91-F-4

Pricing Options with Curved Boundaries

by

Naoto Kunitomo

and

Masayuki Ikeda

March 1991

Pricing Options with Curved Boundaries*

by

Naoto Kunitomo**

and

Masayuki Ikeda***

March 1991

*The research was supported by a grant of the Japan Securities Scholarship Foundation at the Faculty of Economics, University of Tokyo.

** Faculty of Economics, University of Tokyo, Bunkyo-ku, Hongo 7-3-1, Tokyo 113, Japan.

*** Faculty of Economics, Tohoku University, Kawauchi, Aoba-ku, Sendai 980, Japan.

Abstract

This paper gives a general valuation formula for European options when the underlying asset price follows the geometric Brownian motion process with curved boundaries. Our valuation formula is derived by generalizing the well-known formula for Brownian motion by Paul Levy to the case of the geometric Brownian motion process with curved boundaries. Although the general option pricing formula is expressed as an infinite series, our numerical examples suggest that the convergence of the series is quite rapid. Based on our general formula, we derive various valuation formula for complex options with upper and (or) lower curved boundaries. Our results include some valuation formulae already known for the options with a lower boundary by Merton (1973) and for the path dependent options by Goldman, Sossin and Gatto (1979) as special cases. We also discuss some possible applications for valuing corporate securities and the practical problem of hedging.

1. Introduction

Recently, various types of option contracts have been introduced in financial markets. The payoff of the ordinary European option is uniquely determined by the underlying asset price at its maturity date and does not depend upon its historical path. However, other types of option contracts have been appeared in financial markets and also discussed in textbooks on options such as Cox and Rubinstein (1985), for instance. Among them, there is a type of option contracts when the underlying asset price process is restricted by an absorbing barrier. In this type of option contracts, they are nullified when the underlying asset price reaches at a predetermined price, which is called the knockout price. It seems that there exist some option contracts of this kind in Tokyo financial markets according to a recent issue of the NIKKEI FINANCIAL JOURNAL. Merton (1973) has already presented the pricing formula for the options whose underlying asset price is restricted by a floor absorption barrier. In other words, he derived the valuation formula for the down and out call option, which expires whenever the underlying asset price falls and touches the knockout price level. Cox and Rubinstein (1985) also refers to the valuation formula for the up and out put option, which is nullified whenever the underlying asset prices goes up and touches the predetermined fixed upper knockout level.

The main purpose of this paper is to develop a new method of the option valuation when the underlying asset price is restricted by two curved absorbing barriers. In order to derive the valuation formula for the option with two curved boundaries, we first extend the Levy formula, which has been well-known in probability theory. (See Levy (1948) or Hida (1974).) Based on the generalized Levy formula (see our Theorem 2.1 in Section 2), we give the general pricing formula for the option with two curved boundaries (see our Theorems 3.1 and 3.2 in Section 3). The resulting option formula is expressed as an infinite series of the log-normal densities in the general case. However, our numerical examples indicate that the convergence of the infinite series is quite rapid in most cases. This implies that our general option formula may be useful in practical situations. As a special case, we can derive the option formula when the underlying asset is restricted by a floor absorbing barrier, which has been obtained by Merton (1973). Also by using the generalized Levy formula we give the option pricing formulae for some path-dependent options derived by Goldman, Sossin, and Gatto (1979).

This paper is organized as follows. In Section 2, we derive a generalization of the Levy formula, which has not been known in probability theory as well as in finance. Then we shall give the general pricing formula for the options with two curved boundaries in Section 3. We also examine the convergence property of the option formula by a number of numerical experiments. In Section 4, we derive some valuation formulae for the options with one absorbing barrier and one type of the path-dependent options for illustrative purposes. We also discuss other possible applications and a practical problem of hedging. Then we give some concluding comments in Section 5. The proof of our main theorem is given in Appendix.

2. A Generalization of Levy Formula

Let the underlying asset price S(t) at t follow the geometric Brownian Motion process

$$dS = \mu S dt + \sigma S dW \ , \tag{2.1}$$

where W(t) stands for the standard Brownian Motion, μ is the drift parameter and σ is the volatility parameter. For this process of asset price we shall consider the European options with two curved absorbing barriers. Let the upper and the lower absorbing barriers in the interval [0,t] be $Be^{\delta_1 s}$ and $Ae^{\delta_2 s}$, respectively. For the sake of simplicity we assume $B \geqq A$ and $Be^{\delta_1 t} \geqq Ae^{\delta_2 t}$, that is, two curved boundaries do not intersect in the interval [0,t]. The option contracts in this situation are described by Figure 1. If the underlying asset price $\{S_t\}$ starts at S_0 and hit the lower absorbing barrier at C for the first time, then the option contracts are nullified. Similarly, if $\{S_t\}$ starts at S_0 and hit the upper absorbing barrier at D for the first time, the option contracts are also nullified.

<Figure 1 should be around here.>

In order to determine the equilibrium option price with two knockout prices, we need the transition density function of the stochastic process $\{S_t\}$ with two curved boundaries. Let

(2.2) $$L(t) = \min_{0 \leqq s \leqq t} S(s)$$

be the minimum asset price in [0,t]. Also let

(2.3) $$M(t) = \max_{0 \leqq s \leqq t} S(s)$$

be the maximum asset price in [0, t]. Then we have the following result on the joint probability of three random variables (L(t), M(t), S(t)). The proof is given in Appendix.

Theorem 2.1 : Suppose $\{S(t)\}$ follows the geometric Brownian motion given by (2.1) with $S(0) = S_0$ and $I \in [Ae^{\delta_2 T}, Be^{\delta_1 T}]$. Then the probability that $Ae^{\delta_2 t} < L(t) \leqq M(t) < Be^{\delta_1 t}$ for all $t \in [0,T]$ and $S(T) \in I$ is

(2.4) $$P_1 = \int_I \Big(\sum_{n=-\infty}^{\infty} k_n(S) \Big) \frac{dS}{S} ,$$

where

(2.5) $$k_n(S) = \Big(\frac{B^n}{A^n}\Big)^{c_{1n}} \Big(\frac{A}{S_0}\Big)^{c_{2n}} \cdot \Phi\Big[\frac{\ln S - \ln(S_0 B^{2n}/A^{2n}) - (\mu - \sigma^2/2)T}{\sigma\sqrt{T}}\Big]$$

$$- \Big(\frac{A^{n+1}}{S_0 B^n}\Big)^{c_{3n}} \cdot \Phi\Big[\frac{\ln S - \ln(A^{2n+2}/B^{2n} S_0) - (\mu - \sigma^2/2)T}{\sigma\sqrt{T}}\Big] ,$$

and $c_{1n} = 2[\mu - \delta_2 - n(\delta_1 - \delta_2)]/\sigma^2 - 1$, $c_{2n} = 2n(\delta_1 - \delta_2)/\sigma^2$, $c_{3n} = 2[\mu - \delta_2 + n(\delta_1 - \delta_2)]/\sigma^2 - 1$, and $\Phi(\cdot)$ is the density function of the standard normal distribution.

This theorem is a generalization of the well-known formula by Levy (1948) in probability theory. (See Hida (1974), for instance.) He derived a similar density function for the standard Brownian motion $\{W(t)\}$ with two flat boundaries. In order to see this point, we make the transformation $X(t) = \ln[S(t)]$. Then $dS/S = dX$ and the drift parameter $\mu' = \mu - \sigma^2/2$ by Ito's Lemma. Let

(2.6) $$\ell(t) = \min_{0 \leqq s \leqq t} X(s)$$

be the minimum of $X(s)$ in $[0,t]$ and

(2.7) $$m(t) = \max_{0 \leqq s \leqq t} X(s)$$

be the maximum of $X(s)$ in $[0,t]$. Then we have the following proposition.

Theorem 2.2 : Suppose X(t) is the Brownian motion with $X(0) = x_0$ and $I \in [\gamma_2+\delta_2 T, \ \gamma_1+\delta_1 T]$. Then the probability that $\gamma_2+\delta_2 t < \ell(t) \leqq m(t) < \gamma_1+\delta_1 t$ for all $t \in [0,T]$ and $X(T) \in I$ is

$$(2.8) \qquad P_2 = \int_I \Big(\sum_{n=-\infty}^{\infty} k_n'(x) \Big) dx,$$

where

$$(2.9) \quad k_n'(x) = \frac{1}{\sqrt{2\pi T}} \exp\Big(-\frac{1}{2T\sigma^2}\{[x-x_0-\mu' T-2n(\gamma_1-\gamma_2)]^2 + 4nT[n(\gamma_1-\gamma_2)(\delta_1-\delta_2)+\gamma_1\delta_2$$

$$- \gamma_2\delta_1-\mu'(\gamma_1-\gamma_2)+x_0(\delta_1-\delta_2)]\}\Big)$$

$$- \frac{1}{\sqrt{2\pi T}} \exp\Big(-\frac{1}{2T\sigma^2}\{[x+x_0-\mu' T-2\gamma_2-2n(\gamma_2-\gamma_1)]^2$$

$$+ 4T[n(\gamma_1-\gamma_2)-\gamma_2+x_0][n(\delta_1-\delta_2)-\delta_2+\mu']\}\Big).$$

Further if we set $x_0 = \delta_1 = \delta_2 = \mu' = 0$ and $\sigma = 1$ in the above formula, the kernel density $k_n'(x)$ becomes

$$(2.10) \qquad k_n'(x) = \Phi\Big[\frac{x-2n(\gamma_1-\gamma_2)}{\sqrt{T}}\Big] - \Phi\Big[\frac{x-2\gamma_2+2n(\gamma_1-\gamma_2)}{\sqrt{T}}\Big],$$

which is called the Levy formula in probability theory. Hence our Theorem 2.1 generalizes the Levy formula in two respects. One aspect is that our result is for the geometric Brownian motion. The other aspect is that we use some curved absorbing boundaries instead of two flat absorbing boundaries. In this sense Theorems 2.1 and 2.2 may not be a trivial extension of the Levy formula, which has been well-known in probability theory.

Corollary 2.3 : Suppose S(t) is the geometric Brownian motion given by (2.1) with $S(0) = S_0$. Then (i) for an arbitrary interval $I \in [Ae^{\delta_2 T}, +\infty]$ the probability that $Ae^{\delta_2 t} < L(t)$ for all $t \in [0,T]$ and $S(T) \in I$ is

$$(2.11) \qquad P_3 = \int_I \{\Phi[\frac{\ln S - \ln S_0 - (\mu - \sigma^2/2)T}{\sigma\sqrt{T}}]$$

$$- (\frac{A}{S_0})^{2(\mu-\delta_2-\sigma^2/2)/\sigma^2} \cdot \Phi[\frac{\ln S - \ln(A^2/S_0) - (\mu-\sigma^2/2)T}{\sigma\sqrt{T}}]\} \frac{dS}{S} ,$$

and (ii) for an arbitrary interval $I \in [-\infty, Be^{\delta_1 T}]$ the probability that $M(t) < Be^{\delta_1 t}$ for all $t \in [0,T]$ and $S(T) \in I$ is

$$(2.12) \qquad P_4 = \int_I \{\Phi[\frac{\ln S - \ln S_0 - (\mu - \sigma^2/2)T}{\sigma\sqrt{T}}]$$

$$- (\frac{B}{S_0})^{2(\mu-\delta_1-\sigma^2/2)/\sigma^2} \cdot \Phi[\frac{\ln S - \ln(B^2/S_0) - (\mu-\sigma^2/2)T}{\sigma\sqrt{T}}]\} \frac{dS}{S} .$$

This corollary can be obtained formally by letting B go to infinity or A go to zero in Theorem 2.1. A more rigorous proof can be obtained by a similar argument as in Appendix. As we shall show in Section 4, the result by Merton (1973) on the options with a floor boundary is a direct consequence of the first part of Corollary 2.3.

3. A General Option Formula with Curved Boundaries

We assume that the asset price $S(t)$ for $t\in[0,T]$ is described by the geometric Brownian Motion (2.1) with two absorbing barriers. Let S_t be the current asset price, σ is the volatility parameter, T is the maturity date of option contract, and E stands for its exercise price. Let also $r(t)$ be the risk-free interest rate, which is assumed to be independent of $\{W(s), s\leqq T\}$. In order to evaluate the price of option contract, we shall use the risk-neutralized method developed by Cox and Ross (1976) and Harrison and Kreps (1979). The value of European call option at t in the risk-neutralized method is given by

$$(3.1) \qquad C(t) = e^{-\int_t^T r(s)ds} \mathbf{E}[\max(S(T)-E,\ 0)|S(t)=S]$$

$$= e^{-\int_t^T r(s)ds} \int_E^F S(T)f(S(T))dS(T) - Ee^{-\int_t^T r(s)ds} \int_E^F f(S(T))dS(T),$$

where $F = Be^{\delta_1 T}$ and $\mathbf{E}(\cdot)$ is the expectation operator taken with respect to the risk-neutralized density function $f(S(T))$ of $S(T)$ given $S(t) = S$. Since the integrands in (3.2) are bounded, we can make use of the Lebesgue bounded convergence theorem. Assuming that $r(t) = r$ (constant) in $t\in[0, T]$, we have the following result.

Theorem 3.1: The value of call option at t which is nullified before its maturity date whenever the underlying asset price $\{S(t)\}$ reaches at the upper barrier $B\exp(\delta_1 s)$ or the lower barrier $A\exp(\delta_2 s)$ for any $s\in[t,T]$ is given by

$$(3.2) \qquad C(t) = S \sum_{n=-\infty}^{\infty} \Big\{ \left(\frac{B^n}{A^n}\right)^{c^*_{1n}} \left(\frac{A}{S}\right)^{c_{2n}} [\Phi(d_{1n}) - \Phi(d_{2n})]$$

$$- \left(\frac{A^{n+1}}{B^n S}\right)^{c^*_{3n}} [\Phi(d_{3n}) - \Phi(d_{4n})] \Big\}$$

$$- Ee^{-r\tau} \sum_{n=-\infty}^{\infty} \Big\{ \left(\frac{B^n}{A^n}\right)^{c^*_{1n}-2} \left(\frac{A}{S}\right)^{c_{2n}} [\Phi(d_{1n} - \sigma\sqrt{\tau}) - \Phi(d_{2n} - \sigma\sqrt{\tau})]$$

$$- \left(\frac{A^{n+1}}{B^n S_0}\right)^{c^*_{3n}-2} [\Phi(d_{3n} - \sigma\sqrt{\tau}) - \Phi(d_{4n} - \sigma\sqrt{\tau})] \Big\},$$

where $c^*_{1n} = 2[r-\delta_2-n(\delta_1-\delta_2)]/\sigma^2+1$, $c^*_{3n} = 2[r-\delta_2+n(\delta_1-\delta_2)]/\sigma^2+1$, $F = Be^{\delta_1 T}$, $\tau = T-t$,

$$(3.3) \qquad d_{1n} = \frac{\ln(SB^{2n}/EA^{2n}) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}},$$

$$(3.4) \qquad d_{2n} = \frac{\ln(SB^{2n}/FA^{2n}) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}},$$

$$(3.5) \qquad d_{3n} = \frac{\ln(A^{2n+2}/ESB^{2n}) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}},$$

$$(3.6) \qquad d_{4n} = \frac{\ln(A^{2n+2}/FSB^{2n}) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}},$$

and $\Phi(\cdot)$ is the distribution function of the standard normal density.

Although the general formula in (3.2) looks very complicated, the leading term corresponds to the Black=Scholes formula (Black and Scholes (1973)). To see this point, we notice that there are four terms in (3.2) with $n = 0$. Further, if there is not any upper as well as lower boundary, we take $A = 0$ and $B = +\infty$. Then three terms with d_{2n}, d_{3n}, and d_{4n} disappear, and the resulting expression is identical to the Black=Scholes formula. When there are upper and lower boundaries in the general case, however, we need other terms with $n \neq 0$ because of the reflection principle as in the theory of the standard Brownian motion. In this sense (3.1) is a generalization of the Black=Scholes formula.

Similarly, the price of European put option at t with two absorbing boundaries can be calculated by

$$(3.7) \qquad P(t) = e^{-\int_t^T r(s)ds} \mathbf{E}[\max(E-S(T),\ 0) \mid S(t)=S]$$

$$= Ee^{-\int_t^T r(s)ds} \int_{F'}^{E} f(S(T))dS(T) - e^{-\int_t^T r(s)ds} \int_{F'}^{E} S(T)f(S(T))dS(T),$$

where $F' = Ae^{\delta_2 T}$. Assuming that $r(t) = r$ (constant) in $t \in [0,\ T]$, we have the following result.

Theorem 3.2 : The value of put option at t which is nullified before its maturity date whenever the underlying asset price reaches at the upper barrier $B\exp(\delta_1 s)$ or the lower barrier $A\exp(\delta_2 s)$ for any $s \in [t,T]$ is given by

$$(3.8) \qquad P(t) = -S \sum_{n=-\infty}^{\infty} \{ (\frac{B^n}{A^n})^{c_{1n}^*} (\frac{A}{S})^{c_{2n}} [\Phi(d'_{1n}) - \Phi(d'_{2n})]$$

$$- \left(\frac{A^{n+1}}{B^n S}\right)^{c^*_{3n}} [\Phi(d'_{3n}) - \Phi(d'_{4n})]\}$$

$$+ Ee^{-r\tau} \sum_{n=-\infty}^{\infty} \{ \left(\frac{B^n}{A^n}\right)^{c^*_{1n}-2} \left(\frac{A}{S}\right)^{c_{2n}} [\Phi(d'_{1n} - \sigma\sqrt{\tau}) - \Phi(d'_{2n} - \sigma\sqrt{\tau})]$$

$$- \left(\frac{A^{n+1}}{B^n S}\right)^{c^*_{3n}-2} [\Phi(d'_{3n} - \sigma\sqrt{\tau}) - \Phi(d'_{4n} - \sigma\sqrt{\tau})]\}$$

where

(3.9) $$d'_{1n} = \frac{\ln(SB^{2n}/F'A^{2n}) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} ,$$

(3.10) $$d'_{2n} = \frac{\ln(SB^{2n}/EA^{2n}) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} ,$$

(3.11) $$d'_{3n} = \frac{\ln(A^{2n+2}/F'SB^{2n}) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} ,$$

(3.12) $$d'_{4n} = \frac{\ln(A^{2n+2}/ESB^{2n}) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}} ,$$

and $F' = Ae^{\delta_2 T}$.

We notice that the simple put-call parity relation does not hold in the general case. This is due to the fact that there are some probabilities to hit the upper or lower boundaries in [t,T], but we are not sure when these events would occur. In the term of probability theory, the hitting times of boundaries are stopping times. Therefore, we cannot make use of the usual justification for the put-call parity relation. When we take $n = 0$ in (3.8), there are four terms as in (3.2). As $A \to 0$ and $B \to +\infty$, three terms with d'_{2n}, d'_{3n}, and d'_{4n} disappear and we have the put-call parity relation in the limit.

In general, the option formulae with two curved boundaries in Theorems 3.1 and 3.2 are expressed as infinite series of the weighted normal distribution functions. Although the proof of the above Theorems in Appendix show that these infinite series converge, the rate of convergence could be quite slow at this stage. Then in order to study the speeds of convergence properties of the infinite series in these option formulae, we have conducted a systematic numerical investigation. The numerical study may shed some light on the practical usefulness of the general formulae.

<Insert Table 1 around here.>

Table 1 exhibits the call option premium based on our formula (3.3). To see the effect of two simultaneous absorbing barriers imposed, we assumed some realistic parameter values and various combination of upper and lower knockout prices. In every case, it is assumed that $S = 1000$ (yen), $r = 5\%$ (per annum), and $E = 1000$ (yen). As for the volatility parameter, three cases are considered from 20% to 40% (per annum). For the time to maturity, three values are assigned to τ, i.e., 0.0833, 0.25, and 0.5, which correspond to one, three and six months, respectively. Column (a) assumes $(\delta_1, \delta_2) = (0.1, -0.1)$, i.e., a convex upward upper boundary with a convex downward lower boundary. Column (b) assumes $(\delta_1, \delta_2) = (0, 0)$, which indicates two flat absorbing barriers. Column (c) sets $(\delta_1, \delta_2) =$

(-0.1, 0.1) and the upper barrier is exponentially decaying while the lower boundary is growing exponentially as time elapses. As the extreme case with A = 0 and B = $+\infty$, the ordinary Black Scholes call option value is reported in the first low.

<Insert Figure 2 around here.>

In every case the closer two curved boundaries are, the less the option value is, reflecting the increasing probability of absorption before the maturity date. If two boundaries are apart enough as τ becomes larger, the call premium increases as for the ordinary option. If two boundaries are closely located, however, the value of call option dereases as τ becomes larger because of the high absorption probability. The volatility has a similar two-way effect on the call premium. Increasing volatility enhances the value of option as a hedging instrument, but at the same time the chance of being nullified by hitting the barriers becomes higher in the volatile market. It is noteworthy that as the value of B/A decreases, the decrease of the call premium is slow to a certain level depending on other parameter values, and after that point the decrease of premium becomes vivid. This phenomenon is induced by the tail behaviour of log-normal distribution function. Note that the Levy formula decomposes the restrictd Brownian motion process into infinite number of unrestriced Brownian motions by means of the so-called reflection principle. As the range between upper and lower barriers is widended, the absorbing probability decreases rapidly since the probability of the corresponding unrestricted Brownian motion reaching the further boundary decreases exponentially.

Table 2 picked up 18 premium values from Table 1 to investigate the speed of convergence of the infinite series. We only show the case of column (a), an upward upper barrier and a downward lower barrier case, since we found similar tendency in other cases. The series have been calculated in the order of $n = 0$, $n = +1$, $n = -1$, $n = +2$, $n = -2$, $\cdots$. Notice that all the reported values are rounded to the 10^{-5} digit level.

<Insert Table 2 around here.>

When (A, B) = (400, 1600), in all cases it suffices to take only the first term of $n = 0$. When (A, B) = (900, 1100), the option premium decreases drastically and taking 3 to 6 terms is requred achieving the preciseness in 10^{-5} level. In most cases, however, the increment except the first term is negligible when the τ is one month, i.e., when the time to maturity is short. When (A, B) = (950, 1050), only 50 yen of instaneous change in asset price (5% of current price level) nullifies the option contract and this example shows such contract has negligible values. In this case, in order to achieve the preciseness in 10^{-5} level, we need 6 to 10 terms when τ is six months and 3 to 6 terms when τ is one month, respectively. It should be emphasized, however, that even in this very extreme example, the inclusion of only several terms can approximate the option premium sufficiently for practical purposes. These numerical studies suggest that our general formula is useful although it looks very complicated.

4. Some Applications

4.1 An Option Formula with One Absorbing Barrier

The general option pricing formulae we have obtained in Theorems 3.1 and 3.2 include some option formulae as special cases. The first example in our formulation is the option formula with a curved boundary. Let B go to infinity in Theorem 3.1. Then all terms in the infinite series except $n = 0$ converge to zero. Then we have the following results.

Corollary 4.1 : The down and out call option price at t with a knockout price, which is given by the curve $A\exp(\delta_2 s)$ ($s \in [t,T]$, $A<S$) is

$$(4.1) \qquad C(t) = S\{\Phi[\frac{\ln(S/E)+(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}] - (\frac{A}{S})^{\{2(r-\delta_2)/\sigma^2\}+1}\Phi[\frac{\ln(A^2/SE)+(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}]\},$$

$$- e^{-r\tau}E\{\Phi[\frac{\ln(S/E)+(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}] - (\frac{A}{S})^{\{2(r-\delta_2)/\sigma^2\}-1}\Phi[\frac{\ln(A^2/SE)+(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}]\}.$$

A rigorous proof of this corollary is similar to the one given in Appendix. Notice that Merton's formula [1973] (equation (55) in Page 175), expressed in its error function form is the same as the above formula by setting $A = bE\exp(-\eta\tau)$, and $\delta_2 = \eta$ in (4.1). He sets the upward growing boundary with $\eta > 0$. Also Cox and Rubinstein [1985]'s formula for the flat barrier case can be easily obtained by equating $\delta_2 = 0$ in (4.1).

For the put option with one curved boundary, we let A go to zero. Then all terms in the infinite series except $n = 0$ or $n = -1$ disappear. Again this line of arguments can be justified rigorously as in Appecdix. In this case we have the next result.

Corollary 4.2 : The up and out put option price at t with a knockout price, which is given by the curve by $B\exp(\delta_1 s)$, $(s \in [t, T], B > S_0)$ is given by

$$(4.2) \qquad P(t) = -S\{\Phi[\frac{\ln(E/S)-(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}]$$

$$- (\frac{B}{S})^{[2(r-\delta_1)/\sigma^2]+1}\Phi[\frac{\ln(SE/B^2)-(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}]\}$$

$$+ Ee^{-r\tau}\{\Phi[\frac{\ln(E/S)-(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}]$$

$$- (\frac{B}{S})^{[2(r-\delta_1)/\sigma^2]-1}\Phi[\frac{\ln(SE/B^2)-(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}]\}.$$

When $\delta_1 = 0$, the above put option formula is identical to the one given by Cox and Rubinstein (1985). In addition to the above options in Corollaries 4.1 and 4.2, there can be two more cases when there is an upper or lower boundary. However, it is straightforward to derive the resulting formulae for these cases from Theorems 3.1 and 3.2.

4.2 Option Formulae for Lookback Options

The second application of our formulation is the valuation problem for lookback options first introduced by Goldman, Sossin and Gatto (1979). They considered the pricing of call option, which is the right to sell the asset at the highest realized price at its maturity. Let $L(t)$ be the lowest price during $[0,t]$. Then the maturity payoff of this call option is

$$(4.5) \qquad C(T) = S(T) - L(T).$$

When we take L as the lowest price already realized during [0, t] and define L_T as the lowest price during the remaining future period [t, T], then we have $L(T) = \min(L_T, L)$. By the risk-neutralized method, the present value of this option can be obtained by

$$(4.6) \qquad C(t) = e^{-r\tau}\{\mathbf{E}[S(T) - L_T | L_T > L] \cdot P(L_T > L) + \mathbf{E}[S(T) - L_T | L_T \leqq L] \cdot P(L_T \leqq L)\}$$
$$= e^{-r\tau}\{\mathbf{E}[S(T)] - L \cdot P(L_T > L) - \mathbf{E}[L_T | L_T \leqq L] \cdot P(L_T \leqq L)\},$$

where $\tau = T-t$. Another type of option discussed by Goldman, Sosin and Gatto (1979) is the put option, which is the right to buy the asset at the lowest realized price at its maturity date. The maturity payoff of this option is

$$(4.7) \qquad P(T) = M(T) - S(T),$$

where M(T) is the highest price during [0, T]. When we take M as the highest price already realized during [0. t] and define M_T as the highest price during the remaining future period [t, T], then we have $M(T) = \max(M_T, M)$. By applying the risk-neutralized method, the present value of the put option can be obtained by

$$(4.8) \qquad P(t) = e^{-r\tau}\{\mathbf{E}[M - S(T) | M_T \leqq M] \cdot P(M_T \leqq M) + \mathbf{E}[M_T - S(T) | M_T > M] \cdot P(M_T > M)\}$$
$$= e^{-r\tau}\{M \cdot P(M_T \leqq M) - \mathbf{E}[S(T)] + \mathbf{E}[M_T | M_T > M] \cdot P(M_T > M)\}.$$

In order to evaluate the expected values in C(t) and P(t), we need the distribution function of {L(t)} and {M(t)}. Setting $\delta_2 = 0$ in Corollary 2.3, we have

$$(4.9) \qquad P(L(t) \leqq A) = 1 - P(L(t) > A)$$

$$= \Phi\left[\frac{\ln(A/S)-(\mu-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right] + \left(\frac{A}{S}\right)^{(2\mu/\sigma^2)-1} \Phi\left[\frac{\ln(A/S)+(\mu-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right] .$$

Similarly, setting $\delta_1 = 0$ in Corollary 2.3 we have

$$(4.10) \qquad P(M(t) \leqq B)$$

$$= \Phi\left[\frac{\ln(B/S)-(\mu-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right] - \left(\frac{S}{B}\right)^{-(2\mu/\sigma^2)+1} \Phi\left[\frac{\ln(S/B)-(\mu-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right] .$$

Using the marginal density functions of $L(t)$ and $M(t)$, we obtain the following results by Goldman, Sosis and Gatto (1979).

Corollary 4.3 : (i) The value of the lookback call option at t, which is the right to sell the asset at the highest realized price at the maturity, is given by

$$(4.11) \qquad C(t) = S - Le^{-r\tau}\{\Phi\left[\frac{\ln(S/L)+(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right]$$

$$- \frac{\sigma^2}{2r}\left(\frac{S}{L}\right)^{1-(2r/\sigma^2)} \Phi\left[\frac{\ln(L/S)+(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right]\}$$

$$- S\left(1 + \frac{\sigma^2}{2r}\right)\Phi\left[\frac{\ln(L/S)-(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right] .$$

(ii) The value of the lookback put option at t, which is the right to buy the asset at the lowest realized price at the maturity, is given by

$$(4.12)\qquad P(t) = -S + Me^{-r\tau}\{\Phi[\frac{\ln(M/S)-(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}]$$

$$-\frac{\sigma^2}{2r}(\frac{S}{M})^{1-(2r/\sigma^2)}\Phi[\frac{\ln(S/M)-(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}]\}$$

$$+ S\,(1+\frac{\sigma^2}{2r})\Phi[\frac{\ln(S/M)+(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}]\ .$$

We note that the method of our derivations is simpler than the one given by Goldman, Sosin, and Gatto (1979). In addition to the valuation problem of these lookback options, we can apply our method to other lookback options. For example, we can think of the right to receive the amount equal to the maximum realized price less of fixed exercise price E at its maturity date. Another example may be the right to receive the amount equal to the predetermined exercised price E less of the lowest realized price of the asset. Using the same method as in Corollary 4.5, it is straightforward to derive the pricing formulae for these options.

4.3 Valuing Corporate Securities

In previous sections, we have discussed only the option pricing problems. However, as Black and Cox (1976) pointed out, a similar analysis seems to be applied to a number of problems of valuing corporate securities. In some corporate securities, there may be both natural lower and upper boundaries at which the firm's securities must take on specific values. For instance, Black and Cox (1976) referred to the problem of bonds with safety covenants. The problem of its valuation is similar to the valuing option with a curved lower boundary. Ingersoll (1977) also mentioned to the problem of valuing a callable, convertible, discount bond with a call policy. It is similar to the option pricing problem with a curved upper boundary. There could be similar valuation problems of corporate securities. Our method in this paper could be applied to such problems as long as the boundaries are given exogenously as the exponential functions of time.

4.4 Delta Hedging

At the first glance the hedging arguments for the options with curved boundaries seem to be involved because the general valuation formulae in Theorems 3.1 and 3.2 are rather complicated. However, this is not necessarily the case. In order to understand this problem, we have calculated the option delta from (3.2) for the call option. The resulting formula is given by

$$(4.13)\quad \frac{\partial C(t)}{\partial S} = \sum_{n=-\infty}^{\infty} \{(1-c_{2n})\left(\frac{B^n}{A^n}\right)^{c^*_{1n}}\left(\frac{A}{S}\right)^{c_{2n}}[\Phi(d_{1n}) - \Phi(d_{2n})]$$

$$- (1-c^*_{3n})\left(\frac{A^{n+1}}{B^n S}\right)^{c^*_{3n}}[\Phi(d_{3n}) - \Phi(d_{4n})]\}$$

$$- Ee^{-r\tau}\sum_{n=-\infty}^{\infty} \{(-c_{2n})\left(\frac{B^n}{A^n}\right)^{c^*_{1n}-2}\left(\frac{A}{S}\right)^{c_{2n}}[\Phi(d_{1n}- \sigma\sqrt{\tau}) - \Phi(d_{2n}- \sigma\sqrt{\tau})]$$

$$- (2-c_{2n})\left(\frac{A^{n+1}}{B^n S_0}\right)^{c^*_{3n}-2}[\Phi(d_{3n}- \sigma\sqrt{\tau}) - \Phi(d_{4n}- \sigma\sqrt{\tau})]\}$$

$$- \frac{1}{\sigma\sqrt{\tau}}(1-\frac{E}{F}) \sum_{n=-\infty}^{\infty} \{ (\frac{B^n}{A^n})^{c^*_{1n}} (\frac{A}{S})^{c_{2n}} \Phi(d_{2n}) - (\frac{A^{n+1}}{B^n S})^{c^*_{3n}} \Phi(d_{4n}) \}.$$

Since the speed of convergence in (4.13) is quite rapid as for the general option formulae in Theorems 3.1 and 3.2, we need only several leading terms for the practical implementation of the delta hedging. When we take $n = 0$ in (4.13), there are four terms. As $A \to 0$ and $B \to +\infty$, we have

$$(4.14) \quad \frac{\partial C(t)}{\partial S} \to \Phi(d_{10}),$$

where $\Phi(d_{10})$ is the well-known formula of option delta for the Black=Scholes formulation. In this sense (4.13) is a modification of the usual option delta.

5. Conclusions

This paper presents the general valuation formula for the European options when the underlying asset price follows the geometric Brownian motion process restricted by two absorbing barriers which are the exponential functions of time. In order to obtain the general formula, we have derived a generalization of Levy formula, which gives the joint density function of the geometric Brownian motion process restricted by two absorbing barriers.

Although our option formula is represented as an infinite series, we demonstrated that the convergence rate is quite rapid. Our numerical studies suggest that it suffices to calculate leading two or three terms in most cases. Then we have derived some valuation formulae for the knock-out options as special cases. Also we have shown that the pricing of some path-dependent options are easily obtained by the generalized Levy formula. These examples and other examples mentioned to in Section 4 suggest that our general formulation of the options with curved boundaries may be potentially useful to some problems in finance.

Appendix: Proof of Theorem 2.1

In order to prove Theorem 2.1, first we prove Theorem 2.2 when $\mu' = 0$ and $\sigma = 1$. Let $Y(t)$ be the standard Brownian motion with $E(Y(t)) = 0$, $E(Y(t))^2 = t$, and $Y(0) = 0$. We take real numbers T, γ_1, γ_2, δ_1, and δ_2 such that $\gamma_2 < \gamma_1$, $\gamma_2 + \delta_2 T \leqq \gamma_1 + \delta_1 T$, and $T > 0$. Then the conditional probability that $Y(t) \geqq \gamma_1 + \delta_1 t$ for a smaller t ($t \leqq T$) than any t for which $Y(t) \leqq \gamma_2 + \delta_2 t$ given $Y(T) = y$ ($\leqq \gamma_1 + \delta_1 T$) has been derived by Theorem 4.2 in Anderson (1960), which is given by

$$(A.1) \qquad P_1(T, y) = \sum_{n=1}^{\infty} \exp\Big(-\frac{2}{T}\{n^2\gamma_1(\gamma_1 + \delta_1 T - y) + (n-1)^2\gamma_2(\gamma_2 + \delta_2 T - y)$$

$$- n(n-1)[\gamma_1(\gamma_2 + \delta_2 T - y) + \gamma_2(\gamma_1 + \delta_1 T - y)]\}\Big)$$

$$- \sum_{n=1}^{\infty} \exp\Big(-\frac{2}{T}\{n^2[\gamma_1(\gamma_1 + \delta_1 T - y) + \gamma_2(\gamma_2 + \delta_2 T - y)]$$

$$- n(n-1)\gamma_1(\gamma_2 + \delta_2 T - y) - n(n+1)\gamma_2(\gamma_1 + \delta_1 T - y)\}\Big).$$

Now we want to know the probability of a path touching the upper line before the lower line for $0 \leqq t \leqq T$, where T is a fixed terminal time. The unconditional density of $Y(t)$ at $t = T$ is $\Phi(1/\sqrt{T})$, where $\Phi(\cdot)$ is standard normal density. Let $I \in [\gamma_2 + \delta_2 T, \gamma_1 + \delta_1 T]$ be an interval at time T. Then

$$(A.2) \qquad \int_I \Phi\Big[\frac{y}{\sqrt{T}}\Big] P_1(T, y)\,dy$$

$$= \frac{1}{\sqrt{2\pi T}} \int_I \Big(\sum_{n=1}^{\infty} \exp -\frac{1}{2T}\{[y - 2\gamma_2 - 2n(\gamma_1 - \gamma_2)]^2 + 4T[(\gamma_1 - \gamma_2)n + \gamma_2][(\delta_1 - \delta_2)n + \delta_2]\}$$

$$-\sum_{n=1}^{\infty} \exp -\frac{1}{2T}\{[y - 2n(\gamma_1-\gamma_2)]^2 + 4Tn[n(\gamma_1-\gamma_2)(\delta_1-\delta_2) + (\gamma_1\delta_2-\gamma_2\delta_1)]\}\})dy.$$

Similarly, let $P_2(T,y)$ be the conditional probability that $Y(t) \leqq \gamma_2 + \delta_2 t$ for a smaller t ($t \leqq T$) than any t for which $Y(t) \geqq \gamma_1 + \delta_1 t$ given $Y(t) = y$ ($\geqq \gamma_2 + \delta_2 T$). Then $P_2(T,y)$ can be derived simply by replacing (γ_1, δ_1) by $(-\gamma_2, -\delta_2)$. Hence the unconditional probability that $Y(t) \leqq \gamma_2 + \delta_2 t$ for a smaller t ($t \leqq T$) than any t for which $Y(t) \geqq \gamma_1 + \delta_1 t$ is

(A.3) $$\int_I \Phi[\frac{y}{\sqrt{T}}]P_2(T, y)dy$$

$$= \frac{1}{\sqrt{2\pi T}} \int_I (\sum_{n=1}^{\infty} \exp -\frac{1}{2T}\{[y-2\gamma_2 + 2(n-1)(\gamma_1-\gamma_2)]^2 + 4T[(\gamma_1-\gamma_2)n-\gamma_1][(\delta_1-\delta_2)n-\delta_1]\}$$

$$- \sum_{n=1}^{\infty} \exp -\frac{1}{2T}\{[y + 2n(\gamma_1-\gamma_2)]^2 + 4Tn[n(\gamma_1-\gamma_2)(\delta_1-\delta_2) + (\gamma_2\delta_1-\gamma_1\delta_2)]\}\})dy.$$

Then we notice that the joint probability that $Y(T) \in I$ and $\gamma_2 + \delta_2 t < \ell(t) \leqq m(t) < \gamma_1 + \delta_1 t$ for any $t \leqq T$ is given by

(A.4) $$P(T) = P(\ Y(T) \in I\ )$$

$$- \int_I \Phi[\frac{y}{\sqrt{T}}]P_1(T, y)dy - \int_I \Phi[\frac{y}{\sqrt{T}}]P_2(T, y)dy\ .$$

Rearranging each term in (A.2) and (A.3), (A.4) becomes

(A.5) $$\frac{1}{\sqrt{2\pi T}} \int_I (\sum_{n=-\infty}^{\infty} \exp -\frac{1}{2T}\{[y - 2n(\gamma_1-\gamma_2)]^2 + 4Tn[n(\gamma_1-\gamma_2)(\delta_1-\delta_2) + (\gamma_1\delta_2-\gamma_2\delta_1)]\}$$

$$- \sum_{n=-\infty}^{\infty} \exp -\frac{1}{2T}\{[y-2\gamma_2 + 2n(\gamma_1-\gamma_2)]^2 + 4T[n(\gamma_1-\gamma_2)-\gamma_2][n(\delta_1-\delta_2)-\delta_2]\}\}dy.$$

Next let $X(t)$ $(0 \leqq t \leqq T)$ be the Brownian motion starting at x_0 with the drift parameter μ^* and the instantaneous variance σ^2. We shall obtain the joint probability that $X(T) \in I$ and $\gamma_2 + \delta_2 t < \ell(t) \leqq m(t) < \gamma_1 + \delta_1 t$ for any $t \leqq T$. By applying the Maruyama=Girsanov change of measure theorem (Maruyama (1954) and Girsanov (1960)) to $X(t)$, it is given by

$$\text{(A.6)} \qquad P^*(T) = \int_I \frac{1}{\sigma}\exp\left[\frac{\mu^*}{\sigma^2}(x-x_0) - \frac{\mu^{*2}T}{2\sigma^2}\right]k\left[\frac{\gamma_1 - x_0 + \delta_1 T}{\sigma}, \frac{\gamma_2 - x_0 + \delta_2 T}{\sigma}, \frac{x-x_0}{\sigma}\right]dx$$

where

$$\text{(A.7)} \qquad k(\gamma_1 + \delta_1 T, \gamma_2 + \delta_2 T, x)$$

$$= \frac{1}{\sqrt{2\pi T}}\left(\sum_{n=-\infty}^{\infty} \exp -\frac{1}{2T}\{[x-2n(\gamma_1-\gamma_2)]^2 + 4Tn[n(\gamma_1-\gamma_2)(\delta_1-\delta_2) + (\gamma_1\delta_2-\gamma_2\delta_1)]\}\right.$$

$$\left. - \sum_{n=-\infty}^{\infty} \exp -\frac{1}{2T}\{[x-2\gamma_2 + 2n(\gamma_1-\gamma_2)]^2 + 4T[n(\gamma_1-\gamma_2)-\gamma_2][n(\delta_1-\delta_2)-\delta_2]\}\right).$$

Then

$$\text{(A.8)} \qquad P^*(T) = \int_I \left(\sum_{n=-\infty}^{\infty} \Phi\left[\frac{x-x_0-2n(\gamma_1-\gamma_2)-\mu^*T}{\sigma\sqrt{T}}\right]\right.$$

$$\cdot \exp\left\{\frac{2n}{\sigma^2}[\mu^*(\gamma_1-\gamma_2) - n(\gamma_1-\gamma_2)(\delta_1-\delta_2) - (\gamma_1-x_0)\delta_2 + (\gamma_2-x_0)\delta_1]\right\}$$

$$- \sum_{n=-\infty}^{\infty} \Phi\left[\frac{x+x_0-2\gamma_2+2n(\gamma_1-\gamma_2)-\mu^*T}{\sigma\sqrt{T}}\right) \cdot \exp\left\{\frac{2\mu^*}{\sigma^2}[\gamma_2 - x_0 - n(\gamma_1-\gamma_2)]\right\}$$

$$\cdot \exp\{\frac{2}{\sigma^2}[-n(\gamma_1-\gamma_2) + \gamma_2 - x_0][n(\delta_1-\delta_2) - \delta_2]\}\}dx .$$

Finally, we consider the transformation $S(t) = \exp[X(t)]$. Then the drift parameter $\mu = \mu^* + \sigma^2/2$ by Ito's lemmma. Here we notice that the linear boundaries are transformed to the exponential curved boundaries. Let $A = \exp(\gamma_2)$ and $B = \exp(\gamma_1)$. Then the joint probability that $S(T)\in I$ and $Ae^{\delta_2 t} < L(t) \leqq M(t) < Be^{\delta_1 t}$ for any t ($\leqq T$) is given by

$$\text{(A.9)} \qquad P^{**}(T) = \int_I \{ \sum_{n=-\infty}^{\infty} (\frac{B^n}{A^n})^{(2\mu/\sigma^2)-1} [(\frac{A^n}{B^n})^{(\delta_1-\delta_2)} (\frac{S_0}{B})^{\delta_2} (\frac{A}{S_0})^{\delta_1}]^{2n/\sigma^2}$$

$$\cdot \Phi[\frac{\ln S-\ln(S_0B^{2n}/A^{2n})-(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}]$$

$$- \sum_{n=-\infty}^{\infty} (\frac{A^{n+1}}{S_0B^n})^{(2\mu/\sigma^2)-1} (\frac{A^{n+1}}{B^nS_0})^{2[n(\delta_1-\delta_2)-\delta_2]/\sigma^2}$$

$$\cdot \Phi[\frac{\ln S-\ln(A^{2n+2}/S_0B^{2n})-(\mu-\sigma^2/2)T}{\sigma\sqrt{T}}]\}\frac{dS}{S} .$$

□

References

Anderson, T.W. (1960), "A Modification of the Sequential Probability Ratio Test to Reduce the Sample Size." The Annals of Mathematical Statistics, Vol.31, 165-197.

Black, F. and M. Scholes (1973), "The Pricing of Options and Corporate Liabilities," Journal of Political Economy, 81, 637-654.

Black, F. and J.C.Cox (1976), "Valuing Corporate Securities: Some Effects of Bond Indenture Provisions", The Journal of Finance, Vol.XXXI, No.2, 351-367.

Cox, J.C. and M. Rubinstein (1985), Options Markets, Prentice-Hall, Englewood Cliffs, New Jersey.

Cox, J.C. and S. A. Ross (1976), "The Valuation of Options for Alternative Stochastic Processes," Journal of Financial Economics, Vol.3, 145-166.

Girsanov, I.V. (1960), "On Transforming a Certain Class of Stochastic Processes by Absolutely Continuous Substitution of Measures." Theory of Probability and Its Applications, Vol.5, 285-301.

Goldman, M.B., H.B.Sosin, and M.A.Gatto (1979), "Path Dependent Options: Buy at the Low, Sell at the High." Journal of Finance, Vol.34, 1111-27.

Harrison, J.M. and D. Kreps (1979), "Martingales and Arbitrage in Multiperiod Securities Markets." Journal of Economic Theory, Vol.20, 381-408.

Hida, T. (1974), Brownian Motion, (in Japanese), Iwanami-Shoten, Tokyo.

Ingersoll, J. E. (1977), "A Contingent-Claims Valuation of Convertible Securities," Journal of Financial Economics, Vol. 4, 287-322.

Lévy, P. (1948), Processus Stochastique et Mouvement Brownien, Gauthier-Villars, Paris.

Maruyama, G. (1954), "On the Transition Probability Functions of the Markov Process." National Science Report, Ochanomizu University, Vol.5, 10-20.

Merton, R.C. (1973), "Theory of Rational Option Pricing." Bell Journal of Economics and Management Science, Vol.4, 141-83.

Table 1

The Value of Call Option with Two Curved Absorbing Barriers

(S = 1000, r = 0.05, E = 1000)

(1) σ=0.2

		τ=1/12			τ=1/4			τ=1/2		
A	B	(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)
0	+∞	25.12	25.12	25.12	46.15	46.15	46.15	68.89	68.89	68.89
400	1600	25.12	25.12	25.12	46.15	46.15	46.14	68.64	68.14	66.93
500	1500	25.12	25.12	25.12	46.14	46.12	46.07	67.78	66.13	62.75
600	1400	25.12	25.12	25.12	45.97	45.76	45.35	64.63	60.06	52.50
700	1300	25.12	25.12	25.12	44.38	42.99	40.81	55.20	45.65	33.45
800	1200	24.88	24.76	24.58	35.13	30.39	24.67	34.58	22.08	10.86
850	1150	23.21	22.54	21.69	24.52	18.49	12.47	20.88	10.22	2.52
900	1100	16.17	14.40	12.50	11.06	6.21	2.60	7.55	1.79	0.01
930	1070	8.53	6.69	4.96	3.77	1.23	0.15	1.83	0.10	0.00
950	1050	3.39	2.15	1.17	0.76	0.08	0.00	0.23	0.00	0.00

The columns (a),(b), and (c) represents the case when (δ_1, δ_2) = (0.1,-0.1), (δ_1, δ_2) = (0.0, 0.0), and (δ_1, δ_2) = (-0.1, 0.1), respectively.

(2) σ=0.3

		τ=1/12			τ=1/4			τ=1/2		
A	B	(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)
0	+∞	36.59	36.59	36.59	65.83	65.83	65.83	96.35	96.35	96.35
400	1600	36.59	36.59	36.59	65.18	64.77	64.17	85.88	80.06	72.22
500	1500	36.58	36.58	36.58	63.49	62.34	60.75	76.57	67.88	57.31
600	1400	36.56	36.54	36.53	58.47	55.72	52.28	61.48	50.23	38.10
700	1300	36.01	35.84	35.62	46.29	41.31	35.78	40.54	28.90	18.22
800	1200	30.55	29.45	28.21	24.94	19.31	14.02	17.48	9.26	3.54
850	1150	22.14	20.36	18.51	13.01	8.60	4.99	7.26	2.50	0.37
900	1100	10.01	8.31	6.71	3.14	1.29	0.00	0.96	0.08	0.03
930	1070	3.28	2.27	1.45	0.37	0.05	0.00	0.05	0.00	0.00
950	1050	0.58	0.27	0.11	0.01	0.00	0.00	0.00	0.00	0.00

(3) $\sigma=0.4$

		$\tau=1/12$			$\tau=1/4$			$\tau=1/2$		
A	B	(a)	(b)	(c)	(a)	(b)	(c)	(a)	(b)	(c)
0	$+\infty$	48.05	48.05	48.05	85.53	85.53	85.53	123.9	123.9	123.9
400	1600	48.03	48.03	48.02	77.56	75.28	72.52	81.60	71.53	59.60
500	1500	47.89	47.85	47.79	69.54	65.84	61.63	64.85	53.35	41.70
600	1400	46.95	46.72	46.44	55.90	50.76	45.26	45.23	34.22	24.05
700	1300	42.32	41.67	40.52	36.34	30.69	25.15	25.08	16.45	9.44
800	1200	27.63	25.84	24.01	14.81	10.69	7.17	7.34	3.14	0.76
850	1150	16.14	14.35	12.61	5.67	3.26	1.59	1.74	0.37	0.02
900	1100	5.08	3.97	2.99	0.57	0.15	0.00	0.05	0.00	0.00
930	1070	0.88	0.52	0.27	0.01	0.00	0.00	0.00	0.00	0.00
950	1050	0.05	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 2

Convergence of Infinite Series

($S = 1000$, $r = 0.05$, $E = 1000$, $\delta_1 = 0.1$, $\delta_2 = -0.1$)

(1) $\sigma=0.2$

	A = 400, B = 1600		A = 900, B = 1100		A = 950, B = 1050	
n	τ=1/2	τ=1/12	τ=1/2	τ=1/12	τ=1/2	τ=1/12
0	68.63629	25.12067	9.04900	16.17595	1.88621	3.82211
+1	68.63629	25.12067	7.54197	16.17484	-0.26278	3.39105
-1	68.63629	25.12067	7.55348	16.17484	0.26271	3.39233
+2	68.63629	25.12067	7.55348	16.17484	0.22598	3.39233
-2	68.63629	25.12067	7.55348	16.17484	0.22702	3.39233
+3	68.63629	25.12067	7.55348	16.17484	0.22701	3.39233
-3	68.63629	25.12067	7.55348	16 17484	0.22701	3.39233
+4	68.63629	25.12067	7.55348	16.17484	0.22701	3.39233
-4	68.63629	25.12067	7.55348	16.17484	0.22701	3.39233
+5	68.63629	25.12067	7.55348	16.17484	0.22701	3.39233
-5	68.63629	25.12067	7.55348	16.17484	0.22701	3.39233

(2) $\sigma=0.3$

	A = 400, B = 1600		A = 900, B = 1100		A = 950, B = 1050	
n	τ=1/2	τ=1/12	τ=1/2	τ=1/12	τ=1/2	τ=1/12
0	85.88228	36.58566	3.32389	10.15675	0.60315	1.47683
+1	85.88228	36.58566	0.59860	10.00883	-0.58860	0.49392
-1	85.88228	36.58566	0.97271	10.00884	0.24209	0.58034
+2	85.88228	36.58566	0.96384	10.00884	-0.06087	0.57914
-2	85.88228	36.58566	0.96393	10.00884	0.00866	0.57914
+3	85.88228	36.58566	0.96392	10.00884	-0.00029	0.57914
-3	85.88228	36.58566	0.96392	10.00884	0.00051	0.57914
+4	85.88228	36.58566	0.96392	10.00884	0.00047	0.57914
-4	85.88228	36.58566	0.96392	10.00884	0.00048	0.57914
+5	85.88228	36.58566	0.96392	10.00884	0.00048	0.57914
-5	85.88228	36.58566	0.96392	10.00884	0.00048	0.57914

(3) σ=0.4

	A = 400, B = 1600		A = 900, B = 1100		A = 950, B = 1050	
n	τ=1/2	τ=1/12	τ=1/2	τ=1/12	τ=1/2	τ=1/12
0	81.59726	48.03399	1.51228	5.84362	0.26059	0.68685
+1	81.59726	48.03399	-0.71756	5.07620	-0.36595	-0.19725
-1	81.59726	48.03399	0.16767	5.08000	0.26839	0.07544
+2	81.59726	48.03399	0.04559	5.08000	-0.14840	0.04795
-2	81.59726	48.03399	0.05486	5.08000	0.05340	0.04919
+3	81.59726	48.03399	0.05463	5.08000	-0.01587	0.04917
-3	81 59726	48.03399	0.05463	5.08000	0.00315	0.04917
+4	81.59726	48.03399	0.05463	5.08000	-0.00054	0.04917
-4	81.59726	48.03399	0.05463	5.08000	0.00006	0.04917
+5	81.59726	48.03399	0.05463	5.08000	-0.00001	0.04917
-5	81.59726	48.03399	0.05463	5.08000	0.00000	0.04917

Figure 1 : Geometric Brownian Motion with Two Curved Boundaries

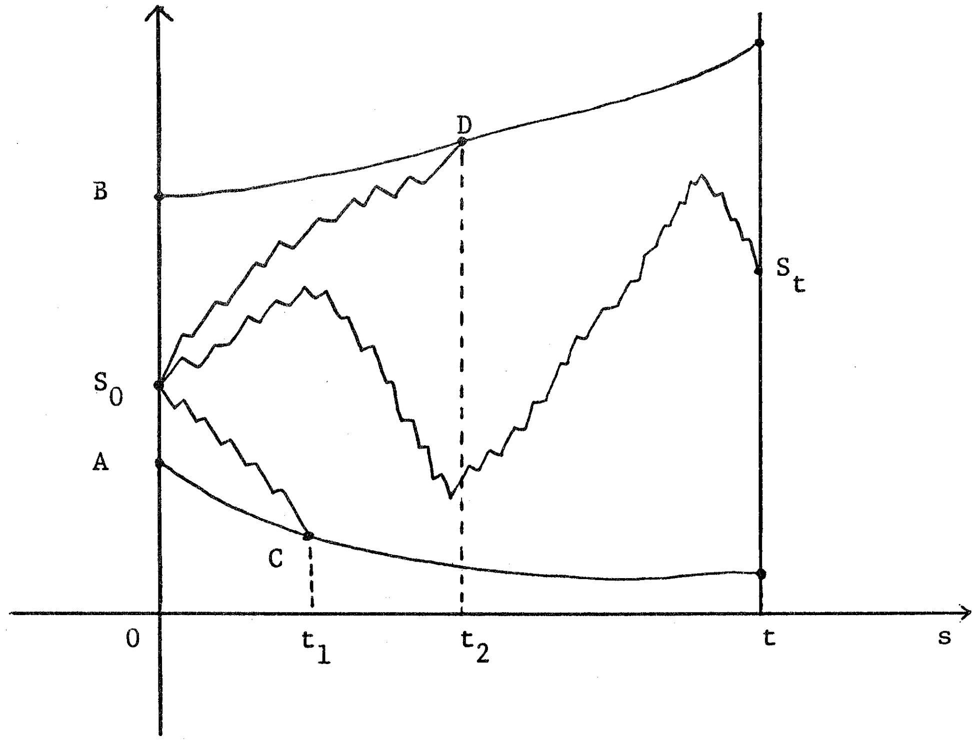
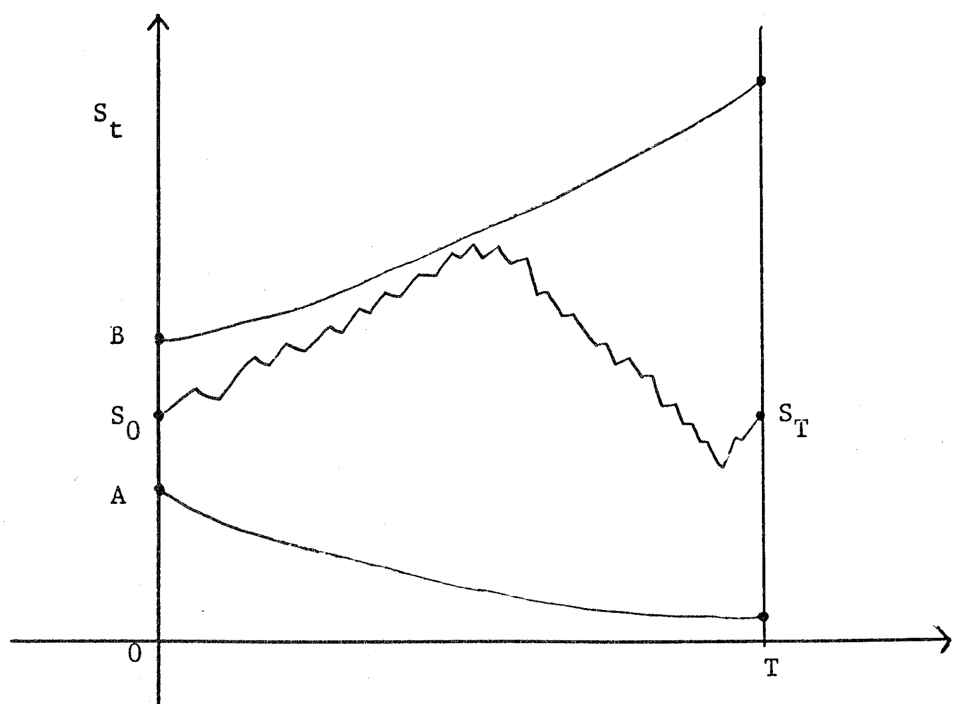
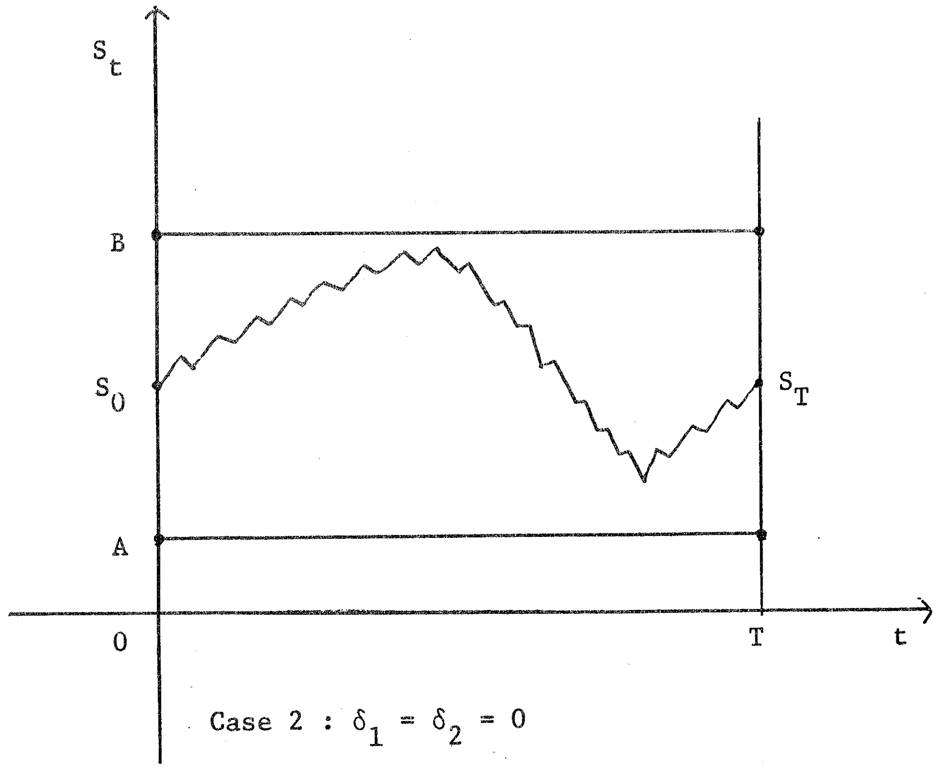


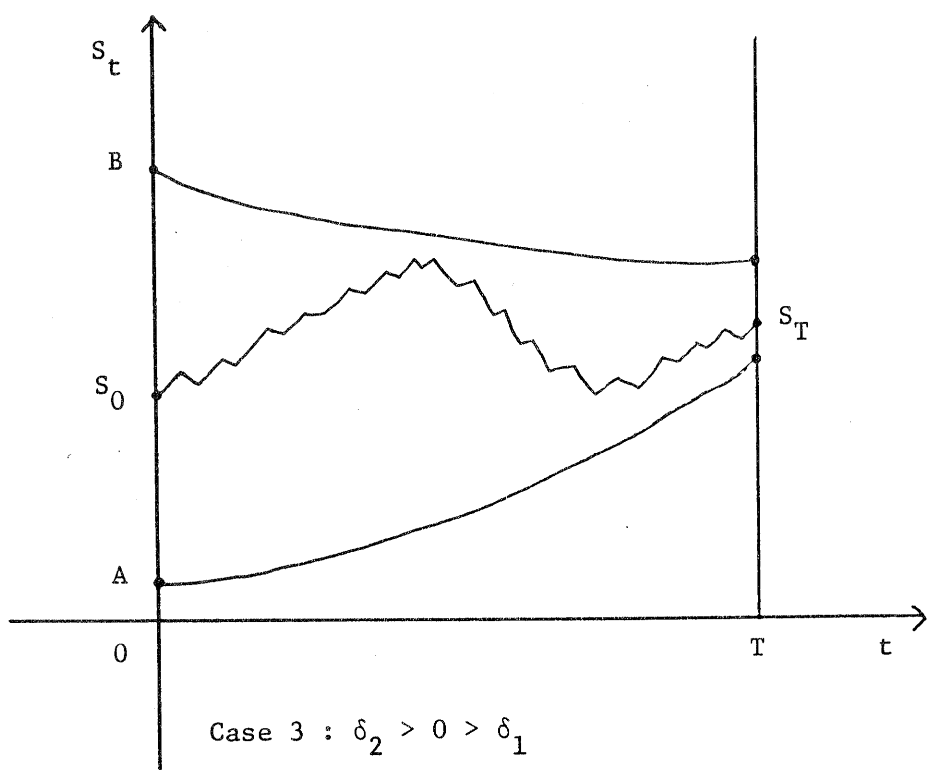
Figure 2 : Three Types of Boundaries



Case 1 : $\delta_2 < 0 < \delta_1$



Case 2 : $\delta_1 = \delta_2 = 0$



Case 3 : $\delta_2 > 0 > \delta_1$