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# A Stationary and Non-stationary Simultaneous Switching Autoregressive Models with an Application to Financial Time Series

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**Abstract**

A common observation among economists on many economic time series including major financial time series is the asymmetrical movement between in the down-ward phase and in the up-ward phase in their sample paths. Since this feature of time irreversibility cannot be described by the standard ARMA and ARIMA time series models, we introduce a stationary and non-stationary Simultaneous Switching Autoregressive (SSAR) models, which are non-linear Markovian switching time series models. We discuss some properties of these time series models and the estimation method for their unknown parameters. We also report a simple empirical result on Nikkei 225 spot and futures indeces by using a non-stationary SSAR model.

**Key Words**

Asymmetry, Non-linearity, Non-stationarity, Simultaneous Switching Autoregressive Model, Time Irreversibility, Financial Time Series

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# 1. Introduction

In the past decade, several non-linear time series models have been proposed by statisticians and econometricians. Granger and Andersen (1978), for instance, have introduced the bilinear time series models. Also Ozaki and Oda (1978), and Tong (1983) have proposed the exponential autoregressive (EXPAR) model and the threshold autoregressive (TAR) model, respectively, in the statistical time series analysis. In particular, a considerable attention has been paid on the TAR model in the past decade by statisticians and econometricians and several related applications have been reported. The statistical details of many non-linear time series models have been discussed by Tong (1990). In the econometric analyses several non-linear time series models including Hamilton (1989), and McCulloch and Tsay (1992) have also been proposed and used in some empirical studies.

In this paper we shall propose an alternative class of non-linear time series models, which is called the Simultaneous Switching Autoregressive (SSAR) time series model. This model is a kind of Markovian switching time series model with a quite distinctive structure of simultaneity. We shall propose this class of statistical models because we have a conviction that the class of standard Autoregressive Moving-average (ARMA) time series model and Autoregressive Integrated Moving-average (ARIMA) time series model cannot describe one important aspect in many economic time series, that is the asymmetrical movement in the up-ward phase (or regime) and in the down-ward phase (or regime). It has been sometimes argued that major economic time series display some kind of asymmetrical movements over various phases of the business cycle. In particular, a number of economists have observed the asymmetrical pattern in the up-ward phase and in the down-ward phase for major financial time series including stock prices. This feature of economic time series can be regarded as one form of time irreversibility discussed in the statistical time series analysis. (See Chapter 4 of Tong (1990), for instance.)

Earlier, Kunitomo and Sato (1994a,b) have introduced the simple stationary SSAR time series model and discussed its statistical properties in some details. For instance, they have shown that even the simplest univariate SSAR model, called SSAR(1), gives us some explanations and descriptions to a very important aspect of the asymmetrical movement of time series in two different phases. This characteristic of economic time series has been observed by a number of economists. However, there has not been any useful time series model incorporating this feature explicitly as far as we know in the econometric literature. The main interest in the study by Kunitomo and Sato (1994a,b) was to link the stationary non-linear time series models to the disequilibrium econometric models. They also have investigated the conditions for ergodicity and the basic properties of the stationary distribution in the stationary SSAR model.

This paper extends the basic SSAR model (denoted by $SSAR_m(p)$) discussed by Kunitomo and Sato (1994b) into two important directions for econometric applications. First, we shall allow that the disturbance terms in the SSAR model can be auto-correlated and have a moving-average structure. By this extension the SSAR model can exhibit more complicated patterns of auto-correlations among economic time series and their differenced data. More importantly, second, we shall consider a class of non-stationary SSAR models, which is useful for the applications to major financial time series. In the past analyses of financial time series data, the linear non-stationary time series models have been often used because the movements of most financial time series are usually too volatile as the realizations of stationary time series. We shall give one convincing economic reasoning why the non-stationary SSAR model introduced in this paper is interesting and useful for its applications in financial time series. There has been a fairly common observation among many economists that many financial time series including stock prices have asymmetrical movements between the up-ward phase and the down-ward phase. However, it is not possible to describe this kind of asymmetrical patterns by the linear non-stationary time series models including the ARIMA time series model. The non-stationary time series model we shall propose can be called the simultaneous switching integrated autoregressive (SSIAR) model because it can be regarded as a non-linear extension of the standard ARIMA model.

In Section 2, we shall introduce the general SSAR model, which is possibly stationary or non-stationary, and investigate some basic properties of a non-stationary univariate SSAR model in some details. Then in Section 3, we shall discuss one justification for the non- stationary SSAR modelling from the view of financial economics, and apply the non-stationary SSAR(1) model for the analysis of Nikkei 225 spot and futures indeces. In Section 4, some concluding remarks on our econometric approach to the non-stationary and non-linear time series modelling will be given. The proofs of some theorems will be given in Appendix.

## 2. A Stationary and Non-stationary SSAR models

### 2.1 The SSAR model

In this section we shall consider the multivariate simultaneous switching autoregressive (SSAR) model with moving-average (MA) disturbances. In the following representation the order of the autoregressive part is one without loss of generality. This is because we can consider the p-th order multivariate SSAR model silimarly, but it can be re-written as the first order multivariate autoregressive form by using the standard Markovian representation well-known in the statistical time series analysis.

Let $\boldsymbol{y}_t$ be an $m \times 1$ vector of time series variables. The model we consider in this section is represented by

$$y_t = \begin{cases} \mu_1 + A y_{t-1} + D_1 u_t & \text{if} \quad e'_m y_t \geq e'_m y_{t-1} \\ \\ \mu_2 + B y_{t-1} + D_2 u_t & \text{if} \quad e'_m y_t < e'_m y_{t-1} \end{cases} , \tag{2.1}$$

where $e'_m = (0, \cdots, 0, 1)$ and $\mu'_i$ $(i = 1, 2)$ are $1 \times m$ vectors of constants, $A$ and $B$ are $m \times m$ matrices, and $D_i$ $(i = 1, 2)$ are $m \times n$ matrices.
The disturbance terms $\{u_t\}$ are a sequence of $I(d)$ process satisfying

$$\Delta^d u_t = \sum_{j=0}^{q} C_j v_{t-j}, \tag{2.2}$$

where $I(d)$ denotes the integrated linear stochastic process,

$$\sum_{j=0}^{q} \|C_j\| < +\infty , \tag{2.3}$$

$\{v_t\}$ are a sequence of martingale differences, and $I(d)$ denotes the integrated linear stochastic process. The order of the moving-average (MA) terms $q$ can be $\infty$ in the general case.

The most important feature of this representation is that the time series variables may take quite different values in two different phases or regimes. This type of statistical time series models could be termed as the threshold models in the recent time series literature. However, since the vector time series and two phases at time $t$ are determined simultaneously, we shall call this type of time series models as the simultaneous switching autoregressive (SSAR) time series models. It will turn out that this simultaneity has not only important economic interpretations, but also a new aspect in the non-linear time series modelling as we shall discuss.

We now consider the basic question whether the stochastic process defined by (2.1), (2.2), and (2.3) does make sense in a proper statistical sense or not. The general answer to this question is negative and we need some additional consitions on the unknown parameters in the SSAR model. This issue has been called the coherency problem. (See Section 4 of Kunitomo and Sato (1994b) in some details.) The conditions of $e'_m y_t \geq e'_m y_{t-1}$ and $e'_m y_t < e'_m y_{t-1}$ can be rewritten as

$$e'_m D_1 u_t \geq e'_m (I_m - A) y_{t-1} - e'_m \mu_1 \quad , \tag{2.4}$$

and

$$e'_m D_2 u_t < e'_m (I_m - B) y_{t-1} - e'_m \mu_2 \quad , \tag{2.5}$$

respectively. When $m = n$, the necessary conditions on coherency for the SSAR model can be summarized by a $1 \times (1 + m)$ vector

$$\begin{aligned} \frac{1}{\sigma_1}[-e'_m \mu_1,\ e'_m (I_m - A)] &= \frac{1}{\sigma_2}[-e'_m \mu_2,\ e'_m (I_m - B)] \\ &= r' , \end{aligned} \tag{2.6}$$

where $\sigma_j$ $(j = 1, 2)$ are unknown scale parameters. Then we have the following proposition.

**Theorem 2.1 :** *Suppose (i) $m = n$ , (ii) $|\boldsymbol{D}_1||\boldsymbol{D}_2| > 0$ , and (iii) the condition (2.6) hold. Then the correspondence between two stochastic processes $\{\boldsymbol{u}_t\}$ and $\{\boldsymbol{y}_t\}$ is one-to-one, and the SSAR model consisting of (2.1), (2.2), and (2.3) is coherent as an econometric model.*

We define the indicator functions by

$$I_t^{(1)} = I\left(\boldsymbol{e}_m'\boldsymbol{y}_t \geq \boldsymbol{e}_m'\boldsymbol{y}_{t-1}\right)$$

and

$$I_t^{(2)} = I\left(\boldsymbol{e}_m'\boldsymbol{y}_t < \boldsymbol{e}_m'\boldsymbol{y}_{t-1}\right),$$

where $I(\omega) = 1$ if the event $\omega$ occurs and $I(\omega) = 0$ otherwise. By use of these notations, it is often more convenient to re-rewrite (2.1) in the following form:

$$\boldsymbol{y}_t = \boldsymbol{\mu}(t) + \boldsymbol{A}(t)\boldsymbol{y}_{t-1} + \boldsymbol{D}(t)\boldsymbol{u}_t \ , \tag{2.7}$$

where

$$\boldsymbol{\mu}(t) = \sum_{i=1}^{2} I_t^{(i)}\boldsymbol{\mu}_i \ , \tag{2.8}$$

$$\boldsymbol{A}(t) = \boldsymbol{A}I_t^{(1)} + \boldsymbol{B}I_t^{(2)} \ , \tag{2.9}$$

and

$$\boldsymbol{D}(t) = \sum_{i=1}^{2} I_t^{(i)}\boldsymbol{D}_i \ . \tag{2.10}$$

There are several special cases of (2.7), which are interesting from the view point of possible econometric applications. Here we shall mention to only three examples in the class of the SSAR models we introduced.

**Example 1 :** We consider the SSAR model when $d = q = 0$. This is the case which Kunitomo and Sato (1994b) have investigated in some details. The disturbance terms $\{\boldsymbol{u}_t\}$ in this case are the sequence of martingale differences and satisfy

$$E(\boldsymbol{u}_t|\mathcal{F}_{t-1}) = \boldsymbol{o} \ , \tag{2.11}$$

and

$$E(\boldsymbol{u}_t\boldsymbol{u}_t'|\mathcal{F}_{t-1}) = \boldsymbol{I}_m \ , \tag{2.12}$$

where $\mathcal{F}_{t-1}$ is the $\sigma$-field generated by the random variables $\{\boldsymbol{y}_s, s \leq t-1\}$. Kunitomo and Sato (1994b) have investigated the conditions for the ergodicity and basic properties of the stationary distributions and their moments. In particular, the necessary and sufficient conditions for the ergodicity when $m = n = 1$ are $A < 1, B < 1$, and $AB < 1$ . It should be noted that the conditions $|A| < 1$ and

$|B| < 1$ are sufficient, but not necessary for the geometric ergodicity of the stationary SSAR model. This illustrates one of interesting differences between the linear time series models and the non-linear time series models. Also we should point out that there are interesting economic interpretations for these differences. For instance, Kunitomo and Sato (1984b) have originally introduced the stationary SSAR model from the reduced form of a disequilibrium econometric model. It seems that the conditions for ergodicity in the disequilibrium econometric model are much weaker than those for the corresponding equilibrium econometric model.

**Example 2** : We can illustrate the possible applications by the multivariate SSAR models. For this purpose, we take $d = 0, m = 2$, $\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = \boldsymbol{o}$, and $\boldsymbol{e}_1'\boldsymbol{A} = \boldsymbol{e}_1'\boldsymbol{B} = (1,0)$, for simplicity. Then by using the coherency condition (2.6) we have the representation

$$
\Delta \boldsymbol{y}_t = \boldsymbol{\alpha}(t)\boldsymbol{r}'\boldsymbol{y}_{t-1} + \boldsymbol{D}(t)\boldsymbol{u}_t \,, \tag{2.13}
$$

where $\boldsymbol{\alpha}(t)' = (0, -\sum_{i=1}^{2} I_t^{(i)}\sigma_i)$ and $\boldsymbol{r}'$ are $1 \times 2$ vectors. The vector $\boldsymbol{r}$ could be called a co-integrated vector in a non-linear sense because the stochastic process defined by $\boldsymbol{r}'\boldsymbol{y}_t$ can be stationary with the additional condition $|\boldsymbol{e}_2'\boldsymbol{r}\sigma_i| < 1$ $(i = 1,2)$. When $\boldsymbol{A} = \boldsymbol{B}$ and $\boldsymbol{D}_1 = \boldsymbol{D}_2$, (2.13) has been called the error-correction representation of a non-stationary linear time series model. (See Engle and Granger (1987), for instance.)

**Example 3** : When $d = 1$, the stochastic process defined by (2.1), (2.2), and (2.3) is non-stationary. In the following analysis of this paper we shall mainly forcus on the non-stationary and univariate case, that is, the SSAR model when $m = n = d = 1$. Thus we are extending the stationary SSAR model discussed in Kunitomo and Sato (1994a,b) to a class of the non-stationary SSAR time series models. Since the integrated moving-average (IMA) process has been a useful non-stationary time series model, we can call the resulting process the simultaneous switching integrated autoregressive (SSIAR) process. In Section 3.1 later, we shall argue that there is a considerable reason why the non-stationary SSAR model we developed is useful to many applications for financial time series.

## 2.2 Characterization of A Non-stationary SSAR model

When $\{\boldsymbol{u}_t\}$ in (2.2) is an $I(1)$ process, the stochastic process $\{\boldsymbol{y}_t\}$ is a non-ergodic process. Hence there are basic questions on the properties of the stochastic process defined by $\{\boldsymbol{y}_t\}$ when $d = 1$. By using the representation of (2.7), the time series model for $\{\Delta\boldsymbol{y}_t\}$ can be written as

(2.14)

$$
\Delta \boldsymbol{y}_t \;=\; D(t)\Delta[D(t)^{-1}\boldsymbol{\mu}(t)]
$$

$$
\begin{aligned}
&+ \quad D(t)D(t-1)^{-1}\Delta \boldsymbol{y}_{t-1} \\
&- \quad D(t)[D(t)^{-1}(I_m - A(t))\boldsymbol{y}_{t-1} - D(t-1)^{-1}(I_m - A(t-1))\boldsymbol{y}_{t-2}] \\
&+ \quad D(t)\Delta \boldsymbol{u}_t \,.
\end{aligned}
$$

Further when $m = 1$ we can simplify some coefficients by the coherency conditions (2.6). In this case we have the relations $\mu(t) = -r_0 D(t)$, and $1 - A(t) = r_1 D(t)$, where $\boldsymbol{r}' = (r_0, r_1)$. Hence we have the following characterization result on $\{\Delta y_t\}$.

**Theorem 2.2 :** *Suppose $d = m = 1$ . Define the non-linear transformation of $\{\Delta y_t\}$ by*

$$T(\Delta y_t) = D(t)^{-1}\Delta y_t \,. \tag{2.15}$$

*Then the tranformed stochastic process $\{T(\Delta y_t)\}$ satisfies*

$$T(\Delta y_t) = A(t-1)T(\Delta y_{t-1}) + \Delta u_t \,. \tag{2.16}$$

The time series model defined by (2.16) has been called the first order threshold autoregressive (TAR) model with MA distrubances in the non-linear time series analysis. From this result we know that $\{\Delta y_t\}$ is slightly different from TAR(1) model with MA disturbances, which has been known to be useful for applications in the recent time series analysis literature.

From the above discussions, we can deduce some properties of the differenced time series $\{\Delta y_t\}$. Thus we can further investigate the univariate non-stationary SSAR model when $d = m = 1$ in some details. For the specific application we shall report in Section 3, we also include the time trend variable in the univariate SSAR model. Thus the non-linear and non-stationary SSAR model to be considered is given by

$$y_t = \begin{cases} A_0 + A_1 t + A_2 y_{t-1} + \sigma_1 u_t & (\text{if} \quad y_t \geq y_{t-1}) \\ \\ B_0 + B_1 t + B_2 y_{t-1} + \sigma_2 u_t & (\text{if} \quad y_t < y_{t-1}) \end{cases} . \tag{2.17}$$

By the same argument used in (2.6), we can obtain the coherency conditions for this model. The resulting conditions can be summarized by

$$-\frac{A_0}{\sigma_1} = -\frac{B_0}{\sigma_2} = r_0 \,, \quad -\frac{A_1}{\sigma_1} = -\frac{B_1}{\sigma_2} = r_1 \,, \quad \frac{1 - A_2}{\sigma_1} = \frac{1 - B_2}{\sigma_2} = r_2 \,. \tag{2.18}$$

Since $\{y_t\}$ is a non-ergodic process, we need to investigate the stochastic process defined by (2.17). For this purpose it is convenient to use the indicator functions $I_t^{(1)} = I(\Delta y_t \geq 0)$ and $I_t^{(2)} = I(\Delta y_t < 0)$. Also we use the notation of $D(t) = \sigma_1 I_t^{(1)} + \sigma_2 I_t^{(2)}$ and re-write the disturbance terms $\{u_t\}$ as

$$u_t = \frac{1}{D(t)}\Delta y_t + r_0 + r_1 t + r_2 y_{t-1} \,. \tag{2.19}$$

Then given the information available at $t-1$, there are four phases for $\Delta y_t$ at $t$ to be considered depending $I_t^{(i)}$ and $I_{t-1}^{(i)}$ $(i=1,2)$. By substituting this equation into (2.7) and re-arranging terms, we have the representation as

$$\Delta y_t = D(t)\left\{-r_1 + \left(-r_2 + \frac{1}{D(t-1)}\right)\Delta y_{t-1} + \Delta u_t\right\} \quad . \tag{2.20}$$

Hence the stochastic process $\{\Delta y_t\}$ has the representation

$$\Delta y_t = \begin{cases} A_1 + A_2 \Delta y_{t-1} + \sigma_1 \Delta u_t & (\text{if} \quad \Delta y_{t-1} \geq 0,\ \Delta y_t \geq 0) \\ A_1 + \left(\frac{\sigma_1}{\sigma_2}\right) B_2 \Delta y_{t-1} + \sigma_1 \Delta u_t & (\text{if} \quad \Delta y_{t-1} < 0,\ \Delta y_t \geq 0) \\ B_1 + \left(\frac{\sigma_2}{\sigma_1}\right) A_2 \Delta y_{t-1} + \sigma_2 \Delta u_t & (\text{if} \quad \Delta y_{t-1} \geq 0,\ \Delta y_t < 0) \\ B_1 + B_2 \Delta y_{t-1} + \sigma_2 \Delta u_t & (\text{if} \quad \Delta y_{t-1} < 0,\ \Delta y_t < 0) \end{cases} . \tag{2.21}$$

For the stochastic process $\{\Delta y_t\}$ defined by (2.21), we can establish the necessary and sufficient conditions for its ergodicity. The proof is given in Appendix.

**Theorem 2.3 :** *Suppose (i) the order of MA terms $q$ on $\{\Delta u_t\}$ is a finite number, (ii) the coherency condition (2.18) holds, (iii) the density function $g(v)$ of $\{v_t\}$ is everywhere positive in $\boldsymbol{R}^1$, and (iv) $E[|v_t|] < +\infty$. Then the Markov chain defined by (2.21) is geometrically ergodic iff*

$$A_2 < 1,\ B_2 < 1,\ A_2 B_2 < 1 \quad . \tag{2.22}$$

It is interesting to see that the conditions given by (2.22) are the same as for the geometric ergodicity as in the stationary SSAR(1) model proven in Kunitomo and Sato (1994b). However, we do not need any additional condition on $\{v_t\}$ in the present case which they have used.

The non-stationary SSAR given by (2.17) is a complicated stochastic process. In order to get some idea on its statistical properties, we did a set of simulation for the simplest case. When $m = d = 1$, the simplest SSAR model can be re-written as

$$y_t - \mu = \begin{cases} A(y_{t-1} - \mu) + \sigma_1 u_t & \text{if} \quad y_t \geq y_{t-1} \\ B(y_{t-1} - \mu) + \sigma_2 u_t & \text{if} \quad y_t < y_{t-1} \end{cases} , \tag{2.23}$$

where we re-define $\mu$ as a location parameter and $\sigma_i$ $(i=1,2)$ as scale parameters. The disturbance terms $\{u_t\}$ follow the random walk process satisfying

$$u_t = u_{t-1} + v_t \quad . \tag{2.24}$$

The innovation terms $\{v_t\}$ in (2.24) are independently and identically distributed random variables and follow $N(0,1)$. The condition on coherency in this case is given by

$$\frac{1-A}{\sigma_1} = \frac{1-B}{\sigma_2} = r \ . \tag{2.25}$$

For the sake of simplicity, we set $\mu = 0$ in our simulations. Although there are four unknown parameters $A$, $B$ and $\sigma_i$ $(i = 1,2)$ in (2.23), there are only three free parameters $A$, $B$ and $r$.

We took several sets of values of these parameters and did a set of simulations in a systematic way. Among them we only present three cases in Figure 2.1. The middle one shows the sample path when $A = B = 0.5$, which means that the non-stationary SSAR(1) model is actually the standard ARIMA(1,1,0) model. When $A \neq B$, we can notice some asymmetrical patterns in the sample paths of the simulated time series. For economic time series, the case when $A = 0.8$ and $B = 0.2$ may be the most interesting one. Even though we use a very simple non-stationary SSAR model, we found that we can get very interesting asymmetrical sample paths of $\{y_t\}$ along the simulated random walk of $\{u_t\}$. This aspect can not be realized by the linear non-stationary time series models such as the ARIMA model. The sample paths of the time series generated by the stationary SSAR models have been investigated by Kunitomo and Sato (1994b).

## 2.3 Maximum Likelihood Estimation

The SSAR model is quite complex as a statistical model in its several aspects when $m = n$ and $d \geq 1$. The first aspect is that it is a kind of the threshold autoregressive model in which the present state variables depend on the past realized values of time series. The other aspect is that there is a simultaneity between the present phase and the present value of the time series variables. The last aspect is that the SSAR model when $d \geq 1$ is a non-linear and non-stationary stochastic process. As it has been discussed for the simple stationary SSAR model by Sato and Kunitomo (1994), the standard least squares estimation method for each phase separately gives a fairly biased estimates for the unknown parameters. The main reason for this is because there is an important simultaneity involved in the SSAR models. Thus, instead of the least squares method, we are proposing to use the maximum likelihood method for the non-stationary SSAR model in this paper.

Under the assumption that the disturbance terms $\{\boldsymbol{v}_t\}$ are independently and normally distributed random variables, the conditional log-likelihood function for $\{\Delta \boldsymbol{y}_t,\ 2 \leq t \leq T\}$ given the initial conditions $\Delta \boldsymbol{y}_1$ and $\boldsymbol{v}_0 = \boldsymbol{v}_{-1} = \cdots = \boldsymbol{v}_{-q} = \boldsymbol{o}$ can be written as

$$\log L_T(\boldsymbol{\theta}) \;=\; -\frac{(T-1)m}{2}\log 2\pi \tag{2.26}$$

$$- \frac{1}{2}\sum_{t=2}^{T} I_t^{(i)} \sum_{i=1}^{2} \log |\boldsymbol{D}_i \boldsymbol{\Omega}(\boldsymbol{\theta}) \boldsymbol{D}_i'|$$
$$- \frac{1}{2}\sum_{t=2}^{T} \boldsymbol{v}_t'(\boldsymbol{\theta})\ \boldsymbol{\Omega}(\boldsymbol{\theta})^{-1}\ \boldsymbol{v}_t(\boldsymbol{\theta})\ ,$$

where $\boldsymbol{\Omega}(\boldsymbol{\theta})$ is the covariance matrix of $\boldsymbol{v}_t$ whose diagonal elements are ones and $\boldsymbol{\theta}$ is a vector of structural parameters appeared in the original SSAR model.

When $m = n = d = 1$ and the error terms $\{u_t\}$ are independently and normally distributed random variables for instance, the conditional log-likelihood function can be further simplified as

$$(2.27)\quad \log L_T(\boldsymbol{\theta}) = -\frac{T-1}{2}\log 2\pi - \frac{1}{2}\sum_{t=2}^{T}\sum_{i=1}^{2} I_t^{(i)} \log \sigma_i^2 - \frac{1}{2}\sum_{t=2}^{T}\sum_{i=1}^{2} I_t^{(i)} \frac{1}{\sigma_i^2}\left[\Delta y_t - \mu_i - \sum_{j=1}^{2} C_{ij} I_{t-1}^{(j)} \Delta y_{t-1}\right]^2 ,$$

where we use the notations that $\sigma_i^2 = D_i D_i'$ and $C_{ij}$ are the parameters appeared in the representation of (2.21).

The maximum likelihood (ML) estimator can be defined as the maximum of $\log L_T(\boldsymbol{\theta})$ with respect to the unknown parameters $\boldsymbol{\theta}$ including the restrictions implied by the coherency conditions (2.6). The asymptotic properties of the ML estimator in the non-stationary SSAR model when $m = d = 1$ can be established by using the method developed in Kunitomo and Sato (1994b), and Sato and Kunitomo (1994). A sketch of the proof is provided in Appendix.

**Theorem 2.4 :** *For the non-stationary SSAR model given by (2.1), (2.2), and (2.3) when $m = d = 1$, suppose (i) the sufficient conditions for the coherency and ergodicity hold, and (ii) the disturbances terms $\{v_t\}$ are independently distributed as $N(0,1)$, (iii) the MA order $q$ is a finite number, and (iv) $D_i > 0$ $(i = 1, 2)$. Also suppose (v) the true parameter vector $\boldsymbol{\theta}$ is an interior point of the parameter space $\boldsymbol{\Theta}$. Then the ML estimators $\hat{\boldsymbol{\theta}}_{ML}$ of unknown parameters $\boldsymbol{\theta}$ are consistent and asymptotically normally distributed as*

$$(2.28)\qquad \sqrt{T}\left(\hat{\boldsymbol{\theta}}_{ML} - \boldsymbol{\theta}\right) \xrightarrow{d} N\left[0,\ I(\boldsymbol{\theta})^{-1}\right] ,$$

*provided*

$$(2.29)\qquad I(\boldsymbol{\theta}) = \operatorname*{plim}_{T\to\infty} \frac{1}{T}\left[-\frac{\partial^2 \log L_T(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}\right]$$

*is a positive definite matrix.*

We also have investigated the finite sample properties of the ML estimator in a systematic way. Because their mathematical expressions are intractable, we have utilized simulation procedures. We generated the simulated time series $\{\Delta y_t\}$ and $\{y_t\}$ for the non-stationary SSAR model when $d = m = 1$ and $q = 0$ . We used the standard normal random numbers for the disturbance terms $\{v_t\}$. Then we obtained the table of the sample mean of the ML estimator from 5,000 replications. Among many tables, we show only the case when $T = 50$ and $T = 500$ in Table 1. From these tables, the bias of the ML estimator is negligible when the sample size is more than 100. Thus we have an evidence for the use of the ML estimation method for the non-stationary SSAR models.

## 3. An Application to Financial Data

### 3.1 Financial Time Series

The main reason to introduce the non-stationary SSAR model is its applicability to economic time series data. Especially, there have been growing interests among econometricians and statisticians to investigate financial time series data by using the statistical time series analysis in the last decade. There have been several interesting features in financial time series data. First, many financial time series such as stock prices, bond prices, interest rates, foreign exchange rates, and their derivatives are often too volatile to use the stationary time series models explained in the statistical time series analysis. Hence the results of the prediction based on the stationary linear time series models are not satisfactory. Second, the distributions of prices and yields are often not well approximated by the Gaussian distribution. Third, some financial time series including stock prices exhibit asymmetrical movements between in the up-ward phase and in the down-ward phase. These features could not be consistent with the standard linear time series models such as the autoregressive integrated moving average (ARIMA) process, which have been sometimes used in econometric applications. We should stress that the non-stationary SSAR model introduced in Section 2.2 has the statistical properties that are consistent to all of the above observations on many financial time series. Thus we hope that the non-stationary and non-linear time series model we introduced in Section 2 would be potentially useful for the applications in many financial data.

### 3.2 A Simple Model of Stock Prices

In this section we first discuss a simple econometric model of stock prices, which leads mathematically to the non-stationary SSAR model. The main reason for the following discussion is not to develop the financial economics, but to illustrate why the SSAR model is useful and applicable to many financial time series. For this purpose, we slightly modify the well-known economic model in financial economics developed by Amihud and Mendelson (1987).

Let the intrinsic value of a security at time t and its observed price be $V_t$ and $P_t$, respectively. We distinguish the intrinsic value of a security and its observed price. There has been some economic reasons why they can be different. (See Amihud and Mendelson (1987), and its references.) Since two values $V_t$ and $P_t$ can be different, we can introduce a partial-adjustment model when the intrisic value $V_t$ at $t$ deviates from the observed past price $P_{t-1}$ at $t-1$ as follows

$$P_t - P_{t-1} = \begin{cases} g_1(V_t - P_{t-1}) & \text{if} \quad V_t - P_{t-1} \geq 0 \\ g_2(V_t - P_{t-1}) & \text{if} \quad V_t - P_{t-1} < 0 \end{cases} , \tag{3.1}$$

where $V_t$ and $P_t$ are in logarithms and the adjustment coefficients $g_i$ satisfy $g_i \geq 0$ $(i = 1, 2)$.

We note that we have modified the adjustment process used in Amihud and Mendelson (1987) in two ways. First, we have omitted the contemporary noise factor in the right hand side. We did this because of the resulting simplicity. Second, we have allowed the adjustment coefficients $g_i$ $(i = 1, 2)$ can take different values. There can be several economic reasons why they can be different. In stead of discussing them, we simply point out that this formulation includes many cases, which are theoretically or practically interesting in financial economics. When $g_1 = g_2$, (3.1) is reduced to the standard linear adjustment model. Further, when $g_1 = g_2 = 1$, $V_t = P_t$ and the intrinsic value of a security is always equal to its observed price. Hence, by using the formulation we have adopted in (3.1) it is possible to examine from the observed time series data if there conditions are reasonable descriptions of reality.

In the recent financial economics, there has been a convention that the logarithms of the intrisic security values $\{V_t\}$ follows an integrated process $I(1)$ with a drift,

$$V_t = V_{t-1} + \sigma e_t + \mu \ , \tag{3.2}$$

where $\mu$ represents the expected daily return and $\{e_t\}$ are a sequence of random variables generated by the linear stationary stochastic process which possessing a MA representation.

By combining (3.1) and (3.2), we can get the representation of $\Delta P_t$ as

$$\Delta P_t = g(t)[\frac{1}{g(t-1)} - 1]\Delta P_{t-1} + g(t)[\mu + \sigma e_t] \ , \tag{3.3}$$

where $g(t) = g_1 I_t^{(1)} + g_2 I_t^{(2)}$ . From this representation, it is obvious that (3.3) is a special case of the non-stationary SSAR model we have discussed in Section 2.1 when $m = n = d = 1$.

By using Theorem 2.3 in Section 2.2, the ergodic region for the process $\{\Delta P_t\}$ with respect to the adjustment coefficients $g_i$ $(i = 1, 2)$ is given in Figure 3.1. We note that the ergodic region when $g_1 \neq g_2$ is quite large in comparison with that when $g_1 = g_2$. This figure may be useful when we interpret the empirical results reported in the next sub-section.

### 3.3 An Empirical Analysis of Spot and Futures Indeces

In this section we shall report a preliminary empirical result using the time series data in the Japanese financial markets. In our data analysis we have used the time series data of Nikkei 225 indeces which are the most popular stock price index traded in Japan. They are the daily data of Nekkei Spot and Futures indeces from January of 1985 to December of 1994. The trade of the Nikkei index Futures started at th end of 1980's in Osaka Stock Exchange, so we have used the data of Nikkei Futures from January of 1990 to December of 1994. All data have been transformed into their logarithms before the estimation of the non-stationary SSAR model. It may be of some interests in financal economics to compare the time series movements of the spot price index and the corresponding futures price index.

Using these data, we have estimated the first order non-stationary univariate SSAR model discussed in Section 2, which could be written as SSIAR(1). The estimation of structural parameters in the SSIAR(1) model has been done by the ML method under the assumption of the normal disturbances. Since we cannot obtain an explicit formula for the ML estimators of unknown parameters, we have used a numerical nonlinear optimization technique with the coherency restrictions on parameters given by (2.18). The resulting estimation results are given in Table 2. The figures of LK stand for the maximized log-likelihood functions. For the purpose of comparison, we also have estimated the standard IAR(1,1) process from our time series data set. In order to make a comparison, we have calculated the likelihood ratio statistic $LR(A = B)$ for testing the null hypothesis

$$H_0 \; : \; A = B \, . \tag{3.4}$$

Under the assumption of the Gaussian disturbances, the likelihood ratio statistic $LR(A = B)$ is asymptotically distributed as $\chi^2(1)$. Thus this test statistic gives us a useful information on the asymmetrical movements of stock prices. The results have been summarized in Table 2 and Figure 3.2.

There are several interesting empirical observations from Table 2 and Figure 3.2. First, the spot stock price index sometimes shows sharp asymmetrical movements either it is in the up-ward phase or in the down-ward phase. This phenomenon has been evident in 1985 and 1987. Actually we have already known that there was a sharp decline in October of 1987. Second, it seems that the futures stock index does not show a significant asymmetry in two phases in comparison with the spot price index. There could be some economic interpretations for this observation. Third, after starting of the active trade of stock index futures in the financial market, there have not been many occasions as were used to when the asymmetrical movements of the price indeces are evident. This finding may be interesting for economists.

These observations from our empirical results are preliminary and further considerations are needed. But clearly it has not been easy to detect these features of the financial time series data by the existing other methods, the linear time series models in particular.

## 4. Conclusions

In this paper, we have focused on one important aspect in many financial economic time series, which has been often ignored in the past econometrics studies. We have argued that the asymmetrical pattern in the movements of time series between the up-ward phase and the down-ward phase often observed by economists can not represented properly by the stationary and non-stationary linear time series models including the standard ARMA and ARIMA process, which have been used in many empirical studies in the past.

Then we have introduced the class of simultaneous switching autoregressive (SSAR) models, which is one type of Markovian switching lnon-inear time series models. It has distinctive properties of simultaneity and time irreversibility. Since Kunitomo and Sato (1994) have investigated the stationary SSAR model, we have focussed on the non-stationary SSAR model and investigated its some properties in the univariate case. In this paper we have proposed the maximum likelihood estimation method for estimating the unknown parameters in the SSAR model. We hope that the results reported in this paper may shed some new lights on the time series properties often observed by many economists and statisticians.

Also we have tried to show that there are some natural reasons why the non-stationary SSAR model introduced in Section 2 is a useful tool to analyze many financial time series in financial markets. We have illustrated this issue by suggesting a very simple model for stock price movements in Section 3.2. The point is that if we allow that the intrisic value of security can be different from the observed price and have an ajustment process, we have non-linear time series models. Of course, there can be many possibilities to describe the financial time series by non-stationary and non-linear time series modelling. At least we can conclude that the non-linear and non-stationary models we introduced in this paper gives an interesting econometric and statistical model, which is useful for the applications.

However, there are several important issues remained to be unsolved. In this paper we have only investigated some special cases of the non-stationary SSAR model. In particular, there are some interesting situations when we have multivariate non-linear time series as illustrated in Example 2 in Section 2.3. Since there can be many non-linear time series models as we indicated in Section 1, a comparison or discrimination of the SSAR model from other statistical models would be necessary. Further investigations should be needed on these problems.

## 5. Mathematical Appendix

In this appendix, we gather some mathematical details which we have ommited in the previous sections.

**Proof of Theorem 2.3** : We shall use the method similar to that used by Liu and Susko (1992) for the TAR(1) model with MA disturbances. However, we note that some changes in their method are necessary and we can establish stronger results than theirs because the non-stationary SSAR model with MA disturbances is different from their model.

**(i) Sufficiency :** Let $x_t = \Delta y_t$ and define $(1+q) \times 1$ vector $\boldsymbol{x}_t$ by

$$\text{(A.1)} \qquad \boldsymbol{x}_t = \begin{pmatrix} x_t \\ v_t \\ v_{t-1} \\ \vdots \\ v_{t-q+1} \end{pmatrix} .$$

Then we consider the Markovian representation for $\boldsymbol{x}_t$. For the sake of simplicity, we set $r_1 = 0$. The condition $x_t \geq 0$ is equivalent to $v_t \geq \boldsymbol{a}_{t-1}^{'}$ where

$$\text{(A.2)} \qquad \boldsymbol{a}_{t-1}^{'} = (r_2 - \frac{1}{D(t-1)}, -c_1, -c_2, \cdots, -c_q) .$$

From (2.20) we have the representation

$$\text{(A.3)} \qquad \boldsymbol{x}_t = \boldsymbol{H}(\boldsymbol{x}_{t-1}, v_t) ,$$

where

$$\text{(A.4)} \qquad \boldsymbol{H}(\boldsymbol{x}_{t-1}, v_t) = \begin{pmatrix} D(t)\boldsymbol{a}_{t-1}^{'}\boldsymbol{x}_{t-1} + D(t)v_t \\ v_t \\ v_{t-1} \\ \vdots \\ v_{t-q+1} \end{pmatrix} .$$

We use the criterion function

$$\text{(A.5)} \qquad G(\boldsymbol{x}) = \sum_{i=1}^{q} h(x_i) ,$$

where $h(x_i) = k|x_i|$ and $\boldsymbol{x} = (x_1, \cdots, x_q)$ for some $k > 0$ . Then

$$\text{(A.6)} \qquad \begin{aligned} E[G(\boldsymbol{x}_t)|\boldsymbol{x}_0] &= E[h(x_t) + \sum_{j=0}^{q-1} h(v_{t-j})|\boldsymbol{x}_0] \\ &\leq c_1 + E[E[g(x_t)|x_{t-1}]|\cdots|\boldsymbol{x}_0] , \end{aligned}$$

because $E[|v_t|] < \infty$ , where $c_1$ is a positive constant.

Let

$$\text{(A.7)} \qquad Q_{t|t-1} = E[h(x_t)|x_{t-1}] = E\left\{h[-D(t)\boldsymbol{a}_{t-1}'\boldsymbol{x}_{t-1} + D(t)v_t|x_{t-1}]\right\}$$

We first consider the case when $x_{t-1} = x > 0$. In this case from (2.20) we have two phases at $t$ given $x > 0$ and so

$$\text{(A.8)} \quad Q_{t|t-1} = k\sigma_1 \int_{z \geq (r_2 - \frac{1}{\sigma_1})x - \theta' \boldsymbol{z}_{t-1}} [-(r_2 - \frac{1}{\sigma_1})x + \theta' \boldsymbol{z}_{t-1} + z] f(z) dz - k\sigma_2 \int_{z < (r_2 - \frac{1}{\sigma_1})x - \theta' \boldsymbol{z}_{t-1}} [-(r_2 - \frac{1}{\sigma_1})x + \theta' \boldsymbol{z}_{t-1} + z] f(z) dz ,$$

where $\boldsymbol{z}_{t-1} = (z_{t-1}, \cdots, z_{t-q})$ . Then by using (2.18), we have

$$Q_{t|t-1} \leq c_2 \times \left(1 + \sum_{i=1}^{q} |v_{t-i}|\right) + kA_2 x \int_{z \geq (r_2 - \frac{1}{\sigma_1})x - \theta' \boldsymbol{z}_{t-1}]} f(z) dz - k\left(\frac{\sigma_2}{\sigma_1}\right) A_2 x \int_{z \geq (r_2 - \frac{1}{\sigma_1})x - \theta' \boldsymbol{z}_{t-1}} f(z) dz ,$$

where $c_2$ is a positive constant.
Here we use the inequality

$$\text{(A.9)} \qquad -\frac{\sigma_2}{\sigma_1} A_2 < 1 .$$

This is because the coherency condition (2.22) implies

$$\text{(A.10)} \qquad 0 < r_2 < \frac{1}{\sigma_1} + \frac{1}{\sigma_2} .$$

By taking

$$\delta = \max\left\{A_2, -\frac{\sigma_2}{\sigma_1} A_2\right\} ,$$

we have the relation

$$\text{(A.11)} \qquad E[G(\boldsymbol{x}_t)|\boldsymbol{x}_{t-1}] \leq c_2 \times \left(1 + \sum_{i=1}^{q} |v_{t-i}|\right) + \delta G(\boldsymbol{x}_{t-1}) ,$$

where $0 < \delta < 1$ . Then we have

$$\text{(A.12)} \qquad E[G(\boldsymbol{x}_t)|\boldsymbol{x}_0] \leq c_2 \times \sum_{k=0}^{t-1} \delta^k \left(1 + \sum_{j=1}^{q} \eta_{t-k-j}\right) + \delta^t G(\boldsymbol{x}_0) ,$$

where $\eta_k = E[|v_k|]I(k > 0) + |v_k|I(k \leq 0)$ . Hence we have established

$$\sup_{t\geq 1} E[G(\boldsymbol{x}_t)|\boldsymbol{x}_0] < +\infty \; . \tag{A.13}$$

The case when $x < 0$ can be similarly treated. Since we can show that the Markov chain defined by (2.21) is weakly-continuous as in Theorem 5.1 of Kunitomo and Sato (1994b, we can show that the additional key condition in Liu and Susko (1992) (their Assumption 2.1) is satisfied in the SSIAR(1) model. Thus we can prove the geometric ergodicity by the arguments given by Liu and Susko (1992).

**(ii) Necessity :** Without loss of generality we take $q = 0$. The essential part is similar to the proof for the TAR(1) model given by Chan et. al. (1985). However, there is one aspect in which we have to modify their proof because the model we are investigating is different from theirs.

We have to consider the situation when the values of parameters are on their boundaries. For an illustration, we consider the case when $A_2 = 1$, $A_1 < 0$ and $B_2 < 1$. By using the coherency condition in this case these conditions imply $1 - \sigma_1 r_2 = 1$, $-\sigma_1 r_1 < 0$ , and $1 - \sigma_2 r_2 < 1$. Then we have $\sigma_1 r_2 = 0$, $\sigma_1 r_1 < 0$, and $\sigma_2 r_2 > 0$. However, they are contradictory when $\sigma_1 > 0$ and $\sigma_2 > 0$.Other boundary cases can be similarly treated. *(QED)*

**Proof of Theorem 2.4** : The most important step in the proof is the martingale property of the partial derivatives of the log-likelihood function summarized in the following lemma. The rest of the proof is very similar to the arguments used in Kunitomo and Sato (1994b), and Sato and Kunitomo (1994). Thus we omit its details. *(QED)*

**Lemma A.1 :** *Let $\boldsymbol{\theta}$ be a vector of unknown parameters in the non-stationary SSAR model given by (2.20) except $r_0$ . Then we have*

$$E\left[\frac{\partial \log L_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}|\mathcal{F}_{t-1}\right] = \frac{\partial \log L_{t-1}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \; , \tag{A.14}$$

*where $\mathcal{F}_{t-1}$ is the $\sigma$-field generated by $\{\boldsymbol{y}_s, s \leq t-1\}$.*

**A Sketch of Proof of Lemma A.1** : Let $\boldsymbol{w}(\boldsymbol{\theta}) = (w_t(\boldsymbol{\theta}))$, where we define the stochastic process $\{w_t(\boldsymbol{\theta})\}$ by

$$\begin{aligned} w_t(\boldsymbol{\theta}) &= D(t)^{-1}\Delta y_t - D(t)^{-1}\mu(t) - D(t)^{-1}\left(\sum_{i,j=1}^{2} c_{ij} I_{t-1}^{(j)} I_t^{(i)}\right)\Delta y_{t-1} \\ &= D(t)^{-1}\Delta y_t + r_1 + [r_2 - D(t-1)^{-1}]\Delta y_{t-1} \; . \end{aligned}$$

Then the log-likelihood function is proportional to

$$L_t(\boldsymbol{\theta}) \propto \frac{1}{2}log|\boldsymbol{\Sigma}_T(\boldsymbol{\theta})| \quad - \quad \frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{2} I_t^{(i)} log|\boldsymbol{\Sigma}_T(\boldsymbol{\theta})|$$
$$- \quad \frac{1}{2}\boldsymbol{w}(\boldsymbol{\theta})' \boldsymbol{\Sigma}_T(\boldsymbol{\theta})^{-1} \boldsymbol{w}(\boldsymbol{\theta}) ,$$

where $\boldsymbol{\Sigma}_T(\boldsymbol{\theta})$ is a $T \times T$ matrix constructed from the MA coefficients on $\{\Delta u_t\}$ and its diagonal elements are ones by the normalization.

Take $\theta_1 = \sigma_1$ for example. In this case

$$\text{(A.15)} \qquad \frac{\partial \log L_t(\boldsymbol{\theta})}{\partial \theta_1} = -\frac{1}{2}\sum_{s=1}^{t} I_s^{(1)} - \boldsymbol{w}(\boldsymbol{\theta})' \boldsymbol{\Sigma}_t(\boldsymbol{\theta})^{-1} \frac{\partial \boldsymbol{w}_t(\boldsymbol{\theta})}{\partial \sigma_1} ,$$

where

$$\text{(A.16)} \qquad \frac{\partial w_t(\boldsymbol{\theta})}{\partial \sigma_1} = -\frac{1}{{\sigma_1}^2} I_t^{(1)} \Delta y_t + \frac{1}{{\sigma_1}^2} I_{t-1}^{(1)} \Delta y_{t-1} ,$$

We use the relation

$$\text{(A.17)} \qquad I_t^{(1)} \Delta y_t = I_t^{(1)}(v_t + c_{t-1}) ,$$

where

$$\text{(A.18)} \qquad c_{t-1} = \sum_{j=1}^{q} \eta_j v_{t-j} - r_1 + \left[-r_2 + \frac{1}{D(t-1)}\right] \Delta y_{t-1} .$$

We also note that we can decomose

$$\text{(A.19)} \qquad \boldsymbol{\Sigma}_t^{-1} = \boldsymbol{H}_t(\boldsymbol{\eta})\boldsymbol{H}_t^{'}(\boldsymbol{\eta}) ,$$

where $\boldsymbol{H}_t$ is a $t \times t$ lower triangular matrix whose diagonal elements are ones. Then using the relation

$$\text{(A.20)} \qquad E\left\{v_t(\boldsymbol{\theta})\boldsymbol{e}_t^{'}\boldsymbol{H}[(-\frac{1}{\sigma_1})I_t^{(1)}(v_t(\boldsymbol{\theta}) + c_{t-1}) + \left(\frac{1}{\sigma_1}\right)^2 I_{t-1}^{(1)} \Delta y_{t-1}]|\mathcal{F}_{t-1}\right\}$$
$$= \quad E\left\{v_t(\boldsymbol{\theta})(-\frac{1}{\sigma_1})I_t^{(1)}(v_t(\boldsymbol{\theta}) + c_{t-1})|\mathcal{F}_{t-1}\right\}$$

we have

$$E\left[\frac{\partial \log L_t(\boldsymbol{\theta})}{\partial \sigma_1}|\mathcal{F}_{t-1}\right] \quad = \quad \frac{\partial \log L_{t-1}(\boldsymbol{\theta})}{\partial \sigma_1}$$
$$+ \quad \frac{1}{\sigma_1} E\left[(v_t(\boldsymbol{\theta})^2 - 1)I_t^{(1)} + c_{t-1}v_t(\boldsymbol{\theta})I_t^{(1)}|\mathcal{F}_{t-1}\right]$$
$$= \quad \frac{\partial \log L_{t-1}(\boldsymbol{\theta})}{\partial \sigma_1} .$$

The last equality is the direct result of calculation because of the normality assumption on $\{v_t\}$. *(QED)*

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Table 1: The mean of the ML estimator of SSIAR model[1,2]

T = 100

| | B = 0.8 | | B = 0.2 | | B = 0.0 | | B = -0.2 | | B = -1.5 | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\hat{A}$ | $\hat{B}$ | $\hat{A}$ | $\hat{B}$ | $\hat{A}$ | $\hat{B}$ | $\hat{A}$ | $\hat{B}$ | $\hat{A}$ | $\hat{B}$ |
| A = | 0.7922 | 0.7921 | 0.7946 | 0.1844 | 0.7932 | -0.0220 | 0.7957 | -0.2181 | 0.7984 | -1.5410 |
| 0.8 | (0.059) | (0.058) | (0.044) | (0.169) | (0.043) | (0.203) | (0.038) | (0.230) | (0.027) | (0.475) |
| A = | 0.1796 | 0.7912 | 0.1966 | 0.2011 | 0.1945 | -0.0107 | 0.1915 | -0.2006 | 0.2036 | -1.5354 |
| 0.2 | (0.173) | (0.044) | (0.117) | (0.115) | (0.110) | (0.144) | (0.100) | (0.155) | (0.061) | (0.336) |
| A = | -0.0215 | 0.7952 | -0.0060 | 0.1943 | -0.0062 | -0.0045 | -0.0008 | -0.1966 | 0.0013 | -1.5138 |
| 0.0 | (0.203) | (0.041) | (0.133) | (0.108) | (0.130) | (0.129) | (0.117) | (0.142) | (0.070) | (0.300) |
| A = | -0.2159 | 0.7979 | -0.2060 | 0.1958 | -0.1957 | -0.0028 | -0.2032 | -0.2049 | -0.1933 | -1.5185 |
| -0.2 | (0.230) | (0.038) | (0.158) | (0.099) | (0.150) | (0.120) | (0.140) | (0.139) | (0.082) | (0.291) |
| A = | -1.5277 | 0.7986 | -1.5065 | 0.2031 | -1.5052 | 0.0044 | -1.4935 | -0.2007 | NA | NA |
| -1.5 | (0.452) | (0.027) | (0.346) | (0.063) | (0.304) | (0.070) | (0.289) | (0.083) | (NA) | (NA) |

[1] The value in the parentheses shows the root mean squared error.
[2] "NA" corresponds to the case when it is not ergodic. We did not have investigated the ML estimator in this case.

T = 500

| | B = 0.8 | | B = 0.2 | | B = 0.0 | | B = -0.2 | | B = -1.5 | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $\hat{A}$ | $\hat{B}$ | $\hat{A}$ | $\hat{B}$ | $\hat{A}$ | $\hat{B}$ | $\hat{A}$ | $\hat{B}$ | $\hat{A}$ | $\hat{B}$ |
| A = | 0.7963 | 0.7964 | 0.7987 | 0.1929 | 0.7998 | -0.0014 | 0.7990 | -0.2033 | 0.7996 | -1.4972 |
| 0.8 | (0.029) | (0.028) | (0.019) | (0.072) | (0.018) | (0.087) | (0.017) | (0.101) | (0.012) | (0.210) |
| A = | 0.1949 | 0.7982 | 0.1992 | 0.1997 | 0.2006 | -0.0011 | 0.1997 | -0.2024 | 0.2004 | -1.5079 |
| 0.2 | (0.075) | (0.020) | (0.052) | (0.052) | (0.048) | (0.062) | (0.043) | (0.066) | (0.027) | (0.139) |
| A = | -0.0042 | 0.7987 | -0.0014 | 0.1998 | -0.0069 | -0.0014 | 0.0025 | -0.1991 | -0.0002 | -1.4945 |
| 0.0 | (0.084) | (0.018) | (0.061) | (0.046) | (0.061) | (0.056) | (0.053) | (0.066) | (0.031) | (0.133) |
| A = | -0.2046 | 0.7994 | -0.1998 | 0.1986 | -0.1987 | 0.0004 | -0.2003 | -0.1969 | -0.2000 | -1.5007 |
| -0.2 | (0.100) | (0.016) | (0.072) | (0.045) | (0.063) | (0.051) | (0.061) | (0.062) | (0.035) | (0.131) |
| A = | -1.5146 | 0.7997 | -1.5081 | 0.1992 | -1.5015 | -0.0020 | -1.5035 | -0.1996 | NA | NA |
| -1.5 | (0.193) | (0.011) | (0.149) | (0.028) | (0.132) | (0.032) | (0.126) | (0.036) | (NA) | (NA) |

Table 2: Results: Nikkei Index (225)

**Spot 1985 – 1989**

| | period | SSIAR(1,1) | | | IAR(1,1) | | |
|---|---|---|---|---|---|---|---|
| | | $a$ | $b$ | LK | $a(=b)$ | LK | $\chi^2$ |
| 1 | 1985.01.04-1985.09.10 | 0.208 | -0.085 | 741.26 | 0.065 | 736.49 | 9.528 ** |
| 2 | 1985.09.11-1986.05.30 | 0.372 | 0.258 | 745.99 | 0.328 | 744.92 | 2.155 |
| 3 | 1986.05.31-1987.02.20 | 0.252 | 0.114 | 643.45 | 0.193 | 642.27 | 2.351 |
| 4 | 1987.02.23-1987.11.07 | 0.233 | -0.962 | 558.63 | -0.198 | 520.06 | 77.148 ** |
| 5 | 1987.11.09-1988.08.03 | 0.191 | 0.190 | 654.21 | 0.191 | 654.21 | 0.000 |
| 6 | 1988.08.04-1989.05.15 | 0.123 | -0.072 | 734.40 | 0.040 | 732.67 | 3.443 * |
| 7 | 1989.05.16-1989.12.29 | 0.134 | 0.099 | 605.28 | 0.119 | 605.24 | 0.097 |

* 10% significance — $\chi^2(1)$
** 1% significance — $\chi^2(1)$

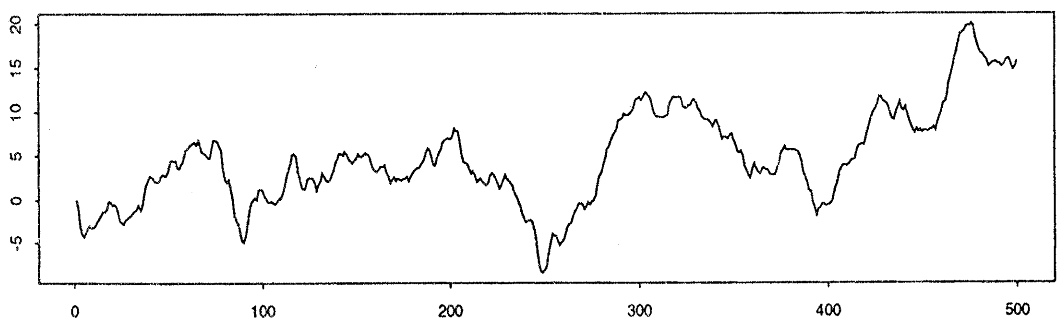
**Spot 1990 – 1994**

| | period | SSIAR(1,1) | | | IAR(1,1) | | |
|---|---|---|---|---|---|---|---|
| | | $a$ | $b$ | LK | $a(=b)$ | LK | $\chi^2$ |
| 1 | 1990.01.04-1990.10.22 | 0.041 | 0.208 | 488.01 | 0.116 | 486.45 | 3.108 * |
| 2 | 1990.10.23-1991.08.15 | -0.038 | 0.063 | 568.90 | 0.028 | 567.94 | 1.925 |
| 3 | 1991.08.16-1992.06.11 | -0.072 | 0.079 | 533.85 | 0.043 | 532.55 | 2.601 |
| 4 | 1992.06.12-1993.03.31 | -0.006 | 0.037 | 530.22 | 0.022 | 530.12 | 0.211 |
| 5 | 1993.04.01-1994.01.24 | 0.043 | -0.073 | 573.47 | -0.015 | 572.90 | 1.134 |
| 6 | 1994.01.25-1994.11.14 | -0.160 | 0.049 | 622.40 | -0.064 | 620.58 | 3.630 * |

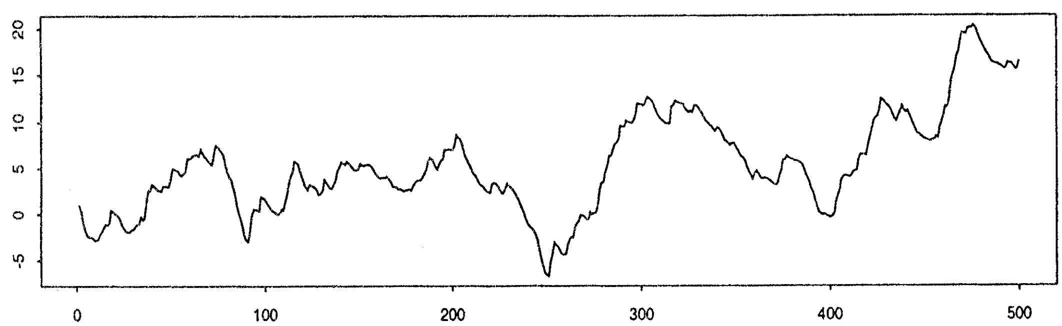
**Futures 1990 – 1994**

| | period | SSIAR(1,1) | | | IAR(1,1) | | |
|---|---|---|---|---|---|---|---|
| | | $a$ | $b$ | LK | $a(=b)$ | LK | $\chi^2$ |
| 1 | 1990.01.04-1990.10.22 | 0.205 | 0.265 | 506.59 | 0.234 | 506.37 | 0.451 |
| 2 | 1990.10.23-1991.08.15 | -0.030 | -0.002 | 561.92 | -0.016 | 561.89 | 0.070 |
| 3 | 1991.08.16-1992.06.11 | 0.088 | 0.192 | 552.34 | 0.143 | 551.72 | 1.257 |
| 4 | 1992.06.12-1993.03.31 | 0.104 | 0.031 | 541.14 | 0.069 | 540.89 | 0.500 |
| 5 | 1993.04.01-1994.01.24 | -0.016 | -0.095 | 567.66 | -0.047 | 567.41 | 0.500 |
| 6 | 1994.01.25-1994.11.14 | -0.135 | -0.227 | 615.97 | -0.178 | 615.71 | 0.520 |


A = 0.8 B = 0.2 r = 1




A = 0.5 B = 0.5 r = 1




A = 0.2 B = 0.8 r = 1

**Figure 2.1:** The sample paths of SSIAR(1)

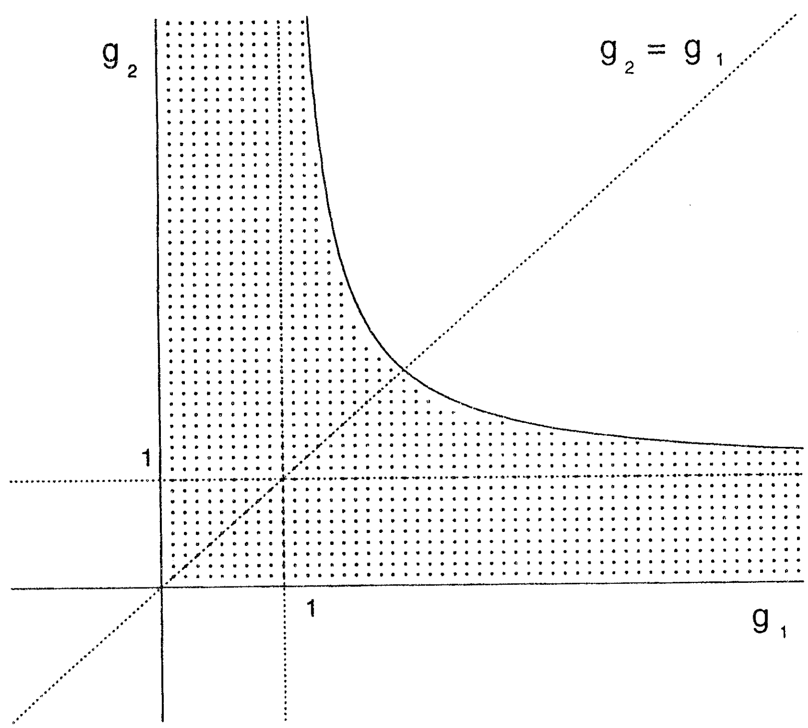



**Figure 3.1:** The region of ergodicity

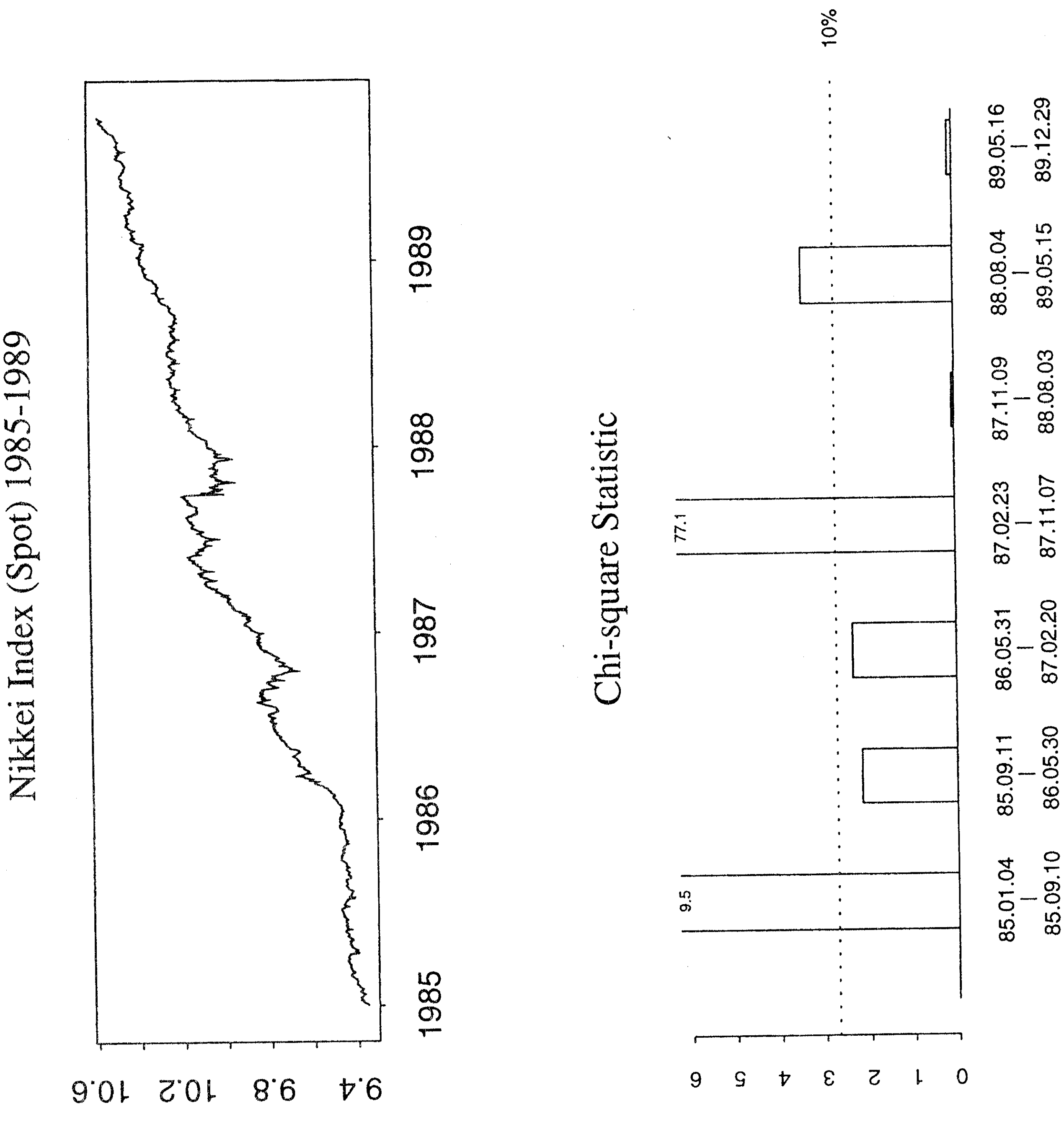



**Figure 3.2:** Result of Nikkei Index 225

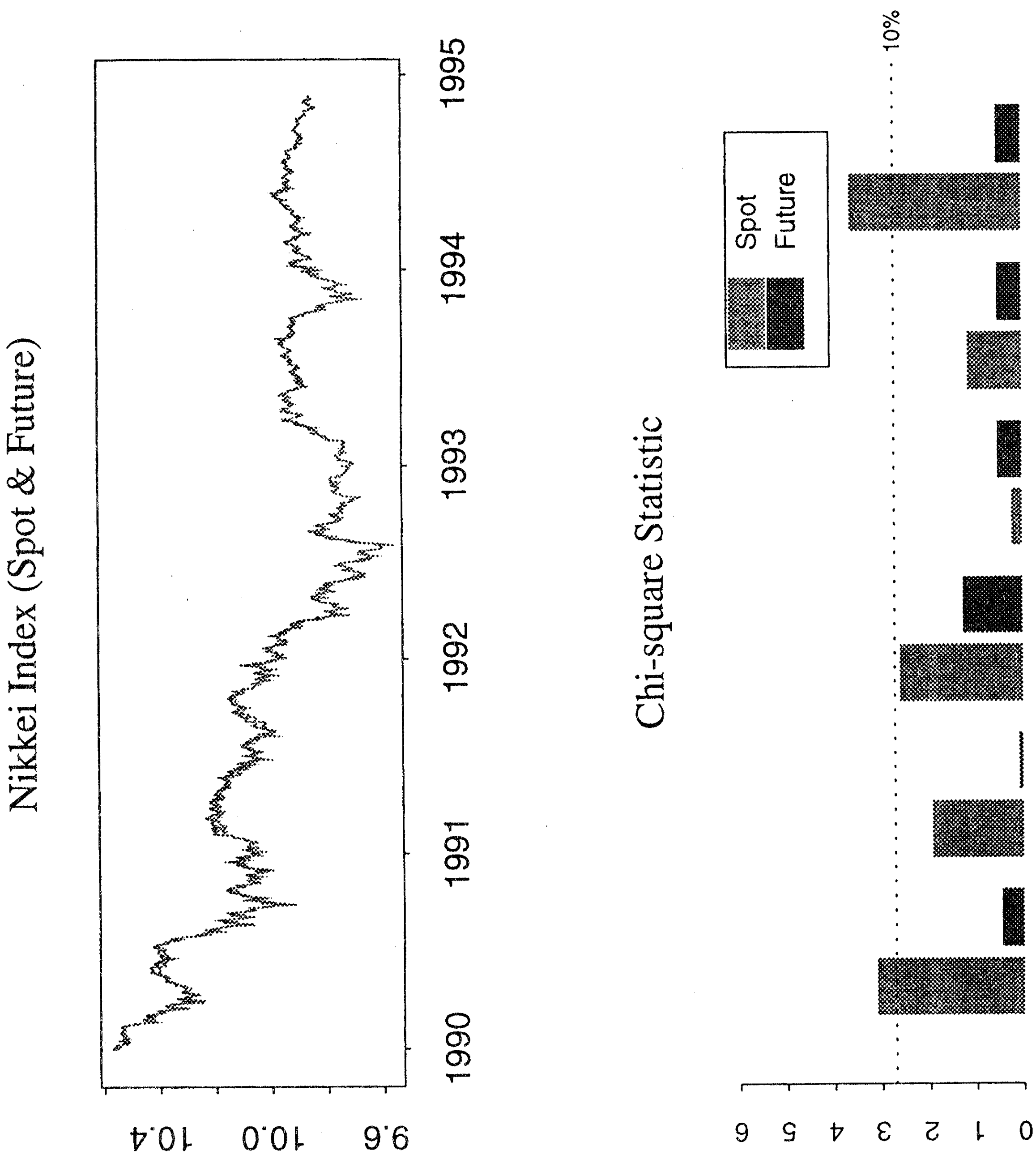



**Figure 3.2:** Result of Nikkei Index 225 (continued)