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Equilibrium Multi-Agent Model with Heterogeneous Views on Fundamental Risks

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Abstract

This paper investigates an equilibrium-based multi-agent optimal consumption and portfolio problem incorporating uncertainties on fundamental risks, where multiple agents have heterogeneous (conservative, neutral, aggressive) views on the risks represented by Brownian motions. Each agent maximizes its expected utility on consumption under its subjective probability measure. Specifically, we formulate the individual optimization problem as a sup-sup-inf problem, which is an optimal consumption and portfolio problem with a choice of a probability measure. Moreover, we provide an expression of the state-price density process in a market equilibrium, which derives the representations of the interest rate and the market price of risk. To the best of our knowledge, this is the first attempt to investigate the multi-agent model with heterogeneous views on the risks by considering a market equilibrium and solving sup-sup-inf problems on the choice of a probability measure. We emphasize that the setting, where each agent has heterogeneous views on different risks, incorporates a special case where each agent has only conservative or neutral views on risks with different degrees of conservativeness. Also, the setting includes the case where the agents have aggressive views on risks, commonly observed as bullish sentiments in the financial markets in the monetary easing after the global financial crisis and particularly in the COVID-19 pandemic. Finally, we present numerical examples of the interest rate model, which show how heterogeneous views of the multiple agents on the risks affect the shape of the yield curve.

Key words: Stochastic control; Optimization under uncertainties; Interest rate model; Application in finance

1 Introduction

This paper investigates an equilibrium-based multi-agent optimal consumption and portfolio problem incorporating uncertainties on fundamental risks, where multiple agents have heterogeneous (conservative, neutral, aggressive) views on the risks represented by Brownian motions. Each agent maximizes its expected utility on the consumption process under its subjective probability measure, reflecting its views on the risks. For instance, Brownian motions may represent domestic and global risks or risks on different asset classes such as stocks, interest rates, commodities, and foreign exchanges.

Specifically, we express the views of the agents on Brownian motions by choice of a probability measure. In detail, when an agent is conservative (aggressive) about some risks, it means that the agent has biases on the Brownian motions so that the biases minimize (maximize) the agent's expected utility. Particularly, the types of the views and their degrees are heterogeneous among both the agents and the risks. Therefore, we formulate sup-sup-inf problems as a combination of the optimal consumption problem and a choice of a probability measure on the expected utility of the agents. To the best of our knowledge, this is the first attempt to investigate the multi-agent model with heterogeneous views on the risks by a market equilibrium approach which solves sup-sup-inf problems on the choice of a probability

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measure.

Uncertainty on fundamental risks is one of the key factors driving asset prices, such as stock, bond, and commodity prices. Particularly, in the worldwide low-interest-rate environment after the global financial crisis and amid the COVID-19 pandemic, the central banks controlled the interest rates at low levels by monetary easing, especially until March 2022 when Federal Reserve Board started to raise the Federal Funds rate to deal with inflation. During the period, the bond yields for different maturities were driven by conservative and aggressive views on the risks of market participants, and the situation remains in Japan where the Bank of Japan continues its monetary easing policy.

For instance, Nishimura et al. [20] and Nakatani et al. [18] utilize a text mining approach to estimate conservative and aggressive sentiment-related factors for the Japanese government bond (JGB) yield curves. In detail, Nishimura et al. [20] and Nakatani et al. [18] identify the sentiment-related factors that affect the yield curve shape. As in Nishimura et al. [20] and Nakatani et al. [18], when aggressive views about the economy prevail, the short-term interest rate rises, and when conservative views spread, the short-term interest rate lowers, which leads to flattening and steepening of the yield curve, respectively. Although, Nishimura et al. [20] and Nakatani et al. [18] supposed a representative agent to estimate the sentiment factors as an aggregate effect of agents in the JGB market, multiple agents with heterogeneous views were not considered in the models. Differences in views among agents are important in the markets where different types of agents are trading, such as foreign investors and domestic investors, and their views affect the asset price. Thus, we provide representations of a state-price density process in equilibrium in a log-utility case, which enables asset pricing under different views on the risks.

For related studies, Saito and Takahashi [28] deal with a sup-inf problem with respect to the best-case and the worst-case scenarios on Brownian motions as a choice of a probability measure by a BSDE approach. Saito and Takahashi [29] consider an optimal investment problem of a single agent under uncertainties on fundamental risks by a Malliavin calculus approach. Our study further extends Saito and Takahashi [28] [29] to a multi-agent case with optimal consumption and portfolio problems to obtain the state-price density process in equilibrium which enables asset pricing that reflects differences in views of agents. In detail, this study considers the individual optimal consumption and investment problems under heterogeneous views on the risks to obtain the state-price density process, which includes the information on the interest rate and the market price of risk in a market equilibrium, by imposing market clearing conditions.

As for literature on uncertainty on probability measures as an application of the robust control (e.g., Petersen et al. [23]), Hansen and Sargent [11] consider a utility maximization problem taking a conservative side on a choice of a probability measure. Beissner et al. [2] deal with the alpha max-min expected utility with an ambiguity of a view on risks of a single agent, which corresponds to a view in between the most aggressive and the most conservative side. We emphasize that our work is different from those approaches in that each agent has different degrees of conservativeness on respective risks and the degrees are heterogeneous among the agents. Moreover, we consider not only the conservative side but also the aggressive side on respective Brownian motions in a multi-agent case, which describes the bullish sentiments observed in the financial markets in the monetary easing environment particularly after the global financial crisis and amid the COVID-19 pandemic.

Particularly, we consider a sup-sup-inf problem for the individual optimization and solve for a market equilibrium under the setting. Although general equilibrium interest rate models for multi-agent settings without uncertainties on risks have been studied (see Karatzas and Shreve [12] for instance), we extend an equilibrium model to a case in which each agent has different views on respective Brownian motions and the sides of the views (conservative, neutral, or aggressive) also vary among the agents.

Moreover, models with stochastic boundaries on agents' views were developed with a single representative agent in Saito and Takahashi [28][29], and applied successfully to empirical research for finance in Nakatani et al. [18] and Nishimura et al. [20] as an estimation of the aggressive and conservative sentiment factors in the JGB market, which seems meaningful from macro and financial economic perspectives.

Specifically, the formulation of Saito and Takahashi [28] motivated by Chen and Epstein [6] is a natural extension in that it incorporates the aggressive side of an agent's views and an agent assigns his/her own view (conservative or aggressive) on each risk (i.e., a Brownian motion) represented by a random interval. In addition, this approach has a nice property that Bellman's principle of optimality holds.

Although in Nakatani et al. [18] and Nishimura et al. [20], the aggressive and conservative sentiment factors were estimated as aggregate views of the market as a whole, it is more realistic and desirable to model and estimate the aggressive and sentiment factors of different types of players in the market. Therefore, the current study has constructed a general equilibrium model where multiple agents have heterogeneous views on risks.

Furthermore, as for applications of stochastic control to optimal portfolio problems, Zhang et al.

[37] propose an optimization approach to construct sparse portfolios with mean-reverting price behaviors. Also, Bannister et al. [1] study a multi-period portfolio selection problem with a mean-standard-deviation criterion. Yan and Wong [34] formulate a time-consistent mean-variance portfolio problem in incomplete markets with stochastic volatility. Ma et al. [17] deal with an optimal portfolio execution problem with stochastic price impact and stochastic net demand pressure. For more applications of stochastic control to optimal portfolio problems, there are several types. Breton et al. [3], Li et al. [14], and Shen et al. [31] deal with multi-agent or mean-field equilibrium in a game theoretic setting. Calafiore [4][5], Gao et al. [10], Liu et al. [15], de Palma and Prigent [22], Pun and Ye [27], Xue et al. [33], Yao et al. [35] investigate optimal portfolio problems under market restrictions. Shen [30] considers the problem with unbounded coefficients, and Lv et al. [16] investigate the case in an incomplete market setting. Costa and de Oliveira [7], Dombrovskii et al. [9], Costa and Araujo [8], Yiu et al. [36], Zhu et al. [38] deal with regime-switching in portfolio optimization. Pu and Zhang [25] and Pun [26] consider robust control for portfolio optimization for instance.

The organization of this paper is as follows. After Section 2 describes the motivation and the setup, Section 3 solves the individual optimization problem under heterogeneous views on risks, and Section 4 provides expressions of the interest rate and the market price of risk in a market equilibrium in a log-utility case. Section 5 presents numerical examples on the term structure of interest rates under heterogeneous views of agents. Due to limitations of space, an exponential utility case is provided in Section 5 of the online supplementary file [13].

2 Motivation and settings

This section describes the settings of the multi-agent model. Firstly, we explain the motivation of the study, the setting of the financial market, and the views of the agents on the fundamental risks. Then, we express the heterogeneous views of the agents by individual optimization problems and describe market-clearing conditions that solutions of the individual optimization problems should satisfy.

The motivation of this study is as follows. We aim to find a state-price density process under heterogeneous views of multiple agents on the risks, which derives expressions of the interest rate and the market price of risk. Firstly, we formulate a multi-agent model where each agent has different views on the risks represented by Brownian motions, namely each agent solves an individual optimal consumption and portfolio problem incorporating conservative, neutral, and aggressive views on the respective Brownian motions. Then, we search for such a state-price density process by first solving individual optimization problems and then imposing the market clearing conditions.

Specifically, in the following, we show that given the searched state-price density process, the individual optimization problems are solved and the results satisfy the market clearing conditions. Moreover, we derive an interest rate model under different views on the risks from the expression of the state-price density process.

2.1 *Multi-agent model with heterogeneous views on fundamental risks*

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and B be a d-dimensional Brownian motion. Let $[0,T], T>0$ be the time horizon and $\{\mathcal{F}_t\}_{0\le t\le T}$ be the augmented filtration generated by B. We call the d-dimensional Brownian motion B the fundamental risks.

We consider a complete market, where K agents invest in d securities and a money market account. Let S^0 and $S^i,\ i=1,\dots,d$ be price processes of the money market account and the risky assets, satisfying stochastic differential equations (SDEs)

$$dS_t^0 = S_t^0 r_t dt,\ S_0^0 = 1,$$
$$dS_t^i = S_t^i\left(b_t^i dt + \sum_{j=1}^d \sigma_{j,t}^i dB_{j,t}\right),$$
$$S_0^i = p^i,\ i=1,\dots,d, \tag{1}$$

where the volatility matrix $\sigma = (\sigma_j^i)_{i,j=1,\dots,d}$ is $\mathbf{R}^{d\times d}$-valued $\{\mathcal{F}_t\}$-progressively measurable process which is invertible a.e. on $[0,T]\times\Omega$, the interest rate process r and $b^i, i=1,\dots,d$ are $\mathbf{R}$-valued $\{\mathcal{F}_t\}$-progressively measurable processes, and $p^i, i=1,\dots,d$ are positive constants.

Let $-\boldsymbol{\theta}_t = \sigma_t^{-1}(\boldsymbol{b}_t - r_t\mathbf{1}_d)$ where $\boldsymbol{b} = (b_1,\dots,b_d)^\top$ and $\mathbf{1}_d = (1,\dots,1)^\top \in \mathbf{R}^d$, and we call $-\boldsymbol{\theta}$ the market price of risk process. We assume that a local martingale $\mathcal{Z}$ given by $\mathcal{Z}_t = \exp\left\{\int_0^t \boldsymbol{\theta}_s^\top dB_s - \frac{1}{2}\int_0^t |\boldsymbol{\theta}_s|^2 ds\right\}$, $t\in[0,T]$, is a martingale.

Then, we define the risk-neutral probability measure $\mathbf{Q}$ by $\mathbf{Q}(A) = \mathbf{E}[\mathcal{Z}_T 1_A],\ A\in\mathcal{F}_T$, where $\mathbf{E}[\cdot]$ denotes the expectation under $\mathbf{P}$, and we note that by Girsanov's theorem $B_t^{\mathbf{Q}} := B_t - \int_0^t \boldsymbol{\theta}_s ds$ is a d-dimensional Brownian motion under $\mathbf{Q}$.

Moreover, we define the state-price density process H_0 by $H_{0,t} = \frac{\mathcal{Z}_t}{S_t^0}$. We note that the state-price density process satisfies an SDE

$$dH_{0,t} = H_{0,t}\left(-r_t dt + \boldsymbol{\theta}_t^\top dB_t\right),\ H_{0,0} = 1. \tag{2}$$

Firstly, we consider K agents who are continuously endowed with income and consume. The agents also invest the rest of its wealth in the d risky assets and the money market account. Let ε^k be the income process of the k-th $(k=1,\dots,K)$ agent, which is a $\mathbf{R}^+$ (or $\mathbf{R}$)-valued $\{\mathcal{F}_t\}$-progressively measurable process. Moreover, let c^k be the consumption process and $\boldsymbol{\pi}^k$ be a portfolio process, which are $\mathbf{R}$ (or $\mathbf{R}^+$)-valued and $\mathbf{R}^d$-valued $\{\mathcal{F}_t\}$-progressively measurable satisfying $\int_0^T |c_t^k| dt < \infty$, $\int_0^T |\sigma_t^\top \boldsymbol{\pi}_t^k|^2 dt < \infty, a.s.$ Specifically, the k-th agent, who has wealth W_t^k at time t, is endowed with income $\varepsilon_t^k dt$ and consume $c_t^k dt$ in the interval $[t, t+dt)$. Moreover, the k-th agent invests in d risky assets with an allocation $\boldsymbol{\pi}^k = (\pi_1^k, \dots, \pi_d^k)^\top$ on value basis and the rest of its wealth in the money market account.

Then, the wealth process W^k corresponding to the initial wealth 0, the endowment process ε^k, the consumption process c^k, and the portfolio process $\boldsymbol{\pi}^k$ is given by

$$\frac{W_t^k}{S_t^0} = \int_0^t \frac{\varepsilon_s^k - c_s^k}{S_s^0} ds + \int_0^t \frac{\boldsymbol{\pi}_s^{k\top}\sigma_s}{S_s^0} dB_s^{\boldsymbol{Q}}$$
$$= \int_0^t \frac{\varepsilon_s^k - c_s^k}{S_s^0} ds + \int_0^t \frac{\boldsymbol{\pi}_s^{k\top}\sigma_s}{S_s^0} (dB_s - \boldsymbol{\theta}_s ds). \quad (3)$$

(e.g. Equation (3.6) in Section 4.3 in Karatzas and Shreve [12].)

Next, we suppose that the agents have heterogeneous (conservative, neutral, aggressive) views on the fundamental risks B. The k-th agent is conservative about the fundamental risks B_j, $j \in \mathcal{J}_1^k$, aggressive about the risks B_j, $j \in \mathcal{J}_2^k$, and neutral about the risks $B_j, j \in \mathcal{J}_3^k$. Here, we set $\{1, \dots, d\} = \mathcal{J}_1^k \cup \mathcal{J}_2^k \cup \mathcal{J}_3^k$, $\mathcal{J}_i^k \cap \mathcal{J}_j^k = \emptyset$, $i \neq j$.

Specifically, let $\boldsymbol{\lambda}^k = (\lambda_1^k, \dots, \lambda_d^k)^\top$ be a set of views of the k-th agent on the Brownian motions $B_j, j = 1, \dots, d$, where we assume that λ_j^k, $j \in \mathcal{J}_1^k$ are $\{\mathcal{F}_t\}$-progressively measurable processes and λ_j^k, $j \in \mathcal{J}_2^k$ are nonrandom processes, satisfying $|\lambda_j^k| \leq \bar{\lambda}_j^k$, $j \in \mathcal{J}_1^k, \mathcal{J}_2^k$. Here, $\bar{\lambda}_j^k$, $j \in \mathcal{J}_1^k, \mathcal{J}_2^k$ are nonnegative and nonrandom, and $\lambda_j^k \equiv 0$, $j \in \mathcal{J}_3^k$. Let $\mathbf{P}^{\boldsymbol{\lambda}^k}$ be the probability measure corresponding to the set of views of the k-th agent $\boldsymbol{\lambda}^k = (\lambda_1^k, \dots, \lambda_d^k)^\top$ defined as $\mathbf{P}^{\boldsymbol{\lambda}^k}(A) = \mathbf{E}[\eta_T^{\boldsymbol{\lambda}^k} 1_A]$, $A \in \mathcal{F}_T$, where $\eta_t^{\boldsymbol{\lambda}^k} = \exp\left\{\sum_{j=1}^d \int_0^t \lambda_{j,s}^k dB_{j,s} - \frac{1}{2}\sum_{j=1}^d \int_0^t |\lambda_{j,s}^k|^2 ds\right\}, 0 \leq t \leq T$.

We note that by Girsanov's theorem, $B^{\boldsymbol{\lambda}^k}$ defined by

$$B_{j,t}^{\boldsymbol{\lambda}^k} := B_{j,t} - \int_0^t \lambda_{j,s}^k ds, \ j \in \mathcal{J}_1^k, \mathcal{J}_2^k,$$
$$B_{j,t}^{\boldsymbol{\lambda}^k} := B_{j,t}, \ j \in \mathcal{J}_3^k,$$

is a d-dimensional Brownian motion under $\mathbf{P}^{\boldsymbol{\lambda}^k}$. We remark that the fundamental risks B is expressed with $B^{\boldsymbol{\lambda}^k}$, the Brownian motion under $\mathbf{P}^{\boldsymbol{\lambda}^k}$, and the set of views $\boldsymbol{\lambda}^k$ in a differential form as

$$dB_t = dB_t^{\boldsymbol{\lambda}^k} + \boldsymbol{\lambda}_t^k dt. \quad (4)$$

This implies that the k-th agent understands dB_t, the instantaneous increment of the Brownian motion under the reference probability measure $\mathbf{P}$, as $dB_t^{\boldsymbol{\lambda}^k}$, the Brownian motion under the subjective probability measure $\mathbf{P}^{\boldsymbol{\lambda}^k}$, with the bias $\boldsymbol{\lambda}_t^k dt$.

2.2 Individual optimization problem

This section describes the subjective probability measure of an agent reflecting the conservative and aggressive views on respective Brownian motions by an individual consumption and portfolio problem with a choice of a probability measure. Let $U^k : \mathbf{R}^+$ (or $\mathbf{R}$) $\to \mathbf{R}$, $k = 1, \dots, K$, be the utility function of the k-th agent on the consumption process c^k, where U^k is twice continuously differentiable with $U^{k'} > 0$ and $U^{k''} < 0$. For a given state-price density process H_0, we consider the following individual optimization problem for the k-th agent

$$\sup_{(c^k, \boldsymbol{\pi}^k) \in \mathcal{A}^k} \sup_{|\lambda_j^k| \leq \bar{\lambda}_j^k, \ j \in \mathcal{J}_2^k} \inf_{|\lambda_j^k| \leq \bar{\lambda}_j^k, \ j \in \mathcal{J}_1^k}$$
$$\mathbf{E}\left[\int_0^T \eta_t^{\boldsymbol{\lambda}^k} U^k(c_t^k) dt\right] \left(= \mathbf{E}^{\boldsymbol{P}^{\boldsymbol{\lambda}^k}}\left[\int_0^T U^k(c_t^k) dt\right]\right), \quad (5)$$

where $\mathcal{A}^k$ is a set of admissible consumption and portfolio processes in Definition 1 below. Particularly, as in Remark 1 in the following, an admissible consumption and portfolio process pair satisfies the budget constraint

$$\mathbf{E}\left[\int_0^T H_{0,t} c_t^k dt\right] \leq \mathbf{E}\left[\int_0^T H_{0,t} \varepsilon_t^k dt\right]. \quad (6)$$

We note that by applying Ito's formula to $H_{0,t} W_t^k = \mathcal{Z}_t \frac{W_t^k}{S_t^0}$, by (3), we obtain

$$H_{0,t} W_t^k - \int_0^t H_{0,s}\left(\varepsilon_s^k - c_s^k\right) ds$$
$$= \int_0^t H_{0,s}\left(\sigma_s^\top \boldsymbol{\pi}_s^k + W_s^k \boldsymbol{\theta}_s\right)^\top dB_s, \ t \in [0, T]. \quad (7)$$

We define the admissibility of the consumption and portfolio process pair $(c^k, \boldsymbol{\pi}^k)$ for the k-th agent ($k = 1, \dots, K$) as follows.

Definition 1 *A consumption and portfolio process pair $(c^k, \boldsymbol{\pi}^k)$ for the k-th agent is admissible, if (7) is a supermartingale, $W_T^k \geq 0, a.s.$, and $\mathbf{E}[\int_0^T U^k(c_t^k)^2 dt] < \infty$. We denote the set of the admissible pairs by $\mathcal{A}^k$.*

Remark 1 *By taking the expectation on both sides in (7) for $t = T$, for an admissible pair $(c^k, \boldsymbol{\pi}^k) \in \mathcal{A}^k$, the budget constraint $\mathbf{E}\left[\int_0^T H_{0,s} c_s^k ds\right] \leq \mathbf{E}\left[\int_0^T H_{0,s} \varepsilon_s^k ds\right]$ in (6) follows.*

The individual optimization problem (5) is solved by the following approach.

2.2.1 Minimization with respect to the conservative view $\boldsymbol{\lambda}_1^k$

In the remaining of Section 2.2 we discuss the individual optimization problem of the k-th agent by assuming $\mathcal{J}_1^k = \{1,\dots,d_1\}$, $\mathcal{J}_2^k = \{d_1+1,\dots,d_1+d_2\}$, $\mathcal{J}_3^k = \{d_1+d_2+1,\dots,d\}$, without loss of generality. We set $\boldsymbol{\lambda}_1^k = (\lambda_1^k,\dots,\lambda_{d_1}^k)^\top$. $\boldsymbol{\lambda}_2^k = (\lambda_{d_1+1}^k,\dots,\lambda_{d_1+d_2}^k)^\top$. Hereafter, for $\boldsymbol{\lambda}^k = (\boldsymbol{\lambda}_1^{k\top},\boldsymbol{\lambda}_2^{k\top},0,\dots,0)^\top$, we denote $\mathbf{E}^{\mathbf{P}^{\boldsymbol{\lambda}^k}}[\cdot]$ by $\mathbf{E}^{\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k}[\cdot]$.

Let $V^{k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k}$ be the expected utility process of the k-th agent defined as $V_t^{k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k} := \mathbf{E}^{\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k}\left[\int_t^T U^k(c_s^k)ds|\mathcal{F}_t\right]$, in particular, $V_0^{k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k} = \mathbf{E}^{\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k}\left[\int_0^T U^k(c_s^k)ds\right]$. Then, $V^{k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k}$ satisfies the backward stochastic differential equation (BSDE)

$$\begin{aligned} dV_t^{k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k} &= -U^k(c_t^k)dt + \sum_{j=1}^d Z_{j,t}dB_{j,t}^{\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k} \\ &= -\left(U^k(c_t^k) + \sum_{j=1}^d \lambda_{j,t}^k Z_{j,t}\right)dt \\ &\quad + \sum_{j=1}^d Z_{j,t}dB_{j,t},\ V_T^{k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k} = 0. \end{aligned} \tag{8}$$

Since $\boldsymbol{\lambda}_1^k$ is bounded, a comparison theorem for a BSDE with a uniform Lipschitz driver (e.g. Theorem 6.2.2 in Pham [24]) applies and it follows that $\hat{\boldsymbol{\lambda}}_1^k := (-\bar{\lambda}_1^k sgn(Z_1),\dots,-\bar{\lambda}_{d_1}^k sgn(Z_{d_1}))^\top$, where Z is a part of the unique solution (V^k,Z) of the BSDE

$$\begin{aligned} &dV_t^k \\ &= -\left(U^k(c_t^k) - \sum_{j=1}^{d_1}\bar{\lambda}_{j,t}^k|Z_{j,t}| + \sum_{j=d_1+1}^{d_1+d_2}\lambda_{j,t}^k Z_{j,t}\right)dt \\ &+ \sum_{j=1}^d Z_{j,t}dB_{j,t},\ V_T^k = 0, \end{aligned} \tag{9}$$

attains $\inf_{\boldsymbol{\lambda}_1^k,|\lambda_j^k|\le\bar{\lambda}_j^k,j=1,\dots,d_1} V_0^{k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k}$.

2.2.2 Maximization with respect to the consumption process c^k

Next, we let $J^k(c^k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k) = \mathbf{E}^{\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k}\left[\int_0^T U^k(c_s^k)ds\right]$. We consider

$$\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\ \inf_{\boldsymbol{\lambda}_1^k,|\lambda_j^k|\le\bar{\lambda}_j^k,j=1,\dots,d_1} J^k(c^k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k), \tag{10}$$

with the budget constraint (6).

Then, the following lemma holds.

Lemma 1 *For given H_0 and $\boldsymbol{\lambda}_2^k$ satisfying $|\lambda_j^k| \le \bar{\lambda}_j^k,\ j = d_1+1,\dots,d_1+d_2$, if $(c^{k,*},\boldsymbol{\lambda}_1^{k,*})$ satisfies*

$$H_{0,t} = \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\frac{U^{k'}(c_t^{k,*})}{U^{k'}(c_0^{k,*})}, \tag{11}$$

with the budget constraint (6) with equality

$$\mathbf{E}\left[\int_0^T H_{0,t}c_t^{k,*}dt\right] = \mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right], \tag{12}$$

and

$$\lambda_j^{k,*} = -\bar{\lambda}_j^k sgn(Z_j),\ j = 1,\dots,d_1, \tag{13}$$

where $Z_j,\ j = 1,\dots,d_1$ is a part of the solution (V^k,Z) of the BSDE

$$\begin{aligned} &dV_t^k \\ &= -\left(U^k(c_t^{k,*}) - \sum_{j=1}^{d_1}\bar{\lambda}_{j,t}^k|Z_{j,t}| + \sum_{j=d_1+1}^{d_1+d_2}\lambda_{j,t}^k Z_{j,t}\right)dt \\ &+ \sum_{j=1}^d Z_{j,t}dB_{j,t},\ V_T^k = 0, \end{aligned} \tag{14}$$

then $(c^{k,*},\boldsymbol{\lambda}_1^{k,*})$ attains the sup-inf in (10).

Proof. For c^k in any admissible pair $(c^k,\boldsymbol{\pi}^k)$, let $\hat{\boldsymbol{\lambda}}_1^k$ be the optimal $\boldsymbol{\lambda}_1^k$ that attains $\inf_{\boldsymbol{\lambda}_1^k}$ for given c^k as in Section 2.2.1. Then, we have

$$\begin{aligned} &\mathbf{E}[\int_0^T \eta_t^{\hat{\boldsymbol{\lambda}}_1^k,\boldsymbol{\lambda}_2^k}U^k(c_t^k)dt] - \mathbf{E}[\int_0^T \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_t^{k,*})dt] \\ &= \mathbf{E}^{\hat{\boldsymbol{\lambda}}_1^k,\boldsymbol{\lambda}_2^k}[\int_0^T U^k(c_t^k)dt] - \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[\int_0^T U^k(c_t^{k,*})dt] \\ &\le \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[\int_0^T U^k(c_t^k) - U^k(c_t^{k,*})dt] \\ &\le \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[\int_0^T U^{k'}(c_t^{k,*})(c_t^k - c_t^{k,*})dt] \\ &= \mathbf{E}[\int_0^T \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^{k'}(c_t^{k,*})(c_t^k - c_t^{k,*})dt] \\ &= U^{k'}(c_0^{k,*})\mathbf{E}[\int_0^T H_{0,t}(c_t^k - c_t^{k,*})dt]. \end{aligned}$$

The first inequality follows from the optimality of $\hat{\boldsymbol{\lambda}}_1^k$ for c^k, the second inequality is due to the concavity of U, and the last equality follows from (11).

By (6) and (12), we have

$$\mathbf{E}\left[\int_0^T H_{0,t}c_t^k dt\right] \le \mathbf{E}\left[\int_0^T H_{0,t}c_t^{k,*}dt\right].$$

Since $U^{k'}(c_0^*) > 0$, we have

$$\mathbf{E}[\int_0^T \eta_t^{\hat{\boldsymbol{\lambda}}_1^k,\boldsymbol{\lambda}_2^k}U^k(c_t^k)dt] - \mathbf{E}[\int_0^T \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_t^{k,*})dt] \le 0,$$

for any admissible c^k. □

We will concretely solve the individual optimization problem (5), which also includes the maximization with respect to the aggressive views $\boldsymbol{\lambda}_2^k$, in the log-utility case in Section 3.

We note that once the optimal consumption process $\bar{c}^{k,*}$ for the individual optimization problem (5) is obtained, by the standard argument for the complete market (e.g. Theorem 4.4.5 in Karatzas and Shreve [12]), the corresponding wealth process $W^{k,*}$ and the optimal portfolio process $\boldsymbol{\pi}^{k,*}$ are given as

$$\begin{aligned} W_t^{k,*} &= \frac{1}{H_{0,t}} \mathbf{E}\left[\int_t^T H_{0,s}(\bar{c}_s^{k,*} - \varepsilon_s^k)ds|\mathcal{F}_t\right], \\ \boldsymbol{\pi}_t^{k,*} &= (\sigma_t^{-1})^\top \left(\frac{1}{H_{0,t}}\boldsymbol{\psi}_t^k - W_t^{k,*}\boldsymbol{\theta}_t\right), \end{aligned} \tag{15}$$

where $\boldsymbol{\psi}^k$ is determined by the martingale representation

$$\mathbf{E}\left[\int_0^T H_{0,s}(\bar{c}_s^{k,*} - \varepsilon_s^k)ds|\mathcal{F}_t\right] = \int_0^t \boldsymbol{\psi}_s^{k\top} dB_s. \tag{16}$$

Remark 2 *This order* $\sup_{\boldsymbol{\lambda}_2^k}\sup_{c^k}\inf_{\boldsymbol{\lambda}_1^k}$ *implies that we consider the most conservative case putting more emphasis on the conservative views. In the proof, we first fix* $\boldsymbol{\lambda}_2^k$ *and solve the* $\inf_{\boldsymbol{\lambda}_1^k}\sup_{(c^k,\boldsymbol{\pi}^k)}$ *part by the saddle point argument, and then maximize the objective function with respect to* $\boldsymbol{\lambda}_2^k$*. The order* $\sup_{\boldsymbol{\lambda}_2^k}\sup_{c^k}\inf_{\boldsymbol{\lambda}_1^k}$ *is interchangeable in the log-utility case in Section 3, since a saddle point argument holds. In detail, given* $\boldsymbol{\lambda}_1^{k,*}$, $(\boldsymbol{\lambda}_2^{k,*}, \bar{c}^{k,*})$ *attains* $\sup_{\boldsymbol{\lambda}_2}\sup_{(c^k,\boldsymbol{\pi}^k)} J^k(c^k, \boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k)$ *and given* $\boldsymbol{\lambda}_2^{k,*}$ *and* $\bar{c}^{k,*}$, $\boldsymbol{\lambda}_1^{k,*}$ *attains* $\inf_{\boldsymbol{\lambda}_1^k} J^k(\bar{c}^{k,*}, \boldsymbol{\lambda}_1^k, \boldsymbol{\lambda}_2^{k,*})$. *Thus,* $\sup_{\boldsymbol{\lambda}_2^k}\sup_{(c^k,\boldsymbol{\pi}^k)}\inf_{\boldsymbol{\lambda}_1^k} J^k(c^k, \boldsymbol{\lambda}_1^k, \boldsymbol{\lambda}_2^k)$ $= \inf_{\boldsymbol{\lambda}_1^k}\sup_{\boldsymbol{\lambda}_2^k}\sup_{(c^k,\boldsymbol{\pi}^k)} J^k(c^k, \boldsymbol{\lambda}_1^k, \boldsymbol{\lambda}_2^k)$ *holds by the saddle point argument. The former follows since given* $\boldsymbol{\lambda}_1^{k,*}$ *and* $\boldsymbol{\lambda}_2^k$*, the maximization with respect to* c^k *and then maximize with respect to* $\boldsymbol{\lambda}_2^k$ *will be done in the proof of Theorem 3 in Section 3, and the latter follows from the same argument in Section 2.2.1. Also, in the exponential utility case in Section 5 of the supplementary file, under the proposed assumptions (Assumption 1 in this paper and Assumption 5 in the supplementary file[13]), the interchangeability of the sup-inf and inf-sup also holds. In a similar way as in the log-utility case, we show that the* $(\boldsymbol{\lambda}_1^{k,*}, (\bar{c}^{k,*}, \boldsymbol{\lambda}_2^{k,*}))$ *is the saddle point, i.e., given* $\boldsymbol{\lambda}_1^{k,*}$, $(\bar{c}^{k,*}, \boldsymbol{\lambda}_2^{k,*})$ *attains the sup-sup, which is proved in the original procedure in solving* $\sup_{\boldsymbol{\lambda}_2}\sup_{c,\pi}\inf_{\boldsymbol{\lambda}_1}$ *(Lemma 1 in this paper and Lemma 11 in the supplementary file[13]), and given* $(\bar{c}^{k,*}, \boldsymbol{\lambda}_2^{k,*})$, $\boldsymbol{\lambda}_1^{k,*}$ *attains the inf, which can be shown by the Malliavin calculus approach focusing on calculation of* $Z_j^{\bar{c}^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^{k,*}}$ *and determination of the sign of* $Z_j^{\bar{c}^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^{k,*}}$ *with the assumptions as in Section 2.2.1 in this paper and Lemma 10 in the supplementary file [13], where* $Z_j^{\bar{c}^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^{k,*}}$, $j = 1, \dots, d_1$ *are part of a solution of BSDE (9).*

2.3 Market clearing conditions

Let $(\bar{c}^{k,*}, \boldsymbol{\pi}^{k,*})$ be the optimal consumption process and the optimal portfolio process for the individual optimization problem (5) of the k-th agent ($k = 1, \dots, K$). For the solutions of the individual optimization problems $\{(\bar{c}^{k,*}, \boldsymbol{\pi}^{k,*})\}_{k=1,\dots,K}$, we aim to find a state-price density process H_0 such that the following market-clearing conditions hold. We set $\varepsilon_t := \sum_{k=1}^K \varepsilon_t^k$, and call ε the aggregate endowment process.

Following the definition of the market equilibrium (e.g. Definition 5.1 in Section 4.5 in Karatzas and Shreve[12]), we call that the market $(\{\varepsilon^k\}_{k=1,\dots,K}, \{S^i\}_{i=0,1,\dots,d})$ is in equilibrium if the solutions of the individual optimization problems for all the agents satisfy the following clearing conditions (17)-(19). Also, we call such H_0 the state-price process in equilibrium.

(1) Clearing of the commodity market

$$\sum_{k=1}^K \bar{c}_t^{k,*} = \varepsilon_t, \; t \in [0, T]. \tag{17}$$

(2) Clearing of the security market

$$\sum_{k=1}^K \boldsymbol{\pi}_t^{k,*} = \mathbf{0}_d, \; t \in [0, T]. \tag{18}$$

(3) Clearing of the money market

$$\sum_{k=1}^K (W_t^{k,*} - \boldsymbol{\pi}_t^{k,*\top}\mathbf{1}_d) = 0, \; t \in [0, T]. \tag{19}$$

2.4 Utility function and aggregate endowment process

In the following, we consider a log-utility case for the utility functions of the agents. We suppose that the utility functions of the agents and the aggregate endowment process are as follows.

2.4.1 Log-utility case

We consider a market such that each agent has a log-utility function U^k given by $U^k(x) = \log x$, and the aggregate endowment process ε satisfies an SDE

$$d\varepsilon_t = \varepsilon_t[\nu_t dt + \boldsymbol{\rho}_t^\top dB_t], \; \varepsilon_0 > 0, \tag{20}$$

where ν is a $\mathbf{R}$-valued $\{\mathcal{F}_t\}$-progressively measurable process with $\mathbf{E}[\int_0^T |\nu_t|^2 dt] < \infty$, and $\boldsymbol{\rho}_t = (\rho_{1,t}, \dots, \rho_{d,t})^\top$ is a nonrandom process satisfying Assumption 1.

Assumption 1 *For* $j = 1, \dots, d$, $\rho_{j,t} > 0$, $\forall t \in [0, T]$.

Remark 3 *This assumption indicates that the instantaneous increment of the endowment process* $d\varepsilon_t$ *has the same sign as the increment of the Brownian motion* dB_t*. Moreover, by (4), this implies that a positive (negative) bias on Brownian motion* B_j *affects positively (negatively) the view of the agent on the aggregate endowment process.*

3 Individual optimization problem (Log-utility case)

In this section, we concretely solve the individual optimization problem (5) in the log-utility case where agent k's utility function is given by a log utility, $U^k(x) = \log x$.

For $k = 1, \dots, K$, we set $\boldsymbol{\lambda}^{k,*} = (\lambda_1^{k,*}, \dots, \lambda_d^{k,*})^\top$, where

$$\lambda_{j,t}^{k,*} = \begin{cases} -\bar{\lambda}_{j,t}^k, j \in \mathcal{J}_1^k \\ +\bar{\lambda}_{j,t}^k, j \in \mathcal{J}_2^k \\ 0, j \in \mathcal{J}_3^k, \end{cases}, \quad 0 \le t \le T. \tag{21}$$

Hereafter, we set $\eta_t^{k,*} := \eta_t^{\boldsymbol{\lambda}^{k,*}}$
$= \exp\left\{\sum_{j=1}^d \int_0^t \lambda_{j,s}^{k,*} dB_{j,s} - \frac{1}{2}\sum_{j=1}^d \int_0^t |\lambda_{j,s}^{k,*}|^2 ds\right\}$.

Specifically, we show that for the given state-price density process H_0 in (22) below, $\lambda_j^{k,*}, j \in \mathcal{J}_1^k, \mathcal{J}_2^k$ of $\boldsymbol{\lambda}^{k,*}$ in (21) and $\bar{c}_t^{k,*} = \frac{\eta_t^{k,*}\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]}{TH_{0,t}}$ with $\boldsymbol{\pi}^{k,*}$ in (15) attain the individual optimization problem (5).

The candidate of the state-price density process H_0 in equilibrium in the log-utility case is obtained as

$$H_{0,t} = \sum_{k=1}^K \frac{Y^k}{\left(\sum_{l=1}^K Y^l\right)} \eta_t^{k,*} \frac{\varepsilon_0}{\varepsilon_t}$$
$$\left(= \sum_{k=1}^K \frac{Y^k}{\left(\sum_{l=1}^K Y^l\right)} \eta_t^{k,*} \frac{U^{k'}(\varepsilon_t)}{U^{k'}(\varepsilon_0)} \right), \tag{22}$$

where $Y^1 = 1$ and Y^k $(k = 2 \dots, K)$ are positive constants satisfying the linear equation (23), whose existence and uniqueness are guaranteed by Proposition 2 below.

Remark 4 *This indicates that the state-price density process is expressed as the weighted average of* $\eta_t^{k,*}\frac{U^{k'}(\varepsilon_t)}{U^{k'}(\varepsilon_0)}$*, where the weight* Y^k *corresponds to* $\frac{y^1}{y^k}$*. Here,* y^k *is the Lagrange multiplier for the budget constraint (6) with equality of the individual optimal consumption and portfolio problem (5), where* $\boldsymbol{\lambda}^k$ *is replaced with* $\boldsymbol{\lambda}^{k,*}$ *in (21).*

With the expression of H_0*, (22) in equilibrium,* $\bar{c}_t^{k,*} = \frac{\eta_t^{k,*}\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]}{TH_{0,t}}$ *is further expressed as* $\bar{c}_t^{k,*} = \frac{\eta_t^{k,*}Y^k}{\sum_{l=1}^K \eta_t^{l,*}Y^l}\varepsilon_t$ *in the same way as in (48) in Section 2 of the online supplementary file [13].*

This implies that the optimal consumption of the k*-th agent is proportional to the aggregate endowment* ε_t *with the weight* $\frac{\eta_t^{k,*}Y^k}{\sum_{l=1}^K \eta_t^{l,*}Y^l}$*, which is the same weight that appears in the expression of the equilibrium interest rate* r *and market price of risk* $-\boldsymbol{\theta}$ *in (31) and (32) in Proposition 5 of Section 4.*

Moreover, $Y^k,\ k = 2, \dots, K$ *are obtained by solving the equation (23) in Proposition 2 below, which is derived by plugging* H_0 *in (22) and* $\bar{c}_t^{k,*} = \frac{\eta_t^{k,*}Y^k}{\sum_{l=1}^K \eta_t^{l,*}Y^l}\varepsilon_t$ *into the budget constraint (6) with equality.*

Proposition 2 *The linear equation*

$$AY = B, \tag{23}$$

where $A = TI - \tilde{A}$*,*

$$\tilde{A} = \begin{pmatrix} \mathbf{E}\left[\int_0^T \frac{\varepsilon_t^2}{\varepsilon_t}\eta_t^{2,*}dt\right] & \mathbf{E}\left[\int_0^T \frac{\varepsilon_t^2}{\varepsilon_t}\eta_t^{3,*}dt\right] & \dots & \mathbf{E}\left[\int_0^T \frac{\varepsilon_t^2}{\varepsilon_t}\eta_t^{K,*}dt\right] \\ \mathbf{E}\left[\int_0^T \frac{\varepsilon_t^3}{\varepsilon_t}\eta_t^{2,*}dt\right] & \mathbf{E}\left[\int_0^T \frac{\varepsilon_t^3}{\varepsilon_t}\eta_t^{3,*}dt\right] & \dots & \mathbf{E}\left[\int_0^T \frac{\varepsilon_t^3}{\varepsilon_t}\eta_t^{K,*}dt\right] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{E}\left[\int_0^T \frac{\varepsilon_t^K}{\varepsilon_t}\eta_t^{2,*}dt\right] & \mathbf{E}\left[\int_0^T \frac{\varepsilon_t^K}{\varepsilon_t}\eta_t^{3,*}dt\right] & \dots & \mathbf{E}\left[\int_0^T \frac{\varepsilon_t^K}{\varepsilon_t}\eta_t^{K,*}dt\right] \end{pmatrix},$$

$$Y = \begin{pmatrix} Y^2 \\ Y^3 \\ \vdots \\ Y^K \end{pmatrix}, B = \begin{pmatrix} \mathbf{E}\left[\int_0^T \frac{\varepsilon_t^2}{\varepsilon_t}\eta_t^{1,*}dt\right] \\ \mathbf{E}\left[\int_0^T \frac{\varepsilon_t^3}{\varepsilon_t}\eta_t^{1,*}dt\right] \\ \vdots \\ \mathbf{E}\left[\int_0^T \frac{\varepsilon_t^K}{\varepsilon_t}\eta_t^{1,*}dt\right] \end{pmatrix},$$

has a unique and strictly positive solution Y*, where* $Y^j > 0, j = 2, \dots, K$*. Here,* I *is the* $K-1$ *dimensional identity matrix.*

Proof. First, we use Hawkins-Simon's condition to show that the linear equation (23) has a unique and nonnegative solution. To confirm Hawkins-Simon's condition, we show that Brauer-Solow's condition in Nikaido [19] (Chapter II, Theorem 6.2's corollary) holds.

We note $\mathbf{E}\left[\int_0^T \frac{\varepsilon_t^l}{\varepsilon_t}\eta_t^{k,*}dt\right] > 0$ for all $l = 1, \dots, K$, $k = 1, \dots, K$. We calculate the k-th ($k = 1, \dots, K-1$) column sums of the matrix $\tilde{A}$,

$$\sum_{l=2}^K \mathbf{E}\left[\int_0^T \frac{\varepsilon_t^l}{\varepsilon_t}\eta_t^{k+1,*}dt\right] = T - \mathbf{E}\left[\int_0^T \frac{\varepsilon_t^1}{\varepsilon_t}\eta_t^{k+1,*}dt\right] < T.$$

Thus, Brauer-Solow's condition, particularly (ii) in the corollary, holds, and then Hawkins-Simon's condition is also satisfied.

Therefore, since all elements of the matrix B are positive, we obtain the unique and nonnegative solution of the linear equation (23). Since Y^k is nonnegative for any k, if $Y^k = 0$ for some k, then the $(k-1)$-th element of AY is

$$-\sum_{l=2}^K \mathbf{E}\left[\int_0^T \frac{\varepsilon_t^k}{\varepsilon_t}\eta_t^{l,*}dt\right] Y^l \le 0.$$

However, since the j-th element of B is strictly positive, this is a contradiction. Thus, $Y^k,\ k = 1, \dots, K$ are strictly positive. □

Moreover, we assume the following.

Assumption 2 *For* $j = 1, \ldots, d$,

$$\rho_{j,u} - \max\left[\max_{l,k\in\{1,\ldots,K\};l\neq k}(\lambda_{j,u}^{l,*} - \lambda_{j,u}^{k,*}),\ \max_{l\in\{1,\ldots,K\}} \lambda_{j,u}^{l,*}\right] > 0,\ \forall u \in [0,T].$$

Assumption 3 *For any* $\boldsymbol{\lambda}^{k,\star} = (\lambda_1^{k,\star}, \ldots, \lambda_d^{k,\star})^\top$ *where*

$$\lambda_{j,t}^{k,\star} = \begin{cases} \lambda_{j,t}^{k,*},\ j \in \mathcal{J}_1^k \\ \lambda_{j,t}^{k},\ j \in \mathcal{J}_2^k \\ 0,\ j \in \mathcal{J}_3^k \end{cases}, \tag{24}$$

with nonrandom processes $\{\lambda_j^k\}_{j\in\mathcal{J}_2^k}$*, we assume*

$$\int_u^s \mathbf{E}^{\boldsymbol{\lambda}^{k,\star}}[D_{j,u}^{\boldsymbol{\lambda}^{k,\star}}\nu_\tau|\mathcal{F}_u]d\tau \geq 0,\ \forall s \in [u,T],\ \forall u \in [0,T],$$

and

$$\mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_u^T \left[D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_s^T \left\{\int_s^t \mathbf{E}_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[D_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\nu_\tau\right]d\tau\right\}dt\right]^\top dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right] = 0,\ \forall u \in [0,T], \tag{25}$$

where $D_{j,u}^{\boldsymbol{\lambda}^{k,\star}}$ *is the Malliavin derivative with respect to* $B_{j,u}^{\boldsymbol{\lambda}^{k,\star}}$.

Assumption 4 *For any* $\boldsymbol{\lambda}^{k,\star}$ *in (24) and any* $\hat{\boldsymbol{\lambda}}^k = (\hat{\lambda}_1^k, \ldots, \hat{\lambda}_d^k)^\top$ *where* $\hat{\lambda}_{j,t}^k = \begin{cases} 0,\ j \in \mathcal{J}_1^k \\ \hat{\lambda}_{j,t}^k,\ j \in \mathcal{J}_2^k \\ 0,\ j \in \mathcal{J}_3^k \end{cases}$,

$\{\hat{\lambda}_j^k\}_{j\in\mathcal{J}_2^k}$ *are positive nonrandom processes, we assume*

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\left\{\int_0^T\int_0^t \mathbf{E}^{\boldsymbol{\lambda}^{k,\star}+\alpha\hat{\boldsymbol{\lambda}}^k}[\nu_s]dsdt - \int_0^T\int_0^t \mathbf{E}^{\boldsymbol{\lambda}^{k,\star}}[\nu_s]dsdt\right\} \geq 0. \tag{26}$$

Remark 5 *These assumptions are interpreted as follows.*

Assumption 2 indicates that the agents solve the optimization problems with the bandwidths of their views, whose levels are within a range dependent on the given volatility of the endowment process, which is natural and plausible since if the current economy is very stable, there are few rooms for agents' sentiment levels and small differences among agents' sentiments. On the contrary, if it is very volatile, there is a large room for agents' sentiment levels, and there can be big differences among agents' sentiments.

Assumption 3 implies that the drift of the endowment process ν *moves in the same direction as the increment of the Brownian motion* $B_{j,u}^{\boldsymbol{\lambda}^{k,\star}}$ *on average in total.*

Also, Assumption 4 indicates that when the aggressive view strengthens, the drift of the aggregate endowment ν *also increases under the subjective probability measure on average in total. Thus, both Assumptions 3 and 4 imply that the view on the drift strengthens when the views on the Brownian motions strengthen. We remark that* ν *given as a sum of Ornstein-Uhlenbeck processes satisfy Assumptions 3 and 4 (see Example 1 in Section 1 of the online supplementary file [13]).*

Then, the following result holds for the individual optimization problem (5) in the log-utility case.

Theorem 3 *Under Assumptions 1-4, given* H_0 *in (22),* $\lambda_j^{k,*}, j \in \mathcal{J}_1^k, \mathcal{J}_2^k$ *of* $\boldsymbol{\lambda}^{k,*}$ *in (21) and* $(\bar{c}^{k,*}, \boldsymbol{\pi}^{k,*})$ *with* $\bar{c}_t^{k,*} = \frac{\eta_t^{\boldsymbol{\lambda}^{k,*}}\mathbf{E}\left[\int_0^T H_{0,s}\varepsilon_s^k ds\right]}{TH_{0,t}}$ *and* $\boldsymbol{\pi}^{k,*}$ *in (15) attain the individual optimization problem (5).*

Proof. First, we fix $k \in \{1, \ldots, K\}$. Without loss of generality, we consider the case where $\mathcal{J}_1^k = \{1, \ldots, d_1\}$, $\mathcal{J}_2^k = \{d_1+1, \ldots, d_1+d_2\}$, and $\mathcal{J}_3^k = \{d_1+d_2+1, \ldots, d\}$, $d_1, d_2 \geq 0$. Set $\boldsymbol{\lambda}_1^k = (\lambda_1^k, \ldots, \lambda_{d_1}^k)^\top$, $\boldsymbol{\lambda}_2^k = (\lambda_{d_1+1}^k, \ldots, \lambda_{d_1+d_2}^k)^\top$, and $\boldsymbol{\lambda}^k = (\boldsymbol{\lambda}_1^{k\top}, \boldsymbol{\lambda}_2^{k\top}, 0, \ldots, 0)^\top$. In the following, we denote $\boldsymbol{E}^{\boldsymbol{\lambda}^k}[\cdot]$ by $\boldsymbol{E}^{\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k}[\cdot]$ and $\eta^{\boldsymbol{\lambda}^k}$ by $\eta^{\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k}$.

Then, by Propositions 6 and 7 in Appendix A, we observe that for given $\boldsymbol{\lambda}_2^k$, $\boldsymbol{\lambda}_1^{k,*} = (-\bar{\lambda}_1^k, \ldots, -\bar{\lambda}_{d_1}^k)^\top$ and $c_t^{k,*} = \frac{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]}{TH_{0,t}}$ attain the sup-inf in (10), and $\sup_{\boldsymbol{\lambda}_2^k,|\lambda_j^k|\leq\bar{\lambda}_j^k,\ j=d_1+1,\ldots,d_1+d_2} J(c^{k,*}, \boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k)$ is attained at $\boldsymbol{\lambda}_2^{k,*} = (\bar{\lambda}_{d_1+1}^k, \ldots, \bar{\lambda}_{d_1+d_2}^k)^\top$.

Next, we show that the optimal consumption and portfolio process pair for the k-th agent $(\bar{c}^{k,*}, \boldsymbol{\pi}^{k,*})$ is admissible. In fact, for $\bar{c}^{k,*}$ and $\boldsymbol{\pi}^{k,*}$ given by (15), the right-hand side of (7) becomes

$$\int_0^t H_{0,s}[\sigma_s^\top\boldsymbol{\pi}_s^{k,*} + W_s^{k,*}\boldsymbol{\theta}_s]^\top dB_s = \int_0^t \boldsymbol{\psi}_s^{k\top}dB_s,$$

which is a martingale by (16). $\mathbf{E}[\int_0^T U^k(\bar{c}_t^{k,*})^2dt] = \mathbf{E}[\int_0^T(\log\bar{c}_t^{k,*})^2dt] < \infty$ also follows from $\mathbf{E}[\int_0^T\int_0^t \nu_s^2 dsdt] < \infty$, the boundedness of $\boldsymbol{\lambda}^{k,*}$, and the expressions of r and $\boldsymbol{\theta}$ as in (31) and (32) in Proposition 5 in Section 4. (For details, see Section 4 of the online supplementary file [13].) Then, since $W_T^{k,*} = 0$ by (15), the optimal consumption and portfolio process pair $(\bar{c}^{k,*}, \boldsymbol{\pi}^{k,*})$ is in $\mathcal{A}^k$.

Therefore, $\lambda_j^{k,*}, j \in \mathcal{J}_1^k, \mathcal{J}_2^k$ of $\boldsymbol{\lambda}^{k,*}$ in (21) and $(\bar{c}^{k,*}, \boldsymbol{\pi}^{k,*})$ attain the individual optimization problem (5) for the given state-price density process H_0 in (22). □

4 Equilibrium interest rate and market price of risk (Log-utility case)

In the following, first, we show in Theorem 4 below that given the state-price density process H_0 in (22) for the log utility, the market is in equilibrium, that is, $\{(\bar{c}^{k,*}, \boldsymbol{\pi}^{k,*})\}_{k=1,\ldots,K}$ obtained in Section 3 satisfies the clearing conditions (17)-(19) in Section 2.3.

Then, we obtain expressions of the equilibrium interest rate and market price of risk in Proposition 5.

Theorem 4 *Under Assumptions 1-4, given the state-price density process H_0 in (22), the clearing conditions (17)-(19) hold for $\{(\bar{c}^{k,*}, \boldsymbol{\pi}^{k,*})\}_{k=1,\ldots,K}$ in Theorem 3.*

Proof. First, we confirm the clearing condition of the commodity market (17). Since $\bar{c}^{k,*}_t = \frac{\eta^{k,*}_t Y^k}{\sum_{l=1}^K \eta^{l,*}_t Y^l}\varepsilon_t$ in Remark 4, the aggregate consumption is

$$\sum_{k=1}^K \bar{c}^{k,*}_t = \sum_{k=1}^K \frac{\eta^{k,*}_t Y^k}{\sum_{l=1}^K \eta^{l,*}_t Y^l}\varepsilon_t = \varepsilon_t.$$

Next, we consider the clearing condition of the stock market (18). Set

$$M^{k,*}_t = \mathbf{E}\left[\int_0^T H_{0,s}(\bar{c}^{k,*}_s - \varepsilon^k_s)ds|\mathcal{F}_t\right].$$

By $\sum_{k=1}^K \bar{c}^{k,*}_t = \varepsilon_t \left(= \sum_{k=1}^K \varepsilon^k_t\right)$, we have

$$\sum_{k=1}^K M^{k,*}_t = \sum_{k=1}^K \mathbf{E}\left[\int_0^T H_{0,s}(\bar{c}^{k,*}_s - \varepsilon^k_s)ds|\mathcal{F}_t\right] = 0, \quad t \in [0,T]. \tag{27}$$

By the martingale representation of $M^{k,*}_t$,

$$0 = \sum_{k=1}^K M^{k,*}_t = \int_0^t \sum_{k=1}^K \boldsymbol{\psi}^{k\top}_s dB_s, \ t \in [0,T].$$

Thus, we obtain

$$\sum_{k=1}^K \boldsymbol{\psi}^k_t = \mathbf{0}_d, \ t \in [0,T]. \tag{28}$$

Similarly, by (15), (27), and (28), since

$$\sum_{k=1}^K W^{k,*}_t = 0, \ t \in [0,T], \tag{29}$$

we have

$$\sigma_t^\top \sum_{k=1}^K \boldsymbol{\pi}^{k,*}_t = \frac{1}{H_{0,t}} \sum_{k=1}^K \boldsymbol{\psi}^k_t = \mathbf{0}_d, \ t \in [0,T].$$

Since $\sigma_t^\top$ is non-singular, we obtain

$$\sum_{k=1}^K \boldsymbol{\pi}^{k,*}_t = \mathbf{0}_d, \ t \in [0,T]. \tag{30}$$

Finally, the clearing condition of the money market (19)

$$\sum_{k=1}^K (W^{k,*}_t - \boldsymbol{\pi}^{k,*\top}_t \mathbf{1}_d) = 0, \ t \in [0,T],$$

follows from (29) and (30). □

Since H_0 in (22) is the state-price density process in equilibrium, by applying Ito's formula to (22) and comparing the result with (2), we obtain expressions of the interest rate and the market price of risk with heterogeneous views on fundamental risks.

Proposition 5 *The interest rate r and the market price of risk $-\boldsymbol{\theta}$ in equilibrium are given by*

$$r_t = \nu_t - |\boldsymbol{\rho}_t|^2 + \boldsymbol{\rho}_t^\top \left[\sum_{k=1}^K \left(\frac{Y^k \eta^{k,*}_t}{\sum_{l=1}^K Y^l \eta^{l,*}_t}\right) \boldsymbol{\lambda}^{k,*}_t\right], \tag{31}$$

$$-\boldsymbol{\theta}_t = \boldsymbol{\rho}_t - \left[\sum_{k=1}^K \left(\frac{Y^k \eta^{k,*}_t}{\sum_{l=1}^K Y^l \eta^{l,*}_t}\right) \boldsymbol{\lambda}^{k,*}_t\right]. \tag{32}$$

Proof.

Applying Ito's formula to (20), we have

$$d\left(\frac{1}{\varepsilon_t}\right) = \left(\frac{1}{\varepsilon_t}\right)\left[\frac{-d\varepsilon_t}{\varepsilon_t} + \frac{d\langle\varepsilon\rangle_t}{\varepsilon_t^2}\right] = \left(\frac{1}{\varepsilon_t}\right)\left[(-\nu_t + |\boldsymbol{\rho}_t|^2)dt - \boldsymbol{\rho}_t^\top dB_t\right].$$

Noting that

$$d\eta^{k,*}_t = \eta^{k,*}_t \boldsymbol{\lambda}^{k,*\top}_t dB_t, \ k = 1, 2, \ldots, K, \tag{33}$$

we have

$$d\left(\sum_{k=1}^K \eta^{k,*}_t Y^k\right) = \left(\sum_{k=1}^K \eta^{k,*}_t Y^k\right)\left[\sum_{k=1}^K \frac{Y^k \eta^{k,*}_t \boldsymbol{\lambda}^{k,*}_t}{\sum_{l=1}^K Y^l \eta^{l,*}_t}\right]^\top dB_t.$$

Recall (22),

$$H_{0,t} = \frac{\sum_{k=1}^K \eta^{k,*}_t Y^k}{\sum_{k=1}^K Y^k}\frac{\varepsilon_0}{\varepsilon_t}. \tag{34}$$

Then, applying Ito's formula to (34), we have

$$dH_{0,t} = H_{0,t}\left(\{-\nu_t + |\boldsymbol{\rho}_t|^2\} - \boldsymbol{\rho}_t^\top \left[\sum_{k=1}^K \frac{Y^k \eta^{k,*}_t \boldsymbol{\lambda}^{k,*}_t}{\sum_{l=1}^K Y^l \eta^{l,*}_t}\right]\right)dt + H_{0,t}\left(\left[\sum_{k=1}^K \frac{Y^k \eta^{k,*}_t \boldsymbol{\lambda}^{k,*}_t}{\sum_{l=1}^K Y^l \eta^{l,*}_t}\right] - \boldsymbol{\rho}_t\right)^\top dB_t. \tag{35}$$

Comparing (35) with (2),

$$dH_{0,t} = H_{0,t}\left(-r_t dt + \boldsymbol{\theta}_t^\top dB_t\right), \tag{36}$$

we have (31) and (32). □

We interpret the expressions of r in (31) and $-\boldsymbol{\theta}$ in (32) as follows. First, $\sum_{k=1}^K \left(\frac{Y^k \eta^{k,*}_t}{\sum_{l=1}^K Y^l \eta^{l,*}_t}\right) \boldsymbol{\lambda}^{k,*}_t$ in (31) represents a weighted average of the views $\boldsymbol{\lambda}^{k,*}_t$ $(k = 1, \ldots, K)$ on the fundamental risks B over the K agents. Then, for the j-th component, if the weighted average of the views $\lambda^{k,*}_{j,t}$ is negative (positive), it affects negatively (positively) the equilibrium interest rate r_t since $\rho_{j,t}$ is positive by Assumption 1. This implies that the net conservative (aggressive) view on a risk makes the interest rate lower (higher), which corresponds to the lower (higher) interest rate.

Next, (32) indicates that the weighted average of the views over the K agents on the j-th risk, $\sum_{k=1}^{K}\left(\frac{Y^k\eta_t^{k,*}}{\sum_{l=1}^{K}Y^l\eta_t^{l,*}}\right)\lambda_{j,t}^{k,*}$, affects positively (negatively) the j-th component of the market price risk $-\theta_{j,t}$ if it is negative (positive). This implies that the net conservative (aggressive) view on a fundamental risk requires a higher (lower) market price of risk in return for the investment on the risk.

5 Numerical examples

In this section, we consider the log-utility case with a proportional endowment allocation. First, Section 5.1 shows the explicit expressions of the agents' equilibrium optimal consumption and portfolios based on Theorem 3, and provide their interpretation in detail. Then, we present its numerical example for two agents having heterogeneous views on a fundamental risk. We note that Proposition 5 gives us the equilibrium interest rate r and market price of risk $-\boldsymbol{\theta}$ appearing in (36), the equilibrium state density process $H_{0,t}$. Then, using this process, Section 5.2 derives the term structure of interest rates, namely $\mathcal{Y}(T) = -\frac{\log P(0,T)}{T}$ with $P(0,T) = \mathbf{E}[H_{0,T}]$ and provide its numerical examples for two agents having heterogeneous views on two fundamental risks.

5.1 *Agents' equilibrium optimal consumption and portfolios*

5.1.1 *Explicit expressions of equilibrium optimal consumption and portfolios*

Let the endowment allocation to each agent, ε^k be proportional to the aggregate endowment process ε. That is, $\varepsilon_t^k = a^k\varepsilon_t$, $a^1,\dots,a^K > 0$, $\sum_{l=1}^{K} a^l = 1$, which determines Y_k as $Y_k = \frac{a_k}{a_1}$ and hence, the equilibrium state density process $H_{0,t}$ in (22) is given as $H_{0,t} = \sum_{k=1}^{K} a^k\eta_t^{k,*}\frac{\varepsilon_0}{\varepsilon_t}$.

Then, given a fixed time $t \in [0,T)$, the optimal consumption process $\bar{c}_t^{k,*}$ in Theorem 3 is explicitly calculated as

$$\bar{c}_t^{k,*} = \varepsilon_t\frac{a^k\eta_t^{k,*}}{\sum_{l=1}^{K} a^l\eta_t^{l,*}} = \frac{W_t^{k,*}}{(T-t)} + a^k\varepsilon_t, \qquad (37)$$

where the optimal wealth of agent k, $W_t^{k,*}$ is given by

$$W_t^{k,*} = a^k\varepsilon_t(T-t)\left(\frac{\eta_t^{k,*} - \sum_{l=1}^{K} a^l\eta_t^{l,*}}{\sum_{l=1}^{K} a^l\eta_t^{l,*}}\right). \qquad (38)$$

Next, agent k's optimal portfolio is expressed as

$$\boldsymbol{\pi}_t^{k,*} = (\sigma_t^{-1})^\top\left(\frac{\boldsymbol{\psi}_t^k}{H_{0,t}} - W_t^{k,*}\boldsymbol{\theta}_t\right), \qquad (39)$$

where

$$\frac{\boldsymbol{\psi}_t^k}{H_{0,t}} = a^k\varepsilon_t(T-t)\frac{(\eta_t^{k,*}\boldsymbol{\lambda}_t^{k,*} - \sum_{l=1}^{K} a^l\eta_t^{l,*}\boldsymbol{\lambda}_t^{l,*})}{\sum_{l=1}^{K} a^l\eta_t^{l,*}}, \qquad (40)$$

and

$$-\boldsymbol{\theta}_t = \boldsymbol{\rho}_t - \sum_{l=1}^{K}\frac{a^l\eta_t^{l,*}}{\sum_{l=1}^{K} a^l\eta_t^{l,*}}\boldsymbol{\lambda}_t^{l,*}. \qquad (41)$$

5.1.2 *Interpretation of the optimal consumption and portfolios*

First, let us provide an interpretation of the optimal consumption in (37). Given each agent's endowment $\varepsilon_t^k = a^k\varepsilon_t$, if an agent's aggressive or conservative sentiment is more realized until time-t than the other agents' sentiments, then the agent's time-t wealth and hence consumption is larger than otherwise.

In detail, we note that given $\varepsilon_t^k = a^k\varepsilon_t$, $\bar{c}_t^{k,*}$ is larger when $W_t^{k,*}$ is larger. Then, we observe that the larger is agent k's density $\eta_t^{k,*}$ relative to $\sum_{l=1}^{K} a^l\eta_t^{l,*}$ (the weighted average of $\eta_t^{l,*}$), $W_t^{k,*}$ in (38) is larger.

As a simple example, with the one-dimensional ($d = 1$) Brownian motion $B_t \equiv B_{1,t}$, if Brownian motion $B_t > 0$ namely, a good state realizes at t, then $\eta_t^{k,*}$ becomes larger than $\eta_t^{k',*}$ ($k' \neq k$) when the agent k is aggressive for the risk B with a positive constant $\boldsymbol{\lambda}^{k,*}$, while the others ($k' \neq k$) are conservative with the same absolute value. Precisely, $\boldsymbol{\lambda}^{k,*} \equiv \boldsymbol{\lambda}_s^{k,*} > 0, \boldsymbol{\lambda}_s^{k',*} = -\boldsymbol{\lambda}^{k,*} < 0$, $\forall s \in [0,t]$, $k' \neq k$ in $\eta_t^{l,*} = \exp\left\{\int_0^t \boldsymbol{\lambda}_s^{l,*}dB_s - \frac{1}{2}\int_0^t |\boldsymbol{\lambda}_s^{l,*}|^2 ds\right\}$, $l = k, k'$.

On the contrary, if Brownian motion $B_t < 0$ namely, a bad state realizes at t, then $\eta_t^{k,*}$ becomes larger when the agent k is conservative for the risk B with a negative constant $\boldsymbol{\lambda}^{k,*}$, while the other agents are aggressive with the same absolute value. Precisely, $\boldsymbol{\lambda}^{k,*} \equiv \boldsymbol{\lambda}_s^{k,*} < 0, \boldsymbol{\lambda}_s^{k',*} = -\boldsymbol{\lambda}^{k,*} > 0$, $\forall s \in [0,t]$, $k' \neq k$.

Next, we provide an interpretation of the optimal portfolio in (39). Firstly, let us note that $-\boldsymbol{\theta}_t$ in (41) is positive ($-\boldsymbol{\theta}_t > 0$) under Assumption 2, and that agent k's optimal portfolio $\boldsymbol{\pi}_t^{k,*}$ is decomposed into the hedging portfolio part $(\sigma_t^{-1})^\top\frac{\boldsymbol{\psi}_t^k}{H_{0,t}}$ and the mean-variance portfolio part $W_t^{k,*}(\sigma_t^{-1})^\top(-\boldsymbol{\theta}_t)$. For simplicity, with only one risky asset and one-dimensional Brownian motion, i.e. $d = 1$, and $\sigma_t > 0$, we explain those separately below.

Firstly, as for the hedging portfolio part $(\sigma_t^{-1})^\top\frac{\boldsymbol{\psi}_t^k}{H_{0,t}}$ expressed as (40), we observe that the larger is $\eta_t^{k,*}\boldsymbol{\lambda}_t^{k,*}$ relative to $\sum_{l=1}^{K} a^l\eta_t^{l,*}\boldsymbol{\lambda}_t^{l,*}$ (the weighted average of $\eta_t^{l,*}\boldsymbol{\lambda}_t^{l,*}$), $\frac{\boldsymbol{\psi}_t^k}{H_{0,t}}$ is positive. Namely, if the agent k's density weighted view at t, that is $\eta_t^{k,*}\boldsymbol{\lambda}_t^{k,*}$, on Brownian motion B is larger than its weighted average of all the agents (i.e., $\sum_{l=1}^{K} a^l\eta_t^{l,*}\boldsymbol{\lambda}_t^{l,*}$), then the hedging portfolio part in the agent k's optimal portfolio, $(\sigma_t^{-1})^\top\frac{\boldsymbol{\psi}_t^k}{H_{0,t}}$ becomes a positive amount.

Secondly, as for $W_t^{k,*}(\sigma_t^{-1})^\top(-\boldsymbol{\theta}_t)$, the mean-variance portfolio part in the agent k's optimal portfolio, given the volatility σ_t of the risky assets and the equilibrium market price of risk $-\boldsymbol{\theta}_t > 0$, this part becomes larger when the optimal wealth $W_t^{k,*}$ in (38) is larger. Thus, similarly to the optimal consumption process $\bar{c}_t^{k,*}$ above, if the agent's aggressive or conservative sentiment is more realized until time-t than the other agents' sentiments, then the agent's time-t wealth and hence mean-variance portfolio part is larger than otherwise.

5.1.3 *Numerical examples of optimal consumption and portfolio*

We consider the log-utility case with $K = 2$ (i.e., $k = 1, 2$) and $d = 1$ (i.e., $B = B_1$), where $\mathcal{J}_1^1 = \{1\}$, $\mathcal{J}_1^2 = \{2\}$. Namely, we suppose that agent 1 is conservative about the fundamental risk B_1, while agent 2 is aggressive about B_1. Thus, $\boldsymbol{\lambda}_t^{1,*} = (\lambda_{1,t}^{1,*}) = (-\bar{\lambda}_{1,t}^1)$, $\boldsymbol{\lambda}_t^{2,*} = (\lambda_{1,t}^{2,*}) = (\bar{\lambda}_{1,t}^2)$. Specifically, we set the parameters $\bar{\lambda}_{j,t}^k$, $k = 1, 2$, $j = 1$ to be constants as $\bar{\lambda}_{1,t}^1 = 0.03$ and $\bar{\lambda}_{1,t}^2 = 0.15$.

We note that since $\boldsymbol{\lambda}^{k,*}$ is a constant,

$$\begin{aligned}\eta_t^{k,*} &= \exp\left(\int_0^t \boldsymbol{\lambda}_s^{k,*} \cdot dB_s - \frac{1}{2}\int_0^t |\boldsymbol{\lambda}_s^{k,*}|^2 ds\right) \\ &= \exp\left(\boldsymbol{\lambda}^{k,*} B_t - \frac{1}{2}|\boldsymbol{\lambda}^{k,*}|^2 t\right). \end{aligned} \tag{42}$$

In addition, we assume each agent k's endowment process ε^k, $k = 1, 2$ to be proportional to the aggregate endowment ε, particularly, $\varepsilon_t^k = a^k \varepsilon_t$ with $a^1 = a^2 = 0.5$, and set the aggregate endowment's expected return and volatility as $\nu = 0.04$ and $\rho_1 = 0.2$, respectively.

In this setting, using equations (37) and (39) with (38), (40), (41), we provide numerical examples of $\frac{\bar{c}_t^{k,*}}{\varepsilon_t}$ and $(\sigma_t)^\top \frac{\boldsymbol{\pi}_t^{k,*}}{\varepsilon_t(T-t)}$ for a given realized Brownian motion $B_t = B_{1,t}$. Concretely, we compute agent k's optimal consumption per unit of the aggregate endowment, which is given by (37) and (38) as

$$\frac{\bar{c}_t^{k,*}}{\varepsilon_t} = \frac{a^k \eta_t^{k,*}}{\sum_{l=1}^K a^l \eta_t^{l,*}}. \tag{43}$$

We note that the time-t optimal consumption of the agent whose view for the realized Brownian motion is more accurate than the other one becomes larger.

In addition, we calculate agent k's optimal investment to the risky asset per unit of the aggregate endowment and remaining period, which represents the exposure to the Brownian motion $B_t = B_{1,t}$ and is given by (39) with (38), (40) as follows:

$$\begin{aligned}&(\sigma_t)^\top \frac{\boldsymbol{\pi}_t^{k,*}}{\varepsilon_t(T-t)} \\ &= a^k \frac{(\eta_t^{k,*}\boldsymbol{\lambda}_t^{k,*} - \sum_{l=1}^K a^l \eta_t^{l,*}\boldsymbol{\lambda}_t^{l,*})}{\sum_{l=1}^K a^l \eta_t^{l,*}} \\ &\quad + a^k(-\boldsymbol{\theta}_t)\left(\frac{\eta_t^{k,*}}{\sum_{l=1}^K a^l \eta_t^{l,*}} - 1\right), \end{aligned} \tag{44}$$

where the first and second terms on the right-hand side of (44) correspond to the hedging and mean-variance part, respectively.

We note that the hedging part of the more conservative agent in terms of $\eta_t^{k,*}\boldsymbol{\lambda}_t^{k,*}$ is a negative amount, while that of the more aggressive agent is a positive amount. Moreover, given the positive market price of risk ($-\boldsymbol{\theta}_t > 0$) under Assumption 2 the time-t mean-variance part of the agent whose view for the realized Brownian motion is more accurate than the other one is larger.

In Tables 1 and 2, we compute those quantities for a given realized Brownian motion $B_{1,t} = 0.7$ and $B_{1,t} = -0.7$ at $t = 5$, respectively. The results are consistent with the interpretation for the optimal consumption and investment provided above as well as in Section 5.1.2. In detail, agent 1, who has a conservative view on B_1, takes a short position on the Brownian motion as the negative amount of the hedging part (i) indicates in both tables. On the other hand, the optimal consumption and the mean-variance part (ii) of agent 1, which reflect the accuracy of the agents' views on the realization of the Brownian motion, are less (more) than those of agent 2 in the case of Table 1 (Table 2), where agent 1's conservative view on the Brownian motion is not realized (is realized).

Table 1
Agents' optimal consumption and investment to the risky asset with $\bar{\lambda}_{1,t}^1 = 0.03$ and $\bar{\lambda}_{1,t}^2 = 0.15$ for a given realized Brownian motion $B_{1,t} = 0.7$ at $t = 5$.

	Optimal	Optimal Investment		
$B_{1,t} = 0.7$	Consumption	(i)hedging	(ii)mean-variance	investment:(i)+(ii)
agent 1	0.482	-0.046	-0.002	-0.049
agent 2	0.518	0.046	0.002	0.049

Table 2
Agents' optimal consumption and investment to the risky asset with $\bar{\lambda}_{1,t}^1 = 0.03$ and $\bar{\lambda}_{1,t}^2 = 0.15$ for a given realized Brownian motion $B_{1,t} = -0.7$ at $t = 5$.

	Optimal	Optimal Investment		
$B_{1,t} = -0.7$	Consumption	(i)hedging	(ii)mean-variance	investment:(i)+(ii)
agent 1	0.545	-0.042	0.007	-0.036
agent 2	0.455	0.042	-0.007	0.036

Moreover, given $\nu - \rho_1^2 = 0$ and $\rho_1 = 0.2$ in the equation (31) under the current parameter setting, a weighted average of both agents' views ($\boldsymbol{\lambda}_t^{k,*}$, $k = 1,2$) on the risk B_1 completely determines the equilibrium short-term interest rate r_t as 1.26% and 1.04% for $B_{1,t} = 0.7$ and $B_{1,t} = -0.7$, respectively, which is much higher than 0.01% and -0.01% for the case that $\bar{\lambda}^1_{1,t} = \bar{\lambda}^2_{1,t} = 0.03$. The optimal consumption and investment in this case are shown in Tables 3 and 4 below. Then, with the equilibrium interest rates, comparing the case $\bar{\lambda}^1_{1,t} = 0.03$ and $\bar{\lambda}^2_{1,t} = 0.15$ (Case (I)) to the case $\bar{\lambda}^1_{1,t} = \bar{\lambda}^2_{1,t} = 0.03$ (Case (II)), agent 2 in Case (I) is much more aggressive to long the larger amount of the risky asset by borrowing money even with the higher interest rate.

Table 3
Agents' optimal consumption and investment to the risky asset with $\bar{\lambda}^1_{1,t} = \bar{\lambda}^2_{1,t} = 0.03$ for a given realized Brownian motion $B_{1,t} = 0.7$ at $t = 5$.

	Optimal	Optimal Investment		
$B_{1,t} = 0.7$	Consumption	(i)hedging	(ii)mean-variance	investment:(i)+(ii)
agent 1	0.490	-0.015	-0.002	-0.017
agent 2	0.510	0.015	0.002	0.017

Table 4
Agents' optimal consumption and investment to the risky asset with $\bar{\lambda}^1_{1,t} = \bar{\lambda}^2_{1,t} = 0.03$ for a given realized Brownian motion $B_{1,t} = -0.7$ at $t = 5$.

	Optimal	Optimal Investment		
$B_{1,t} = -0.7$	Consumption	(i)hedging	(ii)mean-variance	investment:(i)+(ii)
agent 1	0.510	-0.015	0.002	-0.013
agent 2	0.490	0.015	-0.002	0.013

Finally, we remark that each zero yield as an element of the term structure of interest rates is determined by a collection of the equilibrium short-term interest rates r_t $(0 \leq t)$ with the risk-neutral density process defined by the equilibrium market price of risk $-\boldsymbol{\theta}_t$, $\mathcal{Z}_t = \exp\left\{\int_0^t \boldsymbol{\theta}_s^\top dB_s - \frac{1}{2}\int_0^t |\boldsymbol{\theta}_s|^2 ds\right\}$. Precisely, let $\mathcal{Y}(T)$ denote the continuously compounded zero yield with maturity T at the initial time 0. Then, $\mathcal{Y}(T)$ is obtained as

$$\mathcal{Y}(T) = \frac{-\log \mathbf{E}\left[H_{0,T}\right]}{T} = \frac{-\log \mathbf{E}\left[\frac{\mathcal{Z}_T}{S_T^0}\right]}{T} = \frac{-\log \mathbf{E}^{\mathbf{Q}}\left[e^{-\int_0^T r_t dt}\right]}{T}, \tag{45}$$

with the expectation operator $\mathbf{E}^{\mathbf{Q}}[\cdot]$ under the risk-neutral probability measure $\mathbf{Q}$. Then, $\{\mathcal{Y}(T)\}_{0\leq T}$ provides the term structure of interest rates, which will be examined in more detail in the next subsection.

5.2 Term structure of interest rates

This subsection provides numerical examples of the term structure of interest rates with the state-price density process expressed by a weighted average of the density processes of each agent's subjective probability.

Concretely, we consider the log-utility case with $K = 2$ and $d = 2$, where $\mathcal{J}_1^1 = \{1\}, \mathcal{J}_2^1 = \{2\}, \mathcal{J}_1^2 = \{2\}, \mathcal{J}_2^2 = \{1\}$. Namely, agent 1 is conservative about the fundamental risk B_1 and aggressive about the risk B_2, while agent 2 is aggressive about B_1 and conservative about B_2. Thus, $\boldsymbol{\lambda}_t^{1,*} = (\lambda_{1,t}^{1,*}, \lambda_{2,t}^{1,*})^\top = (-\bar{\lambda}^1_{1,t}, \bar{\lambda}^1_{2,t})^\top$, $\boldsymbol{\lambda}_t^{2,*} = (\lambda_{1,t}^{2,*}, \lambda_{2,t}^{2,*})^\top = (\bar{\lambda}^2_{1,t}, -\bar{\lambda}^2_{2,t})^\top$ as in (21). For instance, we can regard B_1 as foreign risks and B_2 as domestic risks. Also, we can consider agent 1 as domestic investors who are conservative about the foreign risks and aggressive about the domestic risks, and agent 2 as foreign investors who are aggressive about the foreign risks and conservative about the domestic risks. We further assume endowment processes for ε^k, $k = 1,2$ to be proportional to ε. Specifically, $\varepsilon_t^k = a^k \varepsilon_t$, $a^1 = a^2 = 0.5$. Then, we obtain $Y^2 = \frac{a^2}{a^1}$, in particular $Y^1 = Y^2 = 1$, from the equation (23) since $\mathbf{E}\left[\int_0^T \eta_t^{k,*} dt\right] = T$. Moreover, we set $\nu = 0.08$, $\rho_1 = \rho_2 = 0.2$.

Next, the price of a zero-coupon bond with maturity T that pays off one unit of cash at T, which is denoted by $P(0,T)$, is calculated as $P(0,T) = \mathbf{E}[H_{0,T}] = \mathbf{E}^{\mathbf{Q}}\left[e^{-\int_0^T r_t dt}\right]$. Specifically, we present the continuously compounded zero yields $\mathcal{Y}(T)$ for $T = 1,3,5,10,20,30,40$, which are given by $\mathcal{Y}(T) = -\frac{\log P(0,T)}{T}$. We compute the price of the zero-coupon bonds $P(0,T)$ by Monte Carlo simulation with 52-time grids per year and 100,000 paths.

Figure 1 shows the yield curves for four different sets of $\bar{\lambda}_j^k$, $k, j = 1,2$ as in Table 5.

Table 5
Parameter sets of $\bar{\lambda}^k_{j,t}$, $k, j = 1,2$, $(\lambda_{1,t}^{1,*} = -\bar{\lambda}^1_{1,t}$, $\lambda_{2,t}^{1,*} = \bar{\lambda}^1_{2,t}$, $\lambda_{1,t}^{2,*} = \bar{\lambda}^2_{1,t}$, $\lambda_{2,t}^{2,*} = -\bar{\lambda}^2_{2,t})$

Parameter	Set 1	Set 2	Set 3	Set 4
$\bar{\lambda}^1_{1,t}$	0.03	0.03	0.15	0.03
$\bar{\lambda}^1_{2,t}$	0.03	0.15	0.03	$0.03 (0 \leq t < 10)$, $0.15 (10 \leq t \leq 40)$
$\bar{\lambda}^2_{1,t}$	0.03	0.03	0.03	0.03
$\bar{\lambda}^2_{2,t}$	0.03	0.03	0.03	0.03

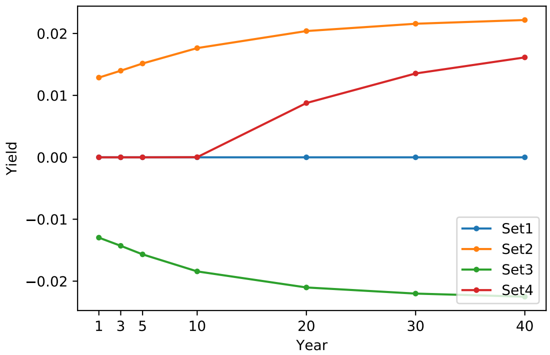

Fig. 1. Yield curves for different parameter sets

Firstly, Set 1 is the base case, where the two agents have opposite views on the fundamental risks with the same degrees. Due to the net effect of the views of the agents in (31), we observe that the yield curve is flat at the zero level, which is implied by $\nu - (\rho_1^2 + \rho_2^2) = 0$. On the other hand, for Sets 2-4, the two agents have opposite views on the fundamental risks, but the degrees of the views are different. Thus, due to the presence of $\eta^{k,*}$ $(k = 1, 2)$ in its expression (31), the interest rate r becomes stochastic and the corresponding yield curve is either upward sloping or downward sloping.

Next, in Sets 2 and 3, we shift the parameters of agent 1. With Set 2, where the parameter on the aggressive side $\bar{\lambda}_2^1$ is higher and the domestic investors become more aggressive about the domestic risks for example, the yield curve shape becomes upward sloping (positive yield). Similarly, with Set 3, where the parameter on the conservative side $\bar{\lambda}_1^1$ is higher and the domestic investors are more conservative about the foreign risks, the yield curve shape becomes downward sloping (negative yield).

Finally, in Set 4, we shift the parameter of agent 1 on the aggressive side $\bar{\lambda}_2^1$ beyond 10 years. This corresponds to the case where the domestic investors become more aggressive about the domestic risks in the future. This example reproduces the yield curve shape observed in the Japanese goverment bond (JGB) market after the financial crisis and amid the COVID-19 pandemic, where the yield curve in the short end is flat around zero levels while the curve is upward sloping in the long end.

6 Concluding remarks and future research

In this study, we have investigated a multi-agent equilibrium model incorporating heterogeneous views of the agents on the fundamental risks represented by Brownian motions. Firstly, we have solved the individual optimal consumption and portfolio problems in which the agents have heterogeneous views on the fundamental risks. The individual optimization problems are formulated as the sup-sup-inf problems, where we consider the sup-inf on the choice of a probability measure and the sup on the optimal consumption and investment.

Particularly, the setting includes the case where the agents have different degrees of conservativeness on the respective Brownian motions and the degrees also vary among the agents. In addition, this sup-sup-inf formulation for the individual optimization problems incorporates the aggressive views of market participants recognized as bullish sentiments in the monetary easing after the global financial crisis and amid the COVID-19 pandemic.

Moreover, we have obtained the state-price density process in a market equilibrium, which yields the expressions of the interest rate and the market price of risk with heterogeneous views on fundamental risks.

Furthermore, we have provided numerical examples for the term structure of interest rates with heterogeneous views on fundamental risks, where the net views of the agents affect the shape of the yield curve. The numerical examples also reproduce the yield curve shape under the monetary easing after the global financial crisis and amid the COVID-19 pandemic. Estimation of the model with market data is one of our future research topics.

6.1 *Remarks on boundaries of the sentiment factors and future research direction*

First of all, we remark that Section 6 of the online supplementary file [13] provides a possible extension to the case where the intervals of agents' views are stochastic in an exponential utility case. Studying it in more detail is our future research topic.

On the contrary, for the log-utility case, as a trade-off by extending the single representative agent cases in Saito and Takahashi [28][29] to the multi-agent case, we need to set nonrandom boundaries to determine the sign of the volatility Z_j, $j = 1, \cdots, d$ in each agent's optimal utility process, which enables us to explicitly solve the general market equilibrium.

Next, let us explain an implication of Assumption 2, and illustrate why the proposed model is meaningful and useful in estimating the multiple agents' different views and their transition.

Firstly, Assumption 2, which indicates that $|\lambda_{j,t}^k| < \rho_{j,t}$ should hold for all k, j, t, is helpful for agents to determine the nonrandom boundaries. That is, the agents solve the optimization problems with the bandwidths of their views, whose levels are within a range dependent on an exogenously given volatility $\boldsymbol{\rho}$ of the aggregate endowment process at each time-t.

In detail, reflecting one's own sentiment, each agent chooses a subjective probability from the family of measures equivalent to the initially given reference probability $\mathbf{P}$. Hence, all agents agree with $\boldsymbol{\rho}$, the given endowment's volatility process representing the economy's variability.

This assumption seems natural and plausible since if the current economy is very stable, there are few rooms for agents' sentiment levels and small differences among agents' sentiments. On the contrary, if it is very volatile, there is a large room for agents' sentiment levels, and there can be big differences among agents' sentiments.

In sum, an exogenously given endowment's volatility $\boldsymbol{\rho}$ representing the variability of our economy's fundamentals is a plausible benchmark for the bandwidth of each agent's view/sentiment on each risk.

Secondly, the proposed multi-agent model with deterministic boundaries seems to be useful in estimating the views of multiple agents and their transition by calibration and/or econometric methods such as a time series analysis. For instance, while the boundary is deterministic, the state-price density process H_0 is stochastic, which leads to reproducing the curvature of the term structure of interest rates (in addition to its slope), which is realistic and helpful for financial institutions and central banks.

Although a serious and detailed empirical analysis with our model is a future research topic, we briefly describe promising and possible examples below.

As the simplest example, in the Japanese long-term interest rate market, domestic life insurance companies and foreign traders such as international investment banks are the main players, and their views on the interest rate hike by the central bank differ: The foreign traders view that the central bank should raise the rate according to central banks in other countries. On the contrary, Japanese investors consider that the bank of Japan still persists in the monetary easing, which helps reduce the government's liability effectively.

In detail, after September 2016, when the Bank of Japan introduced the yield curve control (YCC) policy, the JGBs and the Japanese yen interest rate swap with maturity shorter than 10 years have been scarcely traded. For maturities beyond 10 years, Japanese life insurance companies are the main trading players in the JGB market, while foreign investment banks are in the yen interest rate swap market. Thus, assuming only agent 1 represents the Japanese life insurance companies in the JGB market, we could use its data to estimate agent 1's sentiment factor $\boldsymbol{\lambda}_t^{1,*}$ for $t > 10$ with $\boldsymbol{\lambda}_t^{1,*} = \mathbf{0}$ for $t \leq 10$. Similarly, assuming only agent 2 represents the foreign investment banks in the yen swap market, one could use its data for estimating agent 2's sentiment factor $\boldsymbol{\lambda}_t^{2,*}$ for $t > 10$ with $\boldsymbol{\lambda}_t^{2,*} = \mathbf{0}$ for $t \leq 10$.

In addition, when modeling the term structures of interest rates for the JGB and yen swap to estimate $\boldsymbol{\lambda}_t^{1,*}$ and $\boldsymbol{\lambda}_t^{2,*}$, one may use the equation (31) with $K = 1$ as follows: $r_t^{JGB} = r_t = \nu_t - |\boldsymbol{\rho}_t|^2 + \boldsymbol{\rho}_t^\top \boldsymbol{\lambda}_t^{1,*}$, and $r_t^{swap} = r_t^{JGB} + \boldsymbol{\rho}_t^\top(\boldsymbol{\lambda}_t^{2,*} - \boldsymbol{\lambda}_t^{1,*}) = \nu_t - |\boldsymbol{\rho}_t|^2 + \boldsymbol{\rho}_t^\top \boldsymbol{\lambda}_t^{2,*}$.

Furthermore, we may estimate the sentiment factors of different players in stock markets from the time series of the excess return of stocks, including stock indices and individual stocks.

In detail, by the expression of the excess return of stock $i, i = 1, \ldots, d$, $\boldsymbol{b}_t + \boldsymbol{\delta}_t - r_t \mathbf{1}_d = -\sigma_t \boldsymbol{\theta}_t$ where $\boldsymbol{\delta}_t$ is a given dividend rate of stock $i, i = 1, \ldots, d$, with the expression of the market price of risk $-\boldsymbol{\theta}_t = \boldsymbol{\rho}_t - \sum_{k=1}^K \frac{Y^k \eta_t^k}{\sum_{k=1}^K Y^k \eta_t^k} \boldsymbol{\lambda}_t^{k,*}$ in (32), we may extract the sentiment factors from the time series of the excess return, volatility of stocks, and dividend processes.

Thus, we could estimate the sentiment factors and their transitions of different market participants by focusing on the prices in separate markets where different groups of market participants trade.

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A Proofs

Proposition 6 *Under Assumptions 2 and 3,* $\boldsymbol{\lambda}_1^{k,*} = (-\bar{\lambda}_1^k, \ldots, -\bar{\lambda}_{d_1}^k)^\top$ *and* $c_t^{k,*} = \frac{\eta_t^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} \mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]}{TH_{0,t}}$ *attain the sup-inf in (10).*

Proof. First, it easily follows that assumptions in Lemma 1 are satisfied. Then, by Lemma 1, $\lambda_j^{k,*} = -\bar{\lambda}_j^k sgn(Z_j)$, $j = 1, \ldots, d_1$, and by a Malliavin calculus approach, we can obtain an explicit expression of Z_j $j = 1, \ldots, d_1$ and confirm $sgn(Z_j) = +1, j = 1, \ldots, d_1$, and thus the proposition holds. (For Malliavin calculus approaches to optimal portfolio problems, see Ocone and Karatzas [21] and Takahashi and Yoshida [32], for example). For details, see Section 2 of the online supplementary file [13]. □

Proposition 7 *Under Assumptions 2 and 4,* $\boldsymbol{\lambda}_2^{k,*} = (\bar{\lambda}_{d_1+1}^k, \ldots, \bar{\lambda}_{d_1+d_2}^k)^\top$ *attains the supremum in*

$$\sup_{\boldsymbol{\lambda}_2^k, |\lambda_j^k| \leq \bar{\lambda}_j^k, j=d_1+1,\ldots,d_1+d_2} \mathbf{E}\left[\int_0^T \eta_t^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} \log c_t^{k,*} dt\right], \tag{A.1}$$

where $\boldsymbol{\lambda}_1^{k,*} = (-\bar{\lambda}_1^k, \ldots, -\bar{\lambda}_{d_1}^k)^\top$ *and* $c_t^{k,*} = \frac{\eta_t^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} \mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]}{TH_{0,t}}$.

Proof. Noting that $c_t^{k,*} = \frac{\eta_t^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} \mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]}{TH_{0,t}} =$

$\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\frac{c_0^{k,*}}{H_{0,t}}$, we have

$$\log c_t^{k,*} = \log \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} - \log H_{0,t} + \log c_0^{k,*}.$$

Since $\mathbf{E}\left[\int_0^T \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log c_0^{k,*} dt\right] = T \log c_0^{k,*}$ is independent of $\boldsymbol{\lambda}_2^k$, this optimization problem is equivalent to

$$\begin{aligned} &\sup_{\boldsymbol{\lambda}_2^k} \mathbf{E}\left[\int_0^T \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left(\log \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} - \log H_{0,t}\right) dt\right] \\ &= \sup_{\boldsymbol{\lambda}_2^k} \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T \left(\log \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} - \log H_{0,t}\right) dt\right]. \end{aligned} \tag{A.2}$$

First, we consider

$$\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T \log \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} dt\right].$$

We note that

$$\begin{aligned} d\boldsymbol{B}_{1,t} &= d\boldsymbol{B}_{1,t}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \boldsymbol{\lambda}_{1,t}^{k,*} dt, \\ d\boldsymbol{B}_{2,t} &= d\boldsymbol{B}_{2,t}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \boldsymbol{\lambda}_{2,t}^{k} dt, \end{aligned} \tag{A.3}$$

where

$$\boldsymbol{B}_1 = (B_1, \ldots, B_{d_1})^\top,\ \boldsymbol{B}_2 = (B_{d_1+1}, \ldots, B_{d_1+d_2})^\top,$$
$$\boldsymbol{B}_1^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} = (B_1^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}, \ldots, B_{d_1}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k})^\top,$$
$$\boldsymbol{B}_2^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} = (B_{d_1+1}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}, \ldots, B_{d_1+d_2}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k})^\top.$$

By (A.3), we have

$$\begin{aligned} \log \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} &= \int_0^t \boldsymbol{\lambda}_{1,s}^{k,*\top} d\boldsymbol{B}_{1,s} - \frac{1}{2}\int_0^t |\boldsymbol{\lambda}_{1,s}^{k,*}|^2 ds \\ &+ \int_0^t \boldsymbol{\lambda}_{2,s}^{k\top} d\boldsymbol{B}_{2,s} - \frac{1}{2}\int_0^t |\boldsymbol{\lambda}_{2,s}^{k}|^2 ds \\ &= \int_0^t \boldsymbol{\lambda}_{1,s}^{k,*\top} d\boldsymbol{B}_{1,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \frac{1}{2}\int_0^t |\boldsymbol{\lambda}_{1,s}^{k,*}|^2 ds \\ &+ \int_0^t \boldsymbol{\lambda}_{2,s}^{k\top} d\boldsymbol{B}_{2,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \frac{1}{2}\int_0^t |\boldsymbol{\lambda}_{2,s}^{k}|^2 ds. \end{aligned}$$

Thus, with the nonrandomness of $\boldsymbol{\lambda}_1^{k,*}$ and $\boldsymbol{\lambda}_2^k$,

$$\begin{aligned} &\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T \log \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} dt\right] \\ &= \frac{1}{2}\int_0^T \int_0^t [|\boldsymbol{\lambda}_{1,s}^{k,*}|^2 + |\boldsymbol{\lambda}_{2,s}^{k}|^2] ds dt, \end{aligned}$$

which is increasing as a functional of deterministic process $\boldsymbol{\lambda}_2^k$.

Next, for $\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T - \log H_{0,t} dt\right]$, the following lemma holds.

Lemma 8

$$\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T - \log H_{0,t} dt\right] \tag{A.4}$$

is increasing as a functional of deterministic process $\boldsymbol{\lambda}_2^k$.

Proof. This is confirmed by calculating that the Gateaux derivative with respect to $\boldsymbol{\lambda}_2^k$ is positive. For details, see Section 3 of the online supplementary file [13]. □

Therefore, $\boldsymbol{\lambda}_2^{k,*} = (\bar{\lambda}_{d_1+1}^k, \ldots, \bar{\lambda}_{d_1+d_2}^k)^\top$ attains the supremum in (A.1). □

Online supplement for "Equilibrium multi-agent model with heterogeneous views on fundamental risks"

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Abstract

This supplementary file provides an example of the stochastic process ν satisfying Assumptions 3 and 4 in Section 3, details of the proofs for the Proposition 6 and Lemma 8 in Appendix A, the multi-agent equilibrium in the exponential utility case, and a possible extension to the case of stochastic boundaries on agents' views.

For the convenience of reference, the numbering of the equations, Propositions, and Lemmas is subsequent to that of the main text.

1. Example of stochastic process $\boldsymbol{\nu}$

In this section, we present an example of the stochastic process ν satisfying Assumptions 3 and 4 in Section 3. We consider the following form for ν.

Example 1. *We consider ν described as a sum of Ornstein-Uhlenbeck processes as follows. Under the probability measure* $\mathbf{P}$,

$$\nu_\tau = \sum_{j=1}^{d} X_{j,\tau},$$

$$X_{j,\tau} = X_{j,0} + \int_0^\tau (a_j - b_j X_{j,s})ds + \int_0^\tau \sigma_j dB_{j,s},$$

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where $a_j, b_j \in \boldsymbol{R}, \sigma_j > 0$, which is rewritten as

$$X_{j,\tau} =$$
$$X_{j,0} + \int_0^\tau (a_j - b_j X_{j,s}) ds + \int_0^\tau \sigma_j (dB_{j,s}^{\boldsymbol{\lambda}^{k,\star}} + \lambda_{j,s}^{k,\star} ds),$$

under $\mathbf{P}^{\boldsymbol{\lambda}^{k,\star}}$. We confirm that ν satisfies Assumption 3 as follows.

Taking the Malliavin derivative $D_{j,u}^{\boldsymbol{\lambda}^{k,\star}}$,

$$D_{j,u}^{\boldsymbol{\lambda}^{k,\star}} X_{j,\tau} = -\int_u^\tau b_j D_{j,u}^{\boldsymbol{\lambda}^{k,\star}} X_{j,s} ds + \sigma_j,$$

we obtain

$$D_{j,u}^{\boldsymbol{\lambda}^{k,\star}} X_{j,\tau} = \sigma_j e^{-b_j(\tau-u)}, (u \leq \tau).$$

Then,

$$\int_u^s \mathbf{E}_u^{\boldsymbol{\lambda}^{k,\star}} [D_{j,u}^{\boldsymbol{\lambda}^{k,\star}} \nu_\tau] d\tau = \int_u^s \mathbf{E}_u^{\boldsymbol{\lambda}^{k,\star}} [D_{j,u}^{\boldsymbol{\lambda}^{k,\star}} X_{j,\tau}] d\tau$$
$$= \int_u^s \sigma_j e^{-b_j(\tau-u)} d\tau = \sigma_j \frac{1 - e^{-b_j(s-u)}}{b_j} \geq 0,$$

where we denote $\mathbf{E}^{\boldsymbol{\lambda}^{k,\star}}[\cdot|\mathcal{F}_u]$ by $\mathbf{E}_u^{\boldsymbol{\lambda}^{k,\star}}$.

Also, this Ornstein-Uhlenbeck process also satisfies Assumption 4.

We note that

$$X_{j,s} = X_{j,0} + a_j \int_0^s e^{-b_j(s-u)} du + \sigma_j \int_0^s e^{-b_j(s-u)} dB_{j,u}.$$

Under $\mathbf{P}^{\boldsymbol{\lambda}^{k,\star}+\alpha\hat{\boldsymbol{\lambda}}^k}$,

$$X_{j,s} = X_{j,0} + a_j \int_0^s e^{-b_j(s-u)} du$$
$$+ \sigma_j \int_0^s e^{-b_j(s-u)} [dB_{j,u}^{\boldsymbol{\lambda}^{k,\star}+\alpha\hat{\boldsymbol{\lambda}}^k} + (\lambda_{j,u}^{k,\star} + \alpha\hat{\lambda}_{j,u}^k) du].$$

$B^{\boldsymbol{\lambda}^{k,\star}+\alpha\hat{\boldsymbol{\lambda}}^k}$ under $\mathbf{P}^{\boldsymbol{\lambda}^{k,\star}+\alpha\hat{\boldsymbol{\lambda}}^k}$ has the same distribution as $B^{\boldsymbol{\lambda}^{k,\star}}$ under $\mathbf{P}^{\boldsymbol{\lambda}^{k,\star}}$. Hence, we have

$$\mathbf{E}^{\boldsymbol{\lambda}^{k,\star}+\alpha\hat{\boldsymbol{\lambda}}^k}\left[\int_0^T \int_0^t \sum_{j=1}^d X_{j,s} ds dt\right] = \mathbf{E}^{\boldsymbol{\lambda}^{k,\star}}\left[\int_0^T \int_0^t \left(\sum_{j=1}^d X_{j,s} + \alpha\sigma_j \int_0^s e^{-b_j(s-u)} \hat{\lambda}_{j,u}^k du\right) ds dt\right].$$

Then,

$$\lim_{\alpha\to 0} \frac{1}{\alpha}\left\{\mathbf{E}^{\boldsymbol{\lambda}^{k,\star}+\alpha\hat{\boldsymbol{\lambda}}^k}\left[\int_0^T \int_0^t \sum_{j=1}^d X_{j,s} ds dt\right] - \mathbf{E}^{\boldsymbol{\lambda}^{k,\star}}\left[\int_0^T \int_0^t \sum_{j=1}^d X_{j,s} ds dt\right]\right\}$$
$$= \lim_{\alpha\to 0} \mathbf{E}^{\boldsymbol{\lambda}^{k,\star}}\left[\int_0^T \frac{1}{\alpha}\left\{\int_0^t \left(\sum_{j=1}^d X_{j,s} + \alpha\sigma_j \int_0^s e^{-b_j(s-u)} \hat{\lambda}_{j,u}^k du\right) ds - \int_0^t \sum_{j=1}^d X_{j,s} ds\right\} dt\right]$$
$$= \mathbf{E}^{\boldsymbol{\lambda}^{k,\star}}\left[\int_0^T \int_0^t \sum_{j=1}^d \sigma_j \int_0^s e^{-b_j(s-u)} \hat{\lambda}_{j,u}^k du ds dt\right] > 0.$$

Therefore, the Ornstein-Uhlenbeck process satisfies

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \left\{ \mathbf{E}^{\boldsymbol{\lambda}^{k,\star}+\alpha\hat{\boldsymbol{\lambda}}^k} \left[\int_0^T \int_0^t \nu_s ds dt \right] - \mathbf{E}^{\boldsymbol{\lambda}^{k,\star}} \left[\int_0^T \int_0^t \nu_s ds dt \right] \right\}$$
$$> 0.$$

2. Proof of Proposition 6

In this section, we provide the proof of Proposition 6. Particularly, we use a Malliavin calculus approach. (For Malliavin calculus approaches to optimal portfolio problems, see Ocone and Karatzas [21] and Takahashi and Yoshida [32], for example).

(A) First of all, (11) and (12) in Lemma 1 in the main text hold. In fact, since $c_t^{k,*} = \frac{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]}{T H_{0,t}}$, we have

$$c_0^{k,*} = \frac{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]}{T}, \tag{42}$$

and

$$\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \frac{c_0^{k,*}}{c_t^{k,*}} = H_{0,t}, \tag{43}$$

which corresponds to (11) in the main text in the log utility case. Also,

$$\begin{aligned} &\mathbf{E}\left[\int_0^T H_{0,t} c_t^{k,*} dt\right] \\ &= \frac{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]}{T} \int_0^T \mathbf{E}\left[\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right] dt \\ &= \mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right], \end{aligned} \tag{44}$$

which indicates (12) in the main text.

(B) Thus, by Lemma 1 in the main text, we have only to show (13) in the main text holds, that is, $sgn(Z_j) = 1,\ j = 1, \ldots, d_1$, where

$$\begin{aligned} &dV_t^{k,\hat{\boldsymbol{\lambda}}_1^k,\boldsymbol{\lambda}_2^k} \\ &= -\left(U^k(c_t^{k,*}) - \sum_{j=1}^{d_1} \bar{\lambda}_{j,t}^k |Z_{j,t}| + \sum_{j=d_1+1}^{d_1+d_2} \lambda_{j,t}^k Z_{j,t} \right) dt \\ &+ \sum_{j=1}^{d} Z_{j,t} dB_{j,t},\ V_T^{k,\hat{\boldsymbol{\lambda}}_1^k,\boldsymbol{\lambda}_2^k} = 0. \end{aligned} \tag{45}$$

We note that for $\boldsymbol{\lambda}_1^{k,*} = (-\bar{\lambda}_1^k, \dots, -\bar{\lambda}_{d_1}^k)^\top$, if we show $sgn(Z_j) = 1,\ j = 1, \dots, d_1$, where $Z_j, j = 1, \dots, d_1$ are part of a solution $(V^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}, Z)$ of a BSDE

$$\begin{aligned}
&dV_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \\
&= -\left(U^k(c_t^{k,*}) - \sum_{j=1}^{d_1} \bar{\lambda}_{j,t}^k Z_{j,t} + \sum_{j=d_1+1}^{d_1+d_2} \lambda_{j,t}^k Z_{j,t}\right) dt \\
&+ \sum_{j=1}^{d} Z_{j,t} dB_{j,t}, \\
&= -U^k(c_t^{k,*})dt + \sum_{j=1}^{d} Z_{j,t} dB_{j,t}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k},\ V_T^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} = 0,
\end{aligned} \tag{46}$$

$(V^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}, Z)$ is also a solution of BSDE (45).

Then, by the uniqueness a solution of BSDE (45), it results in $sgn(Z_j) = 1,\ j = 1, \dots, d_1$ for Z in (45).

In the following, we denote $\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k) = (\boldsymbol{\lambda}_1^{k,*\top}, \boldsymbol{\lambda}_2^{k\top}, 0, \dots, 0)^\top$.

Since

$$\begin{aligned}
TY^k &= \mathbf{E}\left[\int_0^T \left(\frac{\sum_{l=1}^K \eta_t^{l,*} Y^l}{\varepsilon_t}\right) \varepsilon_t^k dt\right] \\
&= \frac{\sum_{l=1}^K Y^l}{\varepsilon_0} \mathbf{E}\left[\int_0^T H_{0,t} \varepsilon_t^k dt\right],
\end{aligned} \tag{47}$$

where we used (23) in Proposition 2 in the main text in the first equality and (22) in the main text in the second equality, it follows that

$$\mathbf{E}\left[\int_0^T H_{0,t} \varepsilon_t^k dt\right] = \frac{\varepsilon_0 T Y^k}{\sum_{l=1}^K Y^l},\ k = 1, \dots, K.$$

Hence,

$$\begin{aligned}
c_t^{k,*} &= \frac{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \mathbf{E}\left[\int_0^T H_{0,t} \varepsilon_t^k dt\right]}{T H_{0,t}} \\
&= \frac{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} Y^k}{\left(\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l\right)} \varepsilon_t.
\end{aligned} \tag{48}$$

In the following, we show $sgn(Z_j) = 1,\ j = 1, \dots, d_1$ in Steps 1-3.

Step1: Representation of $Z_j,\ j = 1, \dots, d_1$

We use a Malliavin calculus-based method to investigate the sign of $Z_j\ j = 1, \dots, d_1$.

Noting (46), we let

$$\mathcal{X}^k(T) := \int_0^T U^k(c_s^{k,*})ds = \int_0^T Z_s^\top dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T U^k(c_s^{k,*})ds\right].$$

By taking the Malliavin derivative $D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$ for the both sides, we have

$$D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\mathcal{X}^k(T) = \int_u^T [D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} Z_s]^\top dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + Z_u,$$

where $D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$ is the Malliavin derivative with respect to $B_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$. We first suppose that the conditional expectation $\mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[\int_u^T [D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} Z_s]^\top dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}] = 0$ for all $0 \le u \le T$.

Then, by taking the conditional expectation $\mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$, by which we denote $\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[\cdot|\mathcal{F}_u]$,

$$\begin{aligned} Z_u^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} &= \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\mathcal{X}^k(T)] \\ &= \int_u^T \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_s^{k,*})\right] ds. \end{aligned} \tag{49}$$

In the following, we denote Z as $Z_u^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$ for clarity.

In Step 2, we first calculate $Z^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$ by (49) and later confirm the conditional expectation is zero with the calculated $Z^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$.

Step2: Calculation of $Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$, $j = 1, \ldots, d_1$

Here, we recall Assumption 1 and that we assumed any $\boldsymbol{\lambda}_2^k$ to be nonrandom.

By (48),

$$c_t^{k,*} = \frac{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} Y^k}{\left(\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l\right)} \varepsilon_t = \frac{\frac{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}{\eta_t^{1,*}} Y^k}{\left(1 + \sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l\right)} \varepsilon_t.$$

Taking the log of $c_t^{k,*}$, we have

$$\begin{aligned} \log c_t^{k,*} = \log Y^k + \log \frac{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}{\eta_t^{1,*}} + \log \varepsilon_t \\ - \log\left(1 + \sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l\right). \end{aligned} \tag{50}$$

By (49) and (50), $Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$ is

$$\begin{aligned}
&Z_{j,u}^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \\
&= \int_u^T \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log c_s^{k,*}]ds \\
&= \int_u^T \Bigg\{ \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log \varepsilon_s] \\
&+ \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log \frac{\eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}{\eta_s^{1,*}} \right] \\
&- \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log \left(1 + \sum_{l=2}^K \frac{\eta_s^{l,*}}{\eta_s^{1,*}} Y^l \right) \right] \Bigg\} ds. \qquad (51)
\end{aligned}$$

We consider the first term of the integrand in (51). Under $\mathbf{P}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$, by (20) in the main text, $\log \varepsilon_s$ is given by

$$\begin{aligned}
\log \varepsilon_s &= \log \varepsilon_0 + \int_0^s (\nu_\tau - \frac{1}{2}|\rho_\tau|^2) d\tau + \int_0^s \rho_\tau^\top [dB_\tau^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \boldsymbol{\lambda}_\tau^{k,*}(\boldsymbol{\lambda}_2^k) d\tau] \\
&= \log \varepsilon_0 + \int_0^s (\nu_\tau - \frac{1}{2}|\rho_\tau|^2 + \rho_\tau^\top \boldsymbol{\lambda}_\tau^{k,*}(\boldsymbol{\lambda}_2^k)) d\tau + \int_0^s \rho_\tau^\top dB_\tau^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}.
\end{aligned}$$

Since $\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)$ is nonrandom, taking Malliavin derivative of $\log \varepsilon$ with respect to Brownian motion $B_j^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$,

$$D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log \varepsilon_s = \int_u^s D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \nu_\tau d\tau + \rho_{j,u}. \qquad (52)$$

For the second term of the integrand in (51), by the definition of $\eta_t^{\boldsymbol{\lambda}^k}$,

$$\eta_t^{\boldsymbol{\lambda}^k} = \exp\left\{ \sum_{j=1}^d \int_0^t \lambda_{j,s}^k dB_{j,s} - \frac{1}{2} \sum_{j=1}^d \int_0^t |\lambda_{j,s}^k|^2 ds \right\}, \qquad (53)$$

we have

$$\begin{aligned}
&\log \frac{\eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}{\eta_s^{1,*}} = -\frac{1}{2} \int_0^s (|\boldsymbol{\lambda}_\tau^{k,*}(\boldsymbol{\lambda}_2^k)|^2 - |\boldsymbol{\lambda}_\tau^{1,*}|^2) d\tau \\
&+ \int_0^s (\boldsymbol{\lambda}_\tau^{k,*}(\boldsymbol{\lambda}_2^k) - \boldsymbol{\lambda}_\tau^{1,*})^\top [dB_\tau^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \boldsymbol{\lambda}_\tau^{k,*}(\boldsymbol{\lambda}_2^k) d\tau] \\
&= \frac{1}{2} \int_0^s |\boldsymbol{\lambda}_\tau^{k,*}(\boldsymbol{\lambda}_2^k) - \boldsymbol{\lambda}_\tau^{1,*}|^2 d\tau + \int_0^s (\boldsymbol{\lambda}_\tau^{k,*}(\boldsymbol{\lambda}_2^k) - \boldsymbol{\lambda}_\tau^{1,*})^\top dB_\tau^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}.
\end{aligned}$$

Thus, by nonrandomness of $\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)$ and $\boldsymbol{\lambda}^{1,*}$,

$$D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log \frac{\eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}{\eta_s^{1,*}} = \lambda_{j,u}^{k,*} - \lambda_{j,u}^{1,*}, \ j = 1, \ldots, d_1. \qquad (54)$$

For the third term of the integrand in (51),

$$D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log\left(1+\sum_{l=2}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right) = \frac{D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\sum_{l=2}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l}{\left(1+\sum_{l=2}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)}$$
$$= \frac{\sum_{l=2}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\log\frac{\eta_s^{l,*}}{\eta_s^{1,*}}}{\left(1+\sum_{l=2}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)}.$$

Similarly, by (53), we have

$$\log\frac{\eta_s^{l,*}}{\eta_s^{1,*}} = -\frac{1}{2}\int_0^s (|\boldsymbol{\lambda}_\tau^{l,*}|^2 - |\boldsymbol{\lambda}_\tau^{1,*}|^2)d\tau$$
$$+\int_0^s (\boldsymbol{\lambda}_\tau^{l,*}-\boldsymbol{\lambda}_\tau^{1,*})^\top [dB_\tau^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \boldsymbol{\lambda}_\tau^{k,*}(\boldsymbol{\lambda}_2^k)d\tau].$$

By nonrandomness of $\boldsymbol{\lambda}^{1,*}, \boldsymbol{\lambda}^{l,*}$ and $\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)$,

$$D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\log\frac{\eta_s^{l,*}}{\eta_s^{1,*}} = \lambda_{j,u}^{l,*} - \lambda_{j,u}^{1,*},\ j=1,\ldots,d_1.$$

Thus,

$$D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log\left(1+\sum_{l=2}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)$$
$$= \frac{\sum_{l=2}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l(\lambda_{j,u}^{l,*}-\lambda_{j,u}^{1,*})}{\left(1+\sum_{l=2}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)}. \tag{55}$$

By (52), (54) and (55), (51) is

$$Z_{j,u}^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$$
$$= \int_u^T \left\{ \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_u^s D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\nu_\tau d\tau\right] + \rho_{j,u} + (\lambda_{j,u}^{k,*}-\lambda_{j,u}^{1,*}) - \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\frac{\sum_{l=2}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l(\lambda_{j,u}^{l,*}-\lambda_{j,u}^{1,*})}{\left(1+\sum_{l=2}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)}\right]\right\}ds$$
$$= \int_u^T \left\{ \int_u^s \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\nu_\tau]d\tau + \rho_{j,u} + \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\frac{1+\sum_{l=2;l\neq k}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l}{\left(1+\sum_{l=2}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)}(\lambda_{j,u}^{k,*}-\lambda_{j,u}^{1,*})\right]\right.$$
$$\left. - \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\frac{\sum_{l=2;l\neq k}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l(\lambda_{j,u}^{l,*}-\lambda_{j,u}^{1,*})}{\left(1+\sum_{l=2}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)}\right]\right\}ds$$
$$= \int_u^T \left\{ \int_u^s \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\nu_\tau]d\tau + \rho_{j,u} + \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\frac{(\lambda_{j,u}^{k,*}-\lambda_{j,u}^{1,*}) + \sum_{l=2;l\neq k}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l(\lambda_{j,u}^{k,*}-\lambda_{j,u}^{l,*})}{\left(1+\sum_{l=2}^{K}\frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)}\right]\right\}ds.$$

Using $Y^1 = 1$, we have

$$Z_{j,u}^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} = \int_u^T \left\{ \int_u^s \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[D_{j,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\nu_\tau]d\tau + \rho_{j,u} - \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\frac{\sum_{l=1;l\neq k}^K \frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l(\lambda_{j,u}^{l,*}-\lambda_{j,u}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_s^{l,*}}{\eta_s^{1,*}}Y^l\right)}\right]\right\}ds. \tag{56}$$

Next, with this expression of $Z^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$,

$$Z_{j,s}^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} = \int_s^T \left\{ \int_s^t \mathbf{E}_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[D_{j,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\nu_\tau]d\tau + \rho_{j,s} - \mathbf{E}_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\frac{\sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l\right)}\right]\right\}dt, \tag{57}$$

we will confirm $\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[\int_u^T [D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} Z_s^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}]^\top dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}] = 0$ for all $0 \leq u \leq T$.

By (25) in the main text in Assumption 3 and nonrandomness of ρ_j, we have only to show

$$\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_u^T [D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_s^T \mathbf{E}_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\frac{\sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l\right)}\right]dt]^\top dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right] = 0. \tag{58}$$

For all $0 \leq u \leq s \leq T$, $m = 1, \ldots, d_1$, noting that

$$\begin{aligned} &D_{m,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_s^T \mathbf{E}_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\frac{\sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l\right)}\right]dt \\ &= \int_s^T \mathbf{E}_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[D_{m,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\frac{\sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l\right)}\right]dt, \end{aligned} \tag{59}$$

where we exchanged the order between Malliavin derivative and conditional expectation due to boundedness of $\frac{\sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l\right)}$ as in (65) below, we calculate

$$\begin{aligned} &D_{m,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\frac{\sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l\right)} \\ &= \frac{\sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l\right)} D_{m,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left(\log\sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l(\lambda_{j,s}^{l,*}-\lambda_{j,s}^{k,*}) - \log\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}}Y^l\right)\right), \end{aligned} \tag{60}$$

$$D_{m,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log\left(\sum_{l=1;l\neq k}^{K} \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l(\lambda_{j,s}^{l,*} - \lambda_{j,s}^{k,*})\right) = \frac{\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l(\lambda_{j,s}^{l,*} - \lambda_{j,s}^{k,*})(\lambda_{m,u}^{l,*} - \lambda_{m,u}^{1,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l(\lambda_{j,s}^{l,*} - \lambda_{j,s}^{k,*})\right)}, \tag{61}$$

and

$$D_{m,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \log\left(1+\sum_{l=2}^{K} \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l\right) = \frac{\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l(\lambda_{m,u}^{l,*} - \lambda_{m,u}^{1,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l\right)}. \tag{62}$$

Then,

$$\begin{aligned}
&\left| D_{m,u}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \frac{\sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l(\lambda_{j,s}^{l,*} - \lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l\right)} \right| \\
&\leq 2 \max_{l=1,\ldots,K,0\leq s\leq T} |\lambda_{j,s}^{l,*} - \lambda_{j,s}^{k,*}| \max_{l=1,\ldots,K,0\leq u\leq T} |\lambda_{m,u}^{l,*} - \lambda_{m,u}^{1,*}| \\
&< \infty,
\end{aligned} \tag{63}$$

and thus

$$\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_u^T [D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \int_s^T \mathbf{E}_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\frac{\sum_{l=1;l\neq k}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l(\lambda_{j,s}^{l,*} - \lambda_{j,s}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_t^{l,*}}{\eta_t^{1,*}} Y^l\right)}\right] dt]^\top dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right] = 0. \tag{64}$$

Therefore, $\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}[\int_u^T [D_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} Z_s]^\top dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}] = 0$ for all $0 \leq u \leq T$.

Step3: $sgn(Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}) = 1,\ j = 1, \ldots, d_1$

Using $\frac{\sum_{l=1;l\neq k}^K \frac{\eta_s^{l,*}}{\eta_s^{1,*}} Y^l}{\left(1+\sum_{l=2}^K \frac{\eta_t^{k,*}}{\eta_t^{1,*}} Y^l\right)} \in (0,1)$, we have

$$\begin{aligned}
\frac{\sum_{l=1;l\neq k}^K \frac{\eta_s^{l,*}}{\eta_s^{1,*}} Y^l(\lambda_{j,u}^{l,*} - \lambda_{j,u}^{k,*})}{\left(1+\sum_{l=2}^K \frac{\eta_s^{l,*}}{\eta_s^{1,*}} Y^l\right)} &\leq \max_{l,k\in\{1,\ldots,K\};l\neq k} (\lambda_{j,u}^{l,*} - \lambda_{j,u}^{k,*}) \frac{\sum_{l=1;l\neq k}^K \frac{\eta_s^{l,*}}{\eta_s^{1,*}} Y^l}{\left(1+\sum_{l=2}^K \frac{\eta_s^{l,*}}{\eta_s^{1,*}} Y^l\right)} \\
&< \max_{l,k\in\{1,\ldots,K\};l\neq k} (\lambda_{j,u}^{l,*} - \lambda_{j,u}^{k,*}).
\end{aligned} \tag{65}$$

By Assumptions 1-3, the right hand side of (56) is positive, that is $Z_{j,u}^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} > 0,\ \forall u \in [0,T]$, and thus $sgn(Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}) \equiv 1$.

Therefore,

$$-\bar{\lambda}_j^k sgn(Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}) = -\bar{\lambda}_j^k = \lambda_j^{k,*},\ j = 1, \ldots, d_1.$$

□

3. Proof of Lemma 8

Let $\boldsymbol{\lambda}_2^{l,*} = (\lambda_{d_1+1}^{l,*}, \ldots, \lambda_{d_1+d_2}^{l,*})^\top$,
where $\lambda_{d_1+1}^{l,*}, \ldots, \lambda_{d_1+d_2}^{l,*}$ are defined in (21) in the main text, and $\bar{\boldsymbol{\lambda}}_2^l = (\bar{\lambda}_{d_1+1}^l, \ldots, \bar{\lambda}_{d_1+d_2}^l)^\top$, $l = 1, \ldots, K$. Also, we let $\boldsymbol{B}_3 = (B_{d_1+d_2+1}, \ldots, B_d)^\top$, $\boldsymbol{\rho}_1 = (\rho_1, \ldots, \rho_{d_1})^\top$, $\boldsymbol{\rho}_2 = (\rho_{d_1+1}, \ldots, \rho_{d_1+d_2})^\top$, and $\boldsymbol{\rho}_3 = (\rho_{d_1+d_2+1}, \ldots, \rho_d)^\top$. By (22) in the main text, we have

$$\log H_{0,t} = \log\left(\eta_t^{1,*} + \sum_{l=2}^{K} \eta_t^{l,*} Y^l\right) - \log \varepsilon_t + \log \varepsilon_0 - \log\left(1 + \sum_{l=2}^{K} Y^l\right).$$

Since $\log \varepsilon_0 - \log\left(1 + \sum_{l=2}^{K} Y^l\right)$ is a constant, we have only to consider

$$\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k}\left[\int_0^T \left\{\log \varepsilon_t - \log\left(\eta_t^{1,*} + \sum_{l=2}^{K} \eta_t^{l,*} Y^l\right)\right\} dt\right],$$

which is a part of $\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k}\left[\int_0^T -\log H_{0,t} dt\right]$.

We define $F(\boldsymbol{\lambda}_2^k)$ as follows.

$$\begin{aligned}
F(\boldsymbol{\lambda}_2^k) &= F_1(\boldsymbol{\lambda}_2^k) + F_2(\boldsymbol{\lambda}_2^k), \\
F_1(\boldsymbol{\lambda}_2^k) &= \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k}\left[\int_0^T \log \varepsilon_t dt\right], \\
F_2(\boldsymbol{\lambda}_2^k) &= \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k}\left[\int_0^T -\log\left(\eta_t^{1,*} + \sum_{l=2}^{K} \eta_t^{l,*} Y^l\right) dt\right].
\end{aligned}$$

For any nonrandom $\hat{\boldsymbol{\lambda}}_2^k \leq \bar{\boldsymbol{\lambda}}_2^k$ $(0 \leq \hat{\lambda}_j^k \leq \bar{\lambda}_j^k,\ j = d_1+1, \ldots, d_1+d_2)$,

$$\begin{aligned}
&\lim_{\alpha \to 0} \frac{F(\boldsymbol{\lambda}_2^k + \alpha \hat{\boldsymbol{\lambda}}_2^k) - F(\boldsymbol{\lambda}_2^k)}{\alpha} \\
&= \lim_{\alpha \to 0} \frac{F_1(\boldsymbol{\lambda}_2^k + \alpha \hat{\boldsymbol{\lambda}}_2^k) - F_1(\boldsymbol{\lambda}_2^k)}{\alpha} + \lim_{\alpha \to 0} \frac{F_2(\boldsymbol{\lambda}_2^k + \alpha \hat{\boldsymbol{\lambda}}_2^k) - F_2(\boldsymbol{\lambda}_2^k)}{\alpha}.
\end{aligned}$$

Step 1: Calculation of $\lim_{\alpha \to 0} \frac{F_1(\boldsymbol{\lambda}_2^k + \alpha \hat{\boldsymbol{\lambda}}_2^k) - F_1(\boldsymbol{\lambda}_2^k)}{\alpha}$

For $\log \varepsilon_t$,

$$\begin{aligned}
&\log \varepsilon_t \\
&= \log \varepsilon_0 + \int_0^t (\nu_s - \frac{1}{2}|\boldsymbol{\rho}_s|^2) ds + \int_0^t \boldsymbol{\rho}_s^\top dB_s \\
&= \log \varepsilon_0 + \int_0^t (\nu_s - \frac{1}{2}|\boldsymbol{\rho}_s|^2) ds \\
&+ \int_0^t \boldsymbol{\rho}_{1,s}^\top [d\boldsymbol{B}_{1,s}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k + \alpha \hat{\boldsymbol{\lambda}}_2^k} + \boldsymbol{\lambda}_{1,s}^{k,*} ds] \\
&+ \int_0^t \boldsymbol{\rho}_{2,s}^\top [d\boldsymbol{B}_{2,s}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k + \alpha \hat{\boldsymbol{\lambda}}_2^k} + (\boldsymbol{\lambda}_{2,s}^k + \alpha \hat{\boldsymbol{\lambda}}_{2,s}^k) ds] + \int_0^t \boldsymbol{\rho}_{3,s}^\top d\boldsymbol{B}_{3,s}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k + \alpha \hat{\boldsymbol{\lambda}}_2^k}.
\end{aligned}$$

$(\boldsymbol{B}_1^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}, \boldsymbol{B}_2^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}, \boldsymbol{B}_3^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k})$ under $\mathbf{P}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}$ has the same distribution as $(\boldsymbol{B}_1^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}, \boldsymbol{B}_2^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}, \boldsymbol{B}_3^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k})$ under $\mathbf{P}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$. Hence, we have

$$\begin{aligned}&\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}\bigg[\int_0^T\int_0^t \boldsymbol{\rho}_{1,s}^\top[d\boldsymbol{B}_{1,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}+\boldsymbol{\lambda}_{1,s}^{k,*}ds]dt\bigg]\\&=\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\bigg[\int_0^T\int_0^t \boldsymbol{\rho}_{1,s}^\top[d\boldsymbol{B}_{1,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}+\boldsymbol{\lambda}_{1,s}^{k,*}ds]dt\bigg],\end{aligned}$$

and

$$\begin{aligned}&\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}\bigg[\int_0^T\int_0^t \boldsymbol{\rho}_{2,s}^\top[d\boldsymbol{B}_{2,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}+(\boldsymbol{\lambda}_{2,s}^k+\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k)ds]dt\bigg]\\&=\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\bigg[\int_0^T\int_0^t \boldsymbol{\rho}_{2,s}^\top[d\boldsymbol{B}_{2,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}+(\boldsymbol{\lambda}_{2,s}^k+\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k)ds]dt\bigg].\end{aligned}$$

Hence,

$$\begin{aligned}&\lim_{\alpha\to 0}\frac{F_1(\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k)-F_1(\boldsymbol{\lambda}_2^k)}{\alpha}\\&=\lim_{\alpha\to 0}\frac{1}{\alpha}\bigg\{\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}\bigg[\int_0^T\int_0^t\nu_s dsdt\bigg]-\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\bigg[\int_0^T\int_0^t\nu_s dsdt\bigg]\bigg\}\\&\quad+\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\bigg[\int_0^T\int_0^t\boldsymbol{\rho}_{2,s}^\top\hat{\boldsymbol{\lambda}}_{2,s}^k dsdt\bigg].\end{aligned}\tag{66}$$

For the term containing ν, by Assumption 4,

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\bigg\{\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}\bigg[\int_0^T\int_0^t\nu_s dsdt\bigg]-\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\bigg[\int_0^T\int_0^t\nu_s dsdt\bigg]\bigg\}\geq 0.\tag{67}$$

Step 2: Calculation of $\lim_{\alpha\to 0}\frac{F_2(\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k)-F_2(\boldsymbol{\lambda}_2^k)}{\alpha}$

For $l=1,\ldots,K$,

$$\eta_t^{l,*}=\exp\bigg\{\int_0^t\boldsymbol{\lambda}_{2,s}^{l,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds\bigg\}Z_t^l(\boldsymbol{B}_1^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k},\boldsymbol{B}_2^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}),$$

where we set

$$\begin{aligned}&Z_t^l(\boldsymbol{B}_1^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k},\boldsymbol{B}_2^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k})\\&=\exp\bigg\{\int_0^t\boldsymbol{\lambda}_{1,s}^{l,*\top}(dB_{1,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}+\boldsymbol{\lambda}_{1,s}^{k,*}ds)\\&+\int_0^t\boldsymbol{\lambda}_{2,s}^{l,*\top}(d\boldsymbol{B}_{2,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}+\boldsymbol{\lambda}_{2,s}^k ds)-\frac{1}{2}\int_0^t|\boldsymbol{\lambda}_s^{l,*}|^2ds\bigg\}.\end{aligned}$$

Since $Z_t^l(\boldsymbol{B}_1^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k},\boldsymbol{B}_2^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k})$ under $\mathbf{P}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}$ has the same distribution as $\eta_t^{l,*}=\exp\Big\{\int_0^t\boldsymbol{\lambda}_{1,s}^{l,*\top}(dB_{1,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}+\boldsymbol{\lambda}_{1,s}^{k,*}ds)+\int_0^t\boldsymbol{\lambda}_{2,s}^{l,*\top}(d\boldsymbol{B}_{2,s}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}+\boldsymbol{\lambda}_{2,s}^k ds)-\frac{1}{2}\int_0^t|\boldsymbol{\lambda}_s^{l,*}|^2ds\Big\}$ under $\mathbf{P}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}$, we have

$$
\begin{aligned}
&F_2(\boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k) \\
&= \mathbf{E}^{\boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k}\left[\int_0^T -\log\left(\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l\right) dt\right] \\
&= \mathbf{E}^{\boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k}\Bigg[\int_0^T -\log\Bigg(\exp\left\{\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds\right\} Z_{1,t}(\boldsymbol{B}_1^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}, \boldsymbol{B}_2^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}) \\
&+ \sum_{l=2}^K \exp\left\{\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds\right\} Z_t^l(\boldsymbol{B}_1^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}, \boldsymbol{B}_2^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k+\alpha\hat{\boldsymbol{\lambda}}_2^k}) Y^l\Bigg) dt\Bigg] \\
&= \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T -\log\left(\exp\left\{\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds\right\}\eta_t^{1,*} + \sum_{l=2}^K \exp\left\{\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds\right\}\eta_t^{l,*} Y^l\right) dt\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&F_2(\boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k) - F_2(\boldsymbol{\lambda}_2^k) \\
&= \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T -\log\left(e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds}\eta_t^{1,*} + \sum_{l=2}^K e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds}\eta_t^{l,*} Y^l\right) dt\right] \\
&- \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T -\log\left(\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l\right) dt\right].
\end{aligned}
$$

Thus, we can obtain Gateaux derivative of F_2 as

$$
\begin{aligned}
&\lim_{\alpha\to 0}\frac{F_2(\boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k) - F_2(\boldsymbol{\lambda}_2^k)}{\alpha} \\
&= \lim_{\alpha\to 0}\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\Bigg[\int_0^T \frac{1}{\alpha}\Bigg\{-\log\left(e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds}\eta_t^{1,*} + \sum_{l=2}^K e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds}\eta_t^{l,*} Y^l\right) \\
&+ \log\left(\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l\right)\Bigg\} dt\Bigg]
\end{aligned}
$$

Noting that $\frac{\eta_t^{l,*} Y^l}{\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l} \in (0,1)$, we have

$$
\begin{aligned}
&\left|\frac{d}{d\alpha}\{-\log(e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds}\eta_t^{1,*} + \sum_{l=2}^K e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top}\alpha\hat{\boldsymbol{\lambda}}_{2,s}^k ds}\eta_t^{l,*} Y^l)\}\Big|_{\alpha=0}\right| \\
&= \left|\frac{(\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top}\hat{\boldsymbol{\lambda}}_{2,s}^k ds)\eta_t^{1,*} + \sum_{l=2}^K(\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top}\hat{\boldsymbol{\lambda}}_{2,s}^k ds)\eta_t^{l,*} Y^l}{\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l}\right| \\
&\le \int_0^t \bar{\boldsymbol{\lambda}}_{2,s}^{1\top}\hat{\boldsymbol{\lambda}}_{2,s}^k ds + \sum_{l=2}^K \int_0^t \bar{\boldsymbol{\lambda}}_{2,s}^{l\top}\hat{\boldsymbol{\lambda}}_{2,s}^k ds < \infty,
\end{aligned}
$$

for all $(\omega, t) \in \Omega \times [0, T]$. Then, by the dominated convergence theorem, we have

$$
\begin{aligned}
&\lim_{\alpha \to 0} \frac{F_2(\boldsymbol{\lambda}_2^k + \alpha \hat{\boldsymbol{\lambda}}_2^k) - F_2(\boldsymbol{\lambda}_2^k)}{\alpha} \\
&= \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k}\Bigg[\int_0^T \lim_{\alpha \to 0} \frac{1}{\alpha} \Bigg\{ - \log \left(e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top} \alpha \hat{\boldsymbol{\lambda}}_{2,s}^k ds} \eta_t^{1,*} + \sum_{l=2}^K e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top} \alpha \hat{\boldsymbol{\lambda}}_{2,s}^k ds} \eta_t^{l,*} Y^l \right) \\
&+ \log \left(\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l \right) \Bigg\} dt \Bigg] \\
&= -\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k}\Bigg[\int_0^T \frac{d}{d\alpha} \log \left(e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top} \alpha \hat{\boldsymbol{\lambda}}_{2,s}^k ds} \eta_t^{1,*} + \sum_{l=2}^K e^{\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top} \alpha \hat{\boldsymbol{\lambda}}_{2,s}^k ds} \eta_t^{l,*} Y^l \right) \Bigg|_{\alpha=0} dt \Bigg] \\
&= -\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k}\Bigg[\int_0^T \frac{(\int_0^t \boldsymbol{\lambda}_{2,s}^{1,*\top} \hat{\boldsymbol{\lambda}}_{2,s}^k ds) \eta_t^{1,*} + \sum_{l=2}^K (\int_0^t \boldsymbol{\lambda}_{2,s}^{l,*\top} \hat{\boldsymbol{\lambda}}_{2,s}^k ds) \eta_t^{l,*} Y^l}{\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l} dt \Bigg] \\
&\geq -\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k}\Bigg[\int_0^T \left\{ \int_0^t \max_{l=1,\dots,K} \left(\boldsymbol{\lambda}_{2,s}^{l,*\top} \hat{\boldsymbol{\lambda}}_{2,s}^k \right) ds \times \frac{\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l}{\eta_t^{1,*} + \sum_{l=2}^K \eta_t^{l,*} Y^l} \right\} dt \Bigg] \\
&= -\mathbf{E}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k}\Bigg[\int_0^T \int_0^t \max_{l=1,\dots,K} \left(\boldsymbol{\lambda}_{2,s}^{l,*\top} \hat{\boldsymbol{\lambda}}_{2,s}^k \right) ds dt \Bigg]. \qquad (68)
\end{aligned}
$$

Step 3: Calculation of $\lim_{\alpha \to 0} \frac{F(\boldsymbol{\lambda}_2^k + \alpha \hat{\boldsymbol{\lambda}}_2^k) - F(\boldsymbol{\lambda}_2^k)}{\alpha}$

Therefore, by (66), (67), (68), and Assumption 2, we obtain

$$
\lim_{\alpha \to 0} \frac{F(\boldsymbol{\lambda}_2^k + \alpha \hat{\boldsymbol{\lambda}}_2^k) - F(\boldsymbol{\lambda}_2^k)}{\alpha} > 0.
$$

Hence, (A.4) in the main text is increasing with respect to $\boldsymbol{\lambda}_2^k$. □

4. $\mathbf{E}[\int_0^T \log(\bar{c}_t^{k,*})^2 dt] < \infty$ in the proof of Theorem 3

$\mathbf{E}[\int_0^T \log(\bar{c}_t^{k,*})^2 dt] < \infty$ is confirmed as follows.

$$
\begin{aligned}
(\log \bar{c}_t^{k,*})^2 &= \left(\log \eta_t^{k,*} + \log \mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right] - \log H_{0,t}\right)^2 \\
&\le 3\left((\log \eta_t^{k,*})^2 + (\log H_{0,t})^2 + \left(\log \mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]\right)^2\right) \\
&\le 3\Bigg(\left(-\frac{1}{2}\int_0^t |\boldsymbol{\lambda}_s^{k,*}|^2 ds + \int_0^t \boldsymbol{\lambda}_s^{k,*\top} dB_s\right)^2 + \left(-\int_0^t r_s ds - \frac{1}{2}\int_0^t |\boldsymbol{\theta}_s|^2 ds + \int_0^t \boldsymbol{\theta}_s^\top dB_s\right)^2 \\
&\quad + \left(\log \mathbf{E}[\int_0^T H_{0,t}\varepsilon_t^k dt]\right)^2\Bigg) \\
&\le 9\Bigg(\left(-\frac{1}{2}\int_0^t |\boldsymbol{\lambda}_s^{k,*}|^2 ds\right)^2 + \left(\int_0^t \boldsymbol{\lambda}_s^{k,*\top} dB_s\right)^2 \\
&\quad + \left(\int_0^t r_s ds\right)^2 + \left(-\frac{1}{2}\int_0^t |\boldsymbol{\theta}_s|^2 ds\right)^2 + \left(\int_0^t \boldsymbol{\theta}_s^\top dB_s\right)^2 + \left(\log \mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]\right)^2\Bigg).
\end{aligned} \tag{69}
$$

Thus,

$$
\begin{aligned}
\int_0^T \mathbf{E}[(\log \bar{c}_t^{k,*})^2] dt \le 9\int_0^T \mathbf{E}\Bigg[\Bigg(\left(-\frac{1}{2}\int_0^t |\boldsymbol{\lambda}_s^{k,*}|^2 ds\right)^2 + \left(\int_0^t \boldsymbol{\lambda}_s^{k,*\top} dB_s\right)^2 \\
+ \left(\int_0^t r_s ds\right)^2 + \left(-\frac{1}{2}\int_0^t |\boldsymbol{\theta}_s|^2 ds\right)^2 + \left(\int_0^t \boldsymbol{\theta}_s^\top dB_s\right)^2 + \left(\log \mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]\right)^2\Bigg)\Bigg] dt.
\end{aligned} \tag{70}
$$

Noting that

$$
\begin{aligned}
\int_0^T \mathbf{E}\left[\left(\int_0^t \boldsymbol{\lambda}_s^{k,*\top} dB_s\right)^2\right] dt &= \int_0^T \int_0^t |\boldsymbol{\lambda}_s^{k,*}|^2 ds dt, \\
\int_0^T \mathbf{E}\left[\left(\int_0^t \boldsymbol{\theta}_s^\top dB_s\right)^2\right] dt &= \int_0^T \int_0^t \mathbf{E}\left[|\boldsymbol{\theta}_s|^2\right] ds dt,
\end{aligned} \tag{71}
$$

$$
\begin{aligned}
\int_0^T \mathbf{E}\left[\left(\int_0^t r_s ds\right)^2\right] dt &\le T\int_0^T \int_0^t \mathbf{E}\left[r_s^2\right] ds dt \\
= T\int_0^T \int_0^t \mathbf{E}&\left[\left(\nu_s - |\boldsymbol{\rho}_s|^2 + \boldsymbol{\rho}_s^\top \left[\sum_{k=1}^K \left(\frac{Y^k \eta_s^{k,*}}{\sum_{l=1}^K Y^l \eta_s^{l,*}}\right) \boldsymbol{\lambda}_s^{k,*}\right]\right)^2\right] ds dt \\
\le 2T\int_0^T \int_0^t &\left(\mathbf{E}[\nu_s^2] + \mathbf{E}\left[\left(|\boldsymbol{\rho}_s|^2 + \boldsymbol{\rho}_s^\top \left[\sum_{k=1}^K \left(\frac{Y^k \eta_s^{k,*}}{\sum_{l=1}^K Y^l \eta_s^{l,*}}\right) \boldsymbol{\lambda}_s^{k,*}\right]\right)^2\right]\right) ds dt,
\end{aligned} \tag{72}
$$

since $\left(\frac{Y^k \eta_s^{k,*}}{\sum_{l=1}^K Y^l \eta_s^{l,*}}\right) \in (0,1)$, $\boldsymbol{\lambda}^{k,*}$ and $\boldsymbol{\rho}$ are bounded, and $\int_0^T \int_0^t \mathbf{E}[\nu_s^2] ds dt < \infty$, we have $\int_0^T \mathbf{E}[\log(\bar{c}_t^{k,*})^2] dt < \infty$.

5. Exponential utility case

In the exponential utility case, we consider a market where each agent has a exponential-utility function U^k given by $U^k(x) = -\frac{e^{-\gamma^k x}}{\gamma^k}$, $0 < \gamma^k < \infty$, and the aggregate endowment process ε satisfies an SDE $d\varepsilon_t = \nu_t dt + \boldsymbol{\rho}_t^\top dB_t$, where ν is a $\mathbf{R}$-valued $\{\mathcal{F}_t\}$-progressively measurable process with $\int_0^T |\nu_t| dt < \infty$ a.s. and $\mathbf{E}[\int_0^T \exp(-\frac{4}{\sum_{m=1}^K \frac{1}{\gamma^m}} \int_0^t \nu_s ds) dt] < \infty$, and $\boldsymbol{\rho}_t = (\rho_{1,t}, \ldots, \rho_{d,t})^\top$ is a nonrandom process satisfying Assumption 1.

The state-price density process in equilibrium H_0 is searched by first solving the individual optimization problems (5) as the optimal consumption problems presupposing a form of the conservative and aggressive views of the agents (21) in Section 3 in the main text and then by imposing the market clearing conditions, particularly the clearing on the commodity market (17).

In the following, we first provide H_0 obtained in the above way and confirm that the state-price density process is in fact in equilibrium. That is, given the state-price density process H_0, we first solve the individual optimization problems (5) in Proposition 9, and then show that the market is in equilibrium in Proposition 13, namely the solutions of the individual optimization problems satisfy the market clearing conditions (17)-(19).

Hereafter, we assume that ν is driven by the following Ornstein-Uhlenbeck processes. For $a_j, b_j \in \boldsymbol{R}, \sigma_j > 0$, $\nu_t = \sum_{j=1}^d X_{j,t}$, where $X_{j,t} = X_{j,0} + \int_0^t (a_j - b_j X_{j,s}) ds + \int_0^t \sigma_j dB_{j,s}$.

We further assume the following. Let $\Delta = \sum_{m=1}^K \frac{1}{\gamma^m}$.

Assumption 5. *For $j = 1, \ldots, d$,*

$$\rho_{j,u} - \Delta \max\left[\max_{l,k\in\{1,\ldots,K\};l\neq k} (\lambda_{j,u}^{l,*} - \lambda_{j,u}^{k,*}), \max_{l\in\{1,\ldots,K\}} \lambda_{j,u}^{l,*}\right] > 0, \ \forall u \in [0,T].$$

Then, the state-price density process H_0 in equilibrium in the exponential utility case is given by

$$H_{0,t} = \exp\left(-\frac{\varepsilon_t - \varepsilon_0}{\Delta}\right) \prod_{k=1}^K \left(\eta_t^{k,*}\right)^{\frac{1}{\gamma^k \Delta}}, \tag{73}$$

where $\eta^{k,*} = \eta^{\boldsymbol{\lambda}^{k,*}}$, and $\boldsymbol{\lambda}^{k,*}$ is given by (21), i.e. $\boldsymbol{\lambda}^{k,*} = (\lambda_1^{k,*}, \ldots, \lambda_d^{k,*})^\top$, where

$$\lambda_{j,t}^{k,*} = \begin{cases} -\bar{\lambda}_{j,t}^k, j \in \mathcal{J}_1^k \\ +\bar{\lambda}_{j,t}^k, j \in \mathcal{J}_2^k \\ 0, j \in \mathcal{J}_3^k, \end{cases}, \ 0 \leq t \leq T.$$

Namely, the following propositions hold.

Proposition 9. *Under Assumptions 1 and 5, given* H_0 *in (73),* $\lambda_j^{k,*}, j \in \mathcal{J}_1^k, \mathcal{J}_2^k$ *of* $\boldsymbol{\lambda}^{k,*}$ *in (21) and* $(\bar{c}^{k,*}, \boldsymbol{\pi}^{k,*})$ *with* $\bar{c}_t^{k,*} = -\frac{1}{\gamma^k} \log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right) + \bar{c}_0^{k,*}$,

where $\bar{c}_0^{k,*} = \frac{1}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]} \left\{ \mathbf{E}\left[\int_0^T H_{0,t} \varepsilon_t^k dt\right] + \mathbf{E}\left[\int_0^T H_{0,t} \frac{\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right)}{\gamma^k} dt\right] \right\}$*, and* $\boldsymbol{\pi}^{k,*}$ *in (15) attain the individual optimization problem (5), i.e.,*

$$\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k} \sup_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_2^k} \inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} \mathbf{E}\left[\int_0^T \eta_t^{\boldsymbol{\lambda}^k} U^k(c_t^k) dt\right] \left(= \mathbf{E}^{\boldsymbol{P}^{\boldsymbol{\lambda}^k}}\left[\int_0^T U^k(c_t^k) dt\right]\right). \tag{74}$$

Proof.

Hereafter, we assume $\mathcal{J}_1^k = \{1, \ldots, d_1\}$, $\mathcal{J}_2^k = \{d_1+1, \ldots, d_1+d_2\}$, $\mathcal{J}_3^k = \{d_1+d_2+1, \ldots, d\}$, without loss of generality.

Thus, we consider the following individual optimization problem

$$\begin{aligned}
&\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k} \sup_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_2^k} \inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} \mathbf{E}\left[\int_0^T \eta_t^{\boldsymbol{\lambda}^k} U^k(c_t^k) dt\right] \\
&= \sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k} \sup_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_2^k} \inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k, \boldsymbol{\lambda}_1^k, \boldsymbol{\lambda}_2^k) \\
&= \sup_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_2^k} \sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k} \inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k, \boldsymbol{\lambda}_1^k, \boldsymbol{\lambda}_2^k),
\end{aligned} \tag{75}$$

where we set $J^k(c^k, \boldsymbol{\lambda}_1^k, \boldsymbol{\lambda}_2^k) = \mathbf{E}\left[\int_0^T \eta_t^{\boldsymbol{\lambda}^k} U^k(c_t^k) dt\right]$.

In the following, we first consider $\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k} \inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k, \boldsymbol{\lambda}_1^k, \boldsymbol{\lambda}_2^k)$ for given $\boldsymbol{\lambda}_2^k$ satisfying $|\lambda_j^k| \le \bar{\lambda}_j^k$, $j \in \mathcal{J}_2^k$ in Lemma 10, then show that $\boldsymbol{\lambda}_2^{k*}$ attains $\sup_{\boldsymbol{\lambda}_2^k, |\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_2^k} J^k(c^{k*}, \boldsymbol{\lambda}_1^{k*}, \boldsymbol{\lambda}_2^k)$, where $c^{k*}, \boldsymbol{\lambda}_1^{k*}$ attains the first part, i.e., $\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k} \inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k, \boldsymbol{\lambda}_1^k, \boldsymbol{\lambda}_2^k)$, in Lemma 11.

First, the following lemma holds.

Lemma 10. *For given* $\boldsymbol{\lambda}_2^k$ *satisfying* $|\lambda_j^k| \le \bar{\lambda}_j^k$, $j = d_1+1, \ldots, d_1+d_2$, $\boldsymbol{\lambda}_1^{k,*} = (-\bar{\lambda}_1^k, \ldots, -\bar{\lambda}_{d_1}^k)^\top$ *and* $c_t^{k,*} = -\frac{1}{\gamma^k} \log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k}\right) + c_0^{k,*}$ *with* $\boldsymbol{\lambda}_1^{k,*} = (-\bar{\lambda}_1^k, \ldots, -\bar{\lambda}_{d_1}^k)^\top$ *attain the sup-inf problem below:*

$$\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k} \inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k, \boldsymbol{\lambda}_1^k, \boldsymbol{\lambda}_2^k) = J^k(c^{k,*}, \boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k). \tag{76}$$

Proof.

In the exponential-utility case, since

$$U^{k\prime}(c^k) = e^{-\gamma^k c^k},$$

and $c_t^{k,*}$ satisfying the first order condition in Lemma 1 in the main text

$$H_{0,t} = \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \frac{U^{k\prime}(c_t^{k,*})}{U^{k\prime}(c_0^{k,*})},$$

is given by

$$H_{0,t} = \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} e^{-\gamma^k c_t^{k,*}} / U^{k\prime}(c_0^{k,*}),$$

taking log, we have

$$\begin{aligned} c_t^{k,*} &= -\frac{1}{\gamma^k}\left(\log H_{0,t} - \log \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \log U^{k\prime}(c_0^{k,*})\right) \\ &= -\frac{1}{\gamma^k}\log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right) + c_0^{k,*}, \end{aligned} \tag{77}$$

where $c_0^{k,*}$ is obtained as

$$c_0^{k,*} = \frac{1}{\mathbf{E}\left[\int_0^T H_{0,t}dt\right]}\left\{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right] + \mathbf{E}\left[\int_0^T H_{0,t}\frac{\log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right)}{\gamma^k}dt\right]\right\},$$

due to the budget constraint

$$\mathbf{E}\left[\int_0^T H_{0,t}c_t^{k,*}dt\right] = \mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right].$$

By Lemma1 in the main text, we have only to show $sgn(Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}) = 1,\ j = 1,\dots,d_1$ for $\boldsymbol{\lambda}_1^{k,*} = (-\bar{\lambda}_1^k,\dots,-\bar{\lambda}_{d_1}^k)^\top$.

First, we note that the Ornstein-Uhlenbeck process X_j is solved as

$$\begin{aligned} X_{j,s} &= X_{j,0}e^{-b_j s} + a_j\int_0^s e^{-b_j(s-\tau)}d\tau + \sigma_j\int_0^s e^{-b_j(s-\tau)}dB_{j,\tau} \\ &= \phi_{j,s} + \sigma_j\int_0^s e^{-b_j(s-\tau)}dB_{j,\tau}, \end{aligned}$$

where we set the deterministic term $\phi_{j,s}$ as follows:

$$\phi_{j,s} = X_{j,0}e^{-b_j s} + a_j\int_0^s e^{-b_j(s-\tau)}d\tau.$$

Then, ε is expressed as

$$\begin{aligned} \varepsilon_t &= \varepsilon_0 + \sum_{j=1}^d\int_0^t \phi_{j,s}ds + \sum_{j=1}^d\int_0^t \sigma_j\int_0^s e^{-b_j(s-\tau)}dB_{j,\tau}ds + \sum_{j=1}^d\int_0^t \rho_{j,s}dB_{j,s} \\ &= \varepsilon_0 + \sum_{j=1}^d\int_0^t \phi_{j,s}ds + \sum_{j=1}^d\int_0^t\left(\rho_{j,s} + \sigma_j\int_s^t e^{-b_j(\tau-s)}d\tau\right)dB_{j,s}. \end{aligned}$$

In the following, we denote $\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k) = (\boldsymbol{\lambda}_1^{k,*\top}, \boldsymbol{\lambda}_2^{k\top}, 0, \ldots, 0)^\top$.

Step1: Calculation of $Z_j^{c^{k,*}, \boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k}$

We note

$$
\begin{aligned}
Z_{j,u}^{c^{k,*}, \boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} &= \int_u^T \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} \left[D_{j,u}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} U^k(c_s^{k,*}) \right] ds \\
&= \int_u^T \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} \left[e^{-\gamma^k c_s^{k,*}} D_{j,u}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} c_s^{k,*} \right] ds.
\end{aligned}
$$

By

$$
H_{0,t} = \exp\left(-\frac{\varepsilon_t - \varepsilon_0}{\Delta} \right) \prod_{l=1}^K \left(\eta_t^{l,*} \right)^{\frac{1}{\gamma^l \Delta}},
$$

taking log of H_0, we have

$$
\log H_{0,t} = -\frac{\varepsilon_t - \varepsilon_0}{\Delta} + \sum_{l=1}^K \frac{1}{\gamma^l \Delta} \log \eta_t^{l,*}.
$$

Due to

$$
\begin{aligned}
dB_{j,t} &= dB_{j,t}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} + \lambda_{j,t}^{k,*} dt \ (j = 1, \ldots, d_1), \\
dB_{j,t} &= dB_{j,t}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} + \lambda_{j,t}^{k} dt \ (j = d_1 + 1, \ldots, d_1 + d_2), \\
dB_{j,t} &= dB_{j,t}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} \ (j = d_1 + d_2 + 1, \ldots, d),
\end{aligned}
$$

with Brownian motions under $P^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k}$, by (93), Malliavin derivative with respect to $B_j^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k}$, $j = 1, \ldots, d_1$ is

$$
\begin{aligned}
D_{j,u}^{\boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} c_s^{k,*} &= \frac{1}{\gamma^k \Delta} \left(\rho_{j,u} + \sigma_j \int_u^s e^{-b_j(\tau - u)} d\tau \right) - \sum_{l=1}^K \frac{1}{\gamma^k \gamma^l \Delta} \lambda_{j,u}^{l,*} + \frac{1}{\gamma^k} \lambda_{j,u}^{k,*} \\
&= \frac{1}{\gamma^k \Delta} \Bigg\{ \left(\rho_{j,u} + \sigma_j \int_u^s e^{-b_j(\tau - u)} d\tau \right) - \sum_{l=1}^K \frac{1}{\gamma^l} \lambda_{j,u}^{l,*} + \Delta \lambda_{j,u}^{k,*} \Bigg\} \\
&= \frac{1}{\gamma^k \Delta} \Bigg\{ \left(\rho_{j,u} + \sigma_j \int_u^s e^{-b_j(\tau - u)} d\tau \right) - \sum_{l=1}^K \frac{1}{\gamma^l} \lambda_{j,u}^{l,*} + \left(\sum_{l=1}^K \frac{1}{\gamma^l} \right) \lambda_{j,u}^{k,*} \Bigg\} \\
&= \frac{1}{\gamma^k \Delta} \Bigg\{ \left(\rho_{j,u} + \sigma_j \int_u^s e^{-b_j(\tau - u)} d\tau \right) - \sum_{l=1; l \neq k}^K \frac{1}{\gamma^l} (\lambda_{j,u}^{l,*} - \lambda_{j,u}^{k,*}) \Bigg\}.
\end{aligned}
$$

Hence, we have

$$
Z_{j,u}^{c^{k,*}, \boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k} = \int_u^T \left[\frac{e^{-\gamma^k c_s^{k,*}}}{\gamma^k \Delta} \Bigg\{ \left(\rho_{j,u} + \sigma_j \int_u^s e^{-b_j(\tau - u)} d\tau \right) - \sum_{l=1; l \neq k}^K \frac{1}{\gamma^l} (\lambda_{j,u}^{l,*} - \lambda_{j,u}^{k,*}) \Bigg\} \right] ds. \tag{78}
$$

Step2: Determination of $sgn(Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k})$

Since

$$\sigma_j \int_u^s e^{-b_j(\tau-u)} d\tau = \sigma_j \frac{1-e^{-b_j(s-u)}}{b_j} \geq 0 \text{ (equality holds at } s=u),$$

by Assumptions 1 and 5, the right hand side of (78) is positive, that is $Z_{j,u}^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} > 0,\ \forall u \in [0,T]$, and thus $sgn(Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}) \equiv 1$. Hence, since $\lambda_j^{k,*} = -\bar{\lambda}_j^k$, we have

$$-\bar{\lambda}_j^k sgn(Z_j^{c^{k,*},\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}) = -\bar{\lambda}_j^k = \lambda_j^{k,*}.$$

□

Next, we solve the following problem on $\boldsymbol{\lambda}_2^k$ for $(c^{k,*}(\boldsymbol{\lambda}_2^k), \boldsymbol{\lambda}_1^{k,*}(\boldsymbol{\lambda}_2^k))$:

$$\sup_{\boldsymbol{\lambda}_2^k, |\lambda_j^k| \leq \bar{\lambda}_j^k, j=d_1+1,\ldots,d_1+d_2} \mathbf{E}\left[\int_0^T \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_t^{k,*}(\boldsymbol{\lambda}_2^k)) dt\right]. \tag{79}$$

Lemma 11. $\boldsymbol{\lambda}_2^{k,*} = (\bar{\lambda}_{d_1+1}^k, \ldots, \bar{\lambda}_{d_1+d_2}^k)^\top$ *is optimal in (79).*

Proof. Since the optimal consumption for fixed $\boldsymbol{\lambda}_2^k$ is (93), $\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_t^{k,*}(\boldsymbol{\lambda}_2^k))$ is

$$\begin{aligned}
\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_t^{k,*}(\boldsymbol{\lambda}_2^k)) &= \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \times -\frac{\exp\left(\log\left(U^{k\prime}(c_0^{k,*}(\boldsymbol{\lambda}_2^k)) \frac{H_{0,t}}{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}\right)\right)}{\gamma^k} \\
&= \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \times -\frac{1}{\gamma^k} U^{k\prime}(c_0^{k,*}(\boldsymbol{\lambda}_2^k)) \frac{H_{0,t}}{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}} \\
&= -\frac{1}{\gamma^k} U^{k\prime}(c_0^{k,*}(\boldsymbol{\lambda}_2^k)) H_{0,t}.
\end{aligned}$$

Substituting (93) to

$$\mathbf{E}\left[\int_0^T H_{0,t} c_t^{k,*} dt\right] = \mathbf{E}\left[\int_0^T H_{0,t} \varepsilon_t^k dt\right],$$

we have

$$\mathbf{E}\left[\int_0^T H_{0,t} \left\{ -\frac{\log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right)}{\gamma^k} + c_0^{k,*} \right\} dt\right] = \mathbf{E}\left[\int_0^T H_{0,t} \varepsilon_t^k dt\right].$$

Hence, we obtain

$$c_0^{k,*}(\boldsymbol{\lambda}_2^k) = \frac{1}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]} \left\{ \mathbf{E}\left[\int_0^T H_{0,t} \varepsilon_t^k dt\right] + \mathbf{E}\left[\int_0^T H_{0,t} \frac{\log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right)}{\gamma^k} dt\right] \right\}. \tag{80}$$

Since $c_0^{k,*}(\boldsymbol{\lambda}_2^k)$ is (80), the objective function of $\boldsymbol{\lambda}_2^k$ is

$$
\begin{aligned}
&\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_t^{k,*}(\boldsymbol{\lambda}_2^k)) \\
&= -\frac{H_{0,t}}{\gamma^k}\exp\left(-\gamma^k \frac{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]}\right) \times \exp\left(-\frac{\mathbf{E}\left[\int_0^T H_{0,t}\left\{\log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right)\right\} dt\right]}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]}\right) \\
&= -\Gamma_t \exp\left(\frac{\mathbf{E}\left[\int_0^T H_{0,t}\log \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} dt\right]}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]}\right),
\end{aligned}
$$

where we defined Γ_t independent of $\boldsymbol{\lambda}_2^k$ as

$$
\Gamma_t = \frac{H_{0,t}}{\gamma^k}\exp\left(-\gamma^k \frac{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right]}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]}\right) \times \exp\left(-\frac{\mathbf{E}\left[\int_0^T H_{0,t}\log H_{0,t} dt\right]}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]}\right) > 0.
$$

Moreover, using

$$
\begin{aligned}
&\log \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \\
&= \sum_{j=1}^{d_1}\left(\int_0^t \lambda_{j,s}^{k,*} dB_{j,s} - \frac{1}{2}\int_0^t (\lambda_{j,s}^{k,*})^2 ds\right) + \sum_{j=d_1+1}^{d_1+d_2}\left(\int_0^t \lambda_{j,s}^{k} dB_{j,s} - \frac{1}{2}\int_0^t \lambda_{j,s}^{k,2} ds\right) \\
&= \int_0^t \boldsymbol{\lambda}_{1,s}^{k,*\top} d\boldsymbol{B}_{1,s} - \frac{1}{2}\int_0^t |\boldsymbol{\lambda}_{1,s}^{k,*}|^2 ds + \int_0^t \boldsymbol{\lambda}_{2,s}^{k\top} d\boldsymbol{B}_{2,s} - \frac{1}{2}\int_0^t |\boldsymbol{\lambda}_{2,s}^{k}|^2 ds,
\end{aligned}
$$

we rewrite

$$
\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_t^{k,*}(\boldsymbol{\lambda}_2^k)) = -\Gamma_t F_1(\boldsymbol{\lambda}_1^{k,*}) F_2(\boldsymbol{\lambda}_2^k),
$$

where

$$
F_1(\boldsymbol{\lambda}_1^{k,*}) = \exp\Bigg(\frac{1}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]}\mathbf{E}\Bigg[\int_0^T H_{0,t}\Bigg\{\int_0^t \boldsymbol{\lambda}_{1,s}^{k,*\top} d\boldsymbol{B}_{1,s} - \frac{1}{2}\int_0^t |\boldsymbol{\lambda}_{1,s}^{k,*}|^2 ds\Bigg\} dt\Bigg]\Bigg),
$$

and

$$
F_2(\boldsymbol{\lambda}_2^k) = \exp\Bigg(\frac{1}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]}\mathbf{E}\Bigg[\int_0^T H_{0,t}\Bigg\{\int_0^t \boldsymbol{\lambda}_{2,s}^{k\top} d\boldsymbol{B}_{2,s} - \frac{1}{2}\int_0^t |\boldsymbol{\lambda}_{2,s}^{k}|^2 ds\Bigg\} dt\Bigg]\Bigg).
$$

The optimization problem is equivalent to

$$
\sup_{\boldsymbol{\lambda}_2^k, |\lambda_j^k| \le \bar{\lambda}_j^k, j=d_1+1,\ldots,d_1+d_2} \mathbf{E}\left[\int_0^T -\Gamma_t F_1(\boldsymbol{\lambda}_1^{k,*}) F_2(\boldsymbol{\lambda}_2^k) dt\right]. \tag{81}
$$

Since $\Gamma_t F_1(\boldsymbol{\lambda}_1^{k,*})$ is positive and independent of $\boldsymbol{\lambda}_2^k$, we only have to investigate $F_2(\boldsymbol{\lambda}_2^k)$. If $F_2(\boldsymbol{\lambda}_2^k)$ is minimized, the expected utility is maximized with respect to $\boldsymbol{\lambda}_2^k$. Thus, we have to confirm that

$F_2(\boldsymbol{\lambda}_2^k)$ is decreasing as a functional of the deterministic process $\boldsymbol{\lambda}_2^k = (\lambda_{d_1+1}^k, \dots, \lambda_{d_1+d_2}^k)^\top$, and minimized at $\boldsymbol{\lambda}_2^{k,*}$. Note that H_0 is

$$H_{0,t} = \exp\left(-\frac{1}{\Delta}\left\{ \sum_{j=1}^d \int_0^t \phi_{j,s} ds + \sum_{j=1}^d \int_0^t \left(\rho_{j,s} + \sigma_j \int_s^t e^{-b_j(\tau-s)} d\tau \right) dB_{j,s} \right\}\right)$$
$$\times \prod_{l=1}^K \exp\left(\frac{1}{\gamma^l \Delta} \left\{ \sum_{j=1}^d \left(\int_0^t \lambda_{j,s}^{l,*} dB_{j,s} - \frac{1}{2}\int_0^t (\lambda_{j,s}^{l,*})^2 ds \right)\right\}\right).$$

Set $\boldsymbol{\xi} = (\xi_{d_1+1}, \dots, \xi_{d_1+d_2})^\top$, where

$$\xi_{j,s} = -\frac{1}{\Delta}\left(\rho_{j,s} + \sigma_j \int_s^t e^{-b_j(\tau-s)} d\tau \right) + \sum_{l=1}^K \frac{1}{\gamma^l \Delta} \lambda_{j,s}^{l,*}, \tag{82}$$

$j = d_1 + 1, \dots, d_1 + d_2$. Then, we rewrite H_0 as follows:

$$H_{0,t} = \exp\left(\int_0^t \boldsymbol{\xi}_s^\top d\boldsymbol{B}_{2,s} - \frac{1}{2}\int_0^t |\boldsymbol{\xi}_s|^2 ds \right) \exp\left(\frac{1}{2}\int_0^t |\boldsymbol{\xi}_s|^2 ds \right)$$
$$\times \exp\left(-\frac{1}{\Delta}\left\{ \sum_{i=1}^d \int_0^t \phi_{i,s} ds + \sum_{i=1; i\notin \mathcal{J}_2^k}^d \int_0^t \left(\rho_{i,s} + \sigma_i \int_s^t e^{-b_i(\tau-s)} d\tau \right) dB_{i,s} \right\}\right)$$
$$\times \prod_{l=1}^K \exp\left(\frac{1}{\gamma^l \Delta}\left\{ \sum_{i=1; i\notin \mathcal{J}_2^k}^d \left(\int_0^t \lambda_{i,s}^{l,*} dB_{i,s} - \frac{1}{2}\int_0^t (\lambda_{i,s}^{l,*})^2 ds \right)\right\} - \sum_{j=1}^d \frac{1}{2}\int_0^t (\lambda_{j,s}^{l,*})^2 ds \right)$$
$$= \exp\left(\int_0^t \boldsymbol{\xi}_s^\top d\boldsymbol{B}_{2,s} - \frac{1}{2}\int_0^t |\boldsymbol{\xi}_s|^2 ds \right) H(t, \boldsymbol{B}_2^-),$$

where we set

$$H(t, \boldsymbol{B}_2^-)$$
$$= \exp\left(\frac{1}{2}\int_0^t |\boldsymbol{\xi}_s|^2 ds \right) \exp\left(-\frac{1}{\Delta}\left\{ \sum_{i=1}^d \int_0^t \phi_{i,s} ds + \sum_{i=1; i\notin \mathcal{J}_2^k}^d \int_0^t \left(\rho_{i,s} + \sigma_i \int_s^t e^{-b_i(\tau-s)} d\tau \right) dB_{i,s} \right\}\right)$$
$$\times \prod_{l=1}^K \exp\left(\frac{1}{\gamma^l \Delta}\left\{ \sum_{i=1; i\notin \mathcal{J}_2^k}^d \left(\int_0^t \lambda_{i,s}^{l,*} dB_{i,s} - \frac{1}{2}\int_0^t (\lambda_{i,s}^{l,*})^2 ds \right)\right\} - \sum_{j=1}^d \frac{1}{2}\int_0^t (\lambda_{j,s}^{l,*})^2 ds \right) > 0.$$

We define a probability measure $\tilde{\mathbf{P}}$ with a positive martingale as

$$Z_t^\xi = \exp\left(\int_0^t \boldsymbol{\xi}_s^\top d\boldsymbol{B}_{2,s} - \frac{1}{2}\int_0^t |\boldsymbol{\xi}_s|^2 ds \right).$$

Then, we obtain Brownian motions under $\tilde{\mathbf{P}}$ as follows:

$$\tilde{B}_{i,t} = B_{i,t} \ (i \notin \mathcal{J}_2^k),$$
$$\tilde{B}_{i,t} = B_{i,t} - \int_0^t \xi_{i,s} ds \ (i \in \mathcal{J}_2^k).$$

Set $\tilde{\boldsymbol{B}}_2 = (\tilde{B}_{d_1+1}, \ldots, \tilde{B}_{d_1+d_2})^\top$. Then,

$$
\begin{aligned}
&F_2(\boldsymbol{\lambda}_2^k) \\
&= \exp\left(\frac{1}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]} \left\{\mathbf{E}^{\tilde{\mathbf{P}}}\left[\int_0^T H(t, \tilde{\boldsymbol{B}}_2^-)\left(\int_0^t \boldsymbol{\lambda}_{2,s}^{k\top}(d\tilde{\boldsymbol{B}}_{2,s} + \boldsymbol{\xi}_s ds) - \int_0^t \frac{|\boldsymbol{\lambda}_{2,s}^k|^2}{2} ds\right) dt\right]\right\}\right).
\end{aligned}
$$

Since $\tilde{\boldsymbol{B}}_2^-$ and $\tilde{\boldsymbol{B}}_2$ are independent and

$$
\mathbf{E}^{\tilde{\mathbf{P}}}\left[\int_0^t \boldsymbol{\lambda}_{2,s}^{k\top} d\tilde{\boldsymbol{B}}_{2,s}\right] = 0,
$$

we have

$$
\begin{aligned}
&F_2(\boldsymbol{\lambda}_2^k) \\
&= \exp\left(\frac{1}{\mathbf{E}\left[\int_0^T H_{0,t} dt\right]} \left\{\mathbf{E}^{\tilde{\mathbf{P}}}\left[\int_0^T H(t, \tilde{\boldsymbol{B}}_2^-)\left(\int_0^t \boldsymbol{\lambda}_{2,s}^{k\top}\boldsymbol{\xi}_s ds - \frac{1}{2}\int_0^t |\boldsymbol{\lambda}_{2,s}^k|^2 ds\right) dt\right]\right\}\right).
\end{aligned}
$$

Since $H(t, \tilde{\boldsymbol{B}}_2^-) > 0$, we only have to consider

$$
f_{2,t}(\boldsymbol{\lambda}_2^k) = \int_0^t \left(\boldsymbol{\lambda}_{2,s}^{k\top}\boldsymbol{\xi}_s - \frac{1}{2}|\boldsymbol{\lambda}_{2,s}^k|^2\right) ds.
$$

For any nonrandom $\hat{\boldsymbol{\lambda}}_2^k = (\hat{\lambda}_{d_1+1}, \ldots, \hat{\lambda}_{d_1+d_2})^\top$ with $0 < \hat{\lambda}_j^k \leq \bar{\lambda}_j^k$, $j = d_1+1, \ldots, d_1+d_2$, we calculate

$$
\lim_{\alpha \to 0} \frac{f_{2,t}(\boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k) - f_{2,t}(\boldsymbol{\lambda}_2^k)}{\alpha}.
$$

Since

$$
\begin{aligned}
&f_{2,t}(\boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k) - f_{2,t}(\boldsymbol{\lambda}_2^k) \\
&= \int_0^t \left((\boldsymbol{\lambda}_{2,s}^k + \alpha\hat{\boldsymbol{\lambda}}_{2,s}^k)^\top\boldsymbol{\xi}_s - \frac{|\boldsymbol{\lambda}_{2,s}^k + \alpha\hat{\boldsymbol{\lambda}}_{2,s}^k|^2}{2}\right) ds - \int_0^t \left(\boldsymbol{\lambda}_{2,s}^{k\top}\boldsymbol{\xi}_s - \frac{|\boldsymbol{\lambda}_{2,s}^k|^2}{2}\right) ds \\
&= \int_0^t \left(\alpha\hat{\boldsymbol{\lambda}}_{2,s}^{k\top}\boldsymbol{\xi}_s - \alpha\hat{\boldsymbol{\lambda}}_{2,s}^{k\top}\boldsymbol{\lambda}_{2,s}^k - \frac{|\hat{\boldsymbol{\lambda}}_{2,s}^k|^2}{2}\alpha^2\right) ds,
\end{aligned}
$$

$$
\lim_{\alpha \to 0} \frac{f_{2,t}(\boldsymbol{\lambda}_2^k + \alpha\hat{\boldsymbol{\lambda}}_2^k) - f_{2,t}(\boldsymbol{\lambda}_2^k)}{\alpha} = \int_0^t \left(\hat{\boldsymbol{\lambda}}_{2,s}^{k\top}\boldsymbol{\xi}_s - \hat{\boldsymbol{\lambda}}_{2,s}^{k\top}\boldsymbol{\lambda}_{2,s}^k\right) ds.
$$

If $\boldsymbol{\xi}_s < 0$ for all $0 \leq s \leq t$, then $f_{2,t}(\boldsymbol{\lambda}_2^k)$ is decreasing in $\boldsymbol{\lambda}_2^k$ since the integrand is decreasing in $\boldsymbol{\xi}_s \leq \boldsymbol{\lambda}_{2,s}^k \leq \bar{\boldsymbol{\lambda}}_{2,s}^k$. Hence, $f_{2,t}(\boldsymbol{\lambda}_2^k)$ is minimized at $\boldsymbol{\lambda}_2^k = \bar{\boldsymbol{\lambda}}_2^k$, and thus the expected utility is maximized. In fact, by Assumption 5, $\boldsymbol{\xi}_s < 0$ holds for all $0 \leq s \leq t$. $\square$

The admissibility of $(\bar{c}^{k,*}, \pi^{k,*})$ follows in the same manner as in the proof of Theorem 3 in the main text. Particularly, $\mathbf{E}[\int_0^T U^k(\bar{c}_t^{k,*})^2 dt] < \infty$ follows from the nonrandomness of $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}_s^{l,*}$ and $\mathbf{E}[\int_0^T \exp(-\frac{4}{\Delta}\int_0^t \nu_s ds)dt] < \infty$ (see Remark 6 below for details). Thus, by Lemmas 10 and 11, Proposition 9 holds. □

Remark 6. *For the admissibility of* $(\bar{c}^{k,*}, \pi^{k,*})$, $\mathbf{E}[\int_0^T U^k(\bar{c}_t^{k,*})^2 dt] < \infty$ *is confirmed as follows. First, we note that since*

$$U^k(\bar{c}_t^{k,*}) = -\frac{1}{\gamma^k}\exp(-\gamma^k \bar{c}_t^{k,*}),$$

$$\bar{c}_t^{k,*} = -\frac{1}{\gamma^k}\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right) + \bar{c}_0^{k,*}, \tag{83}$$

we have

$$U^k(\bar{c}_t^{k,*}) = -\frac{K}{\gamma^k}\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right), \tag{84}$$

where $K = \exp(-\gamma^k \bar{c}_0^{k,*})$. *Then,*

$$U^k(\bar{c}_t^{k,*})^2 = \frac{K^2}{(\gamma^k)^2}\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right)^2. \tag{85}$$

Here,

$$\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right)^2 = \exp\left(-\frac{2(\varepsilon_t - \varepsilon_0)}{\Delta}\right)\prod_{l=1}^{K}(\eta_t^{l,*})^{\frac{2}{\gamma^l \Delta}}/(\eta_t^{k,*})^2$$

$$= \exp\left(-\frac{2(\varepsilon_t - \varepsilon_0)}{\Delta} + \sum_{l=1}^{K}\left(-\frac{1}{\gamma^l \Delta}\int_0^t |\boldsymbol{\lambda}_s^{l,*}|^2 ds + \frac{2}{\gamma^l \Delta}\int_0^t \boldsymbol{\lambda}_s^{l,*\top} dB_s\right) + \int_0^t |\boldsymbol{\lambda}_s^{k,*}|^2 ds - 2\int_0^t \boldsymbol{\lambda}_s^{k,*\top} dB_s\right), \tag{86}$$

where

$$\varepsilon_t = \varepsilon_0 + \int_0^t \nu_s ds + \int_0^t \boldsymbol{\rho}_s^\top dB_s. \tag{87}$$

Hence,

$$\mathbf{E}\left[\int_0^T U^k(\bar{c}_t^{k,*})^2 dt\right] \le \frac{K^2}{(\gamma^k)^2}\sqrt{\mathbf{E}\left[\int_0^T \exp\left(-\frac{4}{\Delta}\int_0^t \nu_s ds\right)dt\right]}\sqrt{\mathbf{E}\left[\int_0^T \exp(A_t)dt\right]}, \tag{88}$$

where

$$A_t = -\frac{4}{\Delta}\int_0^t \boldsymbol{\rho}_s^\top dB_s + \sum_{l=1}^{K}\left(-\frac{2}{\gamma^l \Delta}\int_0^t |\boldsymbol{\lambda}_s^{l,*}|^2 ds + \frac{4}{\gamma^l \Delta}\int_0^t \boldsymbol{\lambda}_s^{l,*\top} dB_s\right) + 2\int_0^t |\boldsymbol{\lambda}_s^{k,*}|^2 ds - 4\int_0^t \boldsymbol{\lambda}_s^{k,*\top} dB_s. \tag{89}$$

Since $\boldsymbol{\rho}$ *and* $\boldsymbol{\lambda}_s^{l,*}$ *are nonrandom and* $\mathbf{E}[\int_0^T \exp(-\frac{4}{\Delta}\int_0^t \nu_s ds)dt] < \infty$, *we obtain* $\mathbf{E}[\int_0^T U^k(\bar{c}_t^{k,*})^2 dt] < \infty$.

□

Finally, we show that the clearing conditions (17)-(19) are satisfied.

Proposition 12. *Under Assumptions 1 and 5, given H_0 in (73), the clearing conditions (17)-(19) hold.*

Proof.

First, we confirm the clearing condition of the consumption goods market. Since

$$\bar{c}_t^{k,*} = -\frac{1}{\gamma^k}\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right) + \bar{c}_0^{k,*},$$

where $\bar{c}_0^{k,*} = \frac{1}{\mathbf{E}\left[\int_0^T H_{0,t}dt\right]}\left\{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right] + \mathbf{E}\left[\int_0^T H_{0,t}\frac{\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right)}{\gamma^k}dt\right]\right\}$,

$$\sum_{k=1}^K \bar{c}_t^{k,*} = -\sum_{k=1}^K \frac{1}{\gamma^k}\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right) + \sum_{k=1}^K \bar{c}_0^{k,*}.$$

Note that

$$\begin{aligned}
-\sum_{k=1}^K \frac{1}{\gamma^k}\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right) &= -\sum_{k=1}^K \frac{1}{\gamma^k}(\log H_{0,t} - \log \eta_t^{k,*}) \\
&= -\Delta \log H_{0,t} + \sum_{k=1}^K \frac{1}{\gamma^k}\log \eta_t^{k,*} \\
&= \varepsilon_t - \varepsilon_0,
\end{aligned}$$

since

$$\Delta \log H_{0,t} = -(\varepsilon_t - \varepsilon_0) + \sum_{k=1}^K \frac{1}{\gamma^k}\eta_t^{k,*}.$$

Moreover,

$$\begin{aligned}
\sum_{k=1}^K \bar{c}_0^{k,*} &= \frac{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t dt\right] + \mathbf{E}\left[\int_0^T H_{0,t}\sum_{k=1}^K \frac{\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right)}{\gamma^k}dt\right]}{\mathbf{E}\left[\int_0^T H_{0,t}dt\right]} \\
&= \frac{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t dt\right] + \mathbf{E}\left[\int_0^T H_{0,t}(-\varepsilon_t + \varepsilon_0)dt\right]}{\mathbf{E}\left[\int_0^T H_{0,t}dt\right]} = \varepsilon_0.
\end{aligned}$$

Hence, we have

$$\sum_{k=1}^K \bar{c}_t^{k,*} = \varepsilon_t, \ \ t \in [0, T].$$

The clearing conditions for the security market and the money market also follow in the same way as in the proof of Theorem 4 in the main text. □

□

Furthermore, by applying Ito's formula to (73) and comparing the result with (2), we have the following expressions for the interest rate and the market price of risk with heterogeneous views on the fundamental risks in the exponential utility case.

Proposition 13. *The interest rate r and the market price of risk $-\boldsymbol{\theta}$ in equilibrium are given by* $r_t = \frac{\nu_t}{\Delta} - \frac{|\boldsymbol{\rho}_t|^2}{2\Delta^2} + \sum_{k=1}^{K} \frac{1}{2\gamma^k\Delta^2}\left(\sum_{m\neq k}^{K} \frac{|\boldsymbol{\lambda}_t^{k,*}|^2}{\gamma^m} + 2\boldsymbol{\rho}_t^\top \boldsymbol{\lambda}_t^{k,*}\right)$, *and* $-\boldsymbol{\theta}_t = \frac{1}{\Delta}\boldsymbol{\rho}_t - \sum_{k=1}^{K} \frac{1}{\gamma^k\Delta}\boldsymbol{\lambda}_t^{k,*}$.

Proof.

By $H_{0,t} = \exp\left(-\frac{\varepsilon_t - \varepsilon_0}{\Delta}\right) \prod_{k=1}^{K} \left(\eta_t^{k,*}\right)^{\frac{1}{\gamma^k\Delta}}$,$\log H_0$ is expressed as

$$\log H_0 = -\frac{\varepsilon_t - \varepsilon_0}{\Delta} + \sum_{k=1}^{K} \frac{1}{\gamma^k\Delta} \log \eta_t^{k,*}.$$

Thus, we have

$$\begin{aligned} d\log H_{0,t} &= -\frac{1}{\Delta} d\varepsilon_t + \sum_{k=1}^{K} \frac{1}{\gamma^k\Delta} d\log \eta_t^{k,*} \\ &= -\frac{1}{\Delta}(\nu_t dt + \boldsymbol{\rho}_t^\top dB_t) + \sum_{k=1}^{K} \frac{1}{\gamma^k\Delta}(\boldsymbol{\lambda}_t^{k,*\top} dB_t - \frac{1}{2}|\boldsymbol{\lambda}_t^{k,*}|^2 dt) \\ &= \left[-\frac{1}{\Delta}\nu_t - \frac{1}{2}\sum_{k=1}^{K} \frac{1}{\gamma^k\Delta}|\boldsymbol{\lambda}_t^{k,*}|^2\right] dt + \left[-\frac{1}{\Delta}\boldsymbol{\rho}_t + \sum_{k=1}^{K} \frac{1}{\gamma^k\Delta}\boldsymbol{\lambda}_t^{k,*}\right]^\top dB_t. \end{aligned}$$

On the other hand,

$$dH_{0,t} = H_{0,t}[-r_t dt + \boldsymbol{\theta}_t^\top dB_t], \tag{90}$$

and thus

$$d\log H_{0,t} = \left[-r_t - \frac{1}{2}|\boldsymbol{\theta}_t|^2\right] dt + \boldsymbol{\theta}_t^\top dB_t.$$

By these representations of $d\log H_{0,t}$, the market price of risks $-\theta$ and the interest rate r are given by

$$-\boldsymbol{\theta}_t = \frac{1}{\Delta}\boldsymbol{\rho}_t - \sum_{k=1}^{K} \frac{1}{\gamma^k\Delta}\boldsymbol{\lambda}_t^{k,*},$$

and

$$
\begin{aligned}
r_t &= \frac{\nu_t}{\Delta} + \frac{1}{2}\sum_{k=1}^{K}\frac{1}{\gamma^k\Delta}|\boldsymbol{\lambda}_t^{k,*}|^2 - \frac{1}{2}|\boldsymbol{\theta}_t|^2 \\
&= \frac{\nu_t}{\Delta} + \sum_{k=1}^{K}\frac{|\boldsymbol{\lambda}_t^{k,*}|^2}{2\gamma^k\Delta} - \frac{1}{2}|\frac{1}{\Delta}\boldsymbol{\rho}_t - \sum_{k=1}^{K}\frac{1}{\gamma^k\Delta}\boldsymbol{\lambda}_t^{k,*}|^2 \\
&= \frac{\nu_t}{\Delta} - \frac{|\boldsymbol{\rho}_t|^2}{2\Delta^2} + \sum_{k=1}^{K}\frac{|\boldsymbol{\lambda}_t^{k,*}|^2}{2\gamma^k\Delta}(1 - \frac{1}{\gamma^k\Delta}) + \sum_{k=1}^{K}\frac{\boldsymbol{\rho}_t^\top\boldsymbol{\lambda}_t^{k,*}}{\gamma^k\Delta^2} \\
&= \frac{\nu_t}{\Delta} - \frac{|\boldsymbol{\rho}_t|^2}{2\Delta^2} + \sum_{k=1}^{K}\frac{1}{2\gamma^k\Delta^2}\left(\sum_{m\neq k}^{K}\frac{|\boldsymbol{\lambda}_t^{k,*}|^2}{\gamma^m} + 2\boldsymbol{\rho}_t^\top\boldsymbol{\lambda}_t^{k,*}\right).
\end{aligned}
$$

□

Remark 7. *Since $\boldsymbol{\rho}_t$ and $\boldsymbol{\lambda}_t^{k,*}$ are nonrandom, $\mathcal{Z}_t = \exp\left(\int_0^t \boldsymbol{\theta}_s^\top dB_s - \frac{1}{2}\int_0^t |\boldsymbol{\theta}_s|^2 ds\right)$, $t \in [0,T]$ is in fact a martingale in the exponential case.*

6. Possible extension to an exponential utility case with stochastic boundaries

In this section, we show a possible extension of the model to the case where the boundaries of the views $\bar{\lambda}_j^k$, $j \in \mathcal{J}_1, \mathcal{J}_2$ are stochastic, namely, we assume $\bar{\lambda}_j^k$ are positive $\{\mathcal{F}_t\}$-progressively measurable processes.

In the exponential utility case, we consider a market where each agent has a exponential-utility function U^k given by $U^k(x) = -\frac{e^{-\gamma^k x}}{\gamma^k}$, $0 < \gamma^k < \infty$, and the aggregate endowment process ε is a normal type stochastic process expressed as

$$
\varepsilon_t = \varepsilon_0 + \int_0^t \nu_\tau d\tau + \int_0^t \boldsymbol{\rho}_\tau^\top dB_\tau, \tag{91}
$$

where $\nu, \boldsymbol{\rho}$ are $\{\mathcal{F}_t\}$-progressively measurable processes with $\mathbf{E}[\int_0^T |\nu_\tau|d\tau] < \infty$, $\mathbf{E}[\int_0^T |\boldsymbol{\rho}_\tau|^2 d\tau] < \infty$, and each element of $\boldsymbol{\rho}$ being positive, i.e. $\rho_j > 0$ $(j = 1, \cdots, d)$.

Then, for a given state-price density process H_0 in (73), namely, $H_{0,t} = \exp\left(-\frac{\varepsilon_t - \varepsilon_0}{\Delta}\right)\prod_{k=1}^K \left(\eta_t^{k,*}\right)^{\frac{1}{\gamma^k\Delta}}$, we consider the following individual optimization problem for the k-th agent

$$
\begin{aligned}
&\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\ \sup_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_2^k}\ \inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} \\
&\mathbf{E}\left[\int_0^T \eta_t^{\boldsymbol{\lambda}^k} U^k(c_t^k)dt\right]\left(= \mathbf{E}^{\boldsymbol{P}^{\boldsymbol{\lambda}^k}}\left[\int_0^T U^k(c_t^k)dt\right]\right) \\
&= \sup_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_2^k}\ \sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\ \inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k}\mathbf{E}\left[\int_0^T \eta_t^{\boldsymbol{\lambda}^k} U^k(c_t^k)dt\right].
\end{aligned} \tag{92}
$$

Then, as in the discussion in Section 5, under certain conditions, $\lambda_{j,t}^{k,*} = -\bar{\lambda}_j^k, j \in \mathcal{J}_1^k$, $\lambda_{j,t}^{k,*} = \bar{\lambda}_j^k, j \in \mathcal{J}_2^k$, $\bar{c}_t^{k,*} = -\frac{1}{\gamma^k}\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right) + \bar{c}_0^{k,*}$ with $\bar{c}_0^{k,*} = \frac{1}{\mathbf{E}\left[\int_0^T H_{0,t}dt\right]}\left\{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right] + \mathbf{E}\left[\int_0^T H_{0,t}\frac{\log\left(\frac{H_{0,t}}{\eta_t^{k,*}}\right)}{\gamma^k}dt\right]\right\}$, attain $\sup_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_2^k}\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k)$ and satisfy the clearing conditions (17)-(19) as in Proposition 12 of Section 5.

Hereafter, we assume $\mathcal{J}_1^k = \{1,\ldots,d_1\}$, $\mathcal{J}_2^k = \{d_1+1,\ldots,d_1+d_2\}$, $\mathcal{J}_3^k = \{d_1+d_2+1,\ldots,d\}$, without loss of generality.

In the following, we particularly investigate the conditions under which the individual optimization problem is solved as mentioned. We first consider conditions where for given $\boldsymbol{\lambda}_2^k$, $sgn(Z_j) = +1, j \in j = 1,\ldots,d_1$ for Z_j in (14) of Lemma 1 and then examine conditions where maximization with respect to $\boldsymbol{\lambda}_2^k$ is attained at $\boldsymbol{\lambda}_2^{k,*}$.

6.1. $\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k)$ *for given* $\boldsymbol{\lambda}_2^k$

First, for given $\boldsymbol{\lambda}_2^k, |\lambda_j^k| \le \bar{\lambda}_j^k,\ j\in\mathcal{J}_2^k$, we consider $\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k)$. As in Lemma 10 of Section 5,

$$c_t^{k,*} = -\frac{1}{\gamma^k}\log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right) + c_0^{k,*}, \tag{93}$$

where $c_0^{k,*}$ and $\log H_{0,t}$ are given respectively as

$$c_0^{k,*} = \frac{1}{\mathbf{E}\left[\int_0^T H_{0,t}dt\right]}\left\{\mathbf{E}\left[\int_0^T H_{0,t}\varepsilon_t^k dt\right] + \mathbf{E}\left[\int_0^T H_{0,t}\frac{\log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\right)}{\gamma^k}dt\right]\right\},$$

$$\log H_{0,t} = \frac{1}{\Delta}\left(-(\varepsilon_t-\varepsilon_0) + \sum_{l=1}^K \frac{1}{\gamma^l}\log\eta_t^{l,*}\right),$$

where $\eta^{l,*} = \eta^{\boldsymbol{\lambda}^{l,*}}$ with $\boldsymbol{\lambda}^{l,*}$ given by (21), and $\boldsymbol{\lambda}_1^{k*} = (-\bar{\lambda}_1^k,\ldots,-\bar{\lambda}_{d_1}^k)$ attain $\sup_{(c^k,\boldsymbol{\pi}^k)\in\mathcal{A}^k}\inf_{|\lambda_j^k|\le\bar{\lambda}_j^k,\ j\in\mathcal{J}_1^k} J^k(c^k,\boldsymbol{\lambda}_1^k,\boldsymbol{\lambda}_2^k)$ if $sgn(Z_{j,u}^k) = +1,\ j = 1,\ldots,d_1$ where we denote $Z_{j,t}$ of agent k in (14) as $Z_{j,t}^k$.

Also, we will use the result shown in Appendix A. Namely, $Z_{j,u}^k$ is calculated as follows:

$$Z_{j,u}^k = \int_u^T \mathbf{E}_u^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\left(\frac{1}{\gamma^k}e^{-\gamma^k c_s^{k,*}}\right)\left(\lambda_{j,u}^{k,*} + \left\{\gamma^k D_u^j(c_s^{k,*}) - D_u^j(\log\eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k})\right\}\right)\right]ds, \tag{94}$$

$j = 1,\ldots,d_1$.

To know $sgn(Z^k_{j,u})$, as $\frac{1}{\gamma^k}e^{-\gamma^k c^{k,*}_s} > 0$, we only have to examine the sign of

$$\lambda^{k,*}_{j,u} + \left\{ \gamma^k D^j_u(c^{k,*}_s) - D^j_u(\log \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}) \right\}. \tag{95}$$

Hereafter, we consider a special case of $K = 2$, $d = 1$. Also, we assume that agent 1 is conservative against the risk $B \equiv B_1$, and agent 2 is aggressive against the risk $B \equiv B_1$. Thus, we note $\eta_t^{\boldsymbol{\lambda}_1^{1,*},\boldsymbol{\lambda}_2^1} = \eta_t^{\boldsymbol{\lambda}_1^{1,*},0} = \eta_t^{1,*}$.

Hence, for agent 1, noting that in Lemma 10,

$$c^{1,*}_t = -\frac{1}{\gamma^1}\log\left(H_{0,t}/\eta_t^{\boldsymbol{\lambda}_1^{1,*},0}\right) + c^{1,*}_0, \tag{96}$$

we have

$$c^{1,*}_s = \left(\frac{1}{\gamma^1+\gamma^2}\right)\left(\gamma^2\varepsilon_s + \log\frac{\eta^{1,*}_s}{\eta^{2,*}_s}\right) + \left\{c^{1,*}_0 - \left(\frac{\gamma^2}{\gamma^1+\gamma^2}\right)\varepsilon_0\right\}. \tag{97}$$

Next, suppose that the aggregate endowment process is given as

$$\varepsilon_t = \varepsilon_0 + \int_0^t \nu_\tau d\tau + \int_0^t \rho_\tau dB_\tau, \tag{98}$$

$$\nu_t = \nu_0 + \int_0^t (a - b\nu_s)ds + \int_0^t \sigma dB_s. \tag{99}$$

Then, we have

$$\gamma^1 D_u c^{1,*}_s = \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right) D_u\varepsilon_s + \left(\frac{\gamma^1}{\gamma^1+\gamma^2}\right)\left(D_u\log\eta^{1,*}_s - D_u\log\eta^{2,*}_s\right), \tag{100}$$

and thus,

$$\gamma^1 D_u c^{1,*}_s - D_u\log\eta^{1,*}_s = \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right) D_u\varepsilon_s - \left(\frac{\gamma^2}{\gamma^1+\gamma^2}\right) D_u\log\eta^{1,*}_s - \left(\frac{\gamma^1}{\gamma^1+\gamma^2}\right) D_u\log\eta^{2,*}_s. \tag{101}$$

We also calculate

$$\begin{aligned}&\left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right) D_u\varepsilon_s \\ &= \int_u^s \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right) D_u\nu_\tau d\tau + \int_u^s \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right) D_u\rho_\tau dB_\tau + \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right)\rho_u.\end{aligned} \tag{102}$$

Hence, plugging those into

$$\lambda^{1,*}_u + \left\{\gamma^1 D_u(c^{1,*}_s) - D_u(\log\eta^{1,*}_s)\right\}, \tag{103}$$

we obtain the following:

$$
\begin{aligned}
&\lambda_u^{1,*} + \gamma^1 D_u c_s^{1,*} - D_u \log \eta_s^{1,*} \\
&= \int_u^s \left\{ \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2} \right) D_u \nu_\tau + \left(\frac{\gamma^2}{\gamma^1+\gamma^2} \right) D_u \frac{1}{2} |\lambda_\tau^{1,*}|^2 + \left(\frac{\gamma^1}{\gamma^1+\gamma^2} \right) D_u \frac{1}{2} |\lambda_\tau^{2,*}|^2 \right\} d\tau \\
&+ \int_u^s \left\{ \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2} \right) D_u \rho_\tau - \left(\frac{\gamma^2}{\gamma^1+\gamma^2} \right) D_u \lambda_\tau^{1,*} - \left(\frac{\gamma^1}{\gamma^1+\gamma^2} \right) D_u \lambda_\tau^{2,*} \right\} dB_\tau \\
&+ \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2} \right) \rho_u + \left(\frac{\gamma^1}{\gamma^1+\gamma^2} \right) \lambda_u^{1,*} - \left(\frac{\gamma^1}{\gamma^1+\gamma^2} \right) \lambda_u^{2,*}.
\end{aligned} \tag{104}
$$

Then, to have $-\bar{\lambda}_1^1$ to be optimal for the conservative agent 1 in the random boundary case, that is, to get $sgn(Z^1) = 1$ as in Lemma 10 of Section 5, we need the following conditions:

$$
D_u \nu_\tau + D_u \frac{1}{2\gamma^1} |\lambda_\tau^{1,*}|^2 + D_u \frac{1}{2\gamma^2} |\lambda_\tau^{2,*}|^2 > 0, ((\text{integrand of integration w.r.t } d\tau \text{ in (104) }) > 0), \tag{105}
$$

$$
D_u \rho_\tau = \left(\frac{1}{\gamma^1} \right) D_u \lambda_\tau^{1,*} + \left(\frac{1}{\gamma^2} \right) D_u \lambda_\tau^{2,*}, ((\text{integrand of integration w.r.t } dB_\tau \text{ in (104)}) = 0), \tag{106}
$$

$$
\rho_u > \left(\frac{1}{\gamma^2} \right) \left(\lambda_u^{2,*} - \lambda_u^{1,*} \right), (\text{the last term in (104)} > 0), \tag{107}
$$

6.2. $\sup_{\boldsymbol{\lambda}_2^k, |\lambda_j^k| \le \bar{\lambda}_j^k,\ j \in \mathcal{J}_2^k} J^k(c^{k,*}, \boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k)$

Next, we consider $\sup_{\boldsymbol{\lambda}_2^k, |\lambda_j^k| \le \bar{\lambda}_j^k,\ j \in \mathcal{J}_2^k} J^k(c^{k,*}, \boldsymbol{\lambda}_1^{k,*}, \boldsymbol{\lambda}_2^k)$ for $c^{k,*}$ and $\boldsymbol{\lambda}_1^{k,*}$ that attain the first $\sup_{(c^k, \boldsymbol{\pi}^k) \in \mathcal{A}^k} \inf_{|\lambda_j^k| \le \bar{\lambda}_j^k,\ j \in \mathcal{J}_1^k} J^k(c^k, \boldsymbol{\lambda}_1^k, \boldsymbol{\lambda}_2^k)$ part for given $\boldsymbol{\lambda}_2^k$.

As in Lemma 11 of Section 5, by using (82) with $d = 1,\ d_1 = 0,\ d_2 = 1$ and $K = 2$, we need the following condition for $\bar{\lambda}_2^2$ to be optimal for the aggressive agent 2 in the random boundary case:

$$
\rho_s + D_s \nu_t = \rho_s + \sigma \int_s^t e^{-b(\tau - s)} d\tau > \frac{1}{\gamma^1} \lambda_s^{1,*} + \frac{1}{\gamma^2} \lambda_s^{2,*}, \tag{108}
$$

where we note $D_s \nu_t = \sigma \int_s^t e^{-b(\tau-s)} d\tau \ge 0$. Hence, the next condition is sufficient:

$$
\rho_s > \frac{1}{\gamma^1} \lambda_s^{1,*} + \frac{1}{\gamma^2} \lambda_s^{2,*} \tag{109}
$$

Moreover, noting $\lambda_u^{1,*} < 0$ and $\lambda_u^{2,*} > 0$, the condition $\rho_u > \left(\frac{1}{\gamma^2} \right) \left(\lambda_u^{2,*} - \lambda_u^{1,*} \right)$ is more stringent than $\rho_s > \frac{1}{\gamma^1} \lambda_s^{1,*} + \frac{1}{\gamma^2} \lambda_s^{2,*}$. Thus, if (107) holds, (109) follows.

6.3. Sufficient conditions and implications

Noting $D_s \nu_t = \sigma \int_s^t e^{-b(\tau-s)} d\tau \ge 0$, the next condition is sufficient for (105):

$$
D_u \frac{1}{2\gamma^1} |\lambda_\tau^{1,*}|^2 + D_u \frac{1}{2\gamma^2} |\lambda_\tau^{2,*}|^2 > 0. \tag{110}
$$

Also, the following condition is sufficient for (106):

$$\rho_\tau - c_{\rho,\tau} = \left(\frac{1}{\gamma^1}\right)\lambda^{1,*}_\tau + \left(\frac{1}{\gamma^2}\right)\lambda^{2,*}_\tau;\ c_{\rho,\tau} > 0,\ \text{nonrandom.} \tag{111}$$

Then, both conditions, (107) and (111) imply that

$$\left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right)c_\rho > -\lambda^{1,*} > 0,\ \text{equivalently,}\ \lambda^{1,*} > -\left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right)c_\rho. \tag{112}$$

Namely, $|\lambda^{1,*}|$, the magnitude of agent 1's random conservative view/sentiment should be bounded by the nonrandom process $\left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right)c_\rho$.

In sum, the equations (110), (111) and (112) are sufficient conditions for $\lambda^{1,*} = -\bar{\lambda}^1$ and $\lambda^{2,*} = \bar{\lambda}^2$.

Furthermore, (111) with (112) indicates that

$$\rho - \left(\frac{\gamma^1}{\gamma^1+\gamma^2}\right)c_\rho > \left(\frac{1}{\gamma^2}\right)\lambda^{2,*} > 0. \tag{113}$$

We also note that for a given aggregate endowment's random volatility ρ, we can interpret that those conditions (112) and (113) specify ranges where the agents' views/sentiments vary. That is, given ρ, $c_\rho > 0$, the equations (112) and (113) provide lower and upper limits for the range of agent 1's conservative and agent 2's aggressive views/sentiments, respectively.

Moreover, we remark that as in Proposition 13 of Section 5, we can obtain the equilibrium interest rate r and market price of risk $-\theta$. Namely, let us recall that

$$\begin{aligned} d\log H_{0,t} &= -\frac{1}{\Delta}\left[\nu_t + \frac{1}{2}\sum_{k=1}^{2}\frac{1}{\gamma^k}|\boldsymbol{\lambda}^{k,*}_t|^2\right]dt - \frac{1}{\Delta}\left[\boldsymbol{\rho}_t - \sum_{k=1}^{2}\frac{1}{\gamma^k}\boldsymbol{\lambda}^{k,*}_t\right]^\top dB_t \\ &= -\left[r_t + \frac{1}{2}|\boldsymbol{\theta}_t|^2\right]dt + \boldsymbol{\theta}_t^\top dB_t. \end{aligned}$$

Hence, in the current setting with (111) we have the following:

$-\theta = c_\rho/\Delta = \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right)c_\rho$, and

$$\begin{aligned} r_t &= \frac{1}{\Delta}\left[\nu_t + \frac{1}{2}\sum_{k=1}^{2}\frac{1}{\gamma^k}|\lambda^{k,*}_t|^2\right] - \frac{1}{2}|\theta_t|^2 \\ &= \left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right)\left[\nu_t + \frac{1}{2}\sum_{k=1}^{2}\frac{1}{\gamma^k}|\bar{\lambda}^k_t|^2\right] - \frac{1}{2}\left(\frac{\gamma^1\gamma^2}{\gamma^1+\gamma^2}\right)^2 c^2_{\rho,t}. \end{aligned} \tag{114}$$

As an example, given each agent's ARA parameter $\gamma^k > 0$ $(k = 1, 2)$ and a nonrandom process $c_\rho > 0$, let $M_1 := \left(\frac{\gamma^2}{\gamma^1+\gamma^2}\right)c_\rho - c > 0$ for an arbitrary small constant $c > 0$. We also define each $Y^k > 0$, $k = 1, 2$ as a mean-reverting square-root process:

$$dY^k_t = (a^k_y - b^k_y\, Y^k_t)dt + \sigma^k_y\sqrt{Y^k_t}dB_t;\ Y^k_0 > 0,\ a^k_y,\ b^k_y > 0,\ a^k_y > (\sigma^k_y)^2/2, \tag{115}$$

where $D_u\{\sqrt{Y^k_\tau}\} > 0$ and $D_u\{Y^k_\tau\} > 0$ ($\tau > u$) thanks to Proposition 4.1 and Corollary 4.2 in Alos and Ewald (2008).

Moreover, let $f(y)$ be a smoothly modified function of $\min\{M_1, \sqrt{y}\}$ ($y > 0$) to define $f'(y) \geq 0$ for all $y > 0$ including $y = (M_1)^2$. Then, we set the aggregate endowment volatility ρ as

$$\rho = c_\rho - f(Y^1) + \sqrt{Y^2} > 0. \tag{116}$$

We finally put $\bar{\lambda}^1 = \gamma_1 f(Y^1)$ and $\bar{\lambda}^2 = \gamma_2\sqrt{Y^2}$. Using those $\bar{\lambda}^1$ and $\bar{\lambda}^2$ with (99), the equation (114) explicitly gives us the equilibrium interest rate r.

(**Reference**) Alos, E., and Ewald, C. O. (2008). Malliavin differentiability of the Heston volatility and applications to option pricing. Advances in Applied Probability, 40(1), 144-162.

Appendix A. Derivation of $Z^k_{j,u}$ in (94)

Firstly, for $\boldsymbol{Z}^k_s$ and $\boldsymbol{Z}^{p,k}_s$ in the martingale representations,

$$\int_0^T U^k(c^{k,*}_s)ds = \mathbf{E}^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}\left[\int_0^T U^k(c^{k,*}_s)ds\right] + \int_0^T \boldsymbol{Z}^{k\top}_s dB^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s, \tag{A.1}$$

and

$$\int_0^T \eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s U^k(c^{k,*}_s)ds = \mathbf{E}\left[\int_0^T \eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s U^k(c^{k,*}_s)ds\right] + \int_0^T \boldsymbol{Z}^{p,k\top}_s dB_s, \tag{A.2}$$

the relation

$$\boldsymbol{Z}^k_u = \frac{\boldsymbol{Z}^{p,k}_u}{\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u} - V^{k,\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u \boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}^k_2)_u, \tag{A.3}$$

holds, which is shown as follows. We note the notation $\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}^k_2) = (\boldsymbol{\lambda}^{k,*\top}_1, \boldsymbol{\lambda}^{k\top}_2, 0, \ldots, 0)^\top$.

First, let us recall the definition:

$$V^{k,\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_t = \mathbf{E}^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}\left[\int_t^T U^k(c^{k,*}_s)ds \big| \mathcal{F}_t\right]. \tag{A.4}$$

Then,

$$\begin{aligned}
V^{k,\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_t &= \mathbf{E}^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}\left[\int_t^T U^k(c^{k,*}_s)ds \big| \mathcal{F}_t\right] \\
&= \mathbf{E}^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}\left[\int_0^T U^k(c^{k,*}_s)ds \big| \mathcal{F}_t\right] - \int_0^t U^k(c^{k,*}_s)ds \\
&= \mathbf{E}^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}\left[\int_0^T U^k(c^{k,*}_s)ds\right] + \int_0^t \boldsymbol{Z}^{k\top}_s dB^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s - \int_0^t U^k(c^{k,*}_s)ds. \qquad \text{(A.5)}
\end{aligned}$$

In the second equality, we used the martingale representation theorem:

$$\int_0^T U^k(c_s^{k,*})ds = \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_0^T U^k(c_s^{k,*})ds\right] + \int_0^T \boldsymbol{Z}_s^{k\top} dB_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}. \tag{A.6}$$

Thus,

$$\begin{aligned} dV_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} &= \boldsymbol{Z}_t^{k\top} dB_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} - U^k(c_t^{k,*})dt \\ &= \boldsymbol{Z}_t^{k\top}(dB_t - \boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)_t dt) - U^k(c_t^{k,*})dt \\ &= \boldsymbol{Z}_t^{k\top} dB_t - (U^k(c_t^{k,*}) + \boldsymbol{Z}_t^{k\top}\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)_t)dt. \end{aligned} \tag{A.7}$$

On the other hand, we note

$$V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} = \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_t^T U^k(c_s^{k,*})ds \big| \mathcal{F}_t\right] = \mathbf{E}\left[\int_t^T \frac{\eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}} U^k(c_s^{k,*})ds \big| \mathcal{F}_t\right]. \tag{A.8}$$

This is derived as follows:

$$\begin{aligned} V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} &= \mathbf{E}^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left[\int_t^T U^k(c_s^{k,*})ds \big| \mathcal{F}_t\right] \\ &= \frac{1}{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}\mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_t^T U^k(c_s^{k,*})ds \big| \mathcal{F}_t\right] \\ &= \frac{1}{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}\mathbf{E}\left[\int_t^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_s^{k,*})ds \big| \mathcal{F}_t\right]. \end{aligned} \tag{A.9}$$

The third equality holds as follows.

$$\begin{aligned} \mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_t^T U^k(c_s^{k,*})ds \big| \mathcal{F}_t\right] &= \mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^T U^k(c_s^{k,*})ds \big| \mathcal{F}_t\right] - \mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^t U^k(c_s^{k,*})ds \big| \mathcal{F}_t\right] \\ &= \mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^T U^k(c_s^{k,*})ds \big| \mathcal{F}_t\right] - \mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \big| \mathcal{F}_t\right]\int_0^t U^k(c_s^{k,*})ds \\ &= \mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^T U^k(c_s^{k,*})ds \big| \mathcal{F}_t\right] - \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^t U^k(c_s^{k,*})ds. \end{aligned} \tag{A.10}$$

Here,

$$d\left(\eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^v U^k(c_s^{k,*})ds\right) = d\eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\left(\int_0^v U^k(c_s^{k,*})ds\right) + \eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_v^{k,*})dv. \tag{A.11}$$

Integrating from 0 to t,

$$\begin{aligned} \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^t U^k(c_s^{k,*})ds &= \int_0^t \left(\int_0^v U^k(c_s^{k,*})ds\right) d\eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \int_0^t \eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_v^{k,*})dv \\ &= \int_0^t \left(\int_0^v U^k(c_s^{k,*})ds\right) \eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} \boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)_v^\top dB_v + \int_0^t \eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} U^k(c_v^{k,*})dv, \end{aligned} \tag{A.12}$$

where we used $d\eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} = \eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)_v^\top dB_v$. Therefore, when we note that the first term is a stochastic integral with Brownian motion, we obtain

$$\begin{aligned}\mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_t^T U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] &= \mathbf{E}\left[\eta_T^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^T U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] - \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\int_0^t U^k(c_s^{k,*})ds \\ &= \mathbf{E}\left[\int_0^T \eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_v^{k,*})dv\big|\mathcal{F}_t\right] - \int_0^t \eta_v^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_v^{k,*})dv \\ &= \mathbf{E}\left[\int_t^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right]. \end{aligned} \tag{A.13}$$

Now, we use the martingale representation theorem:

$$\int_0^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds = \mathbf{E}\left[\int_0^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\right] + \int_0^T \boldsymbol{Z}_s^{p,k\top}dB_s. \tag{A.14}$$

Thus,

$$\mathbf{E}\left[\int_0^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] = \mathbf{E}\left[\int_0^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\right] + \int_0^t \boldsymbol{Z}_s^{p,k\top}dB_s. \tag{A.15}$$

Here, since

$$\begin{aligned} V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} &= \frac{1}{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}\mathbf{E}\left[\int_t^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] \\ &= \frac{1}{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}}\left\{\mathbf{E}\left[\int_0^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] - \int_0^t \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\right\}, \end{aligned} \tag{A.16}$$

we have

$$\mathbf{E}\left[\int_0^T \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\big|\mathcal{F}_t\right] = \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \int_0^t \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds. \tag{A.17}$$

We calculate

$$\begin{aligned} & d\left(\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \int_0^t \eta_s^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_s^{k,*})ds\right) \\ &= V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}d\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}dV_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k} + d\langle \eta^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}, V^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\rangle_t + \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_t^{k,*})dt \\ &= V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)_t^\top dB_t + \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}(\boldsymbol{Z}_t^{k\top}dB_t - (U^k(c_t^{k,*}) + \boldsymbol{Z}_t^{k\top}\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)_t)dt) \\ &+ \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\boldsymbol{Z}_t^{k\top}\boldsymbol{\lambda}_t^{k,*}(\boldsymbol{\lambda}_2^k)dt + \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}U^k(c_t^{k,*})dt \\ &= \eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}(V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)_t + \boldsymbol{Z}_t^k)^\top dB_t. \end{aligned} \tag{A.18}$$

Since this volatility coefficient must equal $\boldsymbol{Z}_t^{p,k}$, we obtain

$$\boldsymbol{Z}_t^k = \frac{\boldsymbol{Z}_t^{p,k}}{\eta_t^{\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}} - V_t^{k,\boldsymbol{\lambda}_1^{k,*},\boldsymbol{\lambda}_2^k}\boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}_2^k)_t. \tag{A.19}$$

Then, we have the following expressions for $Z^{p,k}_{j,u}$ and $-V^{k,\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u \boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}^k_2)_u$:

$$\begin{aligned} Z^{p,k}_{j,u} &= \int_u^T \mathbf{E}_u\left[D^j_u\left(\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s \frac{-1}{\gamma^k} e^{-\gamma^k c^{k,*}_s}\right)\right] ds \\ &= \int_u^T \mathbf{E}_u\left[\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s e^{-\gamma^k c^{k,*}_s} D^j_u(c^{k,*}_s) + \frac{-1}{\gamma^k} e^{-\gamma^k c^{k,*}_s} D^j_u(\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s)\right] ds \\ &= \int_u^T \mathbf{E}_u\left[\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s e^{-\gamma^k c^{k,*}_s} \left\{D^j_u(c^{k,*}_s) - \frac{1}{\gamma^k} D^j_u(\log \eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s)\right\}\right] ds, \qquad \text{(A.20)} \\ -V^{k,\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2} \boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}^k_2)_u &= -\mathbf{E}_u\left[\int_u^T \frac{\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s}{\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u}\left(\frac{-1}{\gamma^k} e^{-\gamma^k c^{k,*}_s}\right) ds\right] \boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}^k_2)_u \\ &= \int_u^T \mathbf{E}^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u \left[\left(\frac{1}{\gamma^k} e^{-\gamma^k c^{k,*}_s}\right) ds\right] \boldsymbol{\lambda}^{k,*}(\boldsymbol{\lambda}^k_2)_u, \qquad \text{(A.21)} \end{aligned}$$

where $\mathbf{E}^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u[\cdot] := \mathbf{E}^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}[\cdot|\mathcal{F}_u]$ and the expression for $Z^{p,k}_{j,u}/\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u$ as

$$\begin{aligned} Z^{p,k}_{j,u}/\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u &= \gamma^k \int_u^T \mathbf{E}_u\left[\frac{\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s}{\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u}\left(\frac{1}{\gamma^k} e^{-\gamma^k c^{k,*}_s}\right)\left\{D^j_u(c^{k,*}_s) - \frac{1}{\gamma^k} D^j_u(\log \eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s)\right\}\right] ds \\ &= \gamma^k \int_u^T \mathbf{E}^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u\left[\left(\frac{1}{\gamma^k} e^{-\gamma^k c^{k,*}_s}\right)\left\{D^j_u(c^{k,*}_s) - \frac{1}{\gamma^k} D^j_u(\log \eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s)\right\}\right] ds. \qquad \text{(A.22)} \end{aligned}$$

Hence, we obtain a simple form of $Z^k_{j,u}, j = 1, \ldots, d_1$ as follows:

$$\begin{aligned} Z^k_{j,u} &= Z^{p,k}_{j,u}/\eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u - V^{k,\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u \lambda^{k,*}_{j,u} \\ &= \int_u^T \mathbf{E}^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_u\left[\left(\frac{1}{\gamma^k} e^{-\gamma^k c^{k,*}_s}\right)\left(\lambda^{k,*}_{j,u} + \left\{\gamma^k D^j_u(c^{k,*}_s) - D^j_u(\log \eta^{\boldsymbol{\lambda}^{k,*}_1,\boldsymbol{\lambda}^k_2}_s)\right\}\right)\right] ds. \end{aligned} \qquad \text{(A.23)}$$