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Abstract

This paper considers a multi-agent optimal investment problem with conservative sentiments in an incomplete market by a BSDE approach. Particularly, we formulate the conservative sentiments of the agents by a sup-inf/inf-sup problem where we take infimum on a choice of a probability measure and supremum on trading strategies. To the best of our knowledge, this is the first attempt to investigate a multi-agent equilibrium model in an incomplete setting with heterogeneous views on Brownian motions. Moreover, we show a square-root case and a general case where the trading strategies and the excess return process of the risky asset in equilibrium are explicitly solved. Finally, we present numerical examples of the trading strategies and the expected return process in equilibrium under conservative sentiments, which explain how the sentiments affect the trading strategies of the agents and the expected return process of the risky asset.

Multi-agent system, robust control, portfolio optimization, financial application

1 Introduction

This paper investigates a multi-agent optimal investment problem under an incomplete market setting with heterogeneous views on fundamental risks represented by Brownian motions. Particularly, we consider an exponential utility case, where the degrees of risk aversion and the conservative views on the fundamental risks differ among the agents. Also, we obtain the expected return process of the risky asset and the trading strategies in equilibrium. To the best of our knowledge, this is the first attempt to solve for equilibrium in a multi-agent model under an incomplete market setting with heterogeneous views of the agents. Our work enables asset pricing under heterogeneous conservative views of the agents in an incomplete market setting.

Sentiments of the market participants affect asset prices in financial markets, such as bond prices and stock prices, particularly after the global financial crisis (e.g., Nakatani

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et al. [18] and Nishimura et al. [17]). The number of fundamental risks driving the financial market is generally considered to exceed the number of risky assets as in stochastic volatility models. Thus, considering heterogeneous views of market participants in an incomplete market setting is important. Moreover, this study is useful since how the expected return on the risky asset changes when the sentiment of the market participants varies is essential in constructing a profitable trading strategy. For instance, when the major market participants in the equity market have different views and when their views change, it affects the equity prices through the trading of the market participants. If the sentiment changes are expected, it enables us to construct a trading strategy by predicting how the expected return on the risky asset shifts. Furthermore, policymakers, such as central banks, can estimate the effect of their announcement on the equity market through the change in the bandwidth of the sentiments of market participants in the model.

Since, in an incomplete market, the fundamental risks cannot be replicated with the tradable securities, the market price of risk that defines the risk-neutral probability measure used for securities pricing is not uniquely determined. In this study, we observe how the heterogeneous sentiments affect investment strategies and the expected return process in an incomplete market setting, where the number of Brownian motions is greater than the number of risky assets. Moreover, we obtain each agent's subjective probability measure that incorporates the agent's views on the fundamental risks in pricing.

The motivation of this study is to consider the multi-agent equilibrium optimal investment problem with sentiments in an incomplete market setting, which is an extension of existing studies. Since the incomplete market setting where the number of uncertainties exceeds the number of assets is a more practical situation, which is implemented in modern risky asset price processes such as the stochastic volatility models, it is beneficial for traders and policymakers to utilize the prediction of the expected return process of the risky asset depending on the shifts in the market sentiments.

For related literature, Petersen et al. [24] introduce relative entropy constraints to a stochastic system where the worst case is considered as a choice of a probability measure. Hansen and Sargent [7] apply the robust control theory to finance. They deal with a single agent's utility maximization problem with a choice of a probability measure on the conservative side. Also, as a problem of a choice of a probability measure, Chen and Epstein [2] investigate an optimal consumption problem in finance with ambiguity on risks. On the other hand, Choi and Larsen [3] derive incomplete market equilibrium under exponential utility without a choice of a probability measure. Kizaki et al. [12] consider a multi-agent optimal portfolio problem with conservative and aggressive sentiments in a complete market setting. Our work is different in that we incorporate heterogeneous views on the fundamental risks of multiple agents to obtain the expected return process in equilibrium in an incomplete market setting.

Also, for sentiments in the markets, Nishimura et al. [17] and Nakatani et al. [18] estimate sentiment factors in the interest rate models by using a text mining approach for the Japanese government bond markets. Saito and Takahashi [27] investigate a sup-inf problem on aggressive and conservative sentiments for a given state variable process. Saito and Takahashi [28] solve a sup-sup-inf problem for a single agent, where the agent works on an optimal investment problem under aggressive and conservative sentiments by a Malliavin calculus approach. Our work investigates a multi-agent model with sup-inf/inf-sup problem for individual optimization problems in an incomplete setting, where we solve for an equilibrium expected return process of the risky asset and the subjective probability measure of the agents, which are useful in pricing assets with heterogeneous views on fundamental risks.

Specifically, for optimal portfolio problems on multi-agent systems, Yang et al. [33] investigate principal-agent problems for a contract design with multiple agents, where a principal solves a utility maximization problem. Leung et al. [16] consider a decentralized robust portfolio optimization problem with a cooperative-competitive multi-agent system. (For other studies on multi-agent systems, see e.g., Kumar & Bhattacharya[13], Lee et al.[15], Park et al.[23], Pinto et al.[26], Gharesifard et al. [6] and Yang et al. [34],[35]. For applications of stochastic control to optimal portfolio problems in financial risk management, see, e.g., Cui et al. [5], Kasbekar et al.[11], He et al.[8], Ni et al.[19],[20], [21], Ye and Zhou [36], Lamperski and Cowan [14], Sen [30], Jiang and Fu [9], Wu et al.[32], Aybat et al. [1]). Our study differs in that we investigate market clearing on assets among the agents, who have different views on Brownian motions and risk-aversion parameters, to obtain the expected return process in equilibrium.

The contributions of this study are as follows. To the best of our knowledge, this study is the first attempt to investigate the multi-agent equilibrium under an incomplete market setting where the agents have heterogeneous views on fundamental risks. Kizaki et al. [12] obtained the market equilibrium where the agents have heterogeneous views on fundamental risks but in a complete market setting. This study extends the case to an incomplete market setting, where the number of Brownian motions that drive the market exceeds the number of risky tradable assets. Specifically, with a square-root case, where the standard results for the existence and uniqueness of a solution and the comparison principle for the BSDE with stochastic Lipschitz driver do not apply since the terminal condition is unbounded, we first solve the sup-inf/inf-sup problem for the portfolio (the sup part) and the conservative view (the inf part). Then, we provide a general case where a Gaussian case in Appendix B, in which the existence and uniqueness of a solution and the comparison principle for standard BSDEs hold, is also included.

For the practical contribution, we solve the expected return of the risky asset in an incomplete market. By the expression of the equilibrium expected return, traders can predict the changes in the expected return of stocks by the shift in the market sentiments and construct a profitable trading strategy. Also, policymakers such as central banks can make announcements at the best timing, predicting the effect of their announcement on the return of the risky assets by affecting the bandwidth of the sentiments of the market participants.

The organization of this paper is as follows. After Section 2 introduces the individual optimization problem in an incomplete market, Section 3 shows a square-root case, where the excess return process in equilibrium is explicitly solved. Section 4 provides numerical examples. Section 5 provides a theorem for a general case that also includes a Gaussian case in Appendix B. Finally, Section 6 concludes. Appendix A provides the proof of the theorem in Section 5. Appendix B presents a Gaussian state process case, an example of the general case in Section 5.

2 Setting

In this section, we explain the multi-agent model with heterogeneous views on fundamental risks in an incomplete market. Firstly, we describe the setting of the financial market and then introduce the individual optimization problem of each agent. We consider the following financial market where there are I agents trading one risky asset and a money market account.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let (W_Y, W_S) be a two-dimensional Brownian motion defined on the probability space. Let $\{\mathcal{F}_t\}_{0\le t\le T}$ be the augmented filtration generated by the two-dimensional Brownian motion (W_Y, W_S). Let $\mu_Y, \sigma_Y, \sigma, \rho_S, \hat{\rho}_S := \sqrt{1-\rho_S^2}$, be $\mathcal{R}$-valued $\{\mathcal{F}_t\}$-progressively measurable processes defined on $[0,T]$. Particularly, we assume $\sigma_t > 0, |\rho_{S,t}| \le 1,\ 0 \le t \le T$, and hence $|\hat{\rho}_{S,t}| \le 1$. $x_{i,0}, y_0, i = 1, \dots, I$ are positive constants. Let $\lambda = (\lambda_Y, \lambda_S)^\top$ be a $\mathcal{R}^2$-valued $\{\mathcal{F}_t\}$-progressively measurable process. Also, $\bar{\lambda}_{Y,i}, \bar{\lambda}_{S,i},\ i = 1, \dots, I,\ I \ge 2$ are $\mathcal{R}$-valued $\{\mathcal{F}_t\}$-progressively measurable processes.

Let S_0, S_1 be the price process of the money market account and the risky asset satisfying SDEs

$$\begin{aligned} dS_{0,t} &= rS_{0,t}dt,\ S_{0,0} = 1, \\ dS_{1,t} &= \mu_t S_{1,t}dt + \sigma_t S_{1,t}(\rho_{S,t}dW_{Y,t} + \hat{\rho}_{S,t}dW_{S,t}), \\ S_{1,t} &= p, \end{aligned} \tag{1}$$

where the initial value of the risky asset price p is exogenously given. Specifically, we assume $r \equiv 0$ throughout this study and obtain the expected return process μ in equilibrium, which satisfies market clearing conditions.

Suppose that there exist I agents in the market who trade the money market account and the risky asset aiming to maximize their expected utility on the sum of the wealth and the wealth shock represented by the state process at the terminal time T. Let π_i be the portfolio process satisfying $\int_0^T |\pi_{i,s}\mu_s|ds < \infty,\ \int_0^T \pi_{i,s}^2\sigma_s^2 ds < \infty,\ P-a.s.$, which describes the allocation of the agent i's portfolio on the risky asset on value basis. Then X^{π_i}, the wealth process of agent i, satisfies an SDE

$$\begin{aligned} dX_t^{\pi_i} &= \pi_{i,t}\theta_t\sigma_t dt + \pi_{i,t}\sigma_t(\rho_{S,t}dW_{Y,t} + \hat{\rho}_{S,t}dW_{S,t}), \\ X_0^{\pi_i} &= 0, \end{aligned} \tag{2}$$

where $\theta_t = \frac{\mu_t}{\sigma_t}$.

Let $\mathcal{A}_i$ be a set of admissible strategies which will be specified depending on the respective cases in Section 3 and Appendix B so that arbitrage opportunities are excluded for agent i.

Let $\Lambda_i = \{\lambda = (\lambda_Y, \lambda_S)^\top | |\lambda_{Y,t}| \le \bar{\lambda}_{Y,i,t}, |\lambda_{S,t}| \le \bar{\lambda}_{S,i,t}\}$ with exogenously given $\bar{\lambda}_{Y,i,t}, \bar{\lambda}_{S,i,t} > 0$.

Next, let Y be the state process, which satisfies an SDE

$$dY_t = \mu_{Y,t}dt + \sigma_{Y,t}dW_{Y,t},\ Y_0 = y_0. \tag{3}$$

This state process Y is a source of incompleteness, which cannot be traded in the market and could affect μ, σ, ρ_S, $\hat{\rho}_S$ of the risky asset price process S_1 in (1). We assume that one-time wealth shock Y_T at the terminal T is common among the I agents.

Agent i, $i = 1, \dots, I$ has an exponential utility function u_i on the sum of the terminal wealth $X_T^{\pi_i}$ and the one-time wealth shock Y_T, where $u_i(x) = -\exp(-\gamma_i x)$, $\gamma_i > 0$.

Remark 1 *We remark that although we assume the one-time wealth shock is Y_T for all $i = 1, \dots, I$, which is common among the agents, for simplicity, we can also handle the case where the wealth shock at the terminal T for agent i is a linear functional of Y_T such as $\alpha_i Y_T + \beta_i$, where $\alpha_i, \beta_i \in \mathcal{R}$ are constants, in the same way. This model corresponds to the case where $\alpha_i = 1$ and $\beta_i = 0$, which indicates a positive wealth shock if $Y_T > 0$. When $\alpha_i < 0$ with $Y_T > 0$, this implies a negative wealth shock.*

Also, the agent has conservative views on the fundamental risks related to the risky asset price and the state process. The agent aims to maximize its expected utility by choosing the trading strategy π_i while minimizing with respect to the views λ_Y and λ_S on the fundamental risks W_Y and W_S, respectively. Thus, we consider the following sup-inf/inf-sup problem as the individual optimization problem.

$$\sup_{\pi_i \in \mathcal{A}_i} \inf_{\lambda \in \Lambda_i} E^{P^\lambda}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))], \tag{4}$$

$$\inf_{\lambda \in \Lambda_i} \sup_{\pi_i \in \mathcal{A}_i} E^{P^\lambda}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))], \tag{5}$$

where P^λ is defined as

$$\frac{dP^\lambda}{dP} = \exp\biggl(-\frac{1}{2}\int_0^T (\lambda_{S,t}^2 + \lambda_{Y,t}^2)dt + \int_0^T \lambda_{S,t} dW_{S,t} + \int_0^T \lambda_{Y,t} dW_{Y,t}\biggr).$$

As in Section 3 and Appendix B, when the weak version of Novikov's condition (e.g., Corollary 3.5.14 in Karatzas and Shreve [10]) holds, by Girsanov's theorem, $dW_{S,t} = dW_{S,t}^\lambda + \lambda_{S,t} dt$, $dW_{Y,t} = dW_{Y,t}^\lambda + \lambda_{Y,t} dt$, where $(W_Y^\lambda, W_S^\lambda)$ is a Brownian motion under P^λ. Thus $\lambda_{S,t} dt$ and $\lambda_{Y,t} dt$ indicate the agent i's biases on the instantaneous increment of the fundamental risks $dW_{S,t}$ and $dW_{Y,t}$.

Then, given the optimal trading strategies $\pi_i^*, i = 1, \dots, I$ obtained by the individual optimization problems (4) and (5), we call that the market is in equilibrium if the following market-clearing conditions are satisfied.

$$\sum_{i=1}^I \pi_{i,t}^* = 0,\ 0 \le \forall t \le T. \tag{6}$$

and

$$\sum_{i=1}^I (X_t^{\pi_{i,t}^*} - \pi_{i,t}^*) = 0,\ 0 \le \forall t \le T, \tag{7}$$

where (π_i^*, λ_i^*), $i = 1, \dots, I$ are the pairs that attain the sup-inf/inf-sup problem in (4) and (5) for agent i. (6) is the market clearing condition for the risky asset position, and (7) is the market clearing condition for the money market account, which follows from (2) and (6).

In the following sections, our aim is to find the expected return process μ of S_1 in (1) in equilibrium. Concretely, first presupposing the form of the views on the fundamental risks $\lambda_i^* = (\lambda_{Y,i}^*, \lambda_{S,i}^*)^\top$, $i = 1, \dots, I$, we solve the individual optimization problems (4) and (5). Then, imposing the market clearing conditions (6) and (7), we obtain the candidate of the expected return process in equilibrium. Finally, in theorems, given the candidate of the expected return process μ, we solve the individual optimization problems (4) and (5) and confirm that the market is in equilibrium.

3 The square-root case where the sup-inf/inf-sup individual optimization problem is solved

In this section, we show the second case where the individual optimization problems are concretely solved. We consider the sup-inf/inf-sup problem

$$\sup_{\pi_i \in \mathcal{A}_i} \inf_{\lambda \in \Lambda_i} E^{P^\lambda}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))]$$
$$\inf_{\lambda \in \Lambda_i} \sup_{\pi_i \in \mathcal{A}_i} E^{P^\lambda}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))], \tag{8}$$

with the following square-root process for Y instead of (3).

$$dY_t = (\mu_{Y,1,t} Y_t + \mu_{Y,2,t})dt + \sigma_{Y,t}\sqrt{Y_t} dW_{Y,t}, \tag{9}$$

where $\mu_{Y,1}, \mu_{Y,2}, \sigma_Y$ are nonrandom processes with $\mu_{Y,1,t} \leq 0, \mu_{Y,2,t} \geq 0, \sigma_{Y,t} > 0,\ 0 \leq t \leq T$.

Let π be a $\{\mathcal{F}_t\}$-progressively measurable process with $\int_0^T \pi_s^2 \sigma_s^2 ds < \infty, P{-}a.s.$ We also assume that Λ_i, the set of the views on the fundamental risks λ, has the following square-root form $\Lambda_i = \{\lambda_i = (l_{Y,i,t}\sqrt{Y_t}, l_{S,i,t}\sqrt{Y_t})^\top | -\lambda^\star_{Y,i,t} \leq l_{Y,i,t} \leq \lambda^\star_{Y,i,t}, -\lambda^\star_{S,i,t} \leq l_{S,i,t} \leq \lambda^\star_{S,i,t}\}$, where $\lambda^\star_{Y,i}$ and $\lambda^\star_{S,i}$ are nonnegative random processes.

We consider the set of admissible strategies $\mathcal{A}_i$ as $\mathcal{A}_i = \{\pi_i | X^{\pi_i}\text{is a } Q_i^{\lambda_i^*}\text{-supermartingale}\}$, where $Q_i^{\lambda_i^*}$ is defined as

$$\frac{dQ_i^{\lambda_i^*}}{dP} = \frac{u_i'(X_T^{\pi_i^*} + Y_T)}{E^{P^\lambda}[u_i'(X_T^{\pi_i^*} + Y_T)]} \frac{dP^{\lambda_i^*}}{dP},$$

where $\lambda_{i,t}^* = (-\lambda^\star_{Y,i,t}\sqrt{Y_t}, -\lambda^\star_{S,i,t}\sqrt{Y_t})^\top$, $\pi_{i,t}^* = \frac{1}{\gamma_i \sigma_t}(\theta_t + \rho_{S,t}\lambda^\star_{Y,i,t}\sqrt{Y_t} + \hat{\rho}_{S,t}\lambda^\star_{S,i,t}\sqrt{Y_t} - \gamma_i \rho_{S,t}\sigma_{Y,t} a_{i,t}^* \sqrt{Y_t})$, $\theta_t = \sum_{j=1}^I \frac{1}{\gamma_j}\Gamma(\rho_{S,t}\lambda^\star_{Y,j,t}\sqrt{Y_t}$ $+ \hat{\rho}_{S,t}\lambda^\star_{S,j,t}\sqrt{Y_t} + \gamma_j \rho_{S,t} a_{j,t}^* \sigma_{Y,t}\sqrt{Y_t})$. Here, $\Gamma = \frac{1}{\sum_{k=1}^I \frac{1}{\gamma_k}}$ and $(a_1^*, \dots, a_I^*)$ is a unique solution of Riccati equations (10) defined in Theorem 1 in the following.

The sup-inf/inf-sup problem is solved, and the trading strategies and the expected return process in equilibrium for the square-root case are obtained in the following theorem.

Theorem 1 *Suppose that the system of Riccati equations*

$$-\dot{a}^*_{i,t}$$
$$= \frac{1}{2\gamma_i}(\sum_{j=1}^{I}\frac{1}{\gamma_j}\Gamma(\rho_{S,t}\lambda^\star_{Y,j,t} + \hat{\rho}_{S,t}\lambda^\star_{S,j,t} + \gamma_j\rho_{S,t}a^*_{j,t}\sigma_{Y,t}) - \rho_{S,t}\lambda^\star_{Y,i,t} - \hat{\rho}_{S,t}\lambda^\star_{S,i,t} - \gamma_i\rho_{S,t}a^*_{i,t}\sigma_{Y,t})^2$$
$$+a^*_{i,t}(\mu_{Y,1,t} - \lambda^\star_{Y,i,t}\sigma_{Y,t}) - \frac{1}{2}\gamma_i a^{*2}_{i,t}\sigma^2_{Y,t},$$
$$a^*_{i,T} = 1,\ i = 1, \ldots, I, \tag{10}$$

has a unique solution $(a^*_1, \ldots, a^*_I)$ *in* $[0, T]$.

In addition, we assume that the solution $(a^*_1, \ldots, a^*_I)$ *satisfies the following conditions*

$$\frac{\rho_{S,t}}{\gamma_i}\left(\sum_{j=1}^{I}\frac{1}{\gamma_j}\Gamma(\rho_{S,t}\lambda^\star_{Y,j,t} + \hat{\rho}_{S,t}\lambda^\star_{S,j,t} + \gamma_j\rho_{S,t}\sigma_{Y,t}a^*_{j,t}) - \rho_{S,t}\lambda^\star_{Y,i,t} - \hat{\rho}_{S,t}\lambda^\star_{S,i,t} - \gamma_i\rho_{S,t}\sigma_{Y,t}a^*_{i,t}\right) + \sigma_{Y,t}a^*_{i,t} \geq 0, \tag{11}$$

$$\frac{\hat{\rho}_{S,t}}{\gamma_i}\left(\sum_{j=1}^{I}\frac{1}{\gamma_j}\Gamma(\rho_{S,t}\lambda^\star_{Y,j,t} + \hat{\rho}_{S,t}\lambda^\star_{S,j,t} + \gamma_j\rho_{S,t}\sigma_{Y,t}a^*_{j,t}) - \rho_{S,t}\lambda^\star_{Y,i,t} - \hat{\rho}_{S,t}\lambda^\star_{S,i,t} - \gamma_i\rho_{S,t}\sigma_{Y,t}a^*_{i,t}\right) \geq 0. \tag{12}$$

Then, for the expected return process μ *in (1) given by*

$$\mu_t = \sigma_t\theta_t = \sigma_t\sum_{j=1}^{I}\frac{1}{\gamma_j}\Gamma(\rho_{S,t}\lambda^\star_{Y,j,t}\sqrt{Y_t} + \hat{\rho}_{S,t}\lambda^\star_{S,j,t}\sqrt{Y_t} + \gamma_j\rho_{S,t}a^*_{j,t}\sigma_{Y,t}\sqrt{Y_t}),$$

(π^*_i, λ^*_i) *where* $\pi^*_{i,t} = \frac{1}{\gamma_i\sigma_t}(\theta_t - \rho_{S,t}\lambda^\star_{Y,i,t}\sqrt{Y_t} - \hat{\rho}_{S,t}\lambda^\star_{S,i,t}\sqrt{Y_t} - \gamma_i\rho_{S,t}a^*_{i,t}\sigma_{Y,t}\sqrt{Y_t})$ *and* $\lambda^*_{i,t} = (-\lambda^\star_{Y,i,t}\sqrt{Y_t}, -\lambda^\star_{S,i,t}\sqrt{Y_t})^\top$ *attains the sup-inf/inf-sup problem (8) and the market is in equilibrium.*

Remark 2 *In the numerical example in Section 4, we will show a case where the Riccati equations are numerically solved, and the solutions satisfy the conditions (11) and (12). As we will observe in Section 3.2, the conditions (11) and (12) indicate* $\mathcal{Z}_S$ *and* $\mathcal{Z}_Y$ *are positive, and (33) further implies that the agents' positions are long on the risky asset. Since the agents are conservative with respect to their views on the risks of the risky asset and the endowment,* $\lambda^*_{S,t} = -\bar{\lambda}_{S,t}sgn(\mathcal{Z}^{\lambda^*}_{S,t})$ *and* $\lambda^*_{Y,t} = -\bar{\lambda}_{Y,t}sgn(\mathcal{Z}^{\lambda^*}_{Y,t})$ *indicate that they see less expected return on the risky asset and the endowment process under the subjective probability measure* $P^{\lambda^*_i}$ *than the return under the physical probability measure* P*. We may also consider the opposite case where some agents are short on the risky asset, which corresponds to* $\mathcal{Z}_S$ *and* $\mathcal{Z}_Y$ *being negative, which indicates that the agents see higher drift on the long risky asset positions and the endowment process under* $P^{\lambda^*_i}$ *than the drift under* P.

Proof.

First, an inequality $\sup_{\pi_i\in\mathcal{A}_i}\inf_{\lambda_i\in\Lambda_i} J_i(\lambda_i,\pi_i) \le \inf_{\lambda_i\in\Lambda_i}\sup_{\pi_i\in\mathcal{A}_i} J_i(\lambda_i,\pi_i)$ naturally holds since the admissible set $\mathcal{A}_i$ is independent of λ_i, where $\mathcal{A}_i$ is a set of strategies π_i such that X^{π_i} is a supermartingale under $Q_i^{\lambda_i^*}$.

The opposite side of the inequality $\sup_{\pi_i\in\mathcal{A}_i}\inf_{\lambda_i\in\Lambda_i} J_i(\lambda_i,\pi_i) \ge \inf_{\lambda_i\in\Lambda_i}\sup_{\pi_i\in\mathcal{A}_i} J_i(\lambda_i,\pi_i)$ also holds, which can be proved by showing (λ_i^*,π_i^*) is the saddle point, i.e.,

$$J_i(\lambda_i^*,\pi_i) \le J_i(\lambda_i^*,\pi_i^*) \le J_i(\lambda_i,\pi_i^*),\ \forall \lambda_i\in\Lambda_i, \pi_i\in\mathcal{A}_i. \tag{13}$$

Thus, in this square-root case, the inf-sup and the sup-inf case are solved and proved to coincide.

In the following, we will show that the first part of (13)

$$J_i(\lambda_i^*,\pi_i) \le J_i(\lambda_i^*,\pi_i^*), \tag{14}$$

follows from the convex duality argument and the second part of (13)

$$J_i(\lambda_i^*,\pi_i^*) \le J_i(\lambda_i,\pi_i^*), \tag{15}$$

follows from the martingale representation of R_i under $P^{\lambda_i^*}$, where we define $R_{i,t} = -\exp(-\gamma_i(X_t^{\pi_i^*} + V_{i,t}))$ with V_i satisfying following BSDE.

For $\lambda_{i,t}^* = (-\lambda^\star_{Y,i,t}\sqrt{Y_t}, -\lambda^\star_{S,i,t}\sqrt{Y_t})^\top$, we consider a BSDE under $P^{\lambda_i^*}$

$$\begin{cases} dV_{i,t} = -f_i(Z_{i,t})dt + Z_{i,t}dW_{Y,t}^{\lambda_i^*}, \\ V_{i,T} = Y_T, \end{cases} \tag{16}$$

with

$$f_i(Z_{i,t}) = \frac{1}{2\gamma_i}(\theta_t^{\lambda_i^*} - \gamma_i\rho_{S,t}Z_{i,t})^2 - \frac{1}{2}\gamma_i Z_{i,t}^2, \tag{17}$$

$$\begin{aligned} \theta_t^{\lambda_i^*} &= \theta_t - \rho_{S,t}\lambda^\star_{Y,i,t}\sqrt{Y_t} - \hat{\rho}_{S,i,t}\lambda^\star_{S,i,t}\sqrt{Y_t}, \\ \theta_t &= \sum_{j=1}^{I}\Gamma\frac{1}{\gamma_j}(\rho_{S,t}\lambda^\star_{Y,j,t}\sqrt{Y_t} + \hat{\rho}_{S,t}\lambda^\star_{S,j,t}\sqrt{Y_t} + \gamma_j\rho_{S,t}a^*_{j,t}\sigma_{Y,t}\sqrt{Y_t}), \\ Y_T &= y_0 + \int_0^T((\mu_{Y,1,t} - \lambda^\star_{Y,i,t}\sigma_{Y,t})Y_t + \mu_{Y,2,t})dt + \int_0^T \sigma_{Y,t}\sqrt{Y_t}dW_{Y,t}^{\lambda_i^*}, \end{aligned}$$

which can be solved as follows.

We show that V_i expressed as

$$V_{i,t} = a^*_{i,t}Y_t + b^*_{i,t},\ a^*_{i,T} = 1, b^*_{i,T} = 0,$$

satisfies the BSDE (16), where $a^*_{i,t}, b^*_{i,t}$ are nonrandom processes differentiable with respect to t satisfying Riccati equations (10) and $-\dot{b}^*_{i,t} = a^*_{i,t}\mu_{Y,2,t}$.

Calculating $dV_{i,t}$ and comparing it with BSDE (16),

$$\begin{aligned} dV_{i,t} &= a^*_{i,t}dY_t + Y_t\dot{a}^*_{i,t}dt + \dot{b}^*_{i,t}dt \\ &= \{(a^*_{i,t}(\mu_{Y,1,t} - \lambda^\star_{Y,i,t}\sigma_{Y,t}) + \dot{a}^*_{i,t})Y_t + a^*_{i,t}\mu_{Y,2,t} + \dot{b}^*_{i,t}\}dt + a^*_{i,t}\sigma_{Y,t}\sqrt{Y_t}dW_{Y,t}^{\lambda_i^*}, \end{aligned}$$

we have

$$Z_{i,t} = a^*_{i,t}\sigma_{Y,t}\sqrt{Y_t}. \tag{18}$$

Since f_i in (17) becomes

$$\frac{1}{2\gamma_i}(\sum_{j=1}^{I}\frac{1}{\gamma_j}\Gamma(\rho_{S,t}\lambda^\star_{Y,j,t}\sqrt{Y_t} + \hat{\rho}_{S,t}\lambda^\star_{S,j,t}\sqrt{Y_t} + \gamma_j\rho_{S,t}a^*_{j,t}\sigma_{Y,t}\sqrt{Y_t})$$
$$-\rho_{S,t}\lambda^\star_{Y,i,t}\sqrt{Y_t} - \hat{\rho}_{S,t}\lambda^\star_{S,i,t}\sqrt{Y_t} - \gamma_i\rho_{S,t}Z_{i,t})^2 - \frac{1}{2}\gamma_i Z^2_{i,t},$$

substituting the expression of Z_i in (18), we have

$$(\frac{1}{2\gamma_i}(\sum_{j=1}^{I}\frac{\Gamma}{\gamma_j}(\rho_{S,t}\lambda^\star_{Y,j,t} + \hat{\rho}_{S,t}\lambda^\star_{S,j,t} + \gamma_j\rho_{S,t}a^*_{j,t}\sigma_{Y,t}) - \rho_{S,t}\lambda^\star_{Y,i,t} - \hat{\rho}_{S,t}\lambda^\star_{S,i,t} - \gamma_i\rho_{S,t}a^*_{i,t}\sigma_{Y,t})^2 - \frac{\gamma_i}{2}(a^*_{i,t})^2\sigma^2_{Y,t})Y_t,$$

which is equivalent to

$$-\{(a^*_{i,t}(\mu_{Y,t} - \lambda^\star_{Y,i,t}\sigma_{Y,t}) + \dot{a}^*_{i,t})Y_t + a^*_{i,t}\mu_{Y,2,t} + \dot{b}^*_{i,t}\}.$$

Hence we obtain the system of Riccati equations in (10),

$$\begin{gathered}
-\dot{b}^*_{i,t} = a^*_{i,t}\mu_{Y,2,t}, \\
-\dot{a}^*_{i,t} = \\
\left(\frac{1}{2\gamma_i}(\sum_{j=1}^{I}\frac{\Gamma}{\gamma_j}(\rho_{S,t}\lambda^\star_{Y,j,t} + \hat{\rho}_{S,t}\lambda^\star_{S,j,t} + \gamma_j\rho_{S,t}a^*_{j,t}\sigma_{Y,t}) - \rho_{S,t}\lambda^\star_{Y,i,t} - \hat{\rho}_{S,t}\lambda^\star_{S,i,t} - \gamma_i\rho_{S,t}a^*_{i,t}\sigma_{Y,t})^2 - \frac{\gamma_i}{2}(a^*_{i,t})^2\sigma^2_{Y,t}\right) \\
+a^*_{i,t}(\mu_{Y,t} - \lambda^\star_{Y,i,t}\sigma_{Y,t}).
\end{gathered} \tag{19}$$

By the assumption, the system of Riccati equations has a unique solution $(a^*_1, \ldots, a^*_I)$ in $[0, T]$ that satisfies the conditions (11) and (12), and thus (V_i, Z_i) is a solution of BSDE (16).

Step 1. First, for $\lambda^*_{i,t} = (-\lambda^\star_{Y,i,t}\sqrt{Y_t}, -\lambda^\star_{S,i,t}\sqrt{Y_t})^\top$, we show that π^*_i where $\pi^*_{i,t} = \frac{1}{\gamma_i\sigma_t}(\theta_t - \rho_{S,t}\lambda^\star_{Y,i,t}\sqrt{Y_t} - \hat{\rho}_{S,t}\lambda^\star_{S,i,t}\sqrt{Y_t} - \gamma_i\rho_{S,t}a^*_{i,t}\sigma_{Y,t}\sqrt{Y_t})$ attains the sup.

Concretely, we consider

$$\sup_{\pi_i\in\mathcal{A}_i} E^{P^{\lambda^*_i}}[-\exp(-\gamma_i(X^{\pi_i}_T + Y_T))],$$

where

$$\begin{aligned}
dY_t &= [(\mu_{Y,1,t} - \sigma_{Y,t}\lambda^\star_{Y,i,t})Y_t + \mu_{Y,2,t}]dt + \sigma_{Y,t}\sqrt{Y_t}dW^{\lambda^*_i}_{Y,t}, \\
dX^{\pi_i}_t &= \pi_{i,t}\sigma_t(\theta_t - \rho_{S,t}\lambda^\star_{Y,i,t}\sqrt{Y_t} - \hat{\rho}_{S,i,t}\lambda^\star_{S,i,t}\sqrt{Y_t})dt \\
&\quad +\pi_{i,t}\sigma_t(\rho_{S,t}dW^{\lambda^*_i}_{Y,t} + \hat{\rho}_{S,t}dW^{\lambda^*_i}_{S,t}),
\end{aligned}$$

$$\begin{aligned}
dW^{\lambda^*_i}_{Y,t} &= dW_{Y,t} - \lambda^\star_{Y,i,t}\sqrt{Y_t}dt, \\
dW^{\lambda^*_i}_{S,t} &= dW_{S,t} - \lambda^\star_{S,i,t}\sqrt{Y_t}dt.
\end{aligned}$$

First, we note the following martingale property for R_i, where $R_{i,t} = -\exp(-\gamma_i(X^{\pi^*_i}_t + V_{i,t}))$, and define a new probability measure $Q^{\lambda^*_i}_i$ by R_i.

Lemma 1 *For R_i defined as $R_{i,t} = -\exp(-\gamma_i(X_t^{\pi_i^*} + V_{i,t}))$, R_i is a $P^{\lambda_i^*}$-martingale.*

Proof.

$$
\begin{aligned}
dR_{i,t} &= -\gamma_i R_{i,t} d(X_t^{\pi_i^*} + V_{i,t}) + \frac{1}{2}\gamma_i^2 R_{i,t} d\langle X^{\pi_i^*} + V_i\rangle_t \\
&= -\gamma_i R_{i,t}\Bigg((\pi_{i,t}^* \sigma_t \theta_t^{\lambda_i^*} - \frac{1}{2}\gamma_i((\pi_{i,t}^*\sigma_t\rho_{S,t} + Z_{i,t})^2 + (\pi_{i,t}^*\sigma_t\hat{\rho}_{S,t})^2) \\
&\quad -f_i(Z_{i,t}))dt + (\pi_{i,t}^*\sigma_t\rho_{S,t} + Z_{i,t})dW_{Y,t}^{\lambda_i^*} + \pi_{i,t}^*\sigma_t\hat{\rho}_{S,t}dW_{S,t}^{\lambda_i^*}\Bigg) \\
&= -\gamma_i R_{i,t}\Bigg((\pi_{i,t}^*\sigma_t\rho_{S,t} + Z_{i,t})dW_{Y,t}^{\lambda_i^*} + \pi_{i,t}^*\sigma_t\hat{\rho}_{S,t}dW_{S,t}^{\lambda_i^*}\Bigg),
\end{aligned}
\tag{20}
$$

where the drift part is calculated as

$$
\begin{aligned}
&(\pi_{i,t}^*\sigma_t\theta_t^{\lambda^*} - \frac{1}{2}\gamma_i((\pi_{i,t}^*\sigma_t\rho_{S,t} + Z_{i,t})^2 + (\pi_{i,t}^*\sigma_t\hat{\rho}_{S,t})^2) - f_i(Z_{i,t})) \\
&= -\frac{1}{2}\gamma_i\sigma_t^2(\pi_{i,t}^* - \frac{1}{\gamma_i\sigma_t}(\theta_t^{\lambda_i^*} - \gamma_i\rho_{S,t}Z_{i,t}))^2 = 0,
\end{aligned}
$$

where $Z_{i,t} = a_{i,t}^*\sigma_{Y,t}\sqrt{Y_t}$, and by Theorem 3.2 in Shirakawa [29], the weak version of Novikov condition holds and R_i is a $P^{\lambda_i^*}$-martingale. ■

Next, we define a probability measure $Q_i^{\lambda_i^*}$ by

$$
\frac{dQ_i^{\lambda_i^*}}{dP^{\lambda_i^*}} = \frac{u_i'(X_T^{\pi_i^*} + Y_T)}{E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*} + Y_T)]} = \frac{R_{i,T}}{E^{P^{\lambda_i^*}}[R_{i,T}]},
$$

where

$$
u_i'(x) = \gamma_i \exp(-\gamma_i x).
$$

Since

$$
\begin{aligned}
u_i'(X_t^{\pi_i^*} + V_{i,t}) &= \gamma_i \exp(-\gamma_i(X_t^{\pi_i^*} + V_{i,t})), \\
&= -\gamma_i R_{i,t},
\end{aligned}
$$

and by (20)

$$
d(-\gamma_i R_{i,t}) = -\gamma_i R_{i,t}(-\gamma_i(\pi_{i,t}^*\sigma_t\rho_{S,t} + Z_{i,t})dW_{Y,t}^{\lambda_i^*} - \gamma_i\pi_{i,t}^*\sigma_t\hat{\rho}_{S,t}dW_{S,t}^{\lambda_i^*}),
$$

By Girsanov's theorem, $(W_S^{Q_i^{\lambda_i^*}}, W_Y^{Q_i^{\lambda_i^*}})$ defined by

$$
\begin{aligned}
dW_{Y,t}^{Q_i^{\lambda_i^*}} &= dW_{Y,t}^{\lambda_i^*} + \gamma_i(\pi_{i,t}^*\sigma_t\rho_{S,t} + Z_{i,t})dt, \\
dW_{S,t}^{Q_i^{\lambda_i^*}} &= dW_{S,t}^{\lambda_i^*} + \gamma_i\pi_{i,t}^*\sigma_t\hat{\rho}_{S,t}dt,
\end{aligned}
$$

is a $Q_i^{\lambda_i^*}$-Brownian motion.

Then, by $\pi^*_{i,t} = \frac{1}{\gamma_i \sigma_t}(\theta_t - \rho_{S,t}\lambda^\star_{Y,i,t}\sqrt{Y_t} - \hat{\rho}_{S,t}\lambda^\star_{S,i,t}\sqrt{Y_t} - \gamma_i \rho_{S,t} a^*_{i,t}\sigma_{Y,t}\sqrt{Y_t}) = \frac{1}{\gamma_i \sigma_t}(\theta^{\lambda^*_i}_t - \gamma_i \rho_{S,t} a^*_{i,t}\sigma_{Y,t}\sqrt{Y_t})$,

$$\begin{aligned} &\rho_{S,t} dW^{\lambda^*_i}_{Y,t} + \hat{\rho}_{S,t} dW^{\lambda^*_i}_{S,t} \\ &= \rho_{S,t} dW^{Q^{\lambda^*_i}_i}_{Y,t} + \hat{\rho}_{S,t} dW^{Q^{\lambda^*_i}_i}_{S,t} - \theta^{\lambda^*_i}_t dt, \end{aligned}$$

and thus by

$$\begin{aligned} dX^{\pi_i}_t &= \pi_{i,t}\sigma_t(\theta_t - \rho_{S,t}\lambda^\star_{Y,i,t}\sqrt{Y_t} - \hat{\rho}_{S,i,t}\lambda^\star_{S,i,t}\sqrt{Y_t})dt + \pi_{i,t}\sigma_t(\rho_{S,t} dW^{\lambda^*_i}_{Y,t} + \hat{\rho}_{S,t} dW^{\lambda^*_i}_{S,t}) \\ &= \pi_{i,t}\sigma_t(\theta^{\lambda^*_i}_t dt + \rho_{S,t} dW^{\lambda^*_i}_{Y,t} + \hat{\rho}_{S,t} dW^{\lambda^*_i}_{S,t}), \end{aligned} \tag{21}$$

we have

$$dX^{\pi_i}_t = \pi_{i,t}\sigma_t(\rho_{S,t} dW^{Q^{\lambda^*_i}_i}_{Y,t} + \hat{\rho}_{S,t} dW^{Q^{\lambda^*_i}_i}_{S,t}).$$

By the assumption, for $\pi_i \in \mathcal{A}_i$, X^{π_i} is a $Q^{\lambda^*_i}_i$-supermartingale. Also, $X^{\pi^*_i}$ is a $Q^{\lambda^*_i}_i$-martingale since $E^{Q^{\lambda^*_i}_i}[\int_0^T (\pi^*_{i,t})^2 \sigma^2_t dt] < \infty$, where $\pi^*_{i,t}\sigma_t = \frac{1}{\gamma_i}(\theta_t - \rho_{S,t}\lambda^\star_{Y,i,t}\sqrt{Y_t} - \hat{\rho}_{S,t}\lambda^\star_{S,i,t}\sqrt{Y_t} - \gamma_i \rho_{S,t}\sigma_{Y,t} a^*_{i,t}\sqrt{Y_t})$, $\theta_t = \sum_{j=1}^I \frac{1}{\gamma_j}\Gamma(\rho_{S,t}\lambda^\star_{Y,j,t}\sqrt{Y_t} + \hat{\rho}_{S,t}\lambda^\star_{S,j,t}\sqrt{Y_t} + \gamma_j \rho_{S,t} a^*_{j,t}\sigma_{Y,t}\sqrt{Y_t})$, which is due to the integrability of Y and the fact that $a^*_j,\ j = 1, \ldots, I$ are continuous functions bounded on $[0, T]$.

Then, by using the convex duality argument, we show that π^*_i attains the supremum. See Section 3.1 for details.

Step 2. Next, we show that $\lambda^*_{i,t} = (-\lambda^\star_{Y,i,t}\sqrt{Y_t}, -\lambda^\star_{S,i,t}\sqrt{Y_t})^\top$ attains $\inf_{\lambda_i \in \Lambda_i} E^{P^{\lambda_i}}[-\exp(-\gamma_i(X^{\pi^*_i}_T + Y_T))]$.

Note that by Lemma 1, $R_{i,t} = -\exp(-\gamma_i(X^{\pi^*_i}_t + V_{i,t}))$ is a martingale under $P^{\lambda^*_i}$ satisfying an SDE

$$dR_{i,t} = \mathcal{Z}_{S,i,t} dW^{\lambda^*_i}_{S,t} + \mathcal{Z}_{Y,i,t} dW^{\lambda^*_i}_{Y,t}, \tag{22}$$

where

$$\begin{aligned} \mathcal{Z}_{S,i,t} &= -\gamma_i R_{i,t}(\pi^*_{i,t}\sigma_t\rho_{S,t} + a^*_{i,t}\sigma_{Y,t}\sqrt{Y_t}) \\ \mathcal{Z}_{Y,i,t} &= -\gamma_i R_{i,t}(\pi^*_{i,t}\sigma_t\hat{\rho}_{S,t}). \end{aligned}$$

Then,

$$E^{P^{\lambda^*_i}}[-\exp(-\gamma_i(X^{\pi^*_i}_T + Y_T))] = E^{P^{\lambda^*_i}}[R_{i,T}] = R_{i,0}. \tag{23}$$

By the localization argument in Section 3.2, the inequality holds.

Finally, we can confirm that the market is in equilibrium in the same way as in Lemma 4. ■

3.1 The convex duality argument for the sup part

In this subsection, we provide details of the proof of Theorem 1 for the sup-part in Step 1. Specifically, we show

$$\begin{aligned} & E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i}+Y_T))] \\ & \leq E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i^*}+Y_T))], \forall \pi_i \in \mathcal{A}_i, \end{aligned}$$

by a convex duality argument.

We note that the following properties on the convex duality hold.

Let

$$\tilde{u}_i(y) = \sup_{x\in R}(u_i(x) - xy)$$

for all $y > 0$, where $u_i(x) = -\exp(-\gamma_i x)$.

Then, for all $x \in R$, $y > 0$,

$$u_i(x) \leq \tilde{u}_i(y) + yx, \tag{24}$$

$$\tilde{u}_i(u_i'(x)) + u_i'(x)x = u_i(x). \tag{25}$$

By (24),

$$u_i(X_T^{\pi_i}+Y_T) \leq \tilde{u}_i(E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]\frac{dQ_i^{\lambda_i^*}}{dP^{\lambda_i^*}}) + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]\frac{dQ_i^{\lambda_i^*}}{dP^{\lambda_i^*}}(X_T^{\pi_i}+Y_T),$$

where we set

$$\begin{aligned} x &= X_T^{\pi_i}+Y_T, \\ y &= E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]\frac{dQ_i^{\lambda_i^*}}{dP^{\lambda_i^*}}. \end{aligned}$$

$$\begin{aligned} & E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i}+Y_T))] = E^{P^{\lambda_i^*}}[u_i(X_T^{\pi_i}+Y_T)] \\ & \leq E^{P^{\lambda_i^*}}[\tilde{u}_i(E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]\frac{dQ_i^{\lambda_i^*}}{dP^{\lambda_i^*}})] + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]E^{P^{\lambda_i^*}}[\frac{dQ_i^{\lambda_i^*}}{dP^{\lambda_i^*}}(X_T^{\pi_i}+Y_T)] \\ & = E^{P^{\lambda_i^*}}[\tilde{u}_i(u_i'(X_T^{\pi_i^*}+Y_T))] + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]E^{Q_i^{\lambda_i^*}}[(X_T^{\pi_i}+Y_T)] \qquad (26) \\ & \leq E^{P^{\lambda_i^*}}[\tilde{u}_i(u_i'(X_T^{\pi_i^*}+Y_T))] + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]E^{Q_i^{\lambda_i^*}}[(X_T^{\pi_i^*}+Y_T)] \qquad (27) \\ & = E^{P^{\lambda_i^*}}[\tilde{u}_i(u_i'(X_T^{\pi_i^*}+Y_T))] + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)(X_T^{\pi_i^*}+Y_T)] \qquad (28) \\ & = E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i^*}+Y_T))]. \qquad (29) \end{aligned}$$

(27) follows since X^{π_i} is a $Q_i^{\lambda_i^*}$-supermartingale and $X^{\pi_i^*}$ is a $Q_i^{\lambda_i^*}$-martingale, (26) and (28) from the definition of $Q_i^{\lambda_i^*}$, and (29) from (25).

3.2 The localization argument for the inf-part

This subsection provides the details of the localization argument for the inf-part in Step 2 of the proof for Theorem 1.

First, we define a sequence of stopping times

$$\tau_j := j \wedge \inf\{t \geq 0 || \mathcal{Z}_t| \geq j\},\ j = 1, 2, \ldots, \tag{30}$$

that satisfies

$$\tau_1 \leq \tau_2 \leq \ldots,\ \text{and}\ \lim_{j\to\infty} \tau_j = \infty, \tag{31}$$

in particular $\lim_{j\to\infty} R_{t\wedge\tau_j} = R_t$.

Since

$$dR_{i,t} = \mathcal{Z}_{s,i,t} dW_{S,t}^{\lambda_i^*} + \mathcal{Z}_{Y,i,t} dW_{Y,t}^{\lambda_i^*}, \tag{32}$$

where

$$\begin{aligned} \mathcal{Z}_{S,i,t} &= -\gamma_i R_{i,t}(\pi_{i,t}^* \sigma_t \rho_{S,t} + a_{i,t}^* \sigma_{Y,t} \sqrt{Y_t}), \\ \mathcal{Z}_{Y,i,t} &= -\gamma_i R_{i,t}(\pi_{i,t}^* \sigma_t \hat{\rho}_{S,t}), \end{aligned} \tag{33}$$

we have

$$\begin{aligned} R_{i,t} &= E^{P^{\lambda_i^*}}[R_{i,0}] + \int_0^t \mathcal{Z}_{S,i,s} dW_{S,s}^{\lambda_i^*} + \int_0^t \mathcal{Z}_{Y,i,s} dW_{Y,s}^{\lambda_i^*} \\ &= E^{P^{\lambda_i^*}}[R_{i,0}] + \int_0^t (-(\lambda_{i,s}^* - \lambda_{i,s})^\top (\mathcal{Z}_{S,i,s}, \mathcal{Z}_{Y,i,s})^\top) ds + \int_0^t \mathcal{Z}_{S,i,s} dW_{S,s}^{\lambda_i} + \int_0^t \mathcal{Z}_{Y,i,s} dW_{Y,s}^{\lambda_i}. \end{aligned} \tag{34}$$

Taking the expectation under P^{λ_i} for the stopped process

$$\begin{aligned} R_{i,T\wedge\tau_j} = E^{P^{\lambda_i^*}}[R_{i,0}] &+ \int_0^{T\wedge\tau_j} (-(\lambda_{i,s}^* - \lambda_{i,s})^\top (\mathcal{Z}_{S,i,s}, \mathcal{Z}_{Y,i,s})^\top) ds \\ &+ \int_0^{T\wedge\tau_j} \mathcal{Z}_{S,i,s} dW_{S,s}^{\lambda_i} + \int_0^{T\wedge\tau_j} \mathcal{Z}_{Y,i,s} dW_{Y,s}^{\lambda_i}, \end{aligned} \tag{35}$$

we obtain

$$E^{P^{\lambda_i}}[R_{i,T\wedge\tau_j}] = E^{P^{\lambda_i^*}}[R_{i,0}] + E^{P^{\lambda_i}}[\int_0^{T\wedge\tau_j} (-(\lambda_{i,s}^* - \lambda_{i,s})^\top (\mathcal{Z}_{S,i,s}, \mathcal{Z}_{Y,i,s})^\top) ds] \geq E^{P^{\lambda_i^*}}[R_{i,0}]. \tag{36}$$

Here we used the fact that $-(\lambda_{i,s}^* - \lambda_{i,s}) \geq 0$ and $\mathcal{Z}_{S,i,s}, \mathcal{Z}_{Y,i,s} \geq 0, 0 \leq \forall s \leq T$. We remark that $\mathcal{Z}_{S,i,s}, \mathcal{Z}_{Y,i,s} \geq 0, 0 \leq \forall s \leq T$ follows from conditions (11) and (12) by substituting the expressions of π^* and θ into (33).

By the reverse Fatou's lemma, we have

$$E^{P^{\lambda_i}}[R_{i,T}] = E^{P^{\lambda_i}}[\varlimsup_{j\to\infty} R_{i,T\wedge\tau_j}] \geq \varlimsup_{j\to\infty} E^{P^{\lambda_i}}[R_{i,T\wedge\tau_j}] \geq R_{i,0} = E^{P^{\lambda_i^*}}[R_{i,T}]. \tag{37}$$

Therefore, $\inf_{\lambda_i\in\Lambda_i} J_i(\lambda_i, \pi_i^*)$ is attained at $\lambda_i = \lambda_i^*$.

4 Numerical example

In this section, we present numerical examples of the equilibrium trading strategies and the expected return process in the square-root case in Section 3. We consider the square-root case where $I = 2$, where we call the two agents agent 1 and agent 2. We suppose that agent 1 has conservative views on the fundamental risks, while agent 2 has neutral views, i.e., $\lambda^{\star}_{Y,2}, \lambda^{\star}_{S,2} \equiv 0$.

We consider the following sup-inf/inf-sup problem as described in Section 3, namely,

$$\sup_{\pi \in \mathcal{A}_i} \inf_{\lambda \in \Lambda_i} E^{P^\lambda}[-\exp(-\gamma_i(X^\pi_T + Y_T))],$$
$$\inf_{\lambda \in \Lambda_i} \sup_{\pi \in \mathcal{A}_i} E^{P^\lambda}[-\exp(-\gamma_i(X^\pi_T + Y_T))],$$

where

$$\begin{aligned} dS_{1,t} &= \mu_t S_{1,t} dt + \bar{\sigma}\sqrt{Y_t} S_{1,t}(\rho_S dW_{Y,t} + \hat{\rho}_S dW_{S,t}), \\ S_{1,t} &= p, \end{aligned}$$

$$\begin{aligned} dY_t &= (\mu_{Y,1} Y_t + \mu_{Y,2}) dt + \sigma_Y \sqrt{Y_t} dW_{Y,t} \\ &= -\mu_{Y,1}\left(\frac{\mu_{Y,2}}{-\mu_{Y,1}} - Y_t\right) dt + \sigma_Y \sqrt{Y_t} dW_{Y,t}, \ Y_0 = y, \end{aligned}$$

$\mu_{Y,1} < 0, \mu_{Y,2} > 0$, and

$$\begin{aligned} dX^{\pi_i}_t &= \pi_{i,t}\theta_t \bar{\sigma}\sqrt{Y_t} dt + \pi_{i,t}\bar{\sigma}\sqrt{Y_t}(\rho_S dW_{Y,t} + \hat{\rho}_S dW_{S,t}), \\ X^{\pi_i}_0 &= 0, \end{aligned} \tag{38}$$

$p, y > 0$.

We observe that the state process Y is a mean-reverting process, and when $\rho_S > 0$, the risky asset price process S_1 has a positive correlation with the state process Y.

Then, the conditions (11) and (12) become

(i)

$$\frac{\rho_S}{\gamma_1}\Gamma\left(-\frac{1}{\gamma_2}(\rho_{S,t}\lambda^{\star}_{Y,1} + \hat{\rho}_S \lambda^{\star}_{S,1}) + \rho_S \sigma_Y (a^*_{1,t} + a^*_{2,t})\right) + \hat{\rho}^2_S \sigma_Y a^*_{1,t} \geq 0, \tag{39}$$

(ii)

$$\frac{\hat{\rho}_S}{\gamma_1\gamma_2}\Gamma\left(-(\rho_S\lambda^{\star}_{Y,1} + \hat{\rho}_S\lambda^{\star}_{S,1}) + \rho_S\sigma_{Y,t}(-\gamma_1 a^*_{1,t} + \gamma_2 a^*_{2,t})\right) \geq 0, \tag{40}$$

where $\Gamma = \frac{1}{\frac{1}{\gamma_1} + \frac{1}{\gamma_2}}$ and a^*_1, a^*_2 are solutions of the Riccati equations

$$\begin{aligned} &-\dot{a}^*_{i,t} \\ &= \frac{1}{2\gamma_i}\left(\sum_{j=1}^{2}\frac{1}{\gamma_j}\Gamma(\rho_S\lambda^{\star}_{Y,j} + \hat{\rho}_S\lambda^{\star}_{S,j} + \gamma_j\rho_S a^*_{j,t}\sigma_Y) - \rho_S\lambda^{\star}_{Y,i} - \hat{\rho}_S\lambda^{\star}_{S,i} - \gamma_i\rho_S a^*_{i,t}\sigma_Y\right)^2 \\ &\quad -\frac{1}{2}\gamma_i a^{*,2}_{i,t}\sigma^2_Y + a^*_{i,t}(\mu_{Y,1} - \lambda^{\star}_{Y,i}\sigma_Y), \\ &a^*_{i,T} = 1, \ i = 1, 2. \end{aligned} \tag{41}$$

Moreover, by Theorem 1, the expected return process μ for the risky asset price process S_1 in equilibrium is given by $\mu_t = \sigma_t\theta_t = \bar{\sigma}\sqrt{Y_t}\theta_t$ where

$$\theta_t = \Gamma \sum_{j=1}^{2} \frac{1}{\gamma_j}(\rho_S\lambda^\star_{Y,j} + \hat{\rho}_S\lambda^\star_{S,j} + \gamma_j\rho_S a^*_{j,t}\sigma_Y)\sqrt{Y_t},$$

and (π^*_i, λ^*_i), $i = 1, 2$, are obtained as $\pi^*_{i,t} = \frac{1}{\gamma_i\bar{\sigma}\sqrt{Y_t}}(\theta_t - \rho_S\lambda^\star_{Y,i}\sqrt{Y_t} - \hat{\rho}_S\lambda^\star_{S,i}\sqrt{Y_t}) - \rho_S a^*_{i,t}\frac{\sigma_{Y,t}}{\bar{\sigma}}$ and $\lambda^*_{i,t} = (-\lambda^\star_{Y,i}\sqrt{Y_t}, -\lambda^\star_{S,i}\sqrt{Y_t})^\top$.

In the following, we set the parameters as follows. $\mu_{Y,1} = 1, \mu_{Y,2} = -1, \sigma_Y \equiv 0.2, \lambda^\star_{Y,1} = 0.2, \lambda^\star_{Y,2} = 0, \lambda^\star_{S,1} = 0.2, \lambda^\star_{S,2} = 0, \rho_S = 0.5, \hat{\rho}_S = 0.866, \gamma_1 = 1, \gamma_2 = 20, y_0 = 0.5, T = 1.$ With these parameters, we calculate the optimal portfolio processes π^*_1 and π^*_2 and the expected return process μ for the risky asset price S_1 in equilibrium. Moreover, we compare the result with the case of neutral views for agent 1, where $\lambda^\star_{Y,1} = \lambda^\star_{S,1} = 0$.

Figure 1 presents the optimal portfolio processes of agents 1 and 2 and compares the two cases where agent 1 has conservative or neutral views. Firstly, this shows that the optimal portfolio of agent 1, π^*_1, is positive, and the portfolio of agent 2, π^*_2, is negative, which means that agent 1 has a long position while agent 2 takes a short position. Since the optimal portfolio processes are rewritten as

$$\pi^*_{1,t} = \frac{1}{\gamma_1\bar{\sigma}\sqrt{Y_t}}(\theta_t - \rho_S\lambda^\star_{Y,1}\sqrt{Y_t} - \hat{\rho}_S\lambda^\star_{S,1}\sqrt{Y_t}) - \rho_S\frac{\sigma_Y}{\bar{\sigma}}a^*_{1,t}, \tag{42}$$

$$\pi^*_{2,t} = \frac{1}{\gamma_2\bar{\sigma}\sqrt{Y_t}}\theta_t - \rho_S\frac{\sigma_Y}{\bar{\sigma}}a^*_{2,t},$$

when γ_2 is sufficiently large compared with γ_1, for the position of agent 1, $\frac{1}{\gamma_1\bar{\sigma}\sqrt{Y_t}}(\theta_t - \rho_S\lambda^\star_{Y,1}\sqrt{Y_t} - \hat{\rho}_S\lambda^\star_{S,1}\sqrt{Y_t})$, which is the mean-variance portfolio adjusted by the risk aversion parameter γ_i and the conservative views, is dominant for the position of agent 1, while it is small for agent 2 due to large γ_2 in $\frac{1}{\gamma_2\bar{\sigma}\sqrt{Y_t}}\theta_t$. Thus, the hedging portfolio $-\rho_S\frac{\sigma_Y}{\bar{\sigma}}a^*_{2,t}$, where $\rho_S, \sigma_Y, \bar{\sigma}, a^*_{2,t} > 0$, which is to offset the terminal wealth shock, is governing for agent 2. Since the net position of agents 1 and 2 is zero because of the clearing condition (6), π^*_1 is positive and π^*_2 is negative, meaning that agent 1 takes a long position and agent 2 a short position. We note that in equilibrium, the expected return $\mu_t = \bar{\sigma}\sqrt{Y_t}\theta_t$ is positive since if it is negative, the positions of both agents are short and the market clearing condition (6) is not satisfied.

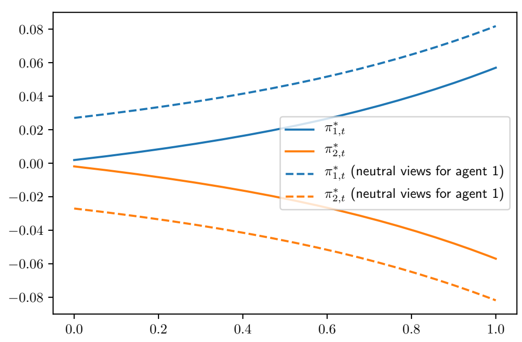

Figure 1: The optimal portfolio processes π^*_1 and π^*_2 when agent 1 has conservative views or neutral views

Secondly, we observe that the long amount of agent 1, π_1^*, is less when agent 1 has conservative views, which is explained as follows. The conservative sentiments make the long position of agent 1 π_1^* less due to the presence of $\lambda_Y^\star$, $\lambda_S^\star$ in the mean-variance term in (42) , which also makes the less short position π_2^* for agent 2 because of the clearing condition.

Finally, Figure 2 exhibits the expectation of the expected return process $E[\mu]$ in both cases with and without conservative views for agent 1. The expectation of the expected return $E[\mu]$ is higher when the agent has conservative views due to the presence of $\lambda_{Y,1}^\star$ and $\lambda_{S,1}^\star$ in the expression of μ in the following;

$$\begin{aligned}\mu_t &= \bar{\sigma}\sqrt{Y_t}\theta_t \\ &= \bar{\sigma}\Gamma\sum_{j=1}^{2}\frac{1}{\gamma_j}(\rho_S\lambda_{Y,j}^\star + \hat{\rho}_S\lambda_{S,j}^\star + \gamma_j\rho_S a_{j,t}^*\sigma_Y)Y_t,\end{aligned}$$

which can also be interpreted that agent 1 requires a higher expected return μ when agent 1 has a conservative view on the risky asset price to take a long position.

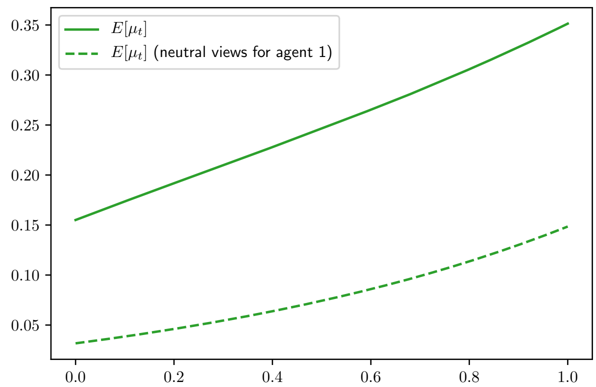

Figure 2: The expectation of the expected return in equilibrium $E[\mu]$ when agent 1 has conservative views or neutral views

Figure 3 describes the simulated solutions of the Riccati ODEs (41). As is easily observed with the comparison principle for the solution of the Riccati ODEs, $a_{1,t}^*$ with the conservative views is smaller than $a_{1,t}^*$ with the neutral views. Here, the conditions (i), (ii) in (39) and (40) are satisfied in both cases where agent 1 has the conservative views and the neutral views.

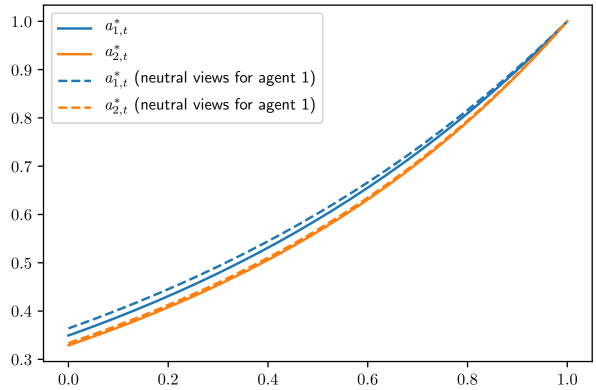

Figure 3: The solutions a_1^* and a_2^* of the Riccati ODEs (41) when agent 1 has conservative views or neutral views

5 The general case where the sup-inf/inf-sup problem is solved

In this section, we provide the excess return process in equilibrium in a general case where the state process is given by (3), and the sup-inf/inf-sup individual optimization problem is solved. This includes the square-root state process case in Section 3, where the existence and uniqueness result and the comparison principle for the BSDE with a stochastic Lipschitz driver do not apply since the terminal condition is unbounded. Moreover, the general case includes a Gaussian state process case in Appendix B, where the existence and uniqueness result and the comparison principle for BSDEs with a standard Lipschitz driver apply. This is proved in a similar way as in the proofs of Theorem 3 in Sections 3, respectively. See Appendix A for details. We let $\Gamma = \frac{1}{\sum_{k=1}^{I}\frac{1}{\gamma_k}}$ in the following.

Theorem 2 *Suppose that there exist* (V_i, Z_i) $i = 1, \ldots, I$*, that satisfy* $E[\sup_{0\le s\le T}|V_{i,s}|^2] < \infty$*,* $E[\int_0^T Z_{i,s}^2 ds] < \infty$ *and BSDEs*

$$dV_{i,t} = -(f_i(Z_{1,t}, \ldots, Z_{I,t}) - \bar{\lambda}_{Y,i,t}Z_{i,t})dt + Z_{i,t}dW_{Y,t},$$
$$V_{i,T} = Y_T,$$

where

$$f_i(Z_{1,t}, \ldots, Z_{I,t})$$
$$= \frac{1}{2\gamma_i}(\theta_t - \rho_{S,t}\bar{\lambda}_{Y,i,t} - \hat{\rho}_{S,i,t}\bar{\lambda}_{S,i,t} - \gamma_i\rho_{S,t}Z_{i,t})^2 - \frac{1}{2}\gamma_i Z_{i,t}^2,$$

and

$$\theta_t = \sum_{j=1}^{I}\frac{1}{\gamma_j}\Gamma(\rho_{S,t}\bar{\lambda}_{Y,i,t} + \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t} + \gamma_j\rho_{S,t}Z_{j,t}).$$

We assume that

$$\left\{\exp(-\frac{1}{2}\int_0^t \gamma_i^2(\pi_{i,s}^*\sigma_s\rho_{S,s}+Z_{i,s})^2+(\gamma_i\pi_{i,s}^*\sigma_s\hat{\rho}_{S,s})^2 ds \right.$$
$$\left. +\int_0^t \gamma_i(\pi_{i,s}^*\sigma_s\rho_{S,s}+Z_{i,s})dW_{Y,s}^{\lambda_i^*}+\gamma_i\pi_{i,s}^*\sigma_s\hat{\rho}_{S,s}dW_{S,s}^{\lambda_i^*})\right\}_{0\le t\le T} \tag{43}$$

is a $P^{\lambda_i^*}$*-martingale, where* $\lambda_{i,t}^*=(-\bar{\lambda}_{Y,i,t},-\bar{\lambda}_{S,i,t})^\top$, $\pi_i^*=\frac{1}{\gamma_i\sigma_t}(\theta_t-\rho_{S,t}\bar{\lambda}_{Y,i,t}-\hat{\rho}_{S,i,t}\bar{\lambda}_{S,i,t}-\gamma_i\rho_{S,t}Z_{i,t})$.

Also, for a probability measure Q_i *defined as*

$$\frac{dQ_i}{dP^{\lambda_i^*}}=\frac{u_i'(X_T^{\pi_i^*}+Y_T)}{E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_T)]},$$

where

$$u_i'(x)=\gamma_i\exp(-\gamma_i x),$$

we assume that

$$E^{Q_i}\left[\int_0^T \pi_{i,t}^{*2}\sigma_t^2 dt\right]<\infty. \tag{44}$$

Moreover, we suppose that for all $\lambda\in\Lambda_i$*, a BSDE*

$$\begin{aligned} d\mathcal{V}_t^\lambda \\ =-(\lambda_{S,t}\mathcal{Z}_{S,t}^\lambda+\lambda_{Y,t}\mathcal{Z}_{Y,t}^\lambda)dt+\mathcal{Z}_{S,t}^\lambda dW_{S,t}+\mathcal{Z}_{Y,t}^\lambda dW_{Y,t}, \\ \mathcal{V}_T^\lambda=\exp(-\gamma_i(X_T^{\pi_i^*}+Y_T)), \end{aligned} \tag{45}$$

has a unique solution $(\mathcal{V},\mathcal{Z}^\lambda)$ *satisfying* $E[\int_0^T(\mathcal{Z}_{S,t}^{\lambda 2}+\mathcal{Z}_{Y,t}^{\lambda 2})dt]<\infty$ *and* $E[\sup_{0\le t\le T}|\mathcal{V}_t^\lambda|^2]<\infty$*. We also suppose that* $\mathcal{V}_0^\lambda$ *is expressed as* $\mathcal{V}_0^\lambda=E^{P^\lambda}[\mathcal{V}_T^\lambda]$ *and* $\mathcal{V}_0^\lambda$ *is minimized at* $(\lambda_{S,i,t}^*,\lambda_{Y,i,t}^*)$ *with respect to* λ*, where* $\lambda_{S,i,t}^*=-\bar{\lambda}_{S,i,t}sgn(\mathcal{Z}_{S,t}^{\lambda_i^*})$, $\lambda_{Y,i,t}^*=-\bar{\lambda}_{Y,i,t}sgn(\mathcal{Z}_{Y,t}^{\lambda_i^*})$.

Furthermore, we assume the following conditions. For $i=1,\dots,I$,

$$\frac{\rho_{S,t}}{\gamma_i}\left(\sum_{j=1}^I\frac{1}{\gamma_j}\Gamma(\rho_{S,t}\bar{\lambda}_{Y,j,t}+\hat{\rho}_{S,t}\bar{\lambda}_{S,j,t}+\gamma_j\rho_{S,t}Z_{i,t})-\rho_{S,t}\bar{\lambda}_{Y,i,t}-\hat{\rho}_{S,t}\bar{\lambda}_{S,i,t}-\gamma_i\rho_{S,t}Z_{i,t}\right)+Z_{i,t}\ge 0, \tag{46}$$

$$\frac{\hat{\rho}_{S,t}}{\gamma_i}\left(\sum_{j=1}^I\frac{1}{\gamma_j}\Gamma(\rho_{S,t}\bar{\lambda}_{Y,j,t}+\hat{\rho}_{S,t}\bar{\lambda}_{S,j,t}+\gamma_j\rho_{S,t}Z_{i,t})\right)\ge 0. \tag{47}$$

Then, (π_i^*,λ_i^*) *given by* $\pi_{i,t}^*=\frac{1}{\gamma_i\sigma_t}(\theta_t-\rho_{S,t}\bar{\lambda}_{Y,i,t}-\hat{\rho}_{S,t}\bar{\lambda}_{S,i,t}-\gamma_i\rho_{S,t}Z_{i,t})$ *and* $\lambda_{i,t}^*=(-\bar{\lambda}_{Y,i,t},-\bar{\lambda}_{S,i,t})^\top$ *attains the sup-inf/inf-sup problem (4), (5) for admissible strategies* $\pi\in\mathcal{A}_i$*, where the set of the admissible strategies is given by* $\mathcal{A}_i=\{\pi|X^\pi$ *is a* Q_i*-supermartingale*$\}$.

6 Conclusion

In this study, we have investigated a multi-agent equilibrium model with heterogeneous views on fundamental risks in an incomplete market setting. We have obtained the expressions of the expected return process in equilibrium in the cases of the square-root state case with the random bound and a general state process with the nonrandom bound, where the sup-inf/inf-sup type individual optimization problems are solved, for the views on Brownian motions. We have also presented numerical examples.

The implications of this study are as follows. Firstly, by utilizing the expected return process in equilibrium, traders can predict how the expected return on the risky asset changes when the sentiments of the market participants shift and construct a profitable trading strategy for investment. Also, policymakers such as central banks can control market sentiments as a result of their announcement of monetary policies so that it affects the risky asset price process they target by influencing the bandwidth of the sentiments of the market participants. Secondly, as a theoretical implication, the result shows that the market equilibrium can be obtained in the incomplete market setting with multiple agents with heterogeneous views on fundamental risks. It implies the possibility of a further extension to the case where the agents have not only conservative views but also aggressive views on the fundamental risks, which is a further extension of the robust control-based individual optimization problem.

For limitations and future research, we have shown that the individual optimization problems are solved in the cases of the square-root state process and a general state process, assuming the one-time wealth shock depending on the state process, which is common among the agents and can be taken as a linear functional of the state process, and supposing the interest rate as zero. Extending the state process to a multi-dimensional one, investigating the case where the one-time wealth shock is a nonlinear functional of the state process, and solving for the equilibrium interest rate along with the excess return process are the next future research topics. Also, applying the model to security pricing under heterogeneous views on fundamental risks in an incomplete market is another future research topic.

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A Proof of Theorem 2

Let

$$J_i(\pi_i, \lambda) = E^{P^\lambda}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))].$$

If (π_i^*, λ_i^*) is a saddle point that satisfies

$$J_i(\pi_i, \lambda_i^*) \le J_i(\pi_i^*, \lambda_i^*) \le J_i(\pi_i^*, \lambda),$$

for all $\pi_i \in \mathcal{A}_i$ and $\lambda \in \Lambda_i$, (π_i^*, λ_i^*) attains the sup-inf (4) and the inf-sup (5).

We show that for given λ_i^*, $\pi = \pi_i^*$ attains the sup by the following convex dual argument.

Proposition 1 *Under assumptions of Theorem 3, for given* $\lambda_i^* = (-\bar{\lambda}_{Y,i,t}, -\bar{\lambda}_{S,i,t})^\top$, $\pi = \pi_i^*$ *attains* $\sup_{\pi \in \mathcal{A}_i} J_i(\pi, \lambda_i^*)$.

Proof.

We consider

$$\sup_{\pi_i \in \mathcal{A}_i} E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))],$$

where

$$\begin{aligned} dY_t &= (\mu_{Y,t} - \sigma_{Y,t}\bar{\lambda}_{Y,i,t})dt + \sigma_{Y,t} dW_{Y,t}^{\lambda_i^*}, \\ dX_t^{\pi_i} &= \pi_t\sigma_t(\theta_t - \rho_{S,t}\bar{\lambda}_{Y,i,t} - \hat{\rho}_{S,i,t}\bar{\lambda}_{S,t})dt + \pi_t\sigma_t(\rho_{S,t} dW_{Y,t}^{\lambda_i^*} + \hat{\rho}_{S,t} dW_{S,t}^{\lambda_i^*}), \end{aligned} \tag{48}$$

$$\begin{aligned} dW_{Y,t}^{\lambda_i^*} &= dW_{Y,t} - (-\bar{\lambda}_{Y,i,t})dt, \\ dW_{S,t}^{\lambda_i^*} &= dW_{S,t} - (-\bar{\lambda}_{S,i,t})dt. \end{aligned}$$

We show that π_i^* attains the sup.

First, we let

$$R_{i,t} = -\exp(-\gamma_i(X_t^{\pi_i^*} + V_{i,t})),$$

where

$$\pi_i^* = \frac{1}{\gamma_i \sigma_t}(\theta_t^{\lambda_i^*} - \gamma_i \rho_{S,t} Z_{i,t}), \tag{49}$$

(V_i, Z_i) $i = 1, \ldots, I$ are solutions of BSDEs

$$\begin{cases} dV_{i,t} = -f_i(Z_{1,t}, \ldots, Z_{I,t})dt + Z_{i,t} dW_{Y,t}^{\lambda_i^*}, \\ V_{i,T} = Y_T, \end{cases}$$

with

$$f_i(Z_{1,t}, \ldots, Z_{I,t}) = \frac{1}{2\gamma_i}(\theta_t^{\lambda_i^*} - \gamma_i \rho_{S,t} Z_{i,t})^2 - \frac{1}{2}\gamma_i Z_{i,t}^2,$$

$$\theta_t^{\lambda_i^*} = \theta_t - \rho_{S,t}\bar{\lambda}_{Y,i,t} - \hat{\rho}_{S,t}\bar{\lambda}_{S,t}.$$

Then,

$$\begin{aligned} dR_{i,t} &= -\gamma_i R_{i,t} d(X_t^{\pi_i^*} + V_{i,t}) + \frac{1}{2}\gamma_i^2 R_{i,t} d\langle X^{\pi_i^*} + V_i \rangle_t \\ &= -\gamma_i R_{i,t}\Big((\pi_{i,t}^* \sigma_t \theta_t^{\lambda_i^*} - \frac{1}{2}\gamma_i((\pi_{i,t}^*\sigma_t \rho_{S,t} + Z_{i,t})^2 + (\pi_t^* \sigma_t \hat{\rho}_{S,t})^2) - f_i(Z_{1,t}, \ldots, Z_{I,t}))dt \\ &\qquad + (\pi_{i,t}^* \sigma_t \rho_{S,t} + Z_{i,t}) dW_{Y,t}^{\lambda_i^*} + \pi_{i,t}^* \sigma_t \hat{\rho}_{S,t} dW_{S,t}^{\lambda_i^*} \Big) \\ &= -\gamma_i R_{i,t}\Big((\pi_{i,t}^* \sigma_t \rho_{S,t} + Z_{i,t}) dW_{Y,t}^{\lambda_i^*} + \pi_{i,t}^* \sigma_t \hat{\rho}_{S,t} dW_{S,t}^{\lambda_i^*} \Big), \end{aligned} \tag{50}$$

since the drift part is

$$\begin{aligned} &(\pi_{i,t}^* \sigma_t \theta_t^{\lambda^*} - \frac{1}{2}\gamma_i((\pi_{i,t}^* \sigma_t \rho_{S,t} + Z_{i,t})^2 + (\pi_{i,t}^* \sigma_t \hat{\rho}_{S,t})^2) - f_i(Z_{1,t}, \ldots, Z_{I,t})) \\ &= -\frac{1}{2}\gamma_i \sigma_t^2 (\pi_{i,t}^* - \frac{1}{\gamma_i \sigma_t}(\theta_t^{\lambda_i^*} - \gamma_i \rho_{S,t} Z_{i,t}))^2 + \frac{1}{2\gamma_i}(\theta_t^{\lambda^*} - \gamma_i \rho_{S,t} Z_{i,t})^2 - \frac{1}{2}\gamma_i Z_{i,t}^2 - f_i(Z_{1,t}, \ldots, Z_{I,t}) = 0. \end{aligned}$$

Next, we define a probability measure Q_i by

$$\frac{dQ_i}{dP^{\lambda_i^*}} = \frac{u_i'(X_T^{\pi_i^*} + Y_T)}{E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*} + Y_T)]}, \tag{51}$$

where

$$u_i'(x) = \gamma_i \exp(-\gamma_i x).$$

We remark that Q_i is well defined since $u_i'(x) > 0$ and $E^{P^{\lambda_i^*}}[\frac{dQ_i}{dP^{\lambda_i^*}}] = 1$. Since

$$\begin{aligned} u_i'(X_t^{\pi_i^*} + V_t) &= \gamma_i \exp(-\gamma_i(X_t^{\pi_i^*} + V_t)), \\ &= -\gamma_i R_t, \end{aligned}$$

and by (50)

$$d(-\gamma_i R_t) = -\gamma_i R_t(-\gamma_i(\pi^*_t \sigma_t \rho_{S,t} + Z_{i,t})dW^{\lambda^*_i}_{Y,t} \\ -\gamma_i \pi^*_t \sigma_t \hat{\rho}_{S,t} dW^{\lambda^*_i}_{S,t}),$$

we can apply Girsanov's theorem because R is a $P^{\lambda^*_i}$-martingale by (43), and $(W^{Q_i}_S, W^{Q_i}_Y)$ defined by

$$dW^{Q_i}_{Y,t} = dW^{\lambda^*_i}_{Y,t} + \gamma_i(\pi^*_t \sigma_t \rho_{S,t} + Z_{i,t})dt, \\ dW^{Q_i}_{S,t} = dW^{\lambda^*_i}_{S,t} + \gamma_i \pi^*_t \sigma_t \hat{\rho}_{S,t} dt,$$

is a Q_i-Brownian motion.

Then, by (49)

$$\rho_{S,t} dW^{\lambda^*_i}_{Y,t} + \hat{\rho}_{S,t} dW^{\lambda^*_i}_{S,t} \\ = \rho_{S,t} dW^{Q_i}_{Y,t} + \hat{\rho}_{S,t} dW^{Q_i}_{S,t} - \theta^{\lambda^*_i}_t dt,$$

and thus by (48)

$$dX^{\pi_i}_t = \pi_t \sigma_t(\rho_{S,t} dW^{Q_i}_{Y,t} + \hat{\rho}_{S,t} dW^{Q_i}_{S,t}).$$

By (44), it follows that for $\pi^*_i \in \mathcal{A}_i$, $X^{\pi^*_i}$ is a Q_i-martingale.

Finally, we show

$$E^{P^{\lambda^*_i}}[-\exp(-\gamma_i(X^{\pi_i}_T + Y_{i,T}))] \\ \leq E^{P^{\lambda^*_i}}[-\exp(-\gamma_i(X^{\pi^*_i}_T + Y_{i,T}))]$$

by a convex duality argument.

We note that the following properties on the convex duality hold.

Let

$$\tilde{u}_i(y) = \sup_{x \in \mathcal{R}}(u_i(x) - xy)$$

for all $y > 0$, where $u_i(x) = -\exp(-\gamma_i x)$.

Then, for all $x \in \mathcal{R}$, $y > 0$,

$$u_i(x) \leq \tilde{u}_i(y) + yx, \tag{52}$$

$$\tilde{u}_i(u'_i(x)) + u'_i(x)x = u_i(x). \tag{53}$$

By (52),

$$u_i(X^{\pi_i}_T + Y_{i,T}) \leq \tilde{u}_i(E^{P^{\lambda^*_i}}[u'_i(X^{\pi_i}_T + Y_{i,T})]\frac{dQ_i}{dP^{\lambda^*_i}}) + E^{P^{\lambda^*_i}}[u'_i(X^{\pi_i}_T + Y_{i,T})]\frac{dQ_i}{dP^{\lambda^*_i}}(X^{\pi_i}_T + Y_{i,T}),$$

where we set

$$x = X^{\pi_i}_T + Y_{i,T}, \\ y = E^{P^{\lambda^*_i}}[u'_i(X^{\pi^*_i}_T + Y_{i,T})]\frac{dQ_i}{dP^{\lambda^*_i}}.$$

Hence

$$
\begin{aligned}
&E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i}+Y_{i,T}))] = E^{P^{\lambda_i^*}}[u_i(X_T^{\pi_i}+Y_{i,T})] \\
\leq\ & E^{P^{\lambda_i^*}}[\tilde{u}_i(E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_{i,T})]\frac{dQ_i}{dP^{\lambda_i^*}})] + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_{i,T})]E^{P^{\lambda_i^*}}[\frac{dQ_i}{dP^{\lambda_i^*}}(X_T^{\pi_i}+Y_{i,T})] \\
=\ & E^{P^{\lambda_i^*}}[\tilde{u}_i(u_i'(X_T^{\pi_i^*}+Y_{i,T}))] + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_{i,T})]E^{Q_i}[(X_T^{\pi_i}+Y_{i,T})] \quad (54)\\
\leq\ & E^{P^{\lambda_i^*}}[\tilde{u}_i(u_i'(X_T^{\pi_i^*}+Y_{i,T}))] + E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_{i,T})]E^{Q_i}[(X_T^{\pi_i^*}+Y_{i,T})] \quad (55)\\
=\ & E^{P^{\lambda_i^*}}[\tilde{u}_i(u_i'(X_T^{\pi_i^*}+Y_{i,T}))] \\
&+E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*}+Y_{i,T})(X_T^{\pi_i^*}+Y_{i,T})] \quad (56)\\
=\ & E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i^*}+Y_{i,T}))]. \quad (57)
\end{aligned}
$$

(55) follows since X^{π_i} is a Q_i-supermartingale and $X^{\pi_i^*}$ is a Q_i-martingale. (54) and (56) are due to the definition of Q_i in (51), and (57) is obtained from (53).

For given π_i^*, we show that $\lambda = \lambda_i^*$ attains the inf by a BSDE approach.

Proposition 2 *Under assumptions of Theorem 3, for given* $\pi_{i,t}^* = \frac{1}{\gamma_i \sigma_t}(\theta_t - \rho_{S,t}\bar{\lambda}_{Y,i,t} - \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t} - \gamma_i\rho_{S,t}Z_{i,t})$, $\lambda = \lambda_i^*$ *attains* $\inf_{\lambda\in\Lambda_i} J_i(\pi_i^*, \lambda)$.

Proof.
Firstly, for $\lambda \in \Lambda_i$, we consider a BSDE

$$
\begin{aligned}
d\mathcal{V}_t^\lambda &= \mathcal{Z}_{S,t}^\lambda(dW_{S,t} - \lambda_{S,t}dt) + \mathcal{Z}_{Y,t}^\lambda(dW_{Y,t} - \lambda_{Y,t}dt) \\
&= -(\lambda_{S,t}\mathcal{Z}_{S,t}^\lambda + \lambda_{Y,t}\mathcal{Z}_{Y,t}^\lambda)dt + \mathcal{Z}_{S,t}^\lambda dW_{S,t} + \mathcal{Z}_{Y,t}^\lambda dW_{Y,t}, \\
\mathcal{V}_T^\lambda &= R_{i,T}.
\end{aligned}
$$

Also, we note that $\mathcal{V}_0^\lambda = E^{P^\lambda}[R_{i,T}]$ and $\mathcal{V}_0^\lambda$ is minimized at $(\lambda_{S,t}^*, \lambda_{Y,t}^*)$, $\lambda_{S,t}^* = -\bar{\lambda}_{S,t}sgn(\mathcal{Z}_{S,t}^{\lambda^*})$, $\lambda_{Y,t}^* = -\bar{\lambda}_{Y,t}sgn(\mathcal{Z}_{Y,t}^{\lambda^*})$, which satisfies $\lambda_{S,t}^*\mathcal{Z}_{S,t}^{\lambda^*} = -\bar{\lambda}_{S,t}|\mathcal{Z}_{S,t}^{\lambda^*}|$, $\lambda_{Y,t}^*\mathcal{Z}_{Y,t}^{\lambda^*} = -\bar{\lambda}_{Y,t}|\mathcal{Z}_{Y,t}^{\lambda^*}|$, by the assumption.

Note that $R_{i,t} = -\exp(-\gamma_i(X_t^{\pi_i^*} + V_{i,t}))$ is a martingale under $P^{\lambda_i^*}$ satisfying an SDE

$$
dR_{i,t} = \mathcal{Z}_{S,i,t}dW_{S,i,t}^{\lambda_i^*} + \mathcal{Z}_{Y,i,t}dW_{Y,i,t}^{\lambda_i^*},
$$

where

$$
\begin{aligned}
\mathcal{Z}_{S,i,t} &= -\gamma_i R_{i,t}(\pi_{i,t}^*\sigma_t\rho_{S,t} + Z_{i,t}) \\
\mathcal{Z}_{Y,i,t} &= -\gamma_i R_{i,t}(\pi_{i,t}^*\sigma_t\hat{\rho}_{S,t}).
\end{aligned}
$$

Hence, to show that $\lambda_{S,t}^* = -\bar{\lambda}_{S,t}$, $\lambda_{Y,t}^* = -\bar{\lambda}_{Y,t}$, we have only to confirm $\mathcal{Z}_{S,i,t}, \mathcal{Z}_{Y,i,t} \geq 0$, namely,

$$
\begin{aligned}
\frac{\rho_{S,t}}{\gamma_i}(\theta_t^{\lambda_i^*} - \gamma_i\rho_{S,t}Z_{i,t}) + Z_{i,t} &\geq 0, \\
\frac{\hat{\rho}_{S,t}}{\gamma_i}(\theta_t^{\lambda_i^*} - \gamma_i\rho_{S,t}Z_{i,t}) &\geq 0,
\end{aligned}
$$

which is satisfied by conditions (46) and (47). ■

B The Gaussian case where the sup-inf/inf-sup individual optimization problem is solved

In this section, we solve the individual optimization problem for given expected return process μ of the risky asset process S_1 in (1) when Y in (3) is a Gaussian process, where $\bar{\lambda}_{Y,i}$, $\bar{\lambda}_{S,i}$, μ_Y, σ_Y and ρ_S are deterministic processes.

The following theorem holds for the expected return process and the trading strategy of the individual optimization problem (4) and (5) in equilibrium for the Gaussian case. We let $\Gamma = \frac{1}{\sum_{k=1}^{I} \frac{1}{\gamma_k}}$.

Theorem 3 *Suppose that the expected return process μ is given by $\mu_t = \sigma_t \theta_t$ where*

$$\theta_t = \sum_{j=1}^{I} \frac{1}{\gamma_j} \Gamma(\rho_{S,t}\bar{\lambda}_{Y,j,t} + \hat{\rho}_{S,t}\bar{\lambda}_{S,j,t} + \gamma_j \rho_{S,t}\sigma_{Y,t}). \tag{58}$$

Also, we assume that the following conditions hold.

$$\frac{\rho_{S,t}}{\gamma_i}\left(\sum_{j=1}^{I} \frac{1}{\gamma_j}\Gamma(\rho_{S,t}\bar{\lambda}_{Y,j,t} + \hat{\rho}_{S,t}\bar{\lambda}_{S,j,t} + \gamma_j\rho_{S,t}\sigma_{Y,t}) - \rho_{S,t}\bar{\lambda}_{Y,i,t} - \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t} - \gamma_i\rho_{S,t}\sigma_{Y,t}\right) + \sigma_{Y,t} \geq 0, \tag{59}$$

$$\begin{aligned}\frac{\hat{\rho}_{S,t}}{\gamma_i}\Bigg(\sum_{j=1}^{I} \frac{1}{\gamma_j}\Gamma(\rho_{S,t}\bar{\lambda}_{Y,j,t} + \hat{\rho}_{S,t}\bar{\lambda}_{S,j,t} + \gamma_j\rho_{S,t}\sigma_{Y,t})\\ -\rho_{S,t}\bar{\lambda}_{Y,i,t} - \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t} - \gamma_i\rho_{S,t}\sigma_{Y,t}\Bigg) \geq 0.\end{aligned} \tag{60}$$

Then, (π_i^*, λ_i^*) given by $\pi_{i,t}^* = \frac{1}{\gamma_i \sigma_t}(\theta_t - \rho_{S,t}\bar{\lambda}_{Y,i,t} - \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t} - \gamma_i\rho_{S,t}\sigma_{Y,t})$ and $\lambda_{i,t}^* = (-\bar{\lambda}_{Y,i,t}, -\bar{\lambda}_{S,i,t})^\top$ attains the sup-inf/inf-sup problem (4), (5) for admissible strategies $\pi \in \mathcal{A}_i$, where the set of the admissible strategies is given by $\mathcal{A}_i = \{\pi | X^\pi \text{is a } Q_i\text{-supermartingale}\}$, where a probability measure Q_i is defined as

$$\frac{dQ_i}{dP^{\lambda_i^*}} = \frac{u_i'(X_T^{\pi_i^*} + Y_T)}{E^{P^{\lambda_i^*}}[u_i'(X_T^{\pi_i^*} + Y_T)]}.$$

Moreover, the market clearing conditions (6) and (7) hold.

Remark 3 *We remark that the following case, where there are two agents and one agent has neutral views, is an example that satisfies the conditions (59) and (60). $I = 2$. $\gamma_1, \gamma_2 > 0$, $\rho_{S,t}, \hat{\rho}_{S,t} > 0$, $\sigma_{Y,t} > 0$. We assume $\bar{\lambda}_{Y,2}, \bar{\lambda}_{S,2} \equiv 0$. Then, the conditions (59) and (60) become*

$$\frac{\rho_{S,t}}{\gamma_1}\left(\sum_{j=1}^{2} \frac{\frac{1}{\gamma_j}}{\sum_{k=1}^{2}\frac{1}{\gamma_k}}(\rho_{S,t}\bar{\lambda}_{Y,j,t} + \hat{\rho}_{S,t}\bar{\lambda}_{S,j,t} + \gamma_j\rho_{S,t}\sigma_{Y,t}) - (\rho_{S,t}\bar{\lambda}_{Y,1,t} + \hat{\rho}_{S,t}\bar{\lambda}_{S,1,t} + \gamma_1\rho_{S,t}\sigma_{Y,t})\right) + \sigma_{Y,t} \geq 0,$$

and

$$\frac{\hat{\rho}_{S,t}}{\gamma_1}\left(\sum_{j=1}^{2} \frac{\frac{1}{\gamma_j}}{\sum_{k=1}^{2}\frac{1}{\gamma_k}}(\rho_{S,t}\bar{\lambda}_{Y,j,t} + \hat{\rho}_{S,t}\bar{\lambda}_{S,j,t} + \gamma_j\rho_{S,t}\sigma_{Y,t}) - (\rho_{S,t}\bar{\lambda}_{Y,1,t} + \hat{\rho}_{S,t}\bar{\lambda}_{S,1,t} + \gamma_1\rho_{S,t}\sigma_{Y,t})\right) \geq 0.$$

Proof.

Let

$$J_i(\pi_i, \lambda) = E^{P^\lambda}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))].$$

If (π_i^*, λ_i^*) is a saddle point that satisfies

$$J_i(\pi_i, \lambda_i^*) \leq J_i(\pi_i^*, \lambda_i^*) \leq J_i(\pi_i^*, \lambda),$$

for all $\pi_i \in \mathcal{A}_i$ and $\lambda \in \Lambda_i$, (π_i^*, λ_i^*) attains the sup-inf in (4) and the inf-sup in (5).

First, we show that for given λ_i^*, $\pi_i = \pi_i^*$ attains the sup as follows using a super-martingale property.

Lemma 2 *Under assumptions of Theorem 3, for given* $\lambda_i^* = (-\bar{\lambda}_{Y,i,t}, -\bar{\lambda}_{S,i,t})^\top$, $\pi_i = \pi_i^*$ *attains* $\sup_{\pi_i \in \mathcal{A}_i} J_i(\pi_i, \lambda_i^*)$.

Proof.

We consider

$$\sup_{\pi_i \in \mathcal{A}_i} E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))],$$

where

$$\begin{aligned} dY_t &= (\mu_{Y,t} - \sigma_{Y,t}\bar{\lambda}_{Y,i,t})dt + \sigma_{Y,t} dW_{Y,t}^{\lambda_i^*}, \\ dX_t^{\pi_i} &= \pi_t \sigma_t(\theta_t - \rho_{S,t}\bar{\lambda}_{Y,i,t} - \hat{\rho}_{S,i,t}\bar{\lambda}_{S,t})dt + \pi_t\sigma_t(\rho_{S,t} dW_{Y,t}^{\lambda_i^*} + \hat{\rho}_{S,t} dW_{S,t}^{\lambda_i^*}), \end{aligned} \tag{61}$$

$$\begin{aligned} dW_{Y,t}^{\lambda_i^*} &= dW_{Y,t} - (-\bar{\lambda}_{Y,i,t})dt, \\ dW_{S,t}^{\lambda_i^*} &= dW_{S,t} - (-\bar{\lambda}_{S,i,t})dt. \end{aligned}$$

We show that π_i^* attains the sup.

First, we let

$$R_{i,t} = -\exp(-\gamma_i(X_t^\pi + V_{i,t})).$$

Here, $V_{i,t},\ i = 1, \ldots, I$ are given by

$$V_{i,t} = Y_T + \int_t^T f_i(\sigma_{Y,t})dt - \int_t^T \sigma_{Y,t} dW_{Y,t}^{\lambda_i^*}$$

with

$$f_i(\sigma_{Y,t}) = \frac{1}{2\gamma_i}(\theta_t^{\lambda_i^*} - \gamma_i\rho_{S,t}\sigma_{Y,t})^2 - \frac{1}{2}\gamma_i\sigma_{Y,t}^2,$$

$$\theta_t^{\lambda_i^*} = \theta_t - \rho_{S,t}\bar{\lambda}_{Y,i,t} - \hat{\rho}_{S,t}\bar{\lambda}_{S,t}.$$

Then,

$$
\begin{aligned}
dR_{i,t} &= -\gamma_i R_{i,t} d(X_t^{\pi_i} + V_{i,t}) + \frac{1}{2}\gamma_i^2 R_{i,t} d\langle X^{\pi_i} + V_i \rangle_t \\
&= -\gamma_i R_{i,t} \Bigg((\pi_{i,t}\sigma_t \theta_t^{\lambda_i^*} - \frac{1}{2}\gamma_i((\pi_{i,t}\sigma_t\rho_{S,t} + \sigma_{Y,t})^2 + (\pi_t \sigma_t \hat{\rho}_{S,t})^2) - f_i(\sigma_{Y,t}))dt \\
&\qquad + (\pi_{i,t}\sigma_t\rho_{S,t} + \sigma_{Y,t}) dW_{Y,t}^{\lambda_i^*} + \pi_{i,t}\sigma_t\hat{\rho}_{S,t} dW_{S,t}^{\lambda_i^*} \Bigg) \\
&= -\gamma_i R_{i,t} \Bigg(-\frac{1}{2}\gamma_i \sigma_t^2 (\pi_{i,t} - \frac{1}{\gamma_i \sigma_t}(\theta_t^{\lambda_i^*} - \gamma_i \rho_{S,t}\sigma_{Y,t}))^2 + (\pi_{i,t}\sigma_t\rho_{S,t} + \sigma_{Y,t}) dW_{Y,t}^{\lambda_i^*} + \pi_{i,t}\sigma_t\hat{\rho}_{S,t} dW_{S,t}^{\lambda_i^*} \Bigg),
\end{aligned}
\tag{62}
$$

since the drift part is

$$
\begin{aligned}
&(\pi_{i,t}\sigma_t \theta_t^{\lambda^*} - \frac{1}{2}\gamma_i((\pi_{i,t}\sigma_t\rho_{S,t} + \sigma_{Y,t})^2 + (\pi_{i,t}\sigma_t\hat{\rho}_{S,t})^2) - f_i(\sigma_{Y,t})) \\
&= -\frac{1}{2}\gamma_i\sigma_t^2(\pi_{i,t} - \frac{1}{\gamma_i\sigma_t}(\theta_t^{\lambda_i^*} - \gamma_i\rho_{S,t}\sigma_{Y,t}))^2 + \frac{1}{2\gamma_i}(\theta_t^{\lambda^*} - \gamma_i\rho_{S,t}\sigma_{Y,t})^2 - \frac{1}{2}\gamma_i\sigma_{Y,t}^2 - f_i(\sigma_{Y,t}) \\
&= -\frac{1}{2}\gamma_i\sigma_t^2(\pi_{i,t} - \frac{1}{\gamma_i\sigma_t}(\theta_t^{\lambda_i^*} - \gamma_i\rho_{S,t}\sigma_{Y,t}))^2,
\end{aligned}
$$

which is maximized at

$$
\pi_i^* = \frac{1}{\gamma_i\sigma_t}(\theta_t^{\lambda_i^*} - \gamma_i\rho_{S,t}\sigma_{Y,t}). \tag{63}
$$

Therefore, R_i is a supermartingale and particularly a martingale when $\pi_i = \pi_i^*$. Hence,

$$
\begin{aligned}
&E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i} + Y_T))] \\
&\le E^{P^{\lambda_i^*}}[-\exp(-\gamma_i(X_T^{\pi_i^*} + Y_T))].
\end{aligned}
$$

■

Next, for given π_i^*, we show that $\lambda = \lambda_i^*$ attains the inf by a BSDE approach.

Lemma 3 *Under assumptions of Theorem 3, for given* $\pi_{i,t}^* = \frac{1}{\gamma_i\sigma_t}(\theta_t - \rho_{S,t}\bar{\lambda}_{Y,i,t} - \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t} - \gamma_i\rho_{S,t}\sigma_{Y,t})$, $\lambda = \lambda_i^* = (-\bar{\lambda}_{Y,i,t}, -\bar{\lambda}_{S,i,t})^\top$ *attains* $\inf_{\lambda\in\Lambda_i} J_i(\pi_i^*, \lambda)$.

Proof.
Firstly, for $\lambda \in \Lambda_i$, we consider a BSDE

$$
\begin{aligned}
d\mathcal{V}_t^\lambda &= \mathcal{Z}_{S,t}^\lambda(dW_{S,t} - \lambda_{S,t}dt) + \mathcal{Z}_{Y,t}^\lambda(dW_{Y,t} - \lambda_{Y,t}dt) \\
&= -(\lambda_{S,t}\mathcal{Z}_{S,t}^\lambda + \lambda_{Y,t}\mathcal{Z}_{Y,t}^\lambda)dt + \mathcal{Z}_{S,t}^\lambda dW_{S,t} + \mathcal{Z}_{Y,t}^\lambda dW_{Y,t}, \\
\mathcal{V}_T^\lambda &= R_{i,T}.
\end{aligned}
$$

Also, we note that $\mathcal{V}_0^\lambda = E^{P^\lambda}[R_{i,T}]$ and $\mathcal{V}_0^\lambda$ is minimized at $(\lambda_{S,t}^*, \lambda_{Y,t}^*)$, $\lambda_{S,t}^* = -\bar{\lambda}_{S,t} sgn(\mathcal{Z}_{S,t}^{\lambda^*})$, $\lambda_{Y,t}^* = -\bar{\lambda}_{Y,t} sgn(\mathcal{Z}_{Y,t}^{\lambda^*})$, which satisfies $\lambda_{S,t}^*\mathcal{Z}_{S,t}^{\lambda^*} = -\bar{\lambda}_{S,t}|\mathcal{Z}_{S,t}^{\lambda^*}|$, $\lambda_{Y,t}^*\mathcal{Z}_{Y,t}^{\lambda^*} = -\bar{\lambda}_{Y,t}|\mathcal{Z}_{Y,t}^{\lambda^*}|$, by the comparison principle for BSDEs (e.g., Theorem 6.2.2 in Pham [25]).

In the following, we first presuppose that $\lambda^*_{S,t}\mathcal{Z}^{\lambda^*}_{S,t} = -\bar{\lambda}_{S,t},\ \lambda^*_{Y,t}\mathcal{Z}^{\lambda^*}_{Y,t} = -\bar{\lambda}_{Y,t}$, then confirm $\mathcal{Z}^{\lambda^*}_{S,t}, \mathcal{Z}^{\lambda^*}_{Y,t} \geq 0$.

Let $R_{i,t} = -\exp(-\gamma_i(X^{\pi^*_i}_t + V_{i,t}))$, where $V_{i,t},\ i = 1, \ldots, I$ are given by $V_{i,t} = Y_T + \int_t^T f_i(\sigma_{Y,s})ds - \int_t^T \sigma_{Y,s}dW^{\lambda^*_i}_{Y,s}$.

Since $R_{i,t} = -\exp(-\gamma_i(X^{\pi^*_i}_t + V_{i,t}))$ is a martingale under $P^{\lambda^*_i}$ satisfying an SDE

$$dR_{i,t} = \mathcal{Z}_{S,i,t}dW^{\lambda^*_i}_{S,i,t} + \mathcal{Z}_{Y,i,t}dW^{\lambda^*_i}_{Y,i,t},$$

where

$$\mathcal{Z}_{S,i,t} = -\gamma_i R_{i,t}(\pi^*_{i,t}\sigma_t\rho_{S,t} + \sigma_{Y,t})$$
$$\mathcal{Z}_{Y,i,t} = -\gamma_i R_{i,t}(\pi^*_{i,t}\sigma_t\hat{\rho}_{S,t}),$$

we have only to confirm $\mathcal{Z}_{S,i,t}, \mathcal{Z}_{Y,i,t} \geq 0$, namely,

$$\frac{\rho_{S,t}}{\gamma_i}(\theta^{\lambda^*_i}_t - \gamma_i\rho_{S,t}\sigma_{Y,t}) + \sigma_{Y,t} \geq 0,$$
$$\frac{\hat{\rho}_{S,t}}{\gamma_i}(\theta^{\lambda^*_i}_t - \gamma_i\rho_{S,t}\sigma_{Y,t}) \geq 0,$$

which is satisfied by conditions (59) and (60). ■

Thus, (π^*_i, λ^*_i) is a saddle point and (π^*_i, λ^*_i) attains the sup-inf in (4) and the inf-sup in (5). Finally, we confirm that when the expected return process μ of the risky asset price process S_1 is given by $\mu_t = \sigma_t\theta_t$ with θ in (58), the market is in equilibrium, that is, the market clearing conditions

$$\sum_{i=1}^{I} \pi^*_{i,t} = 0, \tag{64}$$

and

$$\sum_{i=1}^{I}(X^{\pi^*_{i,t}}_t - \pi^*_{i,t}) = 0 \tag{65}$$

hold.

Lemma 4 *Under assumptions of Theorem 3, for the given expected return process μ, where $\mu_t = \sigma_t\theta_t$ with θ in (58), the market clearing conditions (64) and (65) hold.*

Proof. Since

$$\pi^*_{i,t} = \frac{1}{\gamma_i\sigma_t}(\theta^{\lambda^*_i}_t - \gamma_i\rho_{S,t}\sigma_{Y,t})$$
$$= \frac{1}{\gamma_i\sigma_t}(\theta_t - \rho_{S,t}\bar{\lambda}_{Y,i,t} - \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t} - \gamma_i\rho_{S,t}\sigma_{Y,t}),$$

$$\sigma_t\sum_{i=1}^{I}\pi^*_{i,t} = \sum_{i=1}^{I}\frac{1}{\gamma_i}(\theta_t - \rho_{S,t}\bar{\lambda}_{Y,i,t} - \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t} - \gamma_i\rho_{S,t}\sigma_{Y,t})$$
$$= (\sum_{i=1}^{I}\frac{1}{\gamma_i})\theta_t - \sum_{i=1}^{I}\frac{1}{\gamma_i}(\rho_{S,t}\bar{\lambda}_{Y,i,t} + \hat{\rho}_{S,t}\bar{\lambda}_{S,i,t} + \gamma_i\rho_{S,t}\sigma_{Y,t})$$
$$= 0.$$

Thus, $\sum_{i=1}^{I} \pi_{i,t}^{*} = 0$.

Also, (65) follows from (2) and (64).

Thus, the proof of Theorem 3 is completed. ■