepsilon)}(s,t)\}$. In (3.17), (3.19), (3.22), and (3.24), we can utilize the Fubini-type theorem on the exchanges of integration operations [7]. By expanding the exponential functions, we have

$$
(6.11) \quad P^{(\varepsilon)}(t,T) = \frac{P(0,T)}{P(0,t)}\Big[1 - \varepsilon\int_t^T A(t,u)du - \varepsilon^2\int_t^T B(t,u)du + \varepsilon^2\frac{1}{2}\{\int_t^T A(t,u)du\}^2\Big] + o_p(\varepsilon^2)\ ,
$$

and

$$
(6.12) \quad e^{-\int_0^T r^{(\varepsilon)}(s)ds} = P(0,T)\Big[1 - \varepsilon\int_0^T A(t,t)dt - \varepsilon^2\int_0^T B(t,t)dt + \varepsilon^2\frac{1}{2}\{\int_0^T A(t,t)dt\}^2\Big] + o_p(\varepsilon^2)\ ,
$$

respectively.

Finally, we multiply the stochastic expansions of the discounted factor and the terminal payoff function. Then by rearranging each term in the resulting stochastic expansions, we can obtain the form of (3.26) and (3.38) in Example 1 and Example 2.

6.3 Validity of the Asymptotic Expansion Approach

The validity of the asymptotic expansion approach in this paper can be given along the line based on the remarkable work by Watanabe (1987) on the Malliavin calculus in stochastic analysis. Yoshida (1992) has utilized the results and method originally developed by Watanabe (1987) and given some useful results on the validity of the asymptotic expansions of some functionals on continuous time homogenous diffusions processes. The validity of our method can be obtained by the similar arguments used by Yoshida (1992) and Chapter V of Ikeda and Watanabe (1989) with substantial modifications. This is mainly because the continuous stochastic processes defined by (3.1) for spot interest rates and forward rates are not necessarily Markovian in the usual sense.

Since the rigorous proofs of our claims in this section can be quite lengthy but most parts are quite straightforward extensions of the existing results in stochastic analysis, we shall only give their rough sketch below. Our arguments on the validity of the asymptotic expansion approach for interest rates based contingent claims consist of four steps. The main aim in the following steps will be to check the non-degeneracy condition of the Malliavin-covariance in our situation.

[7] See *Lemma* 4.1 of Ikeda and Watanabe (1989).

[Step 1] : First, we shall prepare some notations. For this purpose, we shall freely use the notations by Ikeda and Watanabe (1989) as a standard textbook. We shall only discuss the validity of the asymptotic expansion approach based on the one-dimensional Wiener space without loss of generality. We only need more complicated notations in the general case. (See Ikeda and Watanabe (1989) for the details.) Let $(\boldsymbol{W}, P)$ be the 1−dimensional Wiener space and let $\boldsymbol{H}$ be the Cameron-Martin subspace of $\boldsymbol{W}$ endowed with the norm

$$|\dot{h}|_H = \int_0^T |\dot{h}_t|^2 dt \tag{6.13}$$

for $h \in \boldsymbol{H}$. The $L_p(\boldsymbol{R})$−norm of $\boldsymbol{R}$−valued Wiener functional g for any $s \in \boldsymbol{R}$, and $p \in (1, \infty)$ is defined by

$$||g||_{p,s} = ||(I - \mathcal{L})^{s/2} g||_p \tag{6.14}$$

where $\mathcal{L}$ is the Ornstein-Uhlenbeck operator in the standard stochastic analysis. An $\boldsymbol{R}$−valued function $g : \boldsymbol{W} \mapsto \boldsymbol{R}$ is called an $\boldsymbol{R}$−valued polynomial functional if $g = \sum_{i=1}^m p([h_1](B), \cdots, [h_k](B))e_i$, where $k, m \in \boldsymbol{Z}^+, h_i \in \boldsymbol{H}, e_i \in \boldsymbol{R}$, $p_i(x_1, \cdots, x_k)$ are polynomials and

$$[h](B) = \int_0^T \dot{h}_t dB(t)$$

for $h \in \boldsymbol{H}$. Let $P(\boldsymbol{R})$ denote the totality of $\boldsymbol{R}$−valued polynomials on the Wiener space (W, P). The Banach space $\boldsymbol{D}_p^s(\boldsymbol{R})$ is the completion of $P(\boldsymbol{R})$ with respect to $||\cdot||_{p,s}$. The dual space of $\boldsymbol{D}_p^s(\boldsymbol{R})$ is $\boldsymbol{D}_q^{-s}(\boldsymbol{R})$, where $s \in \boldsymbol{R}, p > 1$, and $1/p + 1/q = 1$. The space $\boldsymbol{D}^\infty(\boldsymbol{R}) = \cap_{s>0} \cap_{1<p<+\infty} \boldsymbol{D}_p^s(\boldsymbol{R})$ is the set of Wiener test functionals and $\tilde{\boldsymbol{D}}^{-\infty}(\boldsymbol{R}) = \cup_{s>0} \cap_{1<p<+\infty} \boldsymbol{D}_p^{-s}(\boldsymbol{R})$ is a space of generalized Wiener functionals. For $F \in P(\boldsymbol{R})$ and $h \in \boldsymbol{H}$, the derivative of F in the direction of h is defined by

$$< D_h F(B), e > = \frac{d}{d\varepsilon} < F(B + \varepsilon h), e > |_{\varepsilon=0} \tag{6.15}$$

for $e \in \boldsymbol{R}$ and $DF \in P(\boldsymbol{H} \otimes \boldsymbol{R})$ is called the H−derivative of F. It is known that the norm $|| \cdot ||_{p,s}$ is equivalent to the norm $\sum_{k=0}^s ||D^k \cdot ||_p$.

For $F \in \boldsymbol{D}^\infty(\boldsymbol{R})$, we can define the Malliavin-covariance by

$$\sigma(F) = < DF(B), DF(B) >_H \ . \tag{6.16}$$

[Step 2] : We set $n = 1$ and $\varepsilon = 1$ in (3.1) in Step 2. The starting point of our discussion is the result by Morton (1989) on the existence and uniqueness of the solution of the stochastic integral equation (3.1) for forward rate processes.

Theorem 6.1 : *Under Assumption I, there exists a jointly continuous process* $\{f^{(\varepsilon)}(s,t), 0 \le s \le t \le T\}$ *satisfying (3.1) with* $\varepsilon = 1$. *There is at most one solution of (3.1) with* $\varepsilon = 1$.

We shall consider the $H-$derivatives of the forward rate processes $\{f^{(1)}(s,t)\}$. For any $h \in H$, we successively define a sequence of random variables $\{\xi^{(n)}(s,t)\}$ by the integral equation:

(6.17)

$$\begin{aligned}
\xi^{(n+1)}(s,t) &= \int_0^s \left[\partial\sigma(f^{(1)}(v,t),v,t)\int_v^t \sigma(f^{(1)}(v,y),v,y)dy\xi^{(n)}(v,t)\right] dv \\
&+ \int_0^s \left[\sigma(f^{(1)}(v,t),v,t)\int_v^t \partial\sigma(f^{(1)}(v,y),v,y)\xi^{(n)}(v,y)dy\right] dv \\
&+ \int_0^s \partial\sigma(f^{(1)}(v,t),v,t)\xi^{(n)}(v,t)dB(v) \\
&+ \int_0^s \sigma(f^{(1)}(v,t),v,t)\dot{h}_v dv
\end{aligned}$$

where the initial condition is given by $\xi^{(0)}(s,t) = 0$. Then we have the next result by using the standard method in stochastic analysis.

Lemma 6.3 : *For any* $p > 1$ *and* $0 \le s \le t \le T$,

$$E[|\xi^{(n)}(s,t)|^p] < \infty \,, \tag{6.18}$$

and as $n \to \infty$

$$E[\sup_{0\le s\le t\le T} |\xi^{(n+1)}(s,t) - \xi^{(n)}(s,t)|^2] \to 0 \,. \tag{6.19}$$

Proof of Lemma 6.3: **[i]** We use the induction argument for n. We have (6.18) when $n = 1$ because $\sigma(\cdot)$ is bounded and $\dot{h}_v$ is a square-integrable function in (6.17). Suppose (6.18) hold for $n = m$. Then there exist positive constants $M_i (i = 1, \cdots, 4)$ such that

(6.20)

$$\begin{aligned}
|\xi^{(m+1)}(s,t)|^p &\le M_1 \int_0^s |\xi^{(m)}(v,t)|^p dv + M_2[\sup_{0\le u\le s} |\int_0^u \xi^{(m)}(v,t)dB(v)|]^p \\
&\quad + M_3 \int_0^s \int_v^t |\xi^{(m)}(v,y)|^p dy dv + M_4[\int_0^s |\dot{h}_v|^2 dv]^{p/2} \,.
\end{aligned}$$

By a martingale inequality (Theorem III-3.1 of Ikeda and Watanabe (1989)), the expectation of the second term on the right hand side of (6.20) is less than

$$M_3^{'} E[\int_0^s |\xi^{(m)}(v,t)|^2 dv]^{p/2} \le M_3^{''} \int_0^s E[|\xi^{(m)}(v,t)|]^p dv \,, \tag{6.21}$$

where M_3' and M_3'' are positive constants. Because $\dot{h}_v$ is square-integrable, we have (6.18) when $n = m + 1$.

[ii] From (6.17), there exist positive constants $M_i (i = 5, 6, 7)$ such that for $0 \le s \le t$,

(6.22)
$$\begin{aligned} |\xi^{(n+1)}(s,t) - \xi^{(n)}(s,t)|^2 \ &\le\ M_5[\int_0^s |\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)|dv]^2 \\ &+\ M_6[\int_0^s \int_v^t |\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)|dydv]^2 \\ &+\ M_7[\int_0^s \partial\sigma(f^{(1)}(v,t), v, t)|\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)|dB(v)]^2 \\ &\equiv\ \sum_{i=1}^{3} I_i^{(n)}(s,t)\ , \end{aligned}$$

where we have defined $I_i^{(n)}(s,t)$ by the last equality. By using the Cauchy-Schwartz inequality,

$$E[\sup_{0 \le u \le s} I_1^{(n)}(u,t)] \le M_5 s \int_0^s E[|\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)|^2]dv\ . \tag{6.23}$$

By repeating the above argument to the second term of (6.22), we have

$$\begin{aligned} I_2^{(n)}(u,t) \ &\le\ M_6 u \int_0^u [\int_v^t |\xi^{(n)}(v,y) - \xi^{(n-1)}(v,y)|dy]^2 dv \\ &\le\ M_6 ut \int_0^u \int_v^t |\xi^{(n)}(v,y) - \xi^{(n-1)}(v,y)|^2 dydv\ . \end{aligned} \tag{6.24}$$

Then

$$E[\sup_{0 \le u \le s} I_2^{(n)}(u,t)] \le M_6 st \int_0^s \int_v^t E[|\xi^{(n)}(v,y) - \xi^{(n-1)}(v,y)|^2]dydv\ . \tag{6.25}$$

For the third therm of (6.22), we have

$$E[\sup_{0 \le u \le s} I_3^{(n)}(u,t)] \le M_7' \int_0^s E[|\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)|^2]dv \tag{6.26}$$

because of the boundedness of $\partial\sigma(\cdot)$, where M_7' is a positive constant. By using (6.24), (6.25), and (6.26), we have

(6.27)
$$\begin{aligned} E[\sup_{0 \le u \le s} |\xi^{(n+1)}(u,t) - \xi^{(n)}(u,t)|^2] \ &\le\ M_8 \Bigg(\int_0^s E[\sup_{0 \le v \le u} |\xi^{(n)}(v,t) - \xi^{(n-1)}(v,t)|^2]du \\ &+\ \int_0^s \int_u^t E[\sup_{0 \le v \le u} |\xi^{(n)}(v,y) - \xi^{(n-1)}(v,y)|^2]dydu \Bigg)\ , \end{aligned}$$

where M_8 is a positive constant. By defining a sequence of $\{u^{(n)}(s,t)\}$ by

$$u^{(n+1)}(s,t) = E[\sup_{0\le u\le s} |\xi^{(n+1)}(u,t) - \xi^{(n)}(u,t)|^2] ,$$

we have the relation

$$u^{(n+1)}(s,t) \le M_9 \int_0^s [\int_u^t u^{(n)}(u,y)dy + u^{(n)}(u,t)]du , \tag{6.28}$$

where M_9 is a positive constant. If we have an inequality

$$u^{(n+1)}(s,t) \le \frac{1}{(n+1)!}[M_9(t+1)s]^{n+1} , \tag{6.29}$$

we can show (6.19) as $n \to +\infty$. We use the induction argument for $n \ge 1$. When $n = 1$, there exists a positive constant M_9 such that

$$\begin{aligned} u^{(1)}(s,t) &= E[\sup_{0\le u\le s} |\xi^{(1)}(u,t) - \xi^{(0)}(u,t)|^2] \\ &= E[\sup_{0\le u\le s} |\int_0^s \sigma(f^{(1)}(u,t),u,t)\dot{h}_v du|^2] \\ &\le M_9(1+t)s \end{aligned} \tag{6.30}$$

because $\sigma(\cdot)$ is bounded and $\dot{h}_v$ is square- integrable. Suppose (6.29) hold for $n = m$. Then

$$\begin{aligned} u^{(m+1)}(s,t) &\le M_9 \int_0^s [\int_u^t u^{(m)}(u,y)dy + u^{(m)}(u,t)]du \\ &\le M_9 \int_0^s [\int_u^t M_9^m (t+1)^m \frac{s^m}{m!} dy + M_9^m (t+1)^m \frac{s^m}{m!}]du \\ &\le M_9^{m+1}(t+1)^{m+1}\frac{s^{m+1}}{(m+1)!} . \end{aligned} \tag{6.31}$$

(QED)

Because of (6.19), we have

$$\begin{aligned} &\sum_{n=1}^{\infty} P\{\sup_{0\le s\le t\le T} |\xi^{(n+1)}(u,t) - \xi^{(n)}(u,t)| > \frac{1}{2^n}\} \\ \le\ &\sum_{n=1}^{\infty} \frac{1}{n!}[4M_9(T+1)s]^n < +\infty . \end{aligned} \tag{6.32}$$

Then by the Borel-Cantelli lemma, the sequence of random variables $\{\xi^{(n)}(s,t)\}$ converges uniformly on $0 \le s \le t \le T$. Hence we can establish the existence of the H−derivative of $f^{(1)}(s,t)$, which is given by the solution of the stochastic integral equation:

(6.33)

$$\begin{aligned} D_h f^{(1)}(s,t) &= \int_0^s \left[\partial\sigma(f^{(1)}(v,t),v,t)\int_v^t \sigma(f^{(1)}(v,y),v,y)dyD_hf^{(1)}(v,t)\right] dv \\ &+ \int_0^s \left[\sigma(f^{(1)}(v,t),v,t)\int_v^t \partial\sigma(f^{(1)}(v,y),v,y)D_hf^{(1)}(v,y)dy\right] dv \\ &+ \int_0^s \partial\sigma(f^{(1)}(v,t),v,t)D_hf^{(1)}(v,t)dB(v) \\ &+ \int_0^s \sigma(f^{(1)}(v,t),v,t)\dot{h}_v dv \ . \end{aligned}$$

Next, we examine the existence of higer order moments of $D_hf^{(1)}(s,t)$ satisfying (6.33). To do this, we prepare the following inequality.

Lemma 6.4 *: Suppose for* $k_0 > 0, k_1 > 0, A_N > 0$ *and* $0 < s \leq t \leq T$, *a function* $u_N(s,t)$ *satisfies (i)* $0 < u_N(s,t) \leq A_N$ *and (ii)*

$$u_N(s,t) \leq k_0 + k_1\left[\int_0^s u_N(v,t)dv + \int_0^s\int_v^t u_N(v,y)dydv\right] \ . \tag{6.34}$$

Then

$$u_N(s,t) \leq k_0 e^{k_1(1+t)s} \ . \tag{6.35}$$

Proof of Lemma 6.4: By substituting (i) into the right hand side of (6.34), we have

$$\begin{aligned} u_N(s,t) &\leq k_0 + A_Nk_1\left[\int_0^s ds + \int_0^s\int_v^t dydv\right] \\ &\leq k_0 + A_Nk_1(1+t)s \ . \end{aligned} \tag{6.36}$$

By repeating the substitution of (6.36) into the right hand side of (3.34), we have

$$u_N(s,t) \leq k_0\sum_{k=0}^n \frac{1}{k!}[k_1(1+t)]^k + \frac{1}{(n+1)!}A_N[k_1(1+t)s]^{n+1} \ . \tag{6.37}$$

Then we have (3.35) by taking $n \to +\infty$. (QED)

In order to use Lemma 6.4, we consider the truncated random variable

$$\zeta_N(s,t) = \left[D_hf^{(1)}(s,t)\right] I_N(s,t) \ , \tag{6.38}$$

where $I_N(s,t) = 1$ if

$$\sup_{0\le v\le s, v\le y\le t}|D_h f^{(1)}(v,y)| \le N$$

and $I_N(s,t) = 0$ otherwise. By using the boundedness conditions in Assumption I and $\dot{h}_s$ being square-integrable, we can show that there exist positive constants $M_i (i = 10, \cdots, 13)$ such that

$$\begin{aligned} |\zeta_N(s,t)|^p &\le M_{10}\int_0^s |\zeta_N(v,t)|^p dv + M_{11}|\int_0^s \zeta_N(v,t)dB(v)|^p \\ &+ M_{12}\int_0^s\int_v^t |\zeta_N(v,y)|^p dy dv + M_{13}|\int_0^s \sigma(v,t)\dot{h}_v dv|^p . \\ &\equiv \sum_{i=1}^4 J_i^N(s,t) , \end{aligned} \tag{6.39}$$

where we have defined $J_i^N(s,t) (i = 1, \cdots, 4)$ by the last equality. By using a martingale inequality (Theorem III-3.1 of Ikeda- Watanabe (1989)), we have

$$\begin{aligned} E[J_2^N(s,t)] &\le M'_{11}E[\int_0^s |\zeta_N(v,t)|^2 dv]^{p/2} \\ &\le M''_{11}E[\int_0^s |\zeta_N(v,t)|^p dv] , \end{aligned} \tag{6.40}$$

where M'_{11} and M''_{11} are positive constants. Also by the Cauchy-Schwartz inequality, we have

$$J_4^N(s,t) \le M_{13}[\int_0^s \sigma(f^{(1)}(v,t),v,t)^2 dv \int_0^s |\dot{h}_v|^2 dv]^{p/2} , \tag{6.41}$$

which is bounded because $\sigma(\cdot)$ is bounded and $\dot{h}_v$ is square-integrable. If we set $u_N(s,t) = E[|\zeta_N(s,t)|^p]$, then we can directly apply Lemma 6.4. By taking the limit of the expectation function $u_N(s,t)$ as $N \to \infty$, we have the following result.

Lemma 6.5 : *For any* $p > 1$,

$$E[|D_h f^{(1)}(s,t)|^p] < +\infty , \tag{6.42}$$

By this lemma and the equivalence of two norms stated in Step 1, we can establish that

$$f^{(1)}(s,t) \in \cap_{1<p<+\infty}\boldsymbol{D}_p^1(\boldsymbol{R}) . \tag{6.43}$$

Then by repeating the above procedure, we can derive the higher order $H-$derivatives of $f^{(1)}(s,t)$. Hence we can obtain the following result.

Theorem 6.2 : *Suppose Assumption I in Section 3 hold for the forward rate processes. Then for* $0 < s \le t \le T$

$$f^{(1)}(s,t) \in \boldsymbol{D}^{\infty}(\boldsymbol{R}) . \tag{6.44}$$

[Step 3] : Let a stochastic process $\{Y^{(\varepsilon)}(s,t), 0 \le s \le t \le T\}$ be the solution of the stochastic integral equation:

(6.45)
$$Y^{(\varepsilon)}(s,t) = 1 + \varepsilon^2 \int_0^s \left[\partial\sigma(f^{(\varepsilon)}(v,t),v,t)\int_v^t \sigma(f^{(\varepsilon)}(v,y),v,y)dy\right] Y^{(\varepsilon)}(v,t)dv + \varepsilon \int_0^s \partial\sigma(f^{(\varepsilon)}(v,t),v,t)Y^{(\varepsilon)}(v,t)dB(v) .$$

Since the coefficients of $Y^{(\varepsilon)}(s,t)$ on the right hand side of (6.45) are bounded by Assumption I, we can obtain the next result.

Lemma 6.6 : *For any* $1 < p < +\infty, 0 < \varepsilon \le 1,$ *and* $0 < s \le t \le T,$

$$E[|Y^{(\varepsilon)}(s,t)|^p] + E[|Y^{(\varepsilon)-1}(s,t)|^p] < +\infty . \tag{6.46}$$

Proof of Lemma 6.6: We define a sequence of random variables $\{Y_n^{(\varepsilon)}(s,t)\}$ by

(6.47)
$$Y_{n+1}^{(\varepsilon)}(s,t) = 1 + \varepsilon^2 \int_0^s \left[\partial\sigma(f^{(\varepsilon)}(v,t),v,t)\int_v^t \sigma(f^{(\varepsilon)}(v,y),v,y)dy\right] Y_n^{(\varepsilon)}(v,t)dv + \varepsilon \int_0^s \partial\sigma(f^{(\varepsilon)}(v,t),v,t)Y_n^{(\varepsilon)}(v,t)dB(v) ,$$

where the initial condition is given by $Y_0^{(\varepsilon)}(s,t) = 1$. Then by the same argument as the proof of Lemma 6.3, we have

$$E[|Y_n^{(\varepsilon)}(s,t)|^p] < \infty , \tag{6.48}$$

and as $n \longrightarrow \infty$

$$E[\sup_{0 \le s \le t \le T} |Y_{n+1}^{(\varepsilon)}(s,t) - Y_n^{(\varepsilon)}(s,t)|^2] \rightarrow 0 . \tag{6.49}$$

Hence we can establish the existence of the random variables $\{Y^{(\varepsilon)}(s,t)\}$ satisfying (6.45). Then by the same argument as (6.38)-(6.41), we have

$$E[|Y^{(\varepsilon)}(s,t)|^p] < \infty \tag{6.50}$$

for any $p > 1$. Let $Z^{(\varepsilon)}(s,t) = Y^{(\varepsilon)-1}(s,t)$. Then we can show that

$$d[Z^{(\varepsilon)}(s,t)Y^{(\varepsilon)}(s,t)] = 0 \tag{6.51}$$

and

(6.52)

$$Z^{(\varepsilon)}(s,t) = 1 - \varepsilon^2 \int_0^s \left[\partial\sigma(f^{(\varepsilon)}(v,t),v,t) \int_v^t \sigma(f^{(\varepsilon)}(v,y),v,y)dy\right] Z_n^{(\varepsilon)}(v,t)dv$$
$$- \varepsilon \int_0^s \partial\sigma(f^{(\varepsilon)}(v,t),v,t)Z_n^{(\varepsilon)}(v,t)dB(v)$$

by using Itô's Lemma and $Z^{(\varepsilon)}(0,t) = 1$. Hence by the similar argument as on $Y^{(\varepsilon)}(s,t)$, we can establish

$$E[|Z^{(\varepsilon)}(s,t)|^p] < \infty \tag{6.53}$$

for any $p > 1$. (QED)

Now we consider the asymptotic behavior of a functional

$$F^{(\varepsilon)}(s,t) = \frac{1}{\varepsilon}[f^{(\varepsilon)}(s,t) - f^{(0)}(0,t)] \tag{6.54}$$

as $\varepsilon \longrightarrow 0$. By using the stochastic process $\{Y^{(\varepsilon)}(s,t)\}$, the $H-$derivative of $F^{(\varepsilon)}(s,t)$ can be represented as

$$D_h F^{(\varepsilon)}(s,t) = \int_0^s Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(v,t)C^{(\varepsilon)}(v,t)dv\ , \tag{6.55}$$

where

$$\begin{aligned} C^{(\varepsilon)}(v,t) &= \sigma(f^{(\varepsilon)}(v,t),v,t)\dot{h}_v + \varepsilon\sigma(f^{(\varepsilon)}(v,t),v,t) \\ &\times \int_v^t \partial\sigma(f^{(\varepsilon)}(v,y),v,y)D_h f^{(\varepsilon)}(v,y)dy\ . \end{aligned} \tag{6.56}$$

Let

$$a_v^{(\varepsilon)}(s,t) = Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(v,t)C^{(\varepsilon)}(v,t)\ , \tag{6.57}$$

and

(6.58)

$$\begin{aligned} \eta_c^{(\varepsilon)}(s,t) &= c\int_0^s |\varepsilon Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(v,t)\sigma(f^{(\varepsilon)}(v,t),v,t) \\ &\times \int_v^t \partial\sigma(f^{(\varepsilon)}(v,y),v,y)D_h f^{(\varepsilon)}(v,y)dy|^2 dv \\ &+ c\int_0^s |Y^{(\varepsilon)}(s,t)Y^{(\varepsilon)-1}(v,t)\sigma(f^{(\varepsilon)}(v,t),v,t) - \sigma(f^{(0)}(v,t),v,t)|^2 dv\ . \end{aligned}$$

Then the condition in Assumption II in Section 3 is equivalent to the non-degeneracy condition:

$$\Sigma_t = \int_0^s a_v^{(0)}(v,t)^2\, dv > 0 \tag{6.59}$$

because $Y^{(0)}(v,t) = 1$ for $0 \le v \le s \le t$. The next lemma shows that the truncation by $\eta_c^{(\varepsilon)}(s,t)$ is negligible.

Lemma 6.7 *: For $0 < s \leq t \leq T$ and any $n \geq 1$,*

$$\lim_{\varepsilon \to 0} \varepsilon^{-n} P\{|\eta_c^{(\varepsilon)}(s,t)| > \frac{1}{2}\} = 0 \,. \tag{6.60}$$

Proof of Lemma 6.7: We re-write (6.58) as $\eta_c^{(\varepsilon)}(s,t) = \eta_1^{(\varepsilon)} + \eta_2^{(\varepsilon)}$. By using Assumption I, Lemma 6.5, and the Markov inequality, it is straightforward to show that for any $p > 1$ and $c_1 > 0$ there exists a positive constant c_2 such that

$$P\{|\eta_1^{(\varepsilon)}| > c_1\} \leq c_2 \varepsilon^{2p} \,. \tag{6.61}$$

By the Lipschitz continuity of the volatility function $\sigma(\cdot)$, there exist positive constants M_{14} and M_{15} such that

$$|\eta_2^{(\varepsilon)}| \leq M_{14}|f^{(\varepsilon)}(s,t) - f^{(0)}(0,t)| + M_{15}|Y^{(\varepsilon)}(s,t) Y^{(\varepsilon)-1}(v,t) - 1| \,. \tag{6.62}$$

Then by Lemma 10.5 of Ikeda and Watanabe (1989), for a positive c_3 and sufficiently small $\varepsilon > 0$, there exist positive constants c_4 and c_5 such that

$$P\{\sup_{0 \leq s \leq t \leq T} |f^{(\varepsilon)}(s,t) - f^{(0)}(0,t)| > c_3\} \leq c_4 exp(-c_5 \varepsilon^{-2}) \,. \tag{6.63}$$

For the second term of the right hand side of (6.62) for η_2^{ε}, we re-write

$$\eta_{22}^{(\varepsilon)} = M_{15} Y^{(\varepsilon)}(v,t)^{-1} |Y^{(\varepsilon)}(s,t) - Y^{(\varepsilon)}(v,t)| \,, \tag{6.64}$$

where

(6.65)

$$\begin{aligned} Y^{(\varepsilon)}(s,t) - Y^{(\varepsilon)}(v,t) &= \varepsilon^2 \int_v^s \left[\partial\sigma(f^{(\varepsilon)}(u,t),u,t) \int_u^t \sigma(f^{(\varepsilon)}(u,y),u,y) dy\right] Y^{(\varepsilon)}(u,t) du \\ &+ \varepsilon \int_v^s \partial\sigma(f^{(\varepsilon)}(u,t),u,t) Y^{(\varepsilon)}(u,t) dB(u) \,. \end{aligned}$$

Then by Lemma 6.6, for any $p \geq 1$ and $c_6 > 0$ there exists a positive constant c_7 such that

$$P\{|\eta_{22}^{(\varepsilon)}| > c_6\} \leq c_7 \varepsilon^{2p} \,. \tag{6.66}$$

By using (6.61), (6.63), and (6.66), we have (6.60). (QED)

By modifying the method developed by Yoshida (1992) for the present case, we have the key result on the validity of the asymptotic expansion approach in this paper.

Theorem 6.3 : *Under Assumptions I and II in Section 3, the Malliavin-covariance* $\sigma(F^{(\varepsilon)})$ *of* $F^{(\varepsilon)}$ *is uniformly non-degenerate in the sense that there exists* $c_0 > 0$ *such that for any* $c > c_0$ *and any* $p > 1$

$$\sup_{\varepsilon} E[I(|\eta_c^{(\varepsilon)}| \leq 1)\sigma(F^{(\varepsilon)})^{-p}] < +\infty \ , \tag{6.67}$$

where $I(\cdot)$ *is the indicator function.*

Hence we have obtained a truncated version of the non-degeneracy condition of the Malliavin-covariance for the spot interest rates and forward rates processes, which are the solutions of the stochastic integral equation (3.1). The rest of our arguments for the asymptotic expansion approach is based on Theorem 4.1 of Yoshida (1992), which is an extension of Theorem 2.3 of Watanabe (1987) because it gives the validity of the asymptotic expansion of the distribution function of functionals with truncation under the non-degenacy condition on the Malliavin-covariance given by (6.67). Let $\psi : \boldsymbol{R} \rightarrow \boldsymbol{R}$ be a smooth function such that $0 \leq \psi(x) \leq 1, \psi(x) = 1$ for $|x| \leq \frac{1}{2}$, and $\psi(x) = 0$ for $|x| \geq 1$. Then the composite functional $\psi(\eta^{(\varepsilon)})I_A(F^{(\varepsilon)})$ is well-defined for any $A \in \boldsymbol{B}$ in the sense that it is in $\tilde{\boldsymbol{D}}^{-\infty}$, where $\boldsymbol{B}$ is the Borel $\sigma-$field in $\boldsymbol{R}$ and $I_A(\cdot)$ is the indicator function. By using Theorem 6.3, lemmas in this section, and Theorem 4.1 of Yoshida (1992), it has a proper asymptotic expansion as $\varepsilon \rightarrow 0$ uniformly in $\tilde{\boldsymbol{D}}^{-\infty}$. Then we have a proper asymptotic expansion for the density function of our interest by taking the expectation operations.

Also it is straightforward to obtain the similar non-degeneracy conditions as $\Sigma > 0$ for the discounted coupon bond price process and the average interest rate process as we have stated in Theorem 3.2 of Section 3.

[Step 4] : Finally, we should mention that the inversion technique we have used is different from the one used by Yoshida (1992). He has used the Schwartz's type distribution theory for the generalized Wiener functionals while our method is based on the simple inversion technique for the characteristic functions of random variables, which has been standard in the statistical asymptotic theory. Hence what we need to show is that the resulting formulae by our method are equivalent to his final formulae. In the notations of Yoshida (1992), we take $\varphi(x) = 1$ in his Lemma 5.6. Then he has used

$$p_1^{'}(x) = (-1)^k \frac{\partial^k}{\partial x_1 \cdots \partial x_k} \mathbf{E}\left[f_1^i \partial_i I_A(f_0)\right] , \tag{6.68}$$

and

$$p_2^{''}(x) = (-1)^k \frac{\partial^k}{\partial x_1 \cdots \partial x_k} \mathbf{E}\left[\{f_1^i \partial_i \varphi(f_0)\} I_A(f_0)\right] , \tag{6.69}$$

where $I_A(f_0)$ is the indicator function and f_0 corresponds to the random variable of the order $O_p(1)$, which is similar to g_1 in our notation. The differentiation of indicator function has some proper mathematical meaning in the sense of differentiation on the generalized Wiener functionals. (See Watanabe (1987) and Yoshida (1992) for its details.) By the use of the pull-back operation of the generalized Wiener functionals, Yoshida (1992) has obtained the explicit expansion form of the density function for a particular functional in his problem as

$$p_1(x) = p_1^{'}(x) + p_2^{''}(x). \tag{6.70}$$

Since in the second term, however, it is straightforward to show in our framework that

$$p_1(x) = (-1)\frac{d}{dx}\left[\mathbf{E}(g_2|g_1 = x)\phi_\Sigma(x)\right] \tag{6.71}$$

when $k = 1$ by using our notations in this paper. This quantity is exactly what the inversion formula (Lemma 6.2) gives as the second order term in the asymptotic expansion of the density function of the normalized random variable $X_T^{(\varepsilon)}$ in (3.31).

References

[1] Fujikoshi, Y. Morimune, K. Kunitomo, N. and Taniguchi, M. (1982), "Asymptotic Expansions of the Distributions of the Estimates of Coefficients in a Simultaneous Equation System," *Journal of Econometrics*, Vol.18, 191-205.

[2] Heath, D. Jarrow, R. and Morton, A. (1992), "Bond Pricing and the Term Structure of Interest Rates : A New Methodology for Contingent Claims Valuation," *Econometrica*, Vol.60, 77-105.

[3] Ho, T. and Lee, S. (1986), "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance*, Vol.41, 1011-1029.

[4] Hull, J. (1993), *Options, Futures, and Other Derivative Securities*, Second Edition, Prentice-Hall, New Jersey.

[5] Ikeda, N. and Watanabe, S. (1989), *Stochastic Differential Equations and Diffusion Processes*, Second Edition, North-Holland/Kodansha, Tokyo.

[6] Kunitomo, N. and Takahashi, A. (1992), "Pricing Average Options," *Japan Financial Review*, Vol.14, 1-20.

[7] Morton, A.J. (1989), "Arbitrage and Martingales," Unpublished Ph.D. Dissertation, Cornell University.

[8] Takahashi, A. (1995), "Essays on the Valuation Problems of Contingent Claims," Unpublished Manuscript, Haas School of Business, University of California, Berkeley.

[9] Watanabe, S. (1987), "Analysis of Wiener Functionals (Malliavin Calculus) and its Applications to Heat Kernels," *The Annals of Probability*, Vol.15, 1-39.

[10] Yoshida, N. (1992), "Asymptotic Expansion for Statistics Related to Small Diffusions," *Journal of the Japan Statistical Society*, Vol.22, 139-159.

Table 1: Average options on interest rates (T=0.25y)

Strike rate %	5.50	5.00	4.50
(1)Stochastic Expansion	5.36	25.12	63.98
Difference (bp)	-0.034	0.002	0.080
Diff. rate %	-0.56	0.01	0.13
(2)Finite difference	5.36	24.99	63.81
Difference (bp)	0.026	-0.13	-0.09
Diff. rate %	-0.48	-0.52	-0.14
(3)Monte Carlo	5.39	25.12	63.90
(4)European call	16.30	38.05	71.76
(3)/(4) %	33	66	89

Table 2: Average options on interest rates (T=0.50y)

Strike rate %	6.00	5.00	4.00
(1)Stochastic expansion	2.69	32.10	111.54
Difference (bp)	0.005	0.121	-0.010
Diff. rate %	0.20	0.38	-0.09
(2)Finite difference	2.68	31.98	111.34
Difference (bp)	-0.0066	0.002	-0.21
Diff. rate %	-0.25	-0.16	0.23
(3)Monte Carlo	2.69	31.98	111.55
(4)European call	13.86	50.47	119.64
(3)/(4) %	19	63	93

Table 3: Average options on interest rates (T=1.00y)

Strike rate %	6.00	5.00	4.00
(1)Stochastic expansion	8.13	41.37	112.30
Difference (bp)	0.040	-0.030	-0.010
Diff. rate %	0.49	0.07	-0.01
(2)Finite difference	8.06	41.32	112.25
Difference (bp)	-0.03	-0.017	-0.060
Diff. rate %	-0.37	-0.04	-0.05
(3)Monte Carlo	8.09	41.34	112.31
(4)European call	28.14	67.26	129.60
(3)/(4) %	29	62	87