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# PRIVATE OBSERVATION, COMMUNICATION AND COLLUSION

by

KANDORI, Michihiro and MATSUSHIMA, Hitoshi
Faculty of Economics, University of Tokyo

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Faculty of Economics, University of Tokyo

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## ABSTRACT

We examine the possibility of cooperation in a long term relationship, where agents receive *diverse* imperfect information about each other's actions. "Secret price cutting" in the industrial organization literature is a leading example. In a differentiated product market, a firm may not be able to perfectly detect secret price cutting by others, but its own sales may imperfectly indicate what is going on. Since the firms' sales levels are subject to random shocks, they may well end up having diverse expectations: firms with low sales may suspect price cutting while others may not. This causes a serious difficulty in sustaining collusion in such a market. In fact, the characterization of equilibria of this class of games - discounted repeated games where each player receives a *different* signal - has been an open question, despite the large body of literature on repeated games. The present paper shows that communication is a powerful way of resolving the possible confusion among the players in this class of games. In particular, we construct equilibria where players voluntarily communicate what they have observed and prove folk theorems. Our results thus provide a theoretical support for the conventional wisdom that communication is vital in sustaining collusion.



**Keywords:** discounted repeated games, folk theorem, imperfect monitoring, privately observed signals, communication, review strategy

## 1. INTRODUCTION

The present paper analyzes the role of communication and the possibility of cooperation in a long term relationship, when the actions of the players are imperfectly observed. In particular, we consider the situation where the players receive diverse information about the past history and do not share a congruent set of beliefs about what might have happened. In such a situation, we will show that communication is a powerful way of dissolving the possible confusion and coordinating the players' behavior.

"Secret price cutting" is a leading example of the particular situation we analyze in this paper. Consider a small number of firms producing intermediate goods. It is a usual practice in such a market that the effective price of the good is different from the published one, and the former is determined by face-to-face negotiation by the seller and the buyer. This is commonly referred to as "secret price cutting" in the industrial organization literature. As a result, the firms cannot directly observe others' effective prices. However, each firm can observe its own sales, which serves as an imperfect signal about other firms' pricing behavior. If the sales are low, for example, it may be an indication of other firms' secret price cutting. Or, it may just be the case that the market demand is low. An important feature of the market is that *the sales level of each firm is private information and can not be observed by others.* This creates a serious difficulty for firms trying to collude for the following reason. To maintain high prices the firms need to punish potential deviators, and this is easiest when they share common beliefs about when a deviation happened and who might be the deviator. In the above situation, however, the firms typically receive different levels of sales and therefore end up having diverse beliefs about what might have happened.

In fact, the analysis of such a situation is known to be a hard problem in game theory. The situation can be formulated as a repeated game with imperfect monitoring and *privately* observed signals. Note well that the difficulty is not caused by imperfect monitoring *per se*, but by the private observability of signals. The celebrated model of collusion by Green and Porter (1984), in which the market price serves as a *commonly* observable signal, is much more tractable, because the players can easily agree when to punish potential deviators. The study of repeated games with public signals was further extended by Abreu, Pearce and Stacchetti (1986, 1990), and Fudenberg, Levine and Maskin (1989) identified sufficient conditions for the folk theorem to hold in such games. In a sharp contrast, there has been very slow progress in understanding repeated games with private signals. This is rather unfortunate because those models represent a variety of important economic problems, including such a prominent example as secret price cutting.

There are a limited number of previous contributions on this subject: Radner (1986), a series of works by Lehrer (1989, 1990, 1991, 1992a-c), and Fudenberg and Levine (1991). Those papers, however, share a common weakness. Radner analyzes the *no discounting* case, in which any act in a finite period, however long it may be, do not affect the total payoff at all. His work extensively uses this property, and can not readily be extended to the discounted case. In fact, it is known in the study of repeated games with perfect monitoring that there is a substantial difference between the discounted and non-discounted case.[1] Fudenberg and Levine, in contrast, analyze the discounted case, but they assume that the players are only epsilon-rational. When the players are patient, this implies that they do not mind taking suboptimal behavior for a long time, and again it is this rather problematic property that plays a crucial role in their model. Similarly, Lehrer analyzes the no discounting case and/or the discounted epsilon-rational case.

In the present paper, we analyze *perfectly rational* players with *discounting*. Instead of assuming no discounting or irrationality, we

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[1] See Fudenberg and Maskin (1986).

introduce *communication* in the model to overcome the basic difficulty of this subject. We feel that communicating with each other is the most natural way of dissolving confusion among players when they cannot agree on how to maintain collusive behavior. Notice that the conventional wisdom in industrial organization states that communication plays an important role in collusion. In fact, certain kinds of communication are *per se* illegal in the antitrust law . Yet there has been virtually no formal theory to show the role of communication in collusion, and the present paper is hopefully a first step to formulate the conventional wisdom in a general framework.[2]

In particular, we assume that at the end of each period players can communicate what they privately observed. Communication entails no cost so that it is "cheap talk" rather than "signaling". We also assume that the players act strategically when they communicate: they can freely provide false information if it suits their best interest. Nevertheless, we will show that we can construct equilibria in which players reveal their private information truthfully, and show that the folk theorem obtains under a set of mild assumptions.

The use of communication in this class of games was first introduced by Matsushima (1990), who presented some preliminary results and conjectured the possibility of a folk theorem with communication. After the completion of several versions of the present paper, the authors became aware of recent independent contributions by Ben-Porath and Kahneman (1993) and Compte (1994a),

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[2]There are many aspects of communication in collusion, and the present paper does not attempt to formulate all of them. For example, one prominent role of communication is to choose which equilibrium to play. While this probably is one of the most important roles of communication in reality, the well-known literature on cheap talk (Farrell(1988), and Matsui(1991)) shows that this aspect of communication is rather hard to formulate in the standard equilibrium analysis. In contrast, we are able to show within the standard theoretical framework that communication is useful in coordinating player's beliefs about history so that they can choose actions appropriate to collusion.

who also explore the role of communication in repeated games with privately observable signals. Ben-Porath and Kahneman examine the case where each player's action is perfectly observed by a *subset* of other players, and prove a folk theorem when each player's action is observed by at least two players. Compte's paper is more closely related to ours; it examines the same class of games and proves similar folk theorems with communication. Interested readers are strongly advised to read his paper too.

At this juncture let us briefly explain why the analysis of repeated games with privately observed signals has been an open question. In the usual repeated games, players can choose which equilibrium to play depending on the publicly observed signals in each period. This means that after any history, the continuation play is always an equilibrium of the repeated game. This "recursive" structure makes the analysis much easier, and the set of equilibria can be characterized by the dynamic-programming technique introduced by Abreu, Pearce and Stacchetti (1986, 1990). On the other hand, when signals are privately observed, continuation plays are no longer equilibria and the recursive structure is destroyed. To see this, consider an equilibrium where players' actions are $a = (a_1, \ldots, a_n)$ in the first period and each player i receives private signal $\omega_i$ according to distribution $Pr(\omega_1,\ldots,\omega_n|a)$. If each player i conditions his future strategy on $\omega_i$, this means that the continuation play is a (partially) *correlated* equilibrium, rather than a Nash equilibrium. More importantly, if player i deviates from $a_i$ to $a_i'$ in the first period, the continuation play is not even a correlated equilibrium. This is because the players have different beliefs about the distribution of the "correlation device" $(\omega_1,\ldots,\omega_n)$: player i knows that their distribution is changed by his deviation $(Pr(\omega_1,\ldots,\omega_n|a_i,a_i'))$, while others continue to believe that the distribution of $\omega$ is given by the equilibrium action profile a $(Pr(\omega_1,\ldots,\omega_n|a))$. Thus we lose the recursive structure when signals are privately observed, and accordingly there has been no result characterizing the set of equilibria of discounted repeated games with such an information structure.

In the present paper, we overcome this difficulty by introducing communication. Communication generates publicly observable history, and the players can play different equilibria depending on the history of communication. In this way, the recursive structure is recovered, and we are able to use the dynamic programming method developed for the repeated games with publicly observable signals.

The paper is organized as follows. The model is defined in Section 2 and Section 3 summarizes the basic technique from the previous work and provides the overview of basic ideas. *The reader who is not interested in the technical details is advised to read the last part of Section 3* to understand the basic theoretical constructions of our results and the relationship to the existing literature. Section 4 deals with the case with at least three players. The two-player case is somewhat special and requires a different technique. Basically, we show that equilibrium payoffs can be improved by the delay of information release, together with the statistical testing addressed by Radner (1986) and Matsushima (1994). This is explored in Section 5. Concluding remarks are given in the last section.

## 2. THE MODEL

The component game G is defined as follows. Each player $i \in N = \{1,\ldots,n\}$ simultaneously chooses an action $a_i$, and after choosing it she observes her own *private* signal $\omega_i$ which is not observed by the opponents. Let $A_i$ be the finite set of actions for player i, and let $\Omega_i$ be the finite set of possible private signals for player i. We denote $\Pi_{i\in N}A_i = A$ and $\Pi_{i\in N}\Omega_i = \Omega$, and their generic elements are denoted a and $\omega$ respectively. Similar notations will be employed for product sets and vectors in what follows, and the definition will not be repeated. The probability distribution of private signal profile $\omega$ conditional on action profile a is denoted $p(\omega|a)$. We assume this distribution has *full support*, that is, for each $a \in A$ and each $\omega \in \Omega$,

$p(\omega|a) > 0$. Let the marginal distributions for signals $\omega_i$ and $\omega_{-i} = (\omega_1,\ldots\omega_{i-1},\omega_{i+1},\ldots,\omega_n)$ be denoted $p_i(\omega_i|a)$ and $p_{-i}(\omega_{-i}|a)$.

Player i's instantaneous payoff $u_i(a_i,\omega_i)$ is determined by her own action and her own private signal only, i.e., it is independent of the opponents' actions $a_{-i}$ and their private signals $\omega_{-i}$. This formulation makes it sure that the realized payoff $u_i$ reveals no more information than $a_i$ and $\omega_i$ do. In the secret price cutting application, $u_i$ is the profit of firm i, which depends on its price $a_i$ and the quantity sold $\omega_i$. Player i's expected payoff when players choose action profile $a \in A$ is

$$g_i(a) = \sum_{\omega\in\Omega} u_i(a_i,\omega_i)p(\omega|a) \quad .$$

Let $\alpha_i$ be player i's mixed action, $\Delta_i$ the set of player i's mixed actions. Let $g_i(\alpha)$ and $p(\omega|\alpha)$ be player i's expected payoff and the probability of $\omega$ respectively when players conform to mixed action profile $\alpha \in \Delta$.

We will allow players to *communicate* with each other. After choosing actions and observing private signals, the players simultaneously and publicly announce messages $(m_1,\ldots,m_n)$. Let $M_i$ be the finite set of player i's possible messages, which will be specified in what follows.

The infinitely repeated game with discounting associated with the component game G is denoted by $\Gamma(G,\delta)$, where $\delta\in(0,1)$ is the discount factor. A strategy for player i is defined by $s_i = (\sigma_i,\eta_i)$, where $\sigma_i$ and $\eta_i$ specify action and message respectively. In particular, $\sigma_i = (\sigma_i(t))_{t=1}^{\infty}$, $\eta_i = (\eta_i(t))_{t=1}^{\infty}$, $\sigma_i(t):A_i^{t-1}\times\Omega_i^{t-1}\times M^{t-1} \to \Delta_i$, and $\eta_i(t):A_i^{t}\times\Omega_i^{t}\times M^{t-1} \to M_i$. Player i's expected average payoff when players conform to strategy profile $s = (s_1,\ldots,s_n)$ is

$$v_i(s,\delta) = (1-\delta)E[\sum_{t=1}^{\infty} \delta^{t-1}u_i(a_i(t),\omega_i(t))|s],$$

where $E[\cdot|s]$ is the expectation with respect to the probability measure on histories induced by strategy profile s, and $a_i(t)$ and $\omega_i(t)$

are the action and the private signal for player i realized in period t. The solution concept is sequential equilibrium. The set of sequential equilibrium average payoffs is denoted $V(G,\delta)$. Define $V(G) = \lim_{\delta \to 1} V(G,\delta)$.

## 3. CHARACTERIZATION AND OVERVIEW

To analyze the equilibria in the discounted repeated game $\Gamma(G,\delta)$, we employ the method developed by Fudenberg and Levine (1994)[3]. Instead of directly solving the repeated game, this method first considers simple contract problems associated with the stage game. Then, the solutions to those contract problems are utilized to construct the set of equilibrium payoffs of the repeated game. In this section we first present a version of their method modified to fit our framework with private signals and communication. Then, for the reader's convenience, we will provide an intuitive explanation about why this method works. Lastly, non-technical overview of our construction of folk theorems will be given.

We consider the T-time finitely repeated game with no discounting associated with stage game G, which is denoted by $G^T$. We assume before playing $G^T$ players agree to a sidepayment contract $x^T = (x_i^T)_{i \in N}$, where $x_i^T : M^T \to \mathbb{R}$. $x_i^T(m^T)$ is the sidepayment to player i when the history of message profiles $m^T$ is realized. Player i's realized payoff is

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[3] Fudenberg and Levine consider the very general case where there are two types of players, long-run and short-run. The time horizon for a long-run player is infinite, while a short-run player lives only one period and will be replaced by a newcomer. Note well that their model *does* include the usual repeated games played only by long-run players. This becomes a special case of their model when we either let the set of short-lived players empty or make each short-run player a dummy player (her action set being a singleton). Hence their results applies to the standard repeated games only with long-run players.

$$\frac{1}{T}\sum_{t=1}^{T} u_i(a_i(t),\omega_i(t)) + x_i^T(m^T).$$

This defines the finitely repeated game with sidepayments $(G^T, x^T)$, and player i's strategy in this game, $s^T = (\sigma_i^T, \eta_i^T)$, is defined in the same way as in the previous section. Player i's expected payoff under strategy profile $s^T$ is

$$v_i^T(s^T, x_i^T) = E[\frac{1}{T}\sum_{t=1}^{T} u_i(a_i(t)) + x_i^T(m^T) \mid s^T],$$

where $E[\cdot \mid s^T]$ is the expectation with respect to the probability measure on histories induced by strategy profile $s^T$.

For every welfare weight $\lambda \in \mathbb{R}^n \backslash \{0\}$, we introduce the following static optimization problem:

**Problem $(T,\lambda)$:**

$$\max_{(s^T,x^T)} [\sum_{i\in N} \lambda_i v_i^T(s^T, x_i^T)]$$

subject to **(B)** $\sum_{i\in N} \lambda_i x_i^T(m^T) \le 0$ for all $m^T \in M^T$, and

**(IC)** $s^T$ is a sequential equilibrium in $(G^T, x^T)$.

Let $k(T,\lambda)$ denote the optimal value of this problem. Define

$$D(T,\lambda) = \{v \in \mathbb{R}^n \mid \lambda v \le k(T,\lambda)\}, \text{ and } Q(T) = \cap_{\lambda\ne 0} D(T,\lambda).$$

Fudenberg and Levine (1994) show that $Q(T)$ is a subset of $V(G^T)$ whenever $Q(T)$ is full-dimensional. Since $V(G,\delta)$ is approximated by $V(G^T,\delta^T)$ (= the repeated game whose stage game is T-times repetition of G with the discount factor equal to $\delta^T$) when $\delta$ is close to unity, $V(G)$ is equal to $V(G^T)$. Thus, we can conclude that $Q(T)$ is a subset of

V(G) if it is full-dimensional. We finally define $k(\lambda) = \lim_{T\to\infty} k(T,\lambda)$, $D(\lambda) = \{v \in \mathbb{R}^n \mid \lambda v \le k(\lambda)\}$, and $Q = \cap_{\lambda\neq 0} D(\lambda)$.

For each T = 1,2,..., set Q(T) represents the collection of limit equilibrium payoffs (as $\delta \to 1$) in a different class of strategies in the infinitely repeated game. If T = 30 (a month), for example, Q(T) admits the situation where each player utilizes private information (her action and signal) *only* within a given month. On any given day in the first month (say, January), each player chooses her action based on her past actions and private signals, as well as the history of publicly exchanged messages. In February, however, the players abandon what they privately observed in January, and condition their actions on (1) publicly exchanged messages in January and February and (2) private information in February. Hence the players periodically abandon all private information every T periods (i.e., at the beginning of each month), but they (potentially) utilize the *whole* history of publicly exchanged messages. As we will explain in more detail, this linkage to other months is captured by the term $x^T(m^T)$ in Problem $(T,\lambda)$, representing the effect of the message $m^T$ on continuation payoffs.

Formally, for any period t within the $K^{th}$ month (i.e. $KT < t \le (K+1)T$), if both $\sigma_i(t)$ and $\eta_i(t)$ are independent of $(a_i(\tau), \omega_i(\tau))_{\tau \le KT}$, we call $s_i = (\sigma_i, \eta_i)$ a *T-public strategy*. A special case of T-public strategy, which will play an important role in this paper, is the case where the players seriously communicate only every T periods. Finally, if s is a sequential equilibrium in the repeated game and each $s_i$ is T-public, we call s a *T-public perfect equilibrium*. With this definition, the same argument as that of Fudenberg and Levine show that Q(T) is the set of T-public perfect equilibrium payoffs (as $\delta \to 1$). As T increases, more dependence on private information is allowed, so we have $Q(T) \subset Q(T')$ for $T = LT'$, where L is a positive integer.

Now we provide a brief intuition for the Fudenberg-Levine algorithm, which connects the static contract problem $(T,\lambda)$ with the repeated game equilibria. Let us consider the case where the players abandon private information every T periods (i.e., T-public perfect equilibria). We can decompose each player's payoff into the total payoff in the first T periods and the continuation payoff after that. Let W in Figure 1 denote the set of *average* continuation payoffs. In the repeated game, the players take some actions and send messages in the first T periods. According to the exchanged messages $m^T$ in the first T periods, a continuation equilibrium $w(m^T)$ is chosen from set W. This way the players' actions in the first T periods influence the choice of the continuation payoffs. Hence in the construction of an equilibrium, we must choose the continuation payoffs judiciously so that (1) the players are willing to send informative messages and (2)

***** Figure 1 here *****

deviants in the first T periods are punished. Furthermore, if we are interested in the best repeated game equilibrium in a particular direction $\lambda$, we must make the continuation equilibria as close as possible to the boundary of W in the direction of $\lambda$ (See Figure 1(a).). Now notice that the points in set W in Figure 1(a) are *average* (per period) payoffs; if w is an average payoff, the total payoff is $w/(1-\delta)$. This implies that, when the discount factor $\delta$ is close to 1, a very small variation of w creates a huge change in total payoffs. Given that the gains from deviations in the first T periods are bounded, we can then choose $w(m^T)$ from a small neighborhood O in Figure 1(a), and still be able to provide sufficient incentives. Hence, if the discount factor is sufficiently close to unity, we can regard the *total* continuation payoffs being chosen from the half space H defined by the normal vector $\lambda$ (Figure 1(b))[4]. This is basically what is going on in the Fudenberg-Levine algorithm; the 'sidepayment'

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[4] For this to be true, W must be a set with *smooth* boundary. The Fudenberg and Levine algorithm exploits the fact that any smooth subset of the feasible and individually rational payoff set (which is a polyhedron, thus not smooth) can be a subset of the equilibrium payoff set if the discount factor is close enough to one.

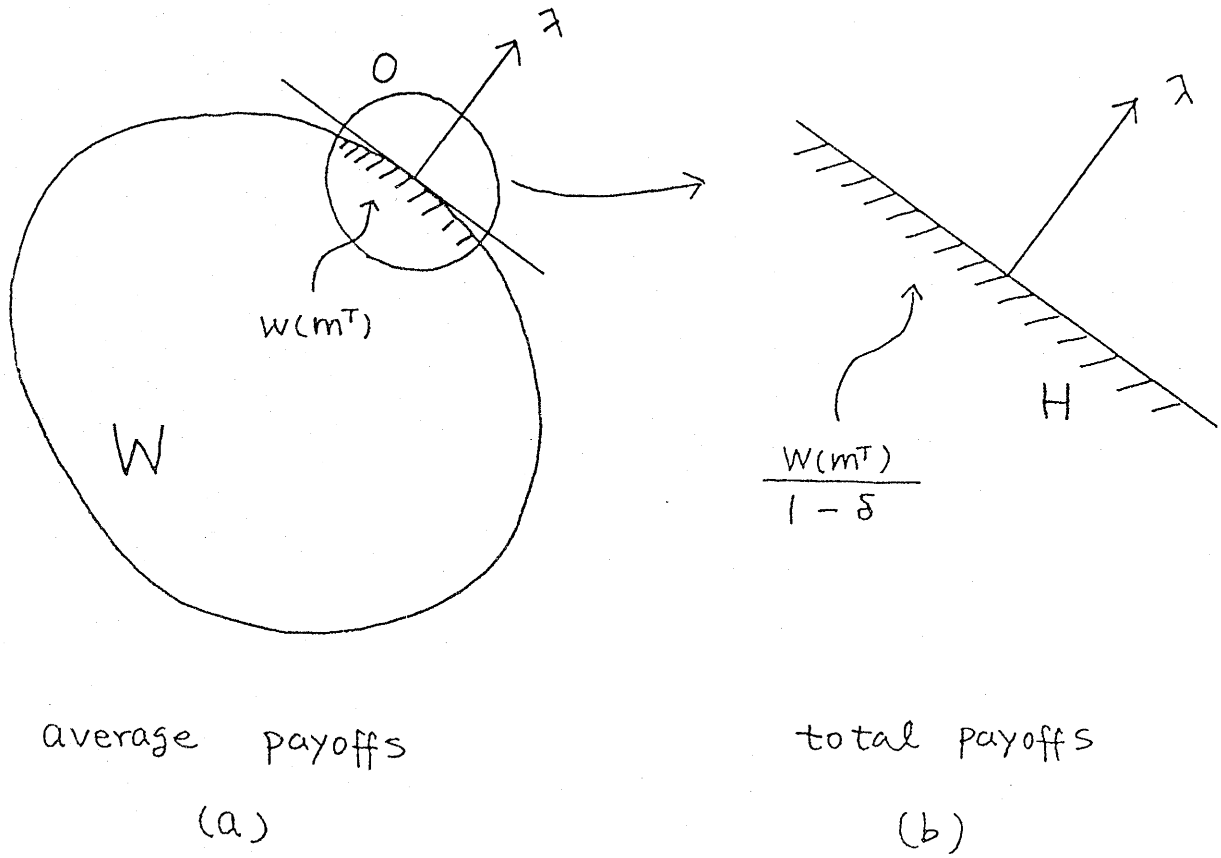


Figure 1.

x in the contract problem $(T,\lambda)$ corresponds to the total continuation payoffs, and the 'budget constraint' (B) indicate that they are chosen from the half space H in Figure 1(b)[5].

With this technique, we will provide several versions of the folk theorem in the following sections. Before going into the technical details, let us sketch the basic ideas. The main idea is based on the following result about the repeated games with publicly observed signals (see Fudenberg, Levine and Maskin (1989)). Roughly speaking, efficiency under publicly observable signals can be achieved if players can be punished by "transfers". That is, if the information structure allows us to tell *which* player is suspect, we can transfer the suspect player's future payoff to the other players. This way we can provide the right incentives without causing any welfare loss. If the signals are privately observed, however, we must induce each player to reveal his signal truthfully, and this imposes certain restrictions on the form of feasible payoff transfers among the players. The easiest way to solicit truthful information is to make each player's future payoff independent of what he communicates. Then he is just indifferent as to what he says, and truthful revelation is a (weak) best response.

This can be done if there are at least three players. A player's private information can be used to determine when and how to *transfer* payoffs among *other* players. This possibility is explored in Section 4. Section 4 also examines the possibility of providing *strict* incentives to tell the truth. We show that when private signals are correlated, there is a way to check if each player is telling the truth. All of this is done by assuming serious communication every period: in other words, we look at 1-public perfect equilibria and we

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[5] This is only a very rough sketch of the basic idea behind the F-L algorithm and not a complete explanation. For example, the discount factor appears in the repeated game problem while it is absent in the static contract problems, and it is not immediately clear why this discrepancy does not cause any problem. For such details, see the Fudenberg and Levine (1994).

utilize the above characterization of Q(1) accordingly.

If there are two players, the above idea cannot be utilized. If what player 1 says induces a transfer between player 1 himself and player 2, he may well send a false message to induce the transfer to himself. Indeed, if the signals are independent (conditional on the action profile), it is shown that there is no way to achieve efficient payoffs in two-player case by means of 1-public perfect equilibrium. Accordingly, we will explore in Section 5 the possibility of a folk theorem by means of T-public perfect equilibrium with $T>1$. That is, if the players communicate seriously only every T periods and their actions can partly depend on their private information, there is a possibility of getting better outcomes. Such a possibility has been demonstrated by Abreu, Milgrom and Pearce (1991), who showed that the delay of the release of public information can enhance efficiency in repeated games. With privately observed signals, it turns out that their method is not directly applicable (Section 5.2 explains the details), and we instead use a new technique. We will show that, as the discount factor $\delta$ tends to 1 and the interval of serious communication T tends to infinity, the folk theorem holds under some conditions. This result assumes the independence of private signals (conditional on the action profile), but otherwise assumes very little about the information structure. On the other hand, it requires some conditions on the stage payoffs. The class of games covered by this folk theorem includes the noisy prisoner's dilemma with private signals.

## 4. Folk Theorem with More Than Two Players

In this section we present folk theorems when the number of players is (strictly) more than two. We start with a set of sufficient conditions for the folk theorem.

### 4.1 Sufficient Conditions

First, we present a set of rather simple sufficient conditions for the folk theorem. Those conditions are easily verified, permit economic interpretation, and the proof of the folk theorem (Theorem 1) is constructive[6]. After proving the simple version, we will present weaker, although more abstract, conditions in the second half of this sub-section.

Let $\Omega_{-i} = \Pi_{k\neq i}\Omega_k$ and $\Omega_{-ij} = \Pi_{k\neq i,j}\Omega_k$, and define vectors

$$p_{-i}(a) \equiv (p(\omega_{-i}|a))_{\omega_{-i}\in\Omega_{-i}} \quad \text{and}$$
$$p_{-ij}(a) \equiv (p(\omega_{-ij}|a))_{\omega_{-ij}\in\Omega_{-ij}},$$

where $p(\omega_{-i}|a)$ and $p(\omega_{-ij}|a)$ are marginal distributions of $p(\omega|a)$. Vectors $p_{-i}(a)$ and $p_{-ij}(a)$ represent the distributions of signals, given action profile a, observed by payer i's opponents and by players i and j's opponents respectively. For brevity, we call the set of players other than i and j "ij-opponents". Conditional distributions given a mixed action profile $\alpha \in \Delta$ are denoted $p_{-i}(\alpha)$ and $p_{-ij}(\alpha)$. Note that $p_{-ij}$ is well defined only when there are more than two players, which will be assumed throughout Section 4.

Let $\mu^i$ be the *minimax* profile (in mixed strategies) for player i:

$$\mu^i_{-i} \in \arg\min_{\alpha_{-i}\in\Delta_{-i}} (\max_{a_i\in A_i} g_i(a_i, \alpha^i_{-i})),$$
$$\mu^i_i \in \arg\max_{a_i\in A_i} g_i(a_i, \mu^i_{-i}).$$

The first assumption we employ is the following.

**(A1)** For all i and $j\neq i$, if there is a mixed strategy $\alpha_j \in \Delta_j$ such that $p_{-j}(\mu^i) = p_{-j}(\mu^i_{-j}, \alpha_j)$, then $g_j(\mu^i) \geq g_j(\mu^i_{-j}, \alpha_j)$.

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[6] In the sense that it explicitly constructs the side-payment scheme in the associated static contract problems.

This assumption will be utilized to provide proper incentives for a player to punish (minimax) another. Assumption (A1) considers a situation where player j has a perfectly undetectable deviation ($\alpha_j$) at the minimax point for player i ($\mu^i$). Then, the assumption requires that such a deviation does not pay. Note well that the 'perfect undetectability' of the deviation $\alpha_j$ in the above sentence has a very strong meaning. It requires that both $\mu^i_j$ and $\alpha_j$ produce exactly the same *distribution* of the signals observed by j's opponents ($\omega_{-j}$). This should not be confused with undetectability in a weaker sense. Under our full support assumption for the signals, any outcome $\omega_{-i}$ can always be realized with a positive probability, irrespective of the action taken. So it is *a fortiori* impossible to determine whether player j is actually punishing i (using $\mu^i_j$) or not ($\alpha_j$) *for sure*. However, if $\mu^i_j$ and $\alpha_j$ produce different distributions of $\omega_{-j}$, the expected reward (future payoffs) for player j, which is a function of $\omega_{-j}$, can change when j switches from $\mu^i_j$ to $\alpha_j$. Hence, if the premise of (A1), $p_{-j}(\mu^i) = p_{-j}(\mu^i_{-j},\alpha_j)$, is violated, there is a possibility to provide an incentive for player j to follow $\mu^i_j$. In what follows we will show that this is indeed the case. We will also explain when this condition is likely to be satisfied shortly.

Now define, for each pair $i \neq j$ and each profile $a \in A$,

$$Q_{ij}(a) = \{p_{-ij}(a_{-i}, a'_i) \mid a'_i \in A_i \backslash \{a_i\}\}.$$

This is a collection of distributions of ij-opponents' signals, generated by player i's deviations from the profile a. Let Ex(A) be the set of strategy profiles which provide the extreme points of the stage payoff set. We denote the convex hull of set X by co(X).

**(A2)** For each pair $i \neq j$ and each $a \in Ex(A)$,

$$p_{-ij}(a) \notin co\{Q_{ij}(a) \cup Q_{ji}(a)\}.$$

***** Figure 2 here *****

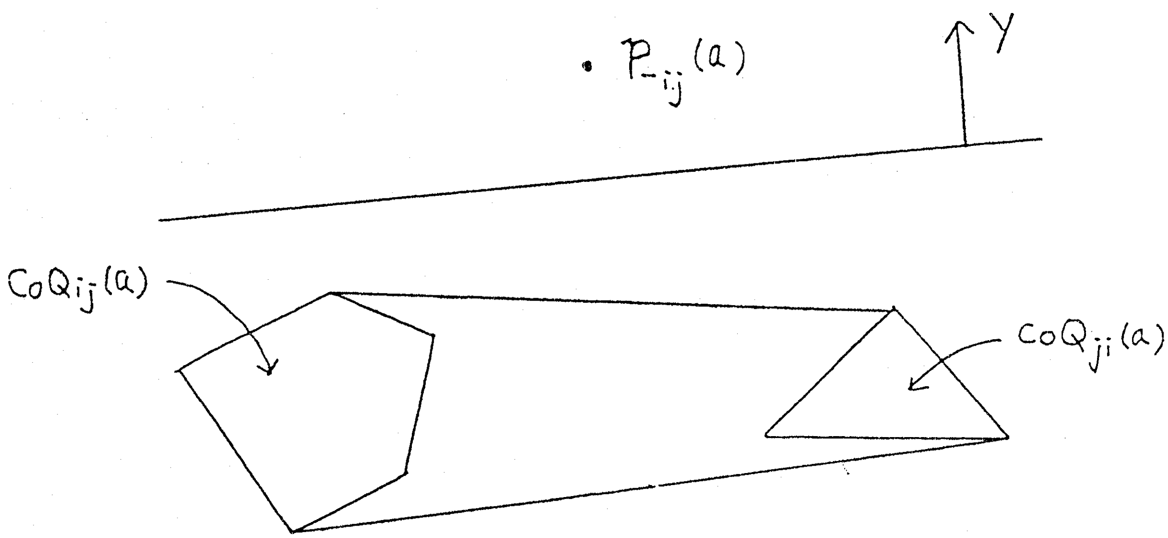


Figure 2.

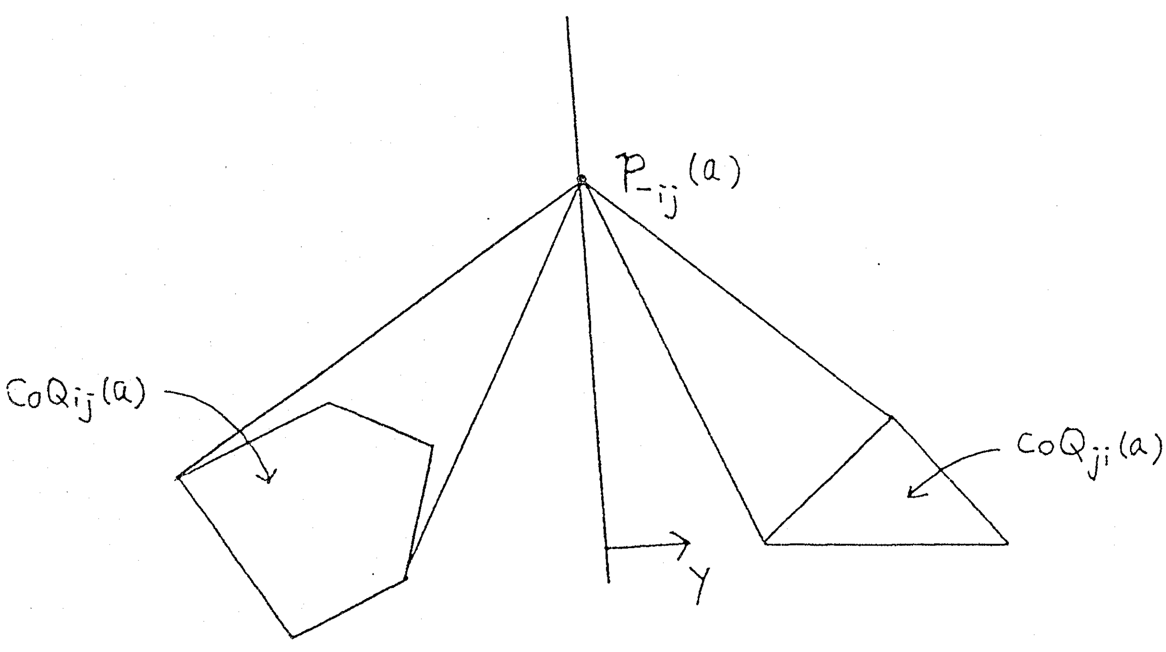


Figure 3.

(A2) says that if either player i or j (but not both) deviates with certain probabilities, the other players can (statistically) detect it. It in particular implies

**(A2')** For each pair $i \neq j$ and $a \in Ex(A)$, $p_{-ij}(a) \notin co(Q_{ij}(a))$.

This says that player i's mixed strategy deviations are statistically detected by ij-opponents. This in turn trivially implies

**(A2")** For each i and $a \in Ex(A)$,

$$p_{-i}(a) \notin co\{p_{-i}(a_{-i}, a'_i) \mid a'_i \in A_i \backslash \{a_i\}\}.$$

That is, any mixed strategy deviation by player i is statistically detected by his opponents.

**(A3)** For each pair $i \neq j$ and each $a \in Ex(A)$,

$$co(Q_{ij}(a) \cup \{p_{-ij}(a)\}) \cap co(Q_{ji}(a) \cup \{p_{-ij}(a)\}) = \{p_{-ij}(a)\}.$$

***** Figure 3 here *****

This says that if either player i or j deviates by a mixed strategy, ij-opponents can statistically tell which one has deviated.

Let us now examine when the information conditions (A1)-(A3) are likely to be satisfied. They are somewhat weaker versions of Fudenberg-Levine-Maskin's "individual full rank" and "pairwise full rank" conditions ((A1) corresponds the former and (A2) and (A3) the latter)[7]. Assumption (A1) is vacuously satisfied when different pure

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[7] Our conditions are weaker because they are stated in terms of *convex combination* of distributions, while Fudenberg, Levine, and Maskin (1989, FLM hereafter) consider *linear combination*. On the other hand, FLM note that their condition, the pairwise full rank condition, needs to be satisfied by *one* (possibly mixed) strategy profile. This is in contrast to our requirement that (A2) and (A3) be

strategy deviations by player j creates linearly independent distributions of the signals $\omega_{-j}$, because the premise of (A1), $p_{-j}(\mu^i) = p_{-j}(\mu^i_{-j},\alpha_j)$, cannot be true in such a case. This linear independence requirement is called individual full rank condition by Fudenberg, Levine and Maskin, and a sufficient (but not necessary) condition for this is that the collection of distributions for $\omega_{-j}$, one for each pure strategy profile $a \in A$, (i.e., $\{p_{-j}(a)\}_{a\in A}$) are linearly independent. This is generically satisfied when the number of possible outcomes of the signals $\omega_{-j}$ exceeds the number of pure strategy profiles. Therefore, if the signal space is rich enough (the number of possible signal outcomes being large compared to the number of pure strategies), the condition (A1) is typically satisfied.

A sufficient condition for (A2) and (A3) is that the signal distributions created by players i and j's deviations are linearly independent (the pairwise full rank condition by Fudenberg-Levine-Maskin). An example which violates this is a symmetric case, where $\Delta a_i$ and $\Delta a_j$ have symmetric effects on the the signal distribution. In the secret price cutting application, for example, (A3) is violated if the quantity sold by firm k, which is its private signal $\omega_k$, depends on the average price charged by its rivals ($[\Sigma_{h\neq k}a_h]/(1-n)$). However, by the same argument as in the previous paragraph, such a counterexample is a knife-edge case, when the signal space is sufficiently rich. When the number of signal outcomes $\omega_{-ij}$ exceeds the number of pure strategy combinations for i and j ($\#[A_i \times A_j]$), we can always satisfy (A2) and (A3) by slightly perturbing the signal distributions. In summary, if each player has rich enough signals, our information conditions (A1)-(A3) are typically satisfied. In the secret price cutting application, those conditions are likely to be satisfied when each firm has many instruments to infer the rival's

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satisfied at *all* extremal profiles. The weaker requirement by FLM works because that assumption ensures that any extremal profile can be approximated by a (mixed strategy) profile where the full rank assumption is satisfied. A similar generalization is possible here, but is omitted for the simplicity of exposition.

price cutting (sales figures of related products, the outcomes of marketing research, etc.).

The following lemma provides the key implications of (A2) and (A3).

**Lemma 1.** Under (A2) and (A3), for each pair $i \neq j$, $\lambda_i \neq 0$, $\lambda_j \neq 0$, and $a \in Ex(A)$, we can construct payment schemes $x_i$, $x_j$ such that

(1) $\lambda_i x_i(\omega_{-ij}) + \lambda_j x_j(\omega_{-ij}) = 0 \quad \forall \omega_{-ij}$,

(2) $E[x_h|a] - E[x_h|a_{-h}, a'_h] \geq g_h(a_{-h}, a'_h) - g_h(a)$, $\forall a'_h$, $h=i,j$ and

(3) $E[x_h|a] - E[x_h|a_{-h}, a'_h] \geq 0 \; \forall a'_h$, $h=i,j$.

This is basically the separating hyperplane theorem, and the proof is left in the appendix. Let us now briefly explain the meaning of the above lemma and how to go about proving the folk theorem (Theorem 1 below)[8]. In the repeated game equilibrium, what ij-opponents say ($m_{-ij}=\omega_{-ij}$) determines the future payoffs of players i and j. That future payoff function is constructed from the transfer rules described in Lemma 1. Lemma 1 states that we can find a transfer rule for each pair of players ($x_i$ and $x_j$) which provides the right incentives (condition (2)) and entails no social welfare loss (condition (1)). Once we define such a transfer rule for each pair, we will 'patch together' all those payment schemes. Note that for each player i, (n-1) payment schemes $x_i^1, \ldots, x_i^{n-1}$ are defined (as there are n-1 pairs which player i belongs to). What we will do is to take the summation of them $X_i = x_i^1 + \cdots + x_i^{n-1}$. This patchwork generates payment schemes $(X_1, \ldots, X_n)$ which satisfy;

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[8] What follows is the intuitive explanation for the case where each element of vector $\lambda$ is strictly positive. Other cases are treated similarly. See the proof of Theorem 1 for the details.

(i) $\lambda_1 X_1 + \cdots + \lambda_n X_n \equiv 0$ (no welfare loss),

(ii) $E[X_h | a] - E[X_h | a_{-h}, a'_h] \geq g_h(a_{-h}, a'_h) - g_h(a)$, $\forall a'_h$ $\forall h$

(providing correct incentives; Lemma 1 (3) is used here.), and

(iii) $X_h$ is independent of $m_h$ (so that each player h has a (weak) incentive to tell the truth).

Those payment schemes correspond to the equilibrium continuation payoffs in the repeated game. This way, we can construct an efficient equilibrium, which is described in the last part of the previous section. This is a rough sketch of the folk theorem stated below (Theorem 1). The formal proof of Theorem 1 is left in the appendix.

Before stating the folk theorem, let us introduce some notation. Let $v^*_i = g_i(\mu^i)$ be the minimax value for player i and define the feasible and individually rational payoff set by

$$V^* = \{v \in co(g(A)) \mid v \geq v^*\}.$$

Now we are ready to present a folk theorem result.

**Theorem 1.** Suppose that there are more than two players and the information structure satisfies conditions (A1), (A2) and (A3). Also suppose that the dimension of $V^*$ is equal to the number of players. Then, any interior point in $V^*$ can be achieved as a sequential equilibrium average payoff profile, if the discount factor $\delta$ is close enough to 1.

Conditions (A2) and (A3) in the above theorem are probably the weakest conditions on the information structure which are imposed *uniformly for all* $a \in Ex(A)$ (the same conditions should apply for all $a \in Ex(A)$) which generate a folk theorem. We can certainly weaken the conditions by imposing (non-uniform) assumptions both on the information structure and the payoffs functions. In what follows, we will explore such a possibility, and derive a folk theorem from weaker assumptions. The readers who are not interested in weakening the informational assumptions may skip the rest of this section and can go directly to Section 5.

Let us define $\alpha(\lambda)$ to be equal to $\mu^i$ (the minimax point for player i) if $\lambda_i < 0$ and $\lambda_j = 0$ (for $j \neq i$), and otherwise let $\alpha(\lambda) = a(\lambda) \in \operatorname{argmax}_{a \in A} \lambda g(a)$. We will provide necessary and sufficient conditions for $\alpha(\lambda)$ to be supported efficiently in the static contract problem in Section 3 (Problem $(T,\lambda)$ for T=1). That is, there exists $x^1$ such that

$$\Sigma_{i \in N} \lambda_i x_i^1(\omega) = 0 \text{ for all } \omega \in \Omega \tag{2}$$

and for every i, every $a_i \in A_i$, and every function $h_i : \Omega_i \to \Omega_i$,

$$\begin{aligned} &g_i(\alpha(\lambda)) + \sum_{\omega \in \Omega} x_i^1(\omega) p(\omega | \alpha(\lambda)) \\ \geq\ &g_i(\alpha_{-i}(\lambda), a_i) + \sum_{\omega \in \Omega} x_i^1(\omega_{-i}, h_i(\omega_i)) p(\omega | \alpha_{-i}(\lambda), a_i). \end{aligned} \tag{3}$$

If these conditions hold for all $\lambda \neq 0$, the result in Section 3 shows that any individually rational feasible outcome can be approximated by sequential equilibria.

Let $y_i$ be a "mixed message rule", which assigns a lottery over $\Omega_i$ to each element of $A_i \times \Omega_i$. Define

$$p^i(\omega | \alpha_{-i}, \alpha_i', y_i) = \sum_{a \in A, \omega_i' \in \Omega_i} p(\omega_{-i}, \omega_i' | a) \alpha_{-i}(a_{-i}) \alpha_i'(a_i) y_i(\omega_i | a_i, \omega_i').$$

The interpretation is that $p^i(\omega | \alpha_{-i}, \alpha_i', y_i)$ is the probability that $m = \omega$ is announced when all players except player i choose $\alpha_{-i}$ and announce honestly, whereas player i chooses according to $\alpha_i'$ and announces according to $y_i$. Now we can prove the following:

**Lemma 2.** Necessary and sufficient conditions for the existence of a sidepayment scheme $x^1$ satisfying (2) and (3) are:

(1) If $p^i(\omega|\alpha_{-i}(\lambda),\alpha_i,y_i) = p(\omega|\alpha(\lambda))$, then $g_i(\alpha(\lambda)) \geq g_i(\alpha_{-i}(\lambda),\alpha_i)$, and

(2) Let $N^* = \{i \in N|\ \lambda_i \neq 0\}$. If there exist $\Delta p$ and $\alpha$ such that

$$p^i(\omega|\alpha_{-i}(\lambda),\alpha_i,y_i) - p(\omega|\alpha(\lambda)) = (\text{sign } \lambda_i)\Delta p \text{ for all } i \in N^*,$$

then it must be the case that

$$\sum_{i \in N^*} |\lambda_i| [g_i(\alpha(\lambda)) - g_i(\alpha_{-i}(\lambda),\alpha_i)] \geq 0.$$

Condition (1) says that perfectly undetectable deviation should not pay. Condition (2) covers the case where punishment is most difficult. Suppose, for example, all welfare weights are nonnegative. Then the equality in condition (2) says that all players with positive welfare weights can change the distribution of announced signals in the same direction $\Delta p$. In this case, there is no way to tell who is deviating. However, the inequality in condition (2) shows that some players would not like to create such a change. Hence, we can transfer sidepayments from other players to such players, whenever $\Delta p$ is detected. The inequality in condition (2) guarantees that just enough transfer can be made to satisfy the budget balancing condition (2), while maintaining incentive constraints (3). A similar interpretation applies to other cases. The conditions we employed before, (A2) and (A3) assure that the situation covered by condition (2) cannot arise[9].

The lemma comes from Ky Fan's theorem in linear algebra, which shows that a system of linear inequalities $Px \geq d$ has a solution x if and only if "$\beta P=0$ and $\beta \geq 0$" implies $\beta d \leq 0$. The detailed proof of Lemma 2 can be found in the appendix. This immediately shows:

**Theorem 2.** Under the conditions in Lemma 2, the folk theorem holds,

[9] (A2) implies that such a $\Delta p$ does not exist when the welfare weights $\lambda_i$, $i \in N^*$, have different signs, while (A3) shows the same when all of them have the same sign.

if the dimension of $V^*$ is equal to the number of players.

### 4.2 Strict Incentives for Truth-Telling

So far, we have only required the weak sense of truthful revelation. Theorem 1 constructs equilibria where each player's message does not affect her future payoffs. Theorem 2 admits the possibility of strict incentives for truth-telling, but this possibility is not explicitly explored in the theorem. We will show below that when private signals are mutually correlated, we can construct a transfer rule such that all players have strict incentives to tell the truth in every period, by adding rather mild informational assumption. For simplicity, we first consider minimax points defined with respect to pure strategies. Let $b^i \in A$ be the minimax point for player i with respect to pure strategies, and let $v^{**}_i = g_i(b^i)$. Accordingly, define the feasible and individually rational payoff set by

$$V^{**} = \{v \in co(g(A)) \mid v \geq v^{**}\}.$$

For $\lambda$ such that $\lambda_i < 0$ for some i and $\lambda_j=0$ for all $j\neq i$, let $a(\lambda) = b^i$, and otherwise $a(\lambda) \in \arg\max_{a\in A} g_i(a)$ as before. We will employ the following condition to provide strict incentives for truth-telling:

**(A4)** For every $a(\lambda)$, $\lambda\neq 0$, every i, every $\omega_i \in \Omega_i$ and every $\omega_i' \in \Omega_i$,

$$\tilde{p}_{-i}(\omega_{-i}|a(\lambda),\omega_i) \neq \tilde{p}_{-i}(\omega_{-i}|a(\lambda),\omega_i') \quad \text{for some } \omega_{-i} \in \Omega_{-i},$$

where $\tilde{p}_{-i}(\omega_{-i}|a(\lambda),\omega_i) = \dfrac{p(\omega|a(\lambda))}{p_i(\omega_i|a(\lambda))}$.

This assumption says that the private signals are correlated, and each realization of a player's private signal induces a different conditional distribution of the opponents' signals.

We define $f_i:\Omega \to \mathbb{R}$ by

$$f_i(\omega) = 2\tilde{p}_{-i}(\omega_{-i}|a(\lambda),\omega_i) - \sum_{\omega'_{-i}\in\Omega_{-i}} \tilde{p}_{-i}(\omega'_{-i}|a(\lambda),\omega_i)^2.$$

To understand this construction, consider the maximization problem of

$$\sum_{\omega_{-i}\in\Omega_{-i}} \{2q(\omega_{-i}) - \sum_{\omega'_{-i}\in\Omega_{-i}} q(\omega'_{-i})^2\} \tilde{p}_{-i}(\omega_{-i}|a(\lambda),\omega_i)$$

with respect to $q:\Omega_{-i} \to \mathbb{R}$. The first order conditions say

$$2\tilde{p}_{-i}(\omega_{-i}|a(\lambda),\omega_i) - 2q(\omega_{-i}) = 0 \quad \text{for all} \quad \omega_{-i} \in \Omega_{-i}.$$

Since the second order conditions are satisfied (i.e., $-2 < 0$), the solution is that $q(\omega_{-i}) = \tilde{p}_{-i}(\omega_{-i}|a(\lambda),\omega_i)$ for all $\omega \in \Omega$. This implies that

$$\sum_{\omega_{-i}\in\Omega_{-i}} f_i(\omega)\tilde{p}_{-i}(\omega_{-i}|a(\lambda),\omega_i) > \sum_{\omega_{-i}\in\Omega_{-i}} f_i(\omega_{-i},\omega'_i)\tilde{p}_{-i}(\omega_{-i}|a(\lambda),\omega_i)$$

for all $\omega_i' \neq \omega_i$. Given any $\lambda$ and $x^1$ satisfying the sufficient condition for the folk theorem, we define another transfer rule $\tilde{x}^1$ by

$$\tilde{x}_i^1(\omega) = x_i^1(\omega) + \rho_i\{f_i(\omega) + B_i\},$$

where $B_i$ is a positive real number such that

$$f_i(\omega) + B_i \geq 0 \quad \text{for all} \quad \omega \in \Omega,$$

and $\rho_i$ is a real number close, but not equal, to zero such that $\rho_i\lambda_i \leq 0$. With this construction, $\sum_{i\in N} \lambda_i\tilde{x}_i^1(\omega)$ is less than, or equal to, zero, and is as close to zero as possible as $\rho_i \to 0$ for each i.

Now, suppose each player has strict incentive not to deviate at each point $a(\lambda)$, $\lambda\neq 0$, under the original sidepayment scheme $x^1$. Then, it is obvious that each player would like (1) to conform to $a(\lambda)$ (because $a_i(\lambda)$ is still the unique best reply, provided that $\rho_i$ is small enough for each i) and (2) to tell the truth, under the modified scheme $\tilde{x}^1$. Thus we have:

**Theorem 3.** Under the conditions (A2") for $a=b^j$ (for any j), (A2), and (A3), any payoff profile $v \in V^{**}$ can be approximately achieved as a sequential equilibrium where the players have strict incentives to tell the truth, as $\delta \to 1$.

**(Proof).** The proof of Lemma 1 actually shows that strict incentives for actions can be provided by some scheme $x^1$ under the above conditions. (For $\lambda_i \neq 0$ and $\lambda_j = 0$ $j \neq i$, where player i is taking a one-shot best response, we could choose $x^1_i \approx 0$ so that player i has a unique best action.) Then, the theorem follows as is explained above. ∎

**Remark:** Supporting mixed action by truth telling is a little more complicated, but possible. First, note that the distribution of the opponents' signals depends not only on a player's realized signal, but also on his realized action. So we must require that each player report both his signal and action, and we can similarly define $f_i(\omega, a_i)$ to support mixed action $\alpha$. Secondly, we must make sure that the players are exactly indifferent among the actions in the support of the mixed actions, and that other actions are inferior. This can be done when the conditions in Lemma 2 are satisfied with $g_i(\alpha)$ replaced with $g'_i(\alpha)=g_i(\alpha)+\rho_i(E[f_i(\omega,a_i)|\alpha_i]+B_i)$, where $E[\cdot|\alpha_i]$ is the expectation under mixed action profile $\alpha_i$. In particular, supporting a mixed strategy minimax point with strict incentives for truth telling is possible, when the collection of vectors $p_{-j}(\mu^i_{-j}, a_j)$, $a_j \in A_j$ are linearly independent (the individual full rank condition of Fudenberg, Levine and Maskin). This is because we can always define $x^1$ to make all actions exactly indifferent with respect to the modified stage payoff g' defined above. Thus we have:

**Theorem 4.** Suppose the collection of vectors $p_{-j}(\mu^i_{-j}, a_j)$, $a_j \in A_j$ are linearly independent for all i and j, and assume (A2) and (A3) are satisfied. Then, any payoff profile $v \in V^*$ can be approximately achieved as a sequential equilibrium where the players have strict incentives to tell the truth, as $\delta \to 1$.

## 5. Independent Private Signals and Two-Player Case

The basic idea in the previous section is to use each player's message to *transfer* payoffs among *others*. This induces truthful revelation of private signals and efficient enforcement of actions. Obviously, the same procedure cannot be employed if there are only two players. Accordingly, we will explore the possibility of efficiency by delaying the release of information. This requires independence of private signals

$$p(\omega|a) = \prod_{i \in N} p_i(\omega_i|a),$$

and a certain restrictions on the payoff structure, which will be discussed in Section 5.3. We begin with a simple example of a noisy prisoners' dilemma.

### 5.1 PRISONERS' DILEMMA

We specify the component game G in the following way: $n = 2$, $A_i = \{c,d\}$ $\Omega_i = \{0,1\}$, and the expected[10] stage payoffs are given by the following table.

[10] It is easy to check that we can specify realized payoff functions $u_i(a_i,\omega_i)$ to generate the expected payoffs in the table.

| | c | d |
|---|---|---|
| c | 1, 1 | $-L_1$, $1+H_2$ |
| d | $1+H_1$, $-L_2$ | 0, 0 |

where $H_i > 0$ and $L_i > 0$ and players 1 and 2 are the row and column players respectively. This is the prisoners' dilemma, where (d,d) is the unique Nash equilibrium but is inefficient. We also specify the message spaces as

$$M_i = \{\text{Pass, Fail}\}$$

and assume that the signals $(\omega_1, \omega_2)$ are conditionally independent and satisfy

$$p_1(1|d,d) \geq p_1(1|c,d) > p_1(1|c,c) > 0, \text{ and}$$

$$p_2(1|d,d) \geq p_2(1|d,c) > p_2(1|c,c) > 0.$$

Here, signal 1 is a bad sign which indicates that the opponent is cheating.

An interpretation of this model is *the exchange of commodities with uncertain qualities*. Player 1's effort level $a_i = c,d$ determines the quality $\omega_2$ of the good he provides to player 2. Here, $\omega_2 = 1$ means that the quality is low, and $\omega_2 = 0$ means high quality. In this interpretation, the quality of player 1's commodity may only be affected by his effort, so we have $P_2(\omega_2|a) = P_2(\omega_2|a_1)$. The symmetric explanation applies to player 2's commodity. In this setting it is natural to assume random shocks to the quality of two commodities are independent, so our assumption of conditional independence is expected to hold.

Let $\xi_1$ and $\xi_2$ be positive but small real numbers, where

$$0 < \xi_1 < p_1(1|c,d) - p_1(1|c,c), \text{ and}$$

$$0 < \xi_2 < p_2(1|d,c) - p_2(1|c,c).$$

We specify $x^T$ in the following way: For $i \neq j$, we set $x_i^T$ is independent of $m_i^T$ and $m_j^{T-1}$ so that $x_i^T = x_i^T(m_j(T))$, with

$$x_i^T(\text{Pass}) = 0, \text{ and}$$

$$x_i^T(\text{Fail}) = -(H_i + \epsilon_i),$$

where $\epsilon_i$, $i=1,2$ are arbitrary positive but small real numbers. We will show below that by choosing T large enough, there exists a sequential equilibrium $\hat{s}^T$ in the finitely repeated game with sidepayments $(G^T, x^T)$ such that for every $i = 1,2$, $v_i^T(\hat{s}^T, x^T)$ is close to $1 = g_i(c,c)$.

Considering the following message strategies:

$$\hat{\eta}_i^T(t) \equiv \text{Pass} \quad \text{for } t \neq T,$$

$$\hat{\eta}_i^T(T) = \begin{cases} \text{Fail} & \text{if } \frac{1}{T}\sum_{t=1}^{T} \omega_i(t) > p_i(1|c,c) + \xi_i \\ \text{Pass} & \text{otherwise.} \end{cases}$$

The first line says that the players' messages in $t = 1,2,\ldots,T-1$ contains no information. Each player i waits until the end of period T and count the total number of bad outcomes. If the frequency of bad outcomes, $\frac{1}{T}\sum_{t-1}^{T} \omega_i(t)$ is below the "threshold" $p_i(1|c,c)+\xi_i$, he announces "Pass", which means his opponent passes the statistical test. Otherwise, he says "Fail".

Next we specify the action plan $\hat{\sigma}_i^T$. We assume that $\hat{\sigma}_i^T$ plays c on the equilibrium path for each $t=1,\ldots,T$. Given the above message rule and the independence of signals, each player accumulates no information during the T periods. So player i's problem effectively reduces to the simultaneous (static) choice of action sequence $(a_i(1),...,a_i(T))$. We will show that $a_i^T = (c,...,c)$ is better than any other action sequence[11].

According to the above strategy profile $\hat{s}^T = (\hat{\sigma}^T, \hat{\eta}^T)$, both players continue to choose "c". It is clear from the law of large numbers that the probability of $\frac{1}{T}\sum_{t=1}^{T}\omega_i(t)$ being around $p_i(1|c,c)$ is close to unity. This implies that given that T is large enough, each player almost surely passes the test and enjoys no penalty, and therefore, $v_i^T(S^T,x^T)$ is approximated by $1 = g_i(c,c)$.

The choice of the message function $\eta_i^T$ is irrelevant to player i's payoff, because $g_i$, $x_i^T$, $\hat{\sigma}_i^T$ and $\hat{s}_j^T$ are independent of $m_i^T$. This implies that announcing messages according to $\hat{\eta}_i^T$ is one of the best responses for player i. From the assumption of independent private signals and the fact that the opponent is sending uninformative messages for $t < T$, player 1 expects in every period t with any history $(a_1^{t-1}, \omega_1^{t-1}, m^{t-1})$ that player 2 has observed $\omega_2^{t-1}$ with probability

$$\prod_{\tau=1}^{t-1} p_2(\omega_2(\tau)\,|\,a_1(\tau),c),$$

which is independent of $(\omega_1^{t-1}, m^{t-1})$. This property also holds for player 2.

Let $f_i^T(h)$ be the probability that player i is penalized when player j continues to choose "c", but player i chooses "c" T-h times and "d" h times. Strategy $\hat{s}_1^T$ is player 1's best response to $\hat{s}_2^T$ if for every h = 1,...,T,

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[11] Formally, let $\hat{\sigma}_i^T(t) = \hat{\sigma}_i^T(t)(a_i^{t-1})$ and let $\hat{\sigma}_i(t) = c$ if $a_i(\tau) = c$ for any $\tau < t$. Otherwise, let $\hat{\sigma}_i(t)$ be the best response given $a_i^{t-1}$, other player's action sequence $a_j^T \equiv (c,...,c)$, message rule $\hat{\eta}_j^T$, and the transfer rule $x_i^T$.

$$1 + x_i^T(\text{Fail}) f_1^T(0) \geq \frac{T-h}{T} + \frac{h}{T}(1+H_1) + x_1^T(\text{Fail}) f_1^T(h),$$

that is,

$$(H_1 + \epsilon_1)(f_1^T(h) - f_1^T(0)) \geq hH_i/T. \tag{4}$$

A similar property also holds for player 2. Matsushima (1994) proved in Theorem in Section 4.2 that by choosing T large enough the above inequalities always hold. This implies in our context that $\hat{s}$ is a sequential equilibrium in $(G^T, x^T)$. The appendix provides a brief explanation of the logical core for the reader's convenience.

Based on the above results, we can derive the folk theorem based as follows. For $\lambda_i \geq 0$, the above "statistical test" ensures that we can make player i choose the desirable action with negligible welfare loss. For $\lambda_i < 0$, we must use "reward" rather than punishment, to satisfy the "budget constraint" $\lambda x^T(m(T)) \leq 0$. Namely, we set $x_i^T(\text{Pass}) = d_i + \epsilon_i$ and $x_i^T(\text{Fail}) = 0$, where $d_i$ is the gain from deviation from the desired action (actually $d_i = L_i$ is the only relevant case). Hence, in the contract problem in Section 3, *any player i with negative weight contributes a "welfare loss" equal to her gain from deviation $d_i$, whereas any player i with positive weight contributes no welfare loss*. From this observation, it is easy to check that the intersection of the half spaces $Q = \cap_{\lambda \neq 0} D(\lambda)$, is equal to the set of feasible and individually rational payoffs (see Figure 4).

***** Figure 4 here *****

Since the set of all individually rational feasible outcomes is full-dimensional, we have the folk theorem for the prisoners' dilemma game with privately observable signals:

**Proposition 1.** Consider the prisoners' dilemma game with privately observable *independent* signals. Any feasible and individually rational payoff profile in can be approximated by sequential equilibria, as $\delta \to 1$.

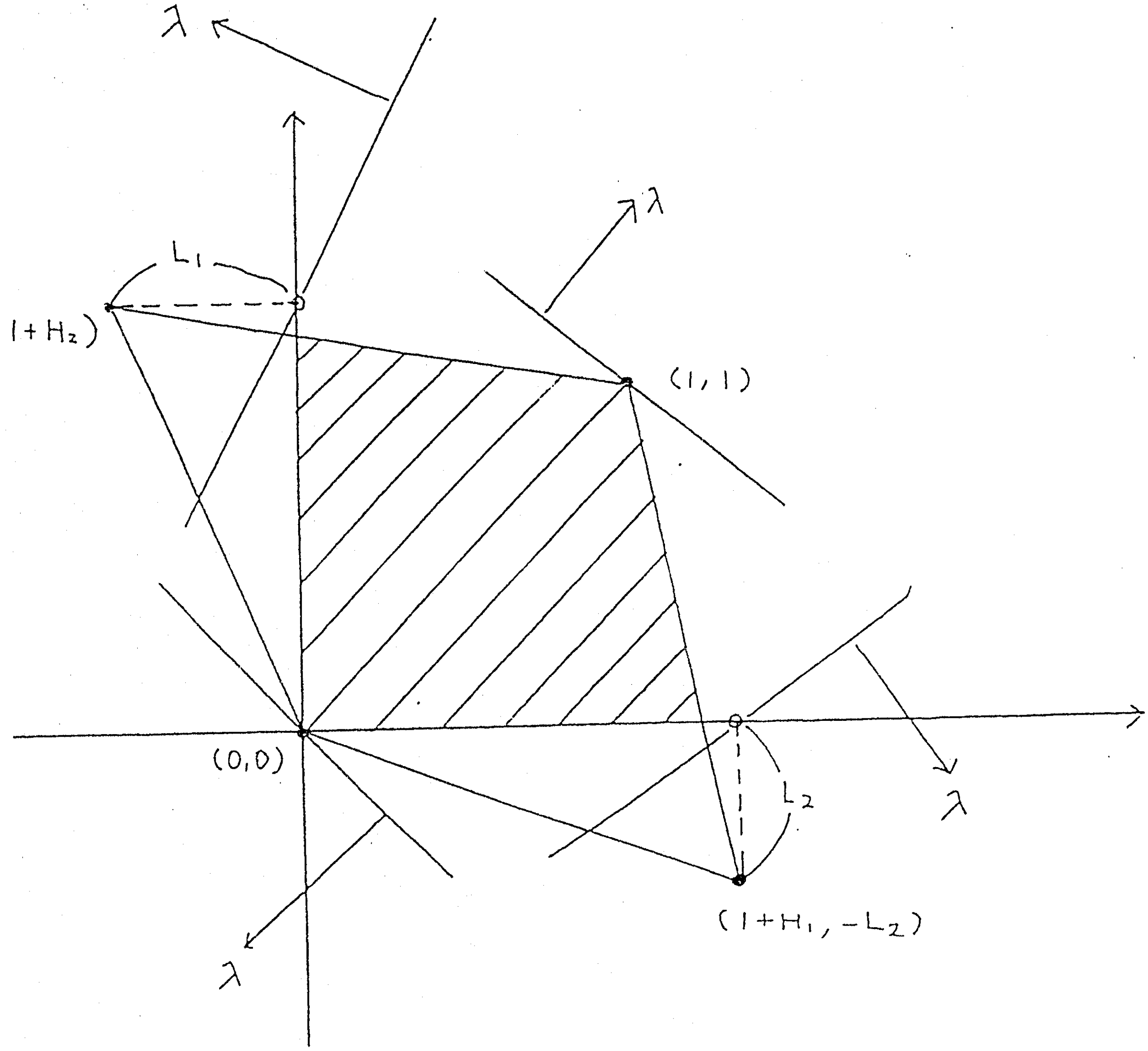


Figure 4.

In what follows, we will generalize this observation.

## 5.2 DISCUSSION

It is our essential device to consider a finitely repeated game $G^T$ as the component game instead of the original G. According to Proposition 3 in Matsushima (1990), we know that in the general two player case Q(1) does not contain any efficient outcome, and therefore, it is impossible to derive the folk theorem by investigating Q(1) only.

Our approach is related to the delay of information release originated by Abreu, Milgrom and Pearce (1991). There are a couple of important differences, and most importantly their method is not directly applicable in the present context. First, the nature of the "statistical tests" are different: They did not use the idea of relative frequency of getting signal "1" (the bad outcome). They instead considered as the punishment region the event that signal "1" was observed *in every period*. Clearly this event rarely occurs even if a player deviates globally. This makes it necessary that the realized punishment $x^1_i$(Pass) - $x^1_i$(Fail) tends to infinity as $T\to\infty$ in their equilibrium, to maintain the magnitude of expected punishment at the required level.

This does not cause any problem in their model with publicly observable signals, but a difficulty arises in our model with private signals for the following reason. First, to induce truth telling about the private signals, we need to make the variations of each player's expected sidepayment independent of what he says (under the assumption of conditional independence). The variations of sidepayments are illustrated by the arrows in Figure 5(a), and such payoff variations require that the set of equilibria should be of full dimension. In contrast, in their model, all players can be simultaneously punished by the reversion to the one-shot Nash equilibrium (0,0), because there is no need for truthful revelation of private information. Hence it suffices to show, in their model, that the one dimensional line segment between (0,0) and (1,1) is (asymptotically) self-generating (see Figure 5(b)).

***** Figure 5 here *****

Secondly, to check the full dimensionality of the equilibria, we must consider the sustainability of (c,d) and a welfare weight $\lambda$ such that $\lambda_1 < 0$ and $\lambda_2 > 0$. The "budget" condition (B) in the contract problem $(T,\lambda)$ requires that $x^T{}_1(\text{Fail})=0$ and $x^T{}_1(\text{Pass})>0$. As we explained above, in their construction $x^T{}_1(\text{Pass})$ must be enhanced as $T\to\infty$, and so must the welfare loss $-\lambda_1 x_1^T(0)\{1 - p_2(1|c,d)^T\}$. This is in contrast to the case of positive welfare weight, where the welfare loss = (probability of punishment)×(realized punishment) vanishes because the probability of punishment tends to zero. In order to minimize the welfare loss, we have to choose T = 1, which make the loss equal to

$$-\lambda_1 x_1^1(\text{Pass})\,p(0|c,d) = \frac{-\lambda_1 L_1 p_2(0|c,d)}{p_2(0|c,d) - p_2(0|d,d)}.$$

Because this value is larger than $-\lambda_1 L_1$, only a strict subset of $V^*$ (including (1,1)) can potentially be sustained by the methods of Abreu, Milgrom and Pearce. However, if $L_1$ is large, this value may be more than $-\lambda_1(1+L_1)$. The condition for this is

$$M_1 p_2(0|d,d) > p_2(0|c,d) - p_2(0|d,d),$$

and this implies that the intersection of the half spaces is empty. In such a case, not only the efficient outcome (1,1), but also any outcome except the one-shot equilibrium outcome (0,0), can not be supported by sequential equilibria, if we were to adopt their method[12].

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[12] Compte (1994a) provides an alternative method, a modification of Abreu, Milgrom and Pearce's construction, to cope with this problem.

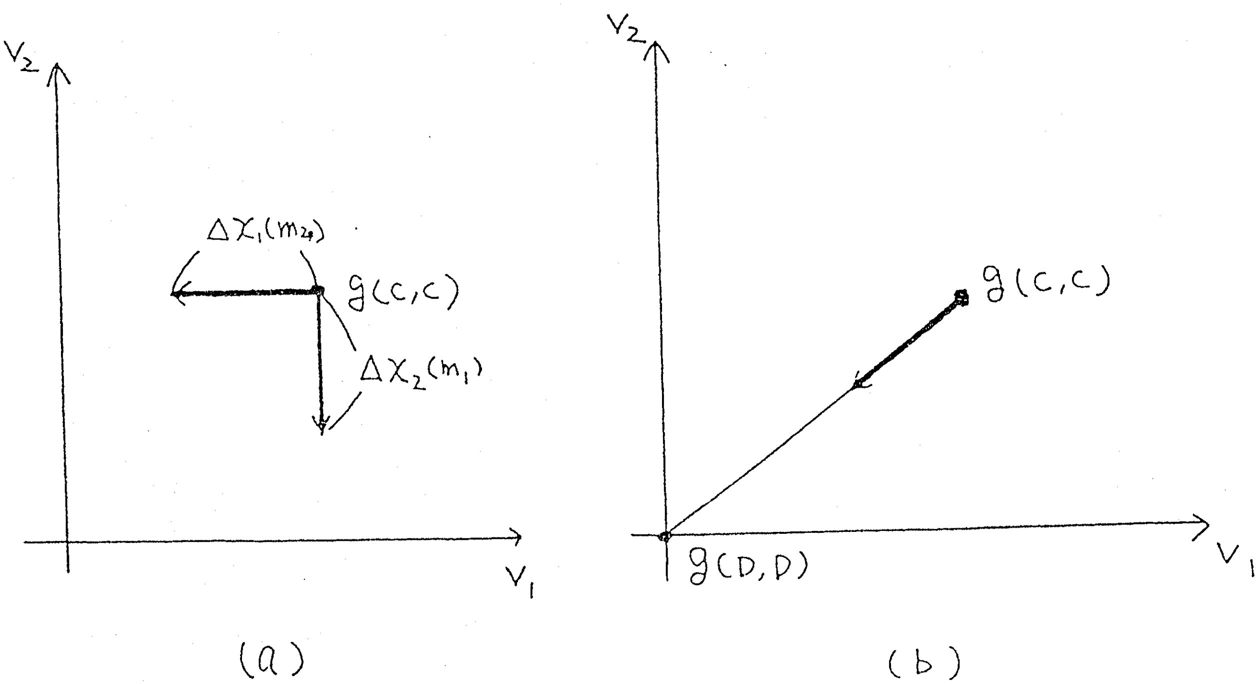


Figure 5.

Note also that the delay of information release in their model is exogenously imposed, while it endogenously arises within our model, where the players can choose to (seriously) communicate every once in a while.

### 5.3 EXTENSION

As we have pointed out in Subsection 5.1, any player i with negative weight contributes a welfare loss equal to her gain from deviation $d_i$, whereas any player i with positive weight entails no welfare loss. This basic idea can be extended to the more general class with independent private signals. Although delaying the release of information is most useful in the two-player case, we will provide a general result for n-player case. As there is no nested relationship between the assumptions here and those in Section 3, this generalization to the n-player case is useful.

First, define a subset A' of A as follows: an action profile $a \in A$ is an element of A' if and only if for every $i \in N$, either

$$g_i(a) \geq g_i(a_{-i}, a_i') \text{ for all } a_i' \in A_i, \text{ or}$$

$$p_{-i}(a) \neq p_{-i}(a_{-i}, \alpha_i) \text{ for any mixed action } \alpha_i \neq a_i.$$

For every subset N' of N and for every $a \in A'$, we define $v(N', a) \in \mathbb{R}^n$ by

$$v_i(N', a) = g_i(a) \text{ for all } i \notin N', \text{ and}$$

$$v_i(N', a) = \max_{a_i' \in A_i} g_i(a_{-i}, a_i') \quad \text{for all } i \in N'.$$

Define subset V(N') of $\mathbb{R}^n$ by the convex hull of $\{v(N', a) | a \in A'\}$, and define

$$V' = \bigcap_{N' \subset N} V(N') .$$

By a similar arguments in Section 5.1, one can show the following:

**Theorem 5.** Consider an n-player game ($n \geq 2$) and suppose the dimension of V' is equal to n. If players' signals $(\omega_1, \ldots, \omega_n)$ are independent given any action profile a, any outcome in V' can be approximated by a sequential equilibrium of $\Gamma(G,\delta)$, if $\delta$ is close enough to 1.

**Proof.** See the appendix.

If we impose some assumptions on the payoff function, the equilibrium set V' in the above theorem coincides with the set of feasible and individually rational payoffs. In particular, for the two player case, the folk theorem for the Prisoner's dilemma in Section 5.1 can be generalized as follows. Recall that $b^i$ refers to the minimax profile for player i with respect to pure strategies, and $V^{**}$ is the set of feasible and individually rational payoffs with respect to $b^i$.

**Corollary.** Assume that there are two players, $\omega_1$ and $\omega_2$ are independent given any pure action profile, and A'=A. If there is $a^0{}_i \in A_i$, i=1,2 such that

$$g_1(b^2{}_1, a^0{}_2) > \max_{v \in V^{**}} v_1, \text{ and}$$

$$g_2(a^0{}_1, b^1{}_2) > \max_{v \in V^{**}} v_2,$$

then any outcome in $V^{**}$ can be approximately by a sequential equilibrium of $\Gamma(G,\delta)$ if $\delta$ is close enough to 1.

The proof of Corollary can also be found in the appendix. Figure 6 illustrates the nature of the assumptions in Corollary.

***** Figure 6 here *****

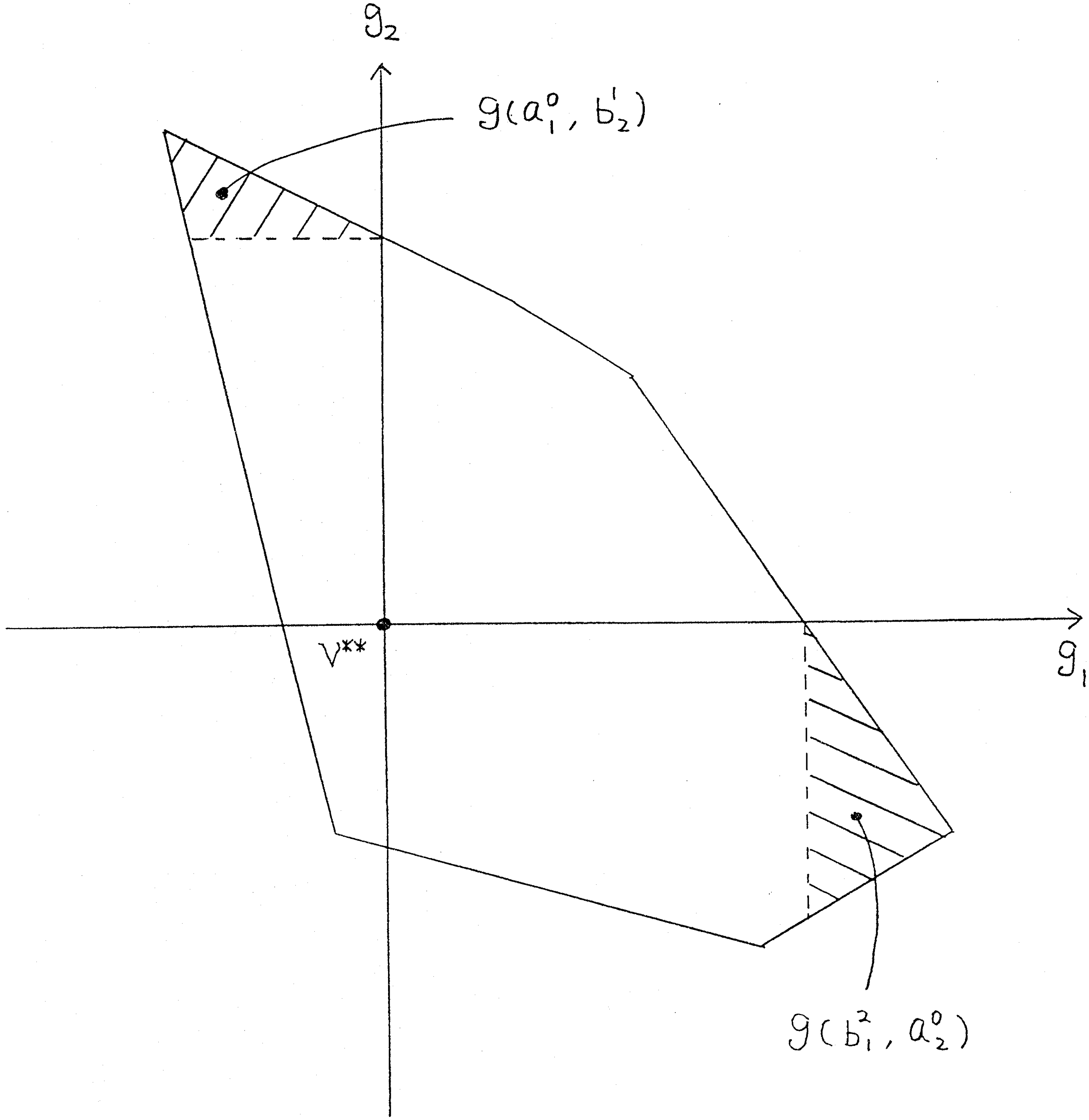


Figure 6.

They require that the Pareto frontier of $V^{**}$ is downward sloping, and there is a "repenting" action $a^0$: If player i chooses the repenting action $a^0_i$ while the opponent is minimaximizing him, the opponent enjoys a payoff which is higher than any of his payoffs in $V^{**}$. These conditions can be satisfied when players have some means to transfer income between them.

## 6. CONCLUDING REMARKS

We have shown in the present paper that communication is an important means to resolve possible confusion among players in the course of collusion during repeated play. Confusion may arise when each player observes a different set of signals about other player's past actions. This class of games, known as repeated games with imperfect monitoring and with *privately observed* signals, includes many important economic applications, such as secret price cutting and exchange of commodities with uncertain quality. The characterization of equilibria in this class of games has been an open question, because the games lack recursive structure and are hard to analyze. We introduced communication to generate publicly observable history, which recovers the recursive structure. We showed that we can construct equilibria in which the players' private information is voluntarily revealed and is utilized to enforce desirable actions.

One thing which we did *not* show is the *necessity* of communication for a folk theorem in this class of games. As we explained above, we do not know how the equilibrium set looks when there is no communication. In principle, there is a possibility that a folk theorem holds even without communication. When we regard a folk theorem as a theory of self-help or cooperation, this may not be a serious problem, as communication is readily available in many cases. On the other hand, if we regard it as a theory of cartel enforcement, it is very important to determine what is possible without communication. This is because certain kinds of communication are *per se* illegal in the antitrust law. If we could show that full collusion is impossible without communication, we would be able to provide a clear-cut theoretical basis for the antitrust law. Thus the characterization of the set of equilibrium without communication is a theoretically challenging and economically important open question[13].

[13] Kandori(1991) and Compte(1994b) consider the case without communication.

## Appendix

**(Proof of Lemma 1).** First, take the case where $\lambda_i$ and $\lambda_j$ have the same sign. By (A2') and (A3), there is a separating hyperplane defined by normal vector y such that (i) $qy < p_{-ij}(a)y$ for all $q \in Q_{ij}(a)$ and (ii) $qy > p_{-ij}(a)y$ for all $q \in Q_{ji}(a)$ (see Figure 3). For a positive number t, define $x_i = ty$ and $x_j = -(\lambda_i/\lambda_j)x_i$ to satisfy the budget balancing condition (1). By making the parameter t arbitrarily large, we can make $(p_{-ij}(a)-q_h)x_h$ an arbitrarily large positive number for h=i,j, $q_i \in Q_{ij}(a)$ and $q_j \in Q_{ji}(a)$. Since $(p_{-ij}(a)-q_h)x_h$ corresponds to the left hand sides of (2) and (3), these conditions are satisfied.

Secondly, suppose $\lambda_i$ and $\lambda_j$ have different signs. By assumption (A2) and the separating hyperplane theorem, there is y such that $qy < p_{-ij}(a)y$ for all $q \in Q_{ij}(a)$ and all $q \in Q_{ji}(a)$ (see Figure 2). The rest of the proof is exactly the same as above.■

**(Proof of Theorem 1).** We use the algorithm explained in Section 3 for T=1. That is, we assume that the players condition their actions on publicly exchanged messages only. At the end of each period, players communicate the signals they received ($M_i = \Omega_i$). In our construction, each player's message does not affect his continuation payoffs (i.e. $x_i = x_i(m_{-i})$), so that he is willing to tell the truth. To this end, we look at a collection of static contract problems for each welfare weights $\lambda \in \mathbb{R}^n\backslash\{0\}$. We examine different cases depending of the signs of $\lambda$.

**Case 1:** Player i is minimaximized ($\lambda_i < 0$, $\lambda_j = 0$ for $j \neq i$).

1) Supported action: $\mu^i$.

2) $x_i \equiv 0$ and player i takes one-shot best response $\mu^i_i$ to $\mu^i_{-i}$.

3) $x_j(\omega_{-j})$ provides player $j \neq i$ correct incentives to take (possibly mixed) strategy $m^i_j$. This is possible by (A1), and the proof is found in Lemma 2 below.

For other welfare weight vectors $\lambda \neq 0$, let $a(\lambda) \in \operatorname{argmax}_{a \in A} \lambda g(a)$.

**Case 2:** Player i is maximized ($\lambda_i > 0$, $\lambda_j = 0$ for $j \neq i$).

1) Supported action: $a(\lambda)$.

2) $x_i \equiv 0$ and player i takes one-shot best response $a_i(\lambda)$ to $a_{-i}(\lambda)$.

3) $x_j(\omega_{-j})$ makes $a_j(\lambda)$ a best response for player $j \neq i$. This is possible by (A2"). The formal proof is similar to Lemma 1 and therefore omitted.

**Case 3:** Otherwise (there are at least two players with non-zero welfare weight).

1) Supported action: $a(\lambda)$.

2) For any player i with zero welfare weight: $x_i(\omega_{-i})$ makes $a_i(\lambda)$ a best response. This is possible by (A2").

3) For *each* pair of players $i \neq j$ whose welfare weights are non-zero, construct a pair of incentive schemes as in Lemma 1.

4) If more than one payment schemes have been constructed for player i, let us finally define $x_i$ to be the *sum* of those schemes. Clearly, $x_i$ does not depend on $\omega_i$ so that telling the truth is a (weak) best response for player i, and $\Sigma_i \lambda_i x_i = 0$ for all $\omega$. Furthermore, the incentive constraints are maintained. If $x^1{}_i, \ldots, x^K{}_i$ are the payment schemes defined for player i in the above construction, each scheme satisfies the incentive constraints

$$E[x^k{}_i | a] - E[x^k{}_i | a_{-i}, a'_i] \geq g_i(a_{-i}, a'_i) - g_i(a) \; \forall a'_i. \tag{1}$$

Part (3) of Lemma 1 assures that the left hand side of (1) is always *nonnegative*. Hence the summation scheme $x_i = \sum_k x^k{}_i$ makes the left hand side even larger, preserving the incentive constraints:

$$E[\sum_k x^k{}_i | a] - E[\sum_k x^k{}_i | a_{-i}, a'_i] \geq g_i(a_{-i}, a'_i) - g_i(a) \; \forall a'_i.$$

Thus we conclude that aside from Case 1, the extremal payoffs in the direction of $\lambda$ can be achieved: $D(1,\lambda) = \{v \mid \lambda v \leq \lambda g(a(\lambda))\}$. For Case 1, where $\lambda$ has a negative element for i and zeros elsewhere, $D(1,\lambda) = \{v \mid v_i \geq v^*{}_i\}$. Therefore $\cap_{\lambda \neq 0} D(1,\lambda) = Q(1) = V^*$ and the full dimensionality of $V^*$ proves the theorem, by means of the characterization in Section 3.■

**(Proof of Lemma 2).** Let $H_i$ be the set of all functions $h_i:\Omega_i \to \Omega_i$, and define, for each $a_i$ and $h_i$

$$p^i(\omega|\alpha_i(\lambda), a_i, h_i) \equiv \sum_{\omega_i' \in h_i^{-1}(\omega_i)} p(\omega_{-i}, \omega_i'|\alpha_{-i}(\lambda), a_i).$$

This is the probability that $m=\omega$ is announced when all players except i play $\alpha_{-i}(\lambda)$ and announce honestly, while player i plays $a_i$ and announces according to $h_i$. Define row vectors $p(\alpha(\lambda)) = (p(\omega|\alpha(\lambda))_{\omega\in\Omega}$, $p(a_i,h_i) = (p^i(\omega|\alpha_{-i}(\lambda),a_i,h_i))_{\omega\in\Omega}$, and let

$$P_i = \begin{pmatrix} \vdots \\ p(\alpha(\lambda)) - p(a_i,h_i) \\ \vdots \end{pmatrix}_{\substack{a_i\in A_i \\ h_i\in H_i}}.$$

This matrix indicate how the distribution of announced signals changes when player i deviates. Let $g_i(a_i,h_i) = g_i(\alpha_{-i}(\lambda),a_i)$, and define the (column) vector of player i's gains from deviations by

$$d_i = \begin{pmatrix} \vdots \\ g_i(\alpha(\lambda)) - g_i(a_i,h_i) \\ \vdots \end{pmatrix}_{\substack{a_i\in A_i \\ h_i\in H_i}}.$$

Also define the column vector of sidepayments for player i by

$$x_i = \begin{pmatrix} \vdots \\ x_i^1(\omega) \\ \vdots \end{pmatrix}_{\omega\in\Omega}.$$

Without loss of generality, assume that $\lambda_i \neq 0$ for i=1,...,k and $\lambda_i = 0$ for i=k+1,...,n. Let I be the #Ω×#Ω identity matrix, and define

$$P = \begin{pmatrix} P_1 & & 0 \\ & \ddots & \\ 0 & & P_k \\ (\text{sign}\lambda_1) I & \cdots & (\text{sign}\lambda_k) I \\ -(\text{sign}\lambda_1) I & \cdots & -(\text{sign}\lambda_k) I \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} |\lambda_1| d_1 \\ \vdots \\ |\lambda_k| d_k \\ 0 \end{pmatrix},$$

where the "0" in the definition of d is a (2#Ω)×1 zero vector. Finally, let

$$x = \begin{pmatrix} |\lambda_1| x_1 \\ \vdots \\ |\lambda_k| x_k \end{pmatrix}.$$

Then, conditions (2) and (3) are equivalent to

$$Px \geq d, \text{ and} \tag{a1}$$

$$P_i x_i \geq d_i \text{ for } i>k, \tag{a2}$$

where "≥" means that weak inequality holds for each element of the vectors. Now we can apply a result in linear algebra, known as Ky Fan's Theorem. It asserts that $Px \geq d$ has a solution x if and only if

$$\text{Any } \beta \geq 0 \text{ with } \beta P = 0 \text{ must satisfy } \beta d \leq 0. \tag{a3}$$

It is easy to see that Ky Fan's theorem implies that (a2) is equivalent to condition (1) in Lemma 2. In this case, the vector β (a3) corresponds to (after a suitable normalization) mixed strategies of player i. For (a1), the vector β in (a3) can be interpreted as $\beta = (\beta_1, \ldots, \beta_k, \beta^+, \beta^-)$, where $\beta_i = (\cdots \beta_i(a_i, h_i) \cdots)_{a_i \in A_i, h_i \in H_i}$ corresponds to player i's mixed strategy over $a_i$ and $h_i$, and $\beta^+$ and $\beta^-$ are nonnegative

$\#\Omega\times 1$ vectors which satisfy $\Delta p = \beta^+ - \beta^-$ for $\Delta p$ in condition (2) of the present lemma (again, all of these assume suitable normalization). For such $\beta$, (a3) corresponds to condition (2) in the lemma. For $\beta = (0,\ldots,0,\beta_i,0,\ldots,0)$, (a3) corresponds to condition (1) in the lemma. It is easy to check that (a3) has no other implications.

**(Statistical test used in Section 5).** To get some intuition, we will separately investigate the following two cases:

**global deviation:** player 1 chooses "d" so many times, i.e., the number of deviation h is closest to T.

**local deviation:** player 1 chooses "d" only a few times, i.e., h is close to 1.

The extreme case of global deviation is that player 1 chooses "d" every time, i.e., H =T. The extreme case of local deviation is that player 1 chooses "d" in period T only, i.e., h = 1. We will investigate these extreme cases only. The basic ideas can be extended to the general case.

Consider the first case, where player 1 chooses "d" every time. It is clear from the law of large numbers that player 1 almost surely fails the test. Then, the left hand side of (4) is close to $H_1 + \epsilon_1$, while the right hand side is $H_i$. From these observations, player 1 has no incentive to deviate all times.

Next, consider the second case, where player 1 chooses "d" in period T only. Note that local deviation gives only a little change on the probability of passing the test. This may make it difficult to prevent local deviation. However, the gain from local deviation is almost negligible as compared to the amount of possible global penalty. This point will make it easy to prevent local deviation. We can check that the latter positive aspect can overcome the former negative aspect as follows. Define an integer $t^*$ by

$$\frac{t^*}{T} < p_2(1\,|\,c,c) + \mu_2 \le \frac{t^*+1}{T}.$$

This means that player 1 is one the verge of failing the test when player 2 has observed $t^*$ bad outcomes by the end of T-1.

If $\sum_{t=1}^{T-1} \omega_2(t)$ is less than $t^*$, then player 1 certainly fails the test regardless of the signal $\omega_2(T)$ in period T. Similarly, if $\sum_{t=1}^{T-1} \omega_2(t)$ is more than $t^*$, then player 2 certainly passes the test. Therefore, player 1's action in period T is relevant only if $\sum_{t=1}^{T-1} \omega_2(t)$ is equal to $t^*$. Let $\Phi$ denote the probability of this event, (the player 1 is just on the verge of punishment at the beginning of time T): $\phi = Pr(\sum_{t=1}^{T-1} \omega_2(t) = t^*)$. The above argument shows that by choosing $a_1(T)$ = d instead of "c", player 1 increases the probability of punishment by

$$\phi\{p_2(1|d,c) - p_2(1|c,c)\},$$

and the expected increase of player 1's penalty is

$$\phi\{p_2(1|d,c) - p_2(1|c,c)\}(H_1 + \epsilon_1).$$

On the other hand, player 1's gain from last period's deviation is $H_1/T$. Hence, all we have to show is that the product $T\Phi$ can be large enough to satisfy

$$H_1 \leq T\phi\{p_2(1|d,c) - p_2(1|c,c)\}(H_1 + \epsilon_1). \qquad \text{(b)}$$

As T tends to infinity, the probability of getting exactly $t^*$ bad outcomes in the T-1 periods ($\Phi$) tends to zero, so the question is how fast it vanishes compared to T. To get some intuition, consider a counter-factual case where $\frac{1}{T-1}\sum_{t=1}^{T-1} \omega_2(t) \equiv Z$ were uniformly distributed.

Then, $\Phi = 1/(T-1)$, so that $T\Phi \to 1$: probability $\Phi$ tends to zero sufficiently slowly even in the uniform distribution case. Actually, the distribution of Z is not uniform, but it is concentrated around $p_2(1|c,c)$ by the law of large numbers. This implies that, by taking the threshold sufficiently close to $P_2(1|c,c)$ as T tends to infinity, we can make sure that (1) $\Phi$ tends to zero sufficiently slower than $1/(T-1)$ so that $T\Phi \to \infty$ as $T \to \infty$, and (2) the probability of passing the test is sufficiently close to 1 ( see Figure 7).

***** Figure 7 here *****

Hence, we can satisfy the incentive constraint for the local deviation (b). Similar argument can be employed to show that any number of deviations do not pay. Hence we can prove that by choosing T large enough, $\hat{s}^T$ is a sequential equilibrium in $(G^T, x^T)$.

**(Proof of Theorem 5).** We will prove the theorem by the algorithm explained in Section 3. Namely, we will consider the optimal contract design problem for the T-period repeated game with sidepayments, called Problem $(T,\lambda)$. Fix $(T,\lambda)$ arbitrarily, and let $N'=\{i\in N|\lambda_i<0\}$ and define the message space by $M^i=\Omega_i^T$ for all i. The message sent by player i in the $t^{th}$ period is denoted $m_i(t)=(m_i(t,1),\ldots,m_i(t,T))$.

Now consider how an arbitrary point $a\in A'$ can be supported in the T-period repeated game with sidepayments. If player i is taking (one-shot) best response at this strategy profile, we set her sidepayment identically equal to zero:

$$x_i^T(m^T) = 0 \text{ for all } m^T \in M^T.$$

If player i is not taking a best response at profile $a \in A'$, the definition of A' implies that

$$p_{-i}(a) \neq p_{-i}(a_{-i}, \alpha_i) \text{ for any mixed action } \alpha_i \neq a_i.$$

Then, by the separating hyperplane theorem, there exist a positive real number $\xi_i > 0$ and a function $k_i : \Omega_{-i} \to \mathbb{R}$ such that

$$\sum_{\omega_{-i} \in \Omega_{-i}} k_i(\omega_{-i}) p_{-i}(\omega_{-i} | a) + 2\xi_i \leq \sum_{\omega_{-i} \in \Omega_{-i}} k_i(\omega_{-i}) p_{-i}(\omega_{-i} | a_{-i}, a'_i)$$

for all $a'_i \neq a_i$. In other words, given $a_{-i}$, the expected value of $k_i$ is uniquely minimized when player i adheres to $a_i$. We denote the minimized expected value by $k^*_i$. We will now construct the sidepayment rule for player i based on the empirical average of $k_i$. The basic idea is as follows. In the first T-1 periods, the players send uninformative fixed messages, which are irrelevant for the sidepayments. In the last period, the players truthfully reveal the entire history of their private signals; $m(T) = (\omega(1), \ldots, \omega(T))$. Then, the empirical average of $k_i$ is calculated by

$$\bar{k}_i(m^T) \equiv \frac{1}{T} \sum_{t=1}^{T} k_i(m_{-i}(T, t)) .$$

If this number falls below $k_i^* + \xi_i$, player i "passes the test". More precisely, with a positive real number $\epsilon_i > 0$, and we define $x_i^T$ as follows:

(Case 1: when $i \notin N'$)

$$x_i^T(m^T) = \begin{cases} 0 & \text{if } \bar{k}_i(m^T) \leq k_i^* + \xi_i \\ -\max_{a'_i \in A_i} g_i(a_{-i}, a'_i) - \varepsilon_i & \text{otherwise.} \end{cases}$$

(Case 2: when $i \in N'$)

$$x_i^T(m^T) = \begin{cases} \max_{a'_i \in A_i} g_i(a_{-i}, a'_i) + \varepsilon_i & \text{if } \bar{k}_i(m^T) \leq k_i^* + \xi_i \\ 0 & \text{otherwise.} \end{cases}$$

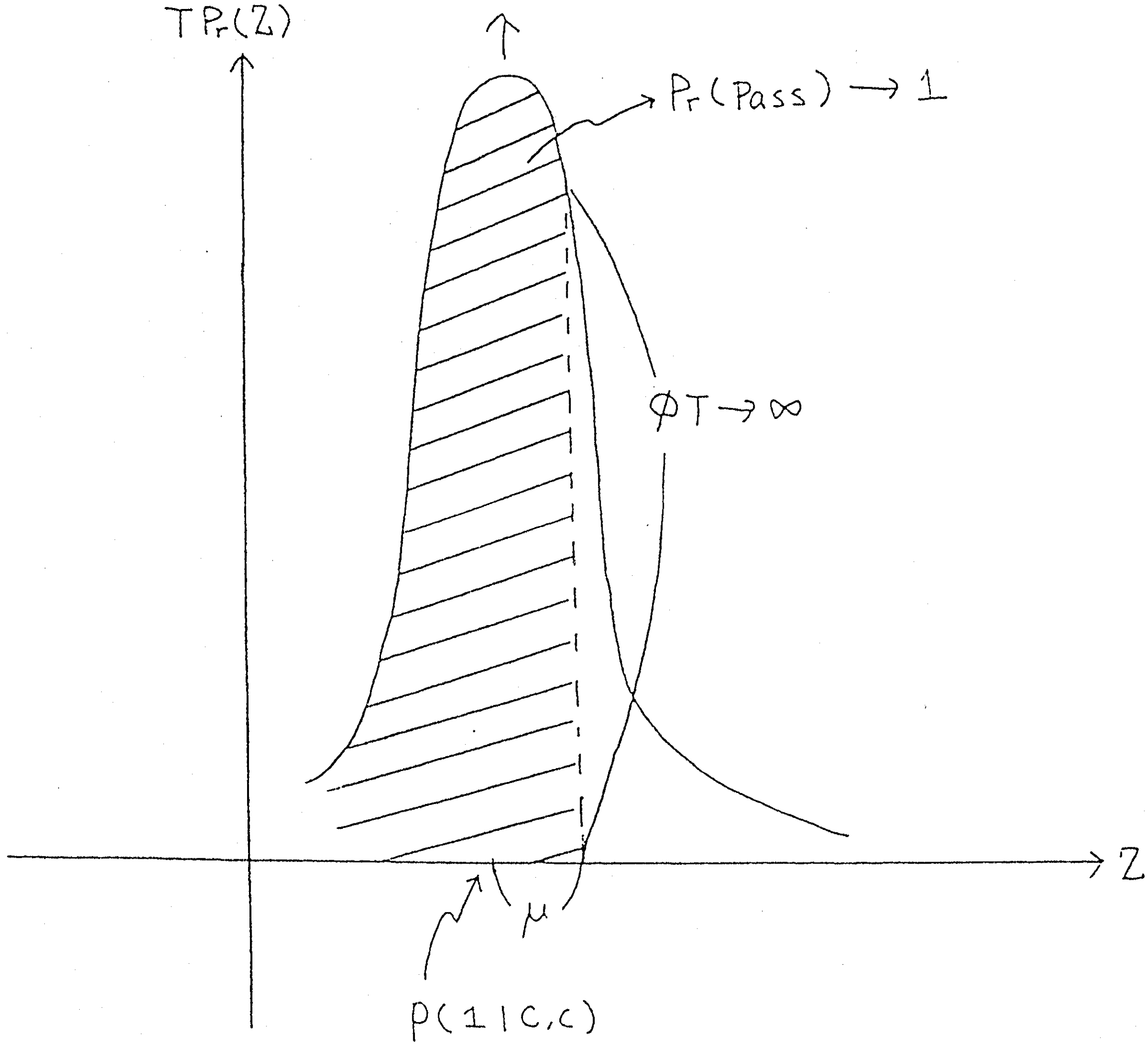


Figure 7.

Note that $x^T$ satisfies the constraint (B) in Problem $(T,\lambda)$, that is, $\lambda x^T(m^T) \leq 0$ for all $m^T$.

The rest of the proof closely follows that of Section 5.1, and the details will not be repeated here. We can construct a sequential equilibrium strategy profile $s^T$ in the T-period repeated game with sidepayment $x^T$, denoted $(G^T,x^T)$, which has the following properties: (1) it assigns the given action profile $a \in A'$ in each period on the equilibrium path irrespective of the exchanged messages (2) it always sends a fixed message profile, say $m^0$, in the first T-1 periods, and (3) it always reveals the history of the private signals in the last period $(m(T)=(\omega(1),\ldots,\omega(T)))$. Taking the given action $a \in A'$ throughout the T periods is shown to be better than any other action rule by Theorem in Section 4.2 in Matsushima (1994). As each player's messages do not affect either other players' actions or her own sidepayment, she has a (weak) incentive to follow the above message plan. From the construction of $x^T$ and $s^T$, the total payoff for player i in $(G^T,x^T)$, denoted $v_i(s^T,x^T)$, is approximated by $g_i(a)$ for all $i \notin N'$ and by $\max_{a_i' \in A_i} g_i(a_{-i},a'_i)+\epsilon_i$ for $i \in N'$, as $T \to \infty$. This is because the players almost always pass the test for a large T. Since $\epsilon_i$ can be made arbitrarily small, we conclude that for all i, $v_i(s^T,x^T)$ is approximated by $v_i(N',a)$.

The above observation implies that, for every subset $N' \subset N$ and every $\lambda$ such that $\lambda_i<0$ for $i \in N'$ and $\lambda_i \geq 0$ for $i \notin N'$, $V(N')$ is a subset of $D(\lambda)$. Hence $V'=\cap_{N' \subset N} V(N')$ is a subset of $Q=\cap_{\lambda \neq 0} D(\lambda)$, which proves the present theorem. ∎

**(Proof of Corollary to Theorem 5).** Fix a subset $N'$ of $N=\{1,2\}$ arbitrarily. From the definitions of $v(N',a)$, $b^1$, $b^2$, $a^0_1$ and $a^0_2$, we have

$$v(N',a) \geq g(a) \text{ for all } A,$$

$$v(N',(b^2_1,b^1_2)) \leq (g_1(b^1),g_2(b^2)) = v^*,$$

$$v_1(N',(b^2{}_1,a^0{}_2)) \geq g_1(b^2{}_1,a^0{}_2) \geq \max_{v \in V*} v_1,$$

$$v_2(N',(b^2{}_1,a^0{}_2)) \leq g_2(b^2) = v^*{}_2,$$

$$v_2(N',(a^0{}_1,b^1{}_2)) \geq g_2(a^0{}_1,b^1{}_2) \geq \max_{v \in V*} v_2, \text{ and}$$

$$v_2(N',(a^0{}_1,b^1{}_2)) \leq g_1(b^1) = v^*{}_1.$$

These imply that $V^{**}$ is a subset of $V(N')$. Hence, we conclude that $V^{**}$ is a subset of $V'$, and Theorem 5 proves this corollary. ■

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