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Optimal Overbooking Strategy in Online Hotel Booking Systems

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Abstract

The revenue management for online booking systems, which incorporates overbooking by the hotels, is particularly important because of increasing cancellations at the last minute and no-shows which cause serious damage to the management of the hotels.

This study proposes a new quantitative overbooking model for online booking systems, where the rival hotels' room charges affect the choice probability of the target hotel. Our model enables a hotel to obtain optimal overbooking strategy and room charge that maximize the expected sales based on its empirical data on the cancellations and the oversale cost per room.

Moreover, we present numerical examples of the optimal overbooking strategy and room charge with actual online booking data of two major luxury hotels in Shinjuku area in Tokyo.

Furthermore, equilibrium room charges of the hotels are considered, where we find the price competition may lead to significantly low levels of room charges and the expected sales.

Keywords: Hotel overbooking, Online booking, Revenue management, Equilibrium room charge

1 Introduction

Against the backdrop of the advancement in information technologies, increasing number of hotel customers reserve rooms through online booking systems. They can conveniently book rooms by comparing several conditions of the hotels. Under the circumstances, the hotel revenue management for online booking systems, where hotels set the room charges at the optimal levels to maximize their expected revenues by taking into account the room charges of the rival hotels, becomes increasingly important.

Whereas the share of the rooms booked through online booking systems increases, the number of cancellations and no-shows also rises due to the convenience of the booking.

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For example, a pattern of cancellation and no-show called juggling, where customers or groups of customers reserve rooms in multiple hotels and then cancel the rooms or do not show up at the last minute except for the one to stay, represents a critical problem for hotels that sell rooms through online booking systems. What makes the situation even worse is that the customers do not prefer to pay in advance because of the possibilities of cancellation for inevitable reasons such as sickness and unpredicted accidents and it is difficult for the hotels to require the guests to pay the cancellation fees or resort to collect them in terms of hospitality when the cancellations actually happen. This causes heavy damage to the revenues of the hotels.

Although hotels overbook their rooms by predicting the number of cancellations at the last minute and no-shows, developing an overbooking strategy is a complicated issue. For instance, if the cancellations and no-shows are more than expected and these generate some vacant rooms, the hotel incurs some loss on sales. On the other hand, if the cancellations and no-shows are less than expected, the hotel get into a situation of oversale. In such a case, the hotel needs to accommodate the guests a higher grade room if available or displace the guests to other hotels by paying for the room charge and offering extra compensation if necessary, which are additional costs for the hotel.

To overcome the difficulties in overbooking, we propose a quantitative overbooking model for online booking systems, which incorporates the cancellations at the last minute and no-shows with the oversale costs which the hotel incurs when the number of customers actually coming on the check-in date exceeds the hotels' capacity. Our study investigates an overbooking strategy that maximizes expected sales of a hotel in an online booking system, where customers choose a hotel from a group comparing the room charge with those of the rival hotels. Particularly, we provide numerical examples using actual online booking data of two major luxury hotels in Shinjuku area in Tokyo, which are crawled from a Japanese online booking website. In addition, we investigate equilibrium room charges and expected sales less the oversale cost of the hotels when they are in a price competition. Hotels can use the overbooking model to determine the overbooking strategy in peak seasons, which maximizes the expected sales less the oversale cost in online booking systems.

For relevant studies on online hotel booking, Vermeulen and Seegers (2009) [32] investigate the impact of online hotel reviews on customer choice. Ling et al. (2014) [19] study on optimal unit commission that a hotel pay to an online travel agency through a sequence game model. Phillips et al. (2015) [22] deal with effects of online reviews on hotels' revenue per available room by a neural network model. Sparks et al. (2016) [27] consider the effects of hotel responses to negative online reviews on the perceptions of potential customers. Li et al. (2017) [15] study the signaling effect of the hotel management response to online customer reviews to engage customers.

Also, for the literature related to hotel revenue management in qualitative perspectives, Kimes and Chase (1998) [12] summarize revenue management practices in different industries including hotels. Upchurch et al. (2002) [31] analyze revenue management practices by having a questionnaire to hotel revenue managers. Emeksiz et al. (2006) [9] propose effective implementation of yield management in hotels. Cetin et al. (2016) [4] interview hotel revenue managers about knowledge, skills and abilities required for revenue management. Abrate and Viglia (2016) [1] investigate price determining factors in dynamic pricing of hotels with data of booking.com.

For quantitative studies on revenue management, Lai and Ng (2005) [14] study revenue management of hotels taking into account the length of stay by a network optimization modeling. Chen and Kachani (2007) [5] consider demand forecasting and revenue management of hotels by a network flow model. Aziz et al. (2011) [3] investigate a dynamic pricing model of hotels by taking into account a room allocation for early discount and forecasting of demand. Mei and Zhan (2013) [20] examine room choice behavior of customers by multinomial logit model by having a questionnaire. Guo et al. (2013) [11] study a dynamic pricing model for hotels based on market segmentation in online reservation systems. Arenoe et al. (2015) [2] deal with equilibrium room charges of hotels in a pricing competition with the multinomial logit model. Ling et al. (2015) [18] examine an optimal room allocation to online travel agencies taking into account the management fees paid to them . Saito et al. (2016) [24] consider expected sales maximization for online booking by the discrete choice model. Solnet et al. (2016) [26] investigate purchasing behavior of hotel customers in Australia by Market basket analysis.

Furthermore, for the studies related to overbooking, Toh (1986) [28] and Toh and Dekay (2002) [29] study practices of no-shows, late cancellations, overbooking and oversales comparing to the cases of airline industry proposing a simple overbooking model for practical use. Dekay et al. (2004) [8] examine penalties on hotel customers for cancellations and no-shows as well as practices of upgrading and walks in the case of oversales by interviewing hotels. Chen et al. (2011) [6] analyze impact of cancellation fee on customer behavior of searching for hotels by multinomial logit model based on experiment on students. Chen and Xie (2013) [7] investigate cancellation policies of U.S. hotels by dividing the hotels into two groups by clustering. Park and Jang (2014) [21] study effects of temporal and monetary sunk costs on travelers' intention to cancel travel products.

Particularly, for quantitative studies on overbooking models, Rothstein (1974) [23] deals with an overbooking problem to determine an optimal booking policy by Markovian sequential decision model. Liberman and Yechiali (1978) [16] deal with an overbooking problem of hotels with stochastic cancellations. Gallego and Ryzin (1994) [10] investigate formulations of revenue management including the case of overbooking. Koide and Ishii (2005) [13] analyze a model for an optimal room allocation for early discount taking into account cancellation and overbooking. Sierag et al. (2015) [25] consider an optimal choice of products to offer for a room type under overbooking.

While all these works focus on the problem of a single hotel irrelevant to the price levels of the other hotels, our study is the first one that deals with overbooking strategies in online booking systems, where the customers choose a hotel to book from a group in the same area taking the room charges of the rival hotels into account. Also, with the actual data crawled from a Japanese online booking website, we calculate the optimal overbooking level and room charge for a certain room type of a hotel, which provides sensible results in accord with intuition. Cancellations at the last minute and no-shows are increasing in contrast with the convenience of the booking in online booking systems, and they have a serious impact on the management of hotels. To cope with the situation, the revenue management that incorporates overbooking by hotels is particularly important. With this model, booking patterns including arrival rates and a price sensitivity of the customers to the choice probabilities, are estimated from the actual data in the online booking system, and in combination with the empirical data of the cancellation rate and the oversale cost per room they own, they can quantitatively obtain the optimal overbooking level and room charge that maximizes their expected sales less the oversale cost and compensate the revenue decrease by the no-shows and cancellations at the last minute.

The organization of the paper is as follows. Section 2 explains the mathematical setting of the overbooking model. Section 3 presents numerical examples of the overbooking strategies and the equilibrium room charges with the actual online booking data of the two hotels in Shinjuku area in Tokyo. Finally, Section 4 concludes. Appendices A & B provide the proofs of the theorems in Sections 2 & 3.

2 The model

This section explains the overbooking model for online booking systems, where an optimal overbooking strategy for a certain room type is defined through an expected sales maximization problem.

Firstly, we consider booking behavior of online customers who visit the website randomly at the frequency following a Poisson process and book a certain room type of a hotel choosing among J hotels in a group. We fix a check-in date T and let $[0, T]$ be the booking period for the check-in date, where 0 is the start date of the booking period. Let $\{N_t\}_{0 \leq t \leq T}$ be the Poisson process with intensity λ, which represents the total number of rooms booked for the J hotels by date t.

Let $L_i \geq 0,\ i = 1, \ldots, J$ be the actual capacity of hotel i for the room type and $L_i^{ob} \geq L_i,\ i = 1, \ldots, J$ be the overbooking level of hotel i, up to which hotel i accepts reservations in the online booking system.

Let $\gamma = (\gamma_1, \ldots, \gamma_J) \in \Pi_{j=1}^J \{0, 1\} \equiv \Gamma$ be the state of room availability of the J hotels for the room type, that is, $\gamma_i = 0$ or 1 indicates that this room type of hotel i is fully occupied or still available. In detail, when $N_T = k'$, there can be $J^{k'}$ patterns of selection orders. Let $\tau = (i_1, i_2, \ldots, i_{k'}) \in \mathcal{S}_{k'},\ i_1, i_2, \ldots, i_{k'} = 1, \ldots, J$ be a selection order where $\mathcal{S}_{k'}$ is the totality of the selection orders with length k'. For $\tau = (i_1, \ldots, i_{k'}) \in \mathcal{S}_{k'}$ and $1 \leq l \leq k'$, we define the room availability of hotel j after l customers arrived and booked in the selection order of τ, $\gamma_j^{l,\tau}$ as

$$\gamma_j^{l,\tau} = \begin{cases} 1, \ \text{if } \sum_{l'=1}^{l} 1_{\{i_{l'} = j\}} < L_j^{ob} \\ 0, \ \text{otherwise} \end{cases}, \ j = 1, \ldots, J,$$

which indicates that hotel j is available until the number of booking for hotel j reaches its limit L_j^{ob}.

Let $x^{(1)}, \ldots, x^{(J)}$ be the room charges of hotels $1, \ldots, J$ and $p_i^{(\gamma)}$ $(i = 1, \ldots, J)$ be the choice probability of hotel i, which is a function of $x^{(1)}, \ldots, x^{(J)}$ and dependent on the room availability γ. For example, $p_i^{(\gamma)}$ can take the following forms.

1. The multinomial logit model:

$$p_i^{(\gamma)} = \frac{e^{-\beta x^{(i)} + \alpha_i} 1_{\{\gamma_i = 1\}}}{\sum_{j=1}^{J} e^{-\beta x^{(j)} + \alpha_j} 1_{\{\gamma_j = 1\}}}, \tag{1}$$

   where $\beta > 0$, $\alpha_j \in \mathbf{R}$.

2. The nested logit model:

$$p_i^{(\gamma)} = \frac{\left(\sum_{j\in C_{k_i}} e^{\frac{-\beta x^{(j)}+\alpha_j}{\nu_{k_i}}} 1_{\{\gamma_i=1\}}\right)^{\nu_{k_i}}}{\sum_{k=1}^{n}\left(\sum_{j\in C_k} e^{\frac{-\beta x^{(j)}+\alpha_j}{\nu_k}} 1_{\{\gamma_j=1\}}\right)^{\nu_k}} \cdot \frac{e^{\frac{-\beta x^{(i)}+\alpha_i}{\nu_{k_i}}} 1_{\{\gamma_i=1\}}}{\sum_{j\in C_{k_i}} e^{\frac{-\beta x^{(j)}+\alpha_j}{\nu_{k_i}}} 1_{\{\gamma_j=1\}}} \tag{2}$$

where $\beta > 0$, $\alpha_j \in \mathbf{R}$, $0 < \nu_k \leq 1,\ k = 1,\ldots,n$.

3. The mixed logit model:

$$p_i^{(\gamma)} = \int_0^\infty \frac{e^{-\beta x^{(i)}+\alpha_i} 1_{\{\gamma_i=1\}}}{\sum_{j=1}^{J} e^{-\beta x^{(j)}+\alpha_j} 1_{\{\gamma_j=1\}}} h(\beta)d\beta, \tag{3}$$

where $\alpha_j \in \mathbf{R}$, $0 \leq h(\beta)$, $\int_0^\infty h(\beta)d\beta = 1$.

We note that in the theory of random utility (e.g. See Train [30]), the choice probabilities $p_i^{(\gamma)}, i = 1,\ldots,J$ in the multinomial logit model (1) correspond to the situation where customers choose the hotel with the highest utility and the customers' utility on hotel j ($j = 1,\ldots,J$) is given by $U_j = -\beta x^{(j)} + \alpha_j + \epsilon_j$. Here, ϵ_j is a random variable where $\{\epsilon_j\}_{j=1,\ldots,J}$ are i.i.d (independent and identically distributed) random variables following an extreme value distribution and the deterministic part of the random utility, $-\beta x^{(j)}+\alpha_j$, is a decreasing function on the room charge $x^{(j)}$, which implies that the higher the room charge is, the lower the utility is. The other characteristics of hotel j, which do not change over time such as reputation and grade, are reflected in α_j. The indicator function in (1) implies that if $\gamma_j = 0$, that is, the room type of the hotel j is fully booked and not available for booking, then hotel j is excluded from the group for the choice.

Similarly, the nested logit model in (2) can be interpreted in terms of the random utility theory, where the joint distribution of $(\epsilon_1,\ldots,\epsilon_J)$ follows a general extreme value distribution. The J hotels are divided into n non-overlapping groups of the same kind, and the dissimilarities within the small groups are represented by $0 \leq \nu_k \leq 1,\ k = 1,\ldots,n$. We note that the nested logit model is a generalization of the multinomial logit model. That is, the model agrees with the multinomial logit model when the dissimilarity parameters are $\nu_k = 1,\ k = 1,\ldots,n$ meaning that the hotels in the small groups are all dissimilar. Note also that when $J = 2$, this model is the same as the multinomial logit model.

The mixed logit model in (3) is also a generalization of the multinomial logit model in (1), where the sensitivity to change in the room charge β in the utilities takes different values among customers with the distribution $h(\beta)$. We shall use the mixed logit model in the numerical examples in the next section and explain this more in detail.

We also assume that the customers' choice behavior is independent of N_T, the total number of bookings for the group by the check-in date T, that is, the conditional probability for the choice order $\tau = (i_1, i_2 \ldots, i_{k'})$ when $N_T = k'$ is

$$\mathbf{P}((i_1, i_2 \ldots, i_{k'})|N_T = k') = p_{i_1}^{(\gamma^{1,\tau})} p_{i_2}^{(\gamma^{2,\tau})} \cdots p_{i_{k'}}^{(\gamma^{k',\tau})}. \tag{4}$$

Hence, the probability for the choice order $(i_1, i_2 \ldots, i_{k'})$ is

$$\mathbf{P}((i_1, i_2 \ldots, i_{k'})) = p_{i_1}^{(\gamma^{1,\tau})} p_{i_2}^{(\gamma^{2,\tau})} \cdots p_{i_{k'}}^{(\gamma^{k',\tau})} \frac{(\lambda T)^{k'}}{k'!} e^{-\lambda T}. \tag{5}$$

Note that since $p_i^{(\gamma)}$ includes $x^{(j)}$ and γ_j, the conditional probability $p_i^{(\gamma)}$, $\mathbf{P}((i_1, i_2 \dots, i_{k'})|N_T = k')$ depends on $x^{(j)}$ and $L_j^{ob}, j = 1, \dots, J$.

Let $R_T^{(i)}$, $i = 1, \dots, J$, be the number of bookings of hotel i by the customers by the check-in date T. Namely, for $N_T = k'$ and $\tau = (i_1, i_2, \dots, i_{k'})$, it is defined as

$$R_T^{(i)} = \sum_{j=1}^{k'} 1_{\{i_j = i\}} \tag{6}$$

and satisfies

$$\sum_{i=1}^{J} R_T^{(i)} = N_T. \tag{7}$$

Next, we consider the expected sales of the hotel i including the cost arising from the oversale. Since the number of hotel i's rooms booked by the check-in date T does not exceed the overbooking level L_i^{ob}, it is also written as

$$\min\left(R_T^{(i)}, L_i^{ob}\right). \tag{8}$$

Let $0 \leq r_i \leq 1$ be the rate of the no-shows and cancellations at the last minute of hotel i, which is a random variable independent of N_T and $R_T^{(i)}, i = 1, \dots, J$ following a discrete probability distribution. Hereafter, we call it the cancellation rate. Then, the number of rooms booked on the check-in date T after the no-shows and cancellations at the last minute is

$$(1 - r_i) \min\left(R_T^{(i)}, L_i^{ob}\right). \tag{9}$$

Let $c_i > 0$ be the oversale cost per room. Hotel i incurs the cost arising from oversale when the number of rooms booked after the no-shows and cancellations exceeds the actual room capacity L_i.

More in detail, i) if the number of booking after the no-shows and cancellations at the last minute falls within the actual capacity, that is,

$$(1 - r_i) \min\left(R_T^{(i)}, L_i^{ob}\right) \leq L_i, \tag{10}$$

there is no oversale cost.

ii) If the number exceeds the actual capacity, namely,

$$(1 - r_i) \min(R_T^{(i)}, L_i^{ob}) > L_i, \tag{11}$$

then

$$\left((1 - r_i) \min\left(R_T^{(i)}, L_i^{ob}\right) - L_i\right) c_i \tag{12}$$

is the oversale cost that hotel i incurs. By putting these cases together, we observe that the sales less the oversale cost is

$$\min\left(L_i, (1 - r_i) \min\left(R_T^{(i)}, L_i^{ob}\right)\right) x^{(i)} - \max\left(\left((1 - r_i) \min\left(R_T^{(i)}, L_i^{ob}\right) - L_i\right), 0\right) c_i. \tag{13}$$

We remark that when hotel i has spare rooms in higher grade room types and they can be used for upgrading, the oversale cost per room c_i is low. Otherwise, the hotel has to displace the guests to other hotels by paying for the room charges and may offer extra compensations. In this case, the oversale cost per room c_i is high.

Finally, we consider the maximization of the expectation of the sales less the oversale cost over $x^{(i)} \in (0, \infty)$ and $L_i^{ob} \in [L_i, \infty)$. Namely,

$$\max_{(x^{(i)}, L_i^{ob}) \in (0,\infty) \times [L_i, \infty)} \mathbf{E}\left[\min\left(L_i, (1-r)\min(R_T^{(i)}, L_i^{ob})\right) x^{(i)} - \max\left((1-r)\min(R_T^{(i)}, L_i^{ob}) - L_i, 0\right) c_i\right]. \tag{14}$$

The following theorem guarantees the existence of the optimal overbooking level and room charge for this maximization problem.

Theorem 1. *Suppose that hotel $J \neq i$ has unlimited number of rooms, i.e. $\gamma_J \equiv 1$, and the cancellation rate r_i follows a discrete distribution with the support $[0, \bar{r}]$ where $0 \leq \bar{r} < 1$. If $p_j^{(\gamma)}, j = 1, \ldots, J$ is given either by the multinomial logit model in (1), the nested logit model in (2), or the mixed logit model in (3) satisfying*

$$\int_0^\infty \frac{1}{\beta} h(\beta) d\beta < \infty \tag{15}$$

, then

$$\mathbf{E}\left[\min\left(L_i, (1-r)\min(R_T^{(i)}, L_i^{ob})\right) x^{(i)} - \max\left((1-r)\min(R_T^{(i)}, L_i^{ob}) - L_i, 0\right) c_i\right] \tag{16}$$

has a maximum point $(x_*^{(i)}, L_{i*}^{ob})$ in $(0, \infty) \times [L_i, \infty)$.

Note that the integrability condition (15) for β of the mixed logit model (3) holds for the lognormal distribution. Moreover, this is also satisfied when the distribution has the support in $(0, \infty)$ or is the triangular distribution on $[0, a], a > 0$

We investigate this maximization problem of hotel i for different levels of the oversale cost per room c_i and the cancellation rate r_i in the numerical examples in the next section.

3 Numerical examples

In this section, we present numerical examples of the optimal overbooking level and room charge of hotels by using actual online booking data of two major luxury hotels in Shinjuku area in Tokyo, which were crawled from a Japanese online booking website.

3.1 Data set

The original dataset includes prices of the accommodation plans and available numbers of accommodation plans for booking for the non-smoking standard twin rooms of Hotels A & B in Shinjuku area in Tokyo. The check-in dates and the booking dates of the accommodation plans in the dataset range from 1st of March 2017 to 26th of April 2017. Table 1 describes basic information of Hotels A & B (number of guest rooms, ratings, and customer reviews). We note that Hotels A & B are luxury hotels with large capacity, high ratings and good customer reviews in the same area.

	Number of rooms	Booking.com rating	Michelin guide 2011 rating	Booking.com review	Rakuten review	Jaran review
Hotel A	806	5 stars	4 pavilions	8.5/10.0	3.88/5.00	3.9/5.0
Hotel B	744	4 stars	3 pavilions	8.6/10.0	4.17/5.00	4.5/5.0

Table 1: Summary of Hotels A & B

Note that the original dataset does not explicitly contain the information on the room charges or the numbers of rooms sold for this room type, while it includes the prices and the number of plans available for booking for the multiple accommodation plans linked to this room type. Therefore, in order to comply with the model, we need to estimate the information from the original dataset in some way.

Considering these points, we first define the representative room charge of this room type of a hotel for each check-in date as follows. For each check-in date and accommodation plan, we calculate the average price of the plan over the corresponding booking dates. Then, we take the minimum of the average prices over the accommodation plans for each check-in date, and define it as the representative room charge of the room type for the check-in date. We use the representative room charges as the room charges in the model.

Also, we define the number of rooms booked for this room type of a hotel on a booking date for a check-in date as follows. We first calculate the change of the available number of plans for an accommodation plan from the previous booking date to the booking date. If the change is negative, we regard this as the number of rooms booked on the booking date, and if this is positive, we consider that this number of new rooms were supplied by the hotel to the booking systems. Note that the number of plans available for booking of an accommodation plan changes in conjunction with all the other accommodation plans linked to the same room type, and hence the numbers are the same among all the accommodation plans.

Remark 1. *Note also that for the accommodation plans, when the number of plans available for booking is more than 10, it is only displayed as "more than 10" in the original dataset and the exact number is unknown. In this case, we regard the available number for the accommodation plan as 10 in the above procedure. Hence when the check-in date is not in a peak period and the hotel has enough inventories, the number of rooms booked can be underestimated, which is a limitation of the original dataset.*

Table 2 summarizes the information on the dataset including the representative room charges and the numbers of rooms booked, which are used in the model parameters estimation.

	HotelA	HotelB
Number of check-in days	61	61
Number of a day before a holiday check-in days	9	9
Number of sold out check-in days	35	39
Number of sold out check-in days, a day before a holiday	6	7
Number of booking days	1,891	1,891
Number of available booking days	976	682
Maximum representative room charge	62,911	59,400
Minimum representative room charge	26,922	20,925
Average representative room charge	40,799	31,917
Maximum representative room charge, a day before a holiday	55,719	59,400
Minimum representative room charge, a day before a holiday	36,355	30,645
Average representative room charge, a day before a holiday	43,447	33,950
Maximum representative room charge, a weekday	62,911	50,202
Minimum representative room charge, a weekday	26,922	20,925
Average representative room charge, a weekday	40,367	31,721
Maximum number of booked rooms per check-in day	20	21
Minimum number of booked rooms per check-in day	0	0
Average number of booked rooms per check-in day	3.97	4.93
Maximum number of booked rooms per check-in day, a day before a holiday	10	10
Minimum number of booked rooms per check-in day, a day before a holiday	0	0
Average number of booked rooms per check-in day, a day before a holiday	3.33	3.00
Maximum number of booked rooms per check-in day, a week day	20	21
Minimum number of booked rooms per check-in day, a week day	0	0
Average number of booked rooms per check-in day, a week day	4.08	5.27

Table 2: Summary of the dataset used for the estimation

3.2 Estimation

Table 3 presents the estimation results of the mixed-logit model (3) by the maximum likelihood method. In detail, we assume the log-normal distribution for β, that is, $\beta = e^X$ where X is a random variable following the normal distribution with the mean $\mu \in \mathbf{R}$ and the standard deviation $\sigma > 0$. In addition, we assume that the intercept $\alpha_j,\ j = 1, \ldots, J$ in (3) takes the form $\alpha_j = \delta_j y + \bar{\alpha}_j$, where y is a dummy variable taking a value 0 or 1; $y = 0$ if the check-in date is a weekday and $y = 1$ if it is a day before a holiday. We label Hotel A and Hotel B as hotel 1 and hotel 2, respectively, and consider the case of $J = 2$. Namely,

$$\begin{aligned} p_i^{(\gamma)} &= \int_0^\infty \frac{e^{-\beta x^{(i)} + \delta_i y + \bar{\alpha}_i} 1_{\{\gamma_i = 1\}}}{\sum_{j=1}^2 e^{-\beta x^{(j)} + \delta_j y + \bar{\alpha}_j} 1_{\{\gamma_j = 1\}}} h(\beta) d\beta, \\ &= \int_0^\infty \frac{e^{-e^{(\mu+\sigma z)} x^{(i)} + \delta_i y + \bar{\alpha}_i} 1_{\{\gamma_i = 1\}}}{\sum_{j=1}^2 e^{-e^{(\mu+\sigma z)} x^{(j)} + \delta_j y + \bar{\alpha}_j} 1_{\{\gamma_j = 1\}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \end{aligned} \tag{17}$$

for $i = 1, 2$. Noting that $p_i^{(\gamma)}$ depends only on the differences $\bar{\alpha}_2 - \bar{\alpha}_1$ and $\delta_2 - \delta_1$ for $\bar{\alpha}_j$ and δ_j, we estimate $\mu, \sigma, \bar{\alpha}_2$ and δ_2, with $\bar{\alpha}_1 = \delta_1 = 0$.

	Estimate	Std. Error	t-value	p-value
$\bar{\alpha}_2$: hotel B:(intercept)	-2.094104	0.27978	-7.4848	7.172e-14 ***
μ: price	-8.161672	0.087479	-93.2985	< 2.2e-16 ***
δ_2: hotel B:holiday	0.849365	0.604821	1.4043	0.1602
σ: sd.price	1.053053	0.243032	4.333	1.471e-05 ***

Note: * - marginal significant at 0.10 level; ** - significant at 0.05 level; *** - significant at 0.01 level.

Table 3: Estimation results of the mixed-logit model.

In the numerical examples, we use the mixed logit model in (3) which is a generalization of the multinomial logit model in (1) in that it incorporates differences of the price sensitivity β among the customers. In detail, we can interpret the mixed logit model in (3) in the framework of the random utility theory (e.g. Train [30]) as follows. Suppose that the utilities on Hotels A & B of person m, $U_{1,m}$ and $U_{2,m}$, are $U_{i,m} = -\beta_m x^{(i)} + \delta_i y + \bar{\alpha}_i + \epsilon_{i,m}$, where $\epsilon_{i,m}$ is the random term of the utility and $\{\epsilon_{i,m}\}$ are i.i.d random variables following an extreme value distribution. β_m is the price sensitivity parameter of person m in the utilities $U_{1,m}$ and $U_{2,m}$, which is a sample of a lognormal random variable $e^X, X \sim N(\mu, \sigma)$. Person m chooses Hotel A if $U_{1,m} > U_{2,m}$, and Hotel B otherwise. Then, it follows that the choice probabilities $p^{(\gamma)}_{1,m} = \mathbf{E}[U_{1,m} > U_{2,m}]$ of Hotels A and $p^{(\gamma)}_{2,m} = \mathbf{E}[U_{2,m} > U_{1,m}]$ of Hotel B by person m under the room availability γ are given by

$$p^{(\gamma)}_{i,m} = \frac{e^{-\beta_m x^{(i)} + \delta_i y + \bar{\alpha}_i} 1_{\{\gamma_i = 1\}}}{\sum_{j=1}^{2} e^{-\beta_m x^{(j)} + \delta_j y + \bar{\alpha}_j} 1_{\{\gamma_j = 1\}}}. \tag{18}$$

Hence, the choice probabilities $p^{(\gamma)}_i$ $(i = 1, 2)$ over all persons are

$$\begin{aligned} p^{(\gamma)}_i &= \int_0^\infty \frac{e^{-\beta x^{(i)} + \delta_i y + \bar{\alpha}_i} 1_{\{\gamma_i = 1\}}}{\sum_{j=1}^{2} e^{-\beta x^{(j)} + \delta_j y + \bar{\alpha}_j} 1_{\{\gamma_j = 1\}}} h(\beta) d\beta \\ &= \int_0^\infty \frac{e^{-e^{(\mu+\sigma z)} x^{(i)} + \delta_i y + \bar{\alpha}_i} 1_{\{\gamma_i = 1\}}}{\sum_{j=1}^{2} e^{-e^{(\mu+\sigma z)} x^{(j)} + \delta_j y + \bar{\alpha}_j} 1_{\{\gamma_j = 1\}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz. \end{aligned} \tag{19}$$

**In the following examples, we set the other parameters $\lambda = 2.1429, T = 14, L_1 = 20, L_2 = \infty, x^{(2)} = 42{,}292$ so that they correspond to the booking data of the check-in date 10th April 2017, when this room type of Hotel A was fully booked after selling 20 rooms and Hotel B still had available rooms for booking after selling 10 rooms with the representative room charges of Hotels A & B for the check-in date JPY 46,680 and JPY 42,292, respectively. λ is estimated as $\lambda = \frac{30}{14}$ by the maximum likelihood method, which agrees with the intensity obtained by setting $\mathbf{E}[N_T] = \lambda T = 30$.

3.3 Optimal overbooking strategy

Firstly, we calculate the optimal overbooking level and room charge of Hotel A in four cases of the cancellation rate and the oversale cost per room. Figure 1 shows the optimal expected sales for different overbooking levels in the four cases; the high/low cancellation rate and the high/low oversale cost per room. The cancellation rates and the oversale costs per room are as follows: the high cancellation rate $(70\%, 50\%, 30\%)$ with the probability $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, the low cancellation rate $(0\%, 10\%, 20\%)$ with the probability $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, the high oversale cost per room JPY 100,000, and the low oversale cost per room JPY 100.

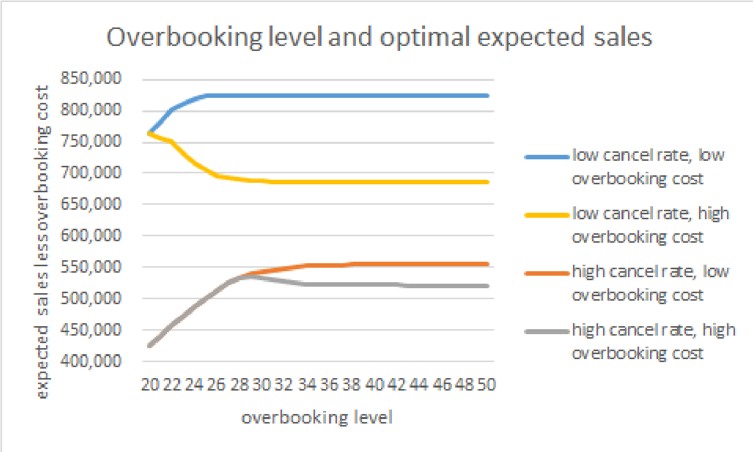

Figure 1: The optimal expected sales less oversale cost for different overbooking levels

The maximum expected sales less the over sale cost and the optimal overbooking level and room charge are as follows.

- The low cancel rate and the low oversale cost per room: JPY 824,200 when $(L_1^{ob}, x^{(1)}) = (25, 43{,}000)$.
- The low cancel rate and the high oversale cost per room: JPY 763,244 when $(L_1^{ob}, x^{(1)}) = (20, 43{,}500)$.
- The high cancel rate and the low oversale cost per room: JPY 555,221 when $(L_1^{ob}, x^{(1)}) = (51, 41{,}500)$.
- The high cancel rate and the high oversale cost per room: JPY 536,454 when $(L_1^{ob}, x^{(1)}) = (29, 42{,}000)$.

We observe that optimal overbooking levels as well as the expected sales less the oversale cost are different among the four cases. The blue and yellow lines for the low cancellation rate show higher expected sales less the oversale cost than the orange and gray lines for the high cancellation rate, because the low cancellation rate implies less sales loss by the no-shows and cancellations at the last minute.

Firstly, with the low cancellation rate, the merit of overbooking is not very much as the blue and yellow lines indicate. More in detail, in the case of the low oversale cost per room, overbooking compensates the sales loss by the no-shows and cancellations at the last minute and this compensation outweighs the cost by the oversale up to 25 rooms of the overbooking level. After that, there is no more sales compensation by raising the overbooking level, and the expect sales less oversale cost only slightly decrease because of the oversale cost. Particularly when the oversale cost per room is high, as the yellow line illustrates, the expected sales less the oversale cost only decrease. Due to the low cancellation rate, there is not much sales loss by the cancellations, and because the oversale cost per room is high, oversale cost outweighs the sales compensation from 20 rooms of the overbooking level.

Secondly, with the high cancellation rate, as the orange and gray lines indicate, it is better for hotels to overbook to cover the decrease in sales by the no-shows and cancellations at the last minute. In the case of the low oversale cost, up to 51 rooms of the overbooking level, the compensation effect outweighs the oversale cost and the orange line is almost flat due to the low oversale cost per room and the little chance for the rooms to be booked up to this level. However, when the oversale cost per room is high, as the gray line shows, after 29 rooms of the overbooking level, overbooking results in high oversale cost and the expected sales less oversale cost start to decline.

We remark that if r_1 is a constant, the optimal overbooking level is $L_1^{ob} = L_1/(1-r_1)$, since as long as $L_1^{ob} \leq L_1/(1-r_1)$, oversale cannot not happen, and it is better for Hotel A to accept as many bookings as possible. For $L_1^{ob} > L_1/(1-r_1)$, the oversale occurs and the hotel incurs more oversale cost as it raises the overbooking level. Hence, it is optimal for Hotel A to set L_1^{ob} at this level if the cancellation rate is a constant. This indicates that if the cancellation rate is at the expectation of the high cancellation rate, 50%, the optimal overbooking level is 40 rooms, and if the rate is at the expectation of the low cancellation rate, 10%, it is 22 rooms.

On the other hand, the optimal overbooking levels for the high cancellation rate are 29 rooms for the high oversale cost per room with 51 rooms for the low oversale cost per room, and the optimal levels for the low cancellation rate are 20 rooms for the high oversale cost per room with 25 rooms for the low oversale cost per room. They imply that when the oversale cost per room is high, the optimal overbooking level is lower than that for the constant cancellation rate, due to the high oversale cost in the case where the cancellation rate takes the low value. They also indicate that when the oversale cost per room is low, the optimal overbooking level is higher than that for the constant cancellation rate in order to compensate the large sales loss in the case where the cancellation rate takes the high value.

Next, Figure 2 illustrates the expected sales less the oversale cost for different room charges in the case of the high cancel rate, $(70\%, 50\%, 30\%)$ with the probability $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and the high oversale cost per room, JPY 100,000, when the overbooking level is set at the optimal level, $L_1^{ob} = 29$.

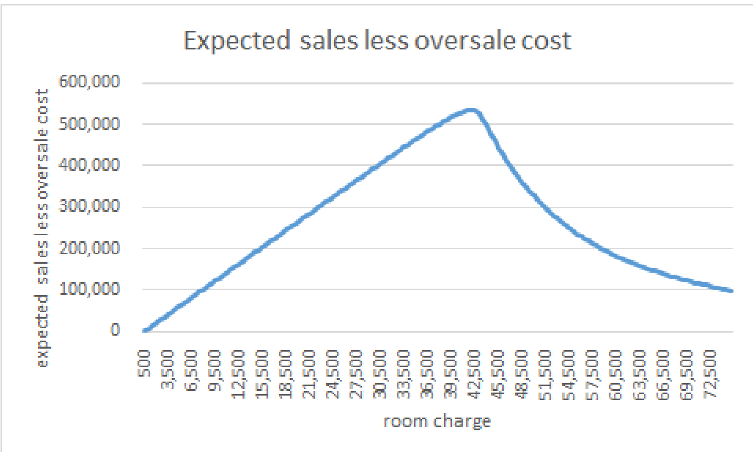

Figure 2: The expected sales less the oversale cost for different room charges

We observe that the expected sales less the oversale cost have a peak of JPY 536,454 at the room charge JPY 42,000. At first, as the room charge increases, the expected sales less the oversale cost also increase because of the higher unit price. After the optimal room charge, the expected sales decline, since the decrease in choice probability outweighs the increase in unit price.

Remark 2. *The oversale cost per room and the distribution of the cancellation rate are determined by hotels based on their own empirical data. As we have observed, using the empirical data of the cancellation rate and the oversale cost per room that they own along with the data from the online booking system, they can obtain the optimal overbooking level and room charge by themselves, which maximizes the expected sales less the oversale cost.*

3.4 Equilibrium room charge

Finally, we consider equilibrium room charges of Hotels A & B to observe how the room charges transition and where they settle if each hotel revises its room charge in response to the optimization of the counterpart. We call (x_1^*, x_2^*) equilibrium room charges if they satisfy

$$x_*^{(1)} = \mathrm{argmax}_{x^{(1)} \in (0,\infty)} f(x^{(1)} | x_*^{(2)}), \tag{20}$$

$$x_*^{(2)} = \mathrm{argmax}_{x^{(2)} \in (0,\infty)} g(x^{(2)} | x_*^{(1)}). \tag{21}$$

Here, $f(x^{(1)}|x^{(2)})$ and $g(x^{(2)}|x^{(1)})$ are the objective functions of hotels 1 and 2, which correspond to Hotels A & B in the following example, when the room charge of the counterpart $x^{(2)}$ or $x^{(1)}$ is given. This indicates that at the equilibrium room charges, hotels 1 and 2 maximize their objective functions at the same time; hotel 1 maximizes its expected sales less the oversale cost at the room charge $x_*^{(1)}$, given the room charge of hotel 2 $x_*^{(2)}$, and hotel 2 also maximizes its objective function at the room charge $x_*^{(2)}$, given the hotel 1's room charge $x_*^{(1)}$.

First, we observe that the existence of the unique equilibrium prices is guaranteed by the following theorem in the case of the multinomial logit model in (1) when the both hotels have an unlimited capacity and hence there is no oversale cost. The proof is provided in Appendix B.

Theorem 2. *Assume that there is no limit for the numbers of rooms available for booking for hotels 1 and 2. Suppose that the choice probabilities of hotels 1 and 2 are given by the multinomial logit model as*

$$\begin{aligned} p_1 &= \frac{1}{1 + e^{\beta(x^{(1)} - x^{(2)}) + \alpha_2 - \alpha_1}}, \\ p_2 &= \frac{1}{1 + e^{\beta(x^{(2)} - x^{(1)}) + \alpha_1 - \alpha_2}}, \end{aligned} \tag{22}$$

and hotels 1 and 2 aim to maximize their expected sales $x^{(1)}\mathbf{E}[R_T^{(1)}]$ and $x^{(2)}\mathbf{E}[R_T^{(2)}]$, respectively. Then, the unique equilibrium room charges $(x_*^{(1)}, x_*^{(2)}) \in (\frac{1}{\beta}, \infty)^2$ exist. Moreover, if hotels 1 and 2 maximize their expected sales iteratively starting from the room charges $(x_0^{(1)}, x_0^{(2)}) \in (0,\infty)^2$, the room charges converge to the equilibrium room charges.

Remark 3. *Since* $\mathbf{E}[R_T^{(1)}] = \lambda T p_1, \mathbf{E}[R_T^{(2)}] = \lambda T p_2$*, hotels 1 and 2 equivalently aim to maximize* $x^{(1)}p_1$ *and* $x^{(2)}p_2$*. Hence, the objective functions in (20) and (21) in the definition of the equilibrium room charges can be taken as* $f(x^{(1)}|x^{(2)}) = x^{(1)}p_1$ *and* $g(x^{(2)}|x^{(1)}) = x^{(2)}p_2$*.*

Then, we calculate the equilibrium room charges in the case of the high cancellation rate and the oversale cost per room in the previous example as follows. First, hotel A maximizes its expected sales less the oversale cost given the initial room charge of Hotel B, and then Hotel B maximizes its objective function given the optimized room charge of Hotel A. They iterate the optimization process 10 times by turns. In each optimization, the hotels assume infinity for the overbooking level of the counterpart since they do not know the actual quantity of inventory of the counterpart. Although this example considers the case of the mixed logit model with the limited capacities and oversale cost, which is different from Theorem 2, we observe that the equilibrium room charges exist, and the optimal room charges converge to the equilibrium levels as a result of the iterative optimizations.

Figures 3 and 4 respectively illustrate transition of the the optimal room charges and that of the expected sales less the oversale cost of Hotels A & B by the iterative optimizations. In Figure 3, starting from the initial room charge of JPY 46,680 and JPY 42,292 for Hotels A & B, respectively, shown as the room charges after the 0-th iteration, Hotel A optimizes its room charge given the initial room charge of Hotel B, and Hotel B does given the optimized room charge of Hotel A. The optimized room charges JPY 42,000 and JPY 30,000 for Hotels A & B are shown as the room charges after the first iteration. In the second iteration, Hotel A optimizes its room charge given the optimized room charge of Hotel B after the first iteration, and then Hotel B does given the optimized room charge of Hotel A. They iterate this process 10 times. We observe that the optimal room charges decline by the iterative optimizations and settle at the equilibrium room charges, JPY 10,500 and JPY 6,500, which are significantly at lower levels than the original room charges. This iterative optimization process can be considered as a price competition between the hotels. Note that the optimal overbooking level of Hotel A is 29 rooms for the first and the second iteration and 28 rooms thereafter. Figure 4 illustrates the corresponding expected sales less the oversale cost. In accordance with the declines of the room charges, the expected sales less oversale cost also decrease for both hotels. In detail, the expected sales less the over sale cost start from JPY 407,992 and JPY 528,751 with the initial room charges, they increase to JPY 536,454 and JPY 620,827 in the first optimization, but thereafter they decrease and settle at JPY 99,654 and JPY 77,084 for Hotels A & B, respectively.

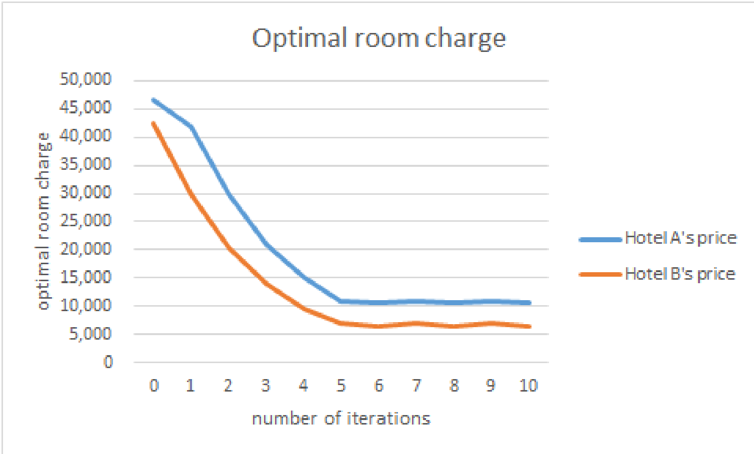

Figure 3: Convergence of the optimal room charges to the equilibrium levels

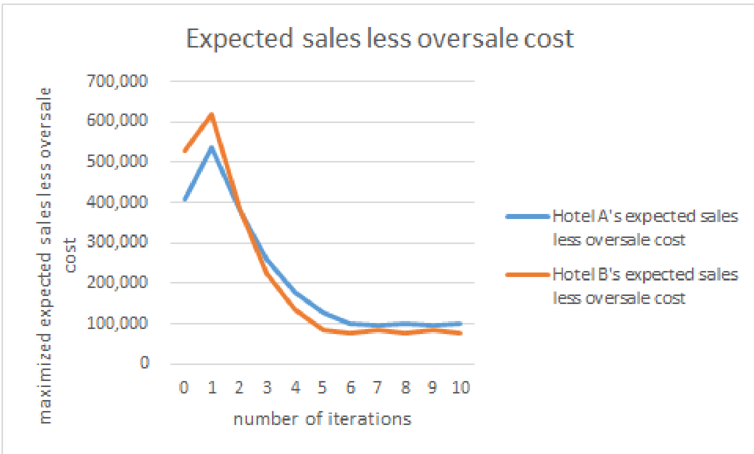

Figure 4: Transition of the expected sales less the oversale cost by the iterative optimizations

For a better understanding of how the optimal room charges converge to the equilibrium room charges, Figure 5 displays the best responses of the hotels, that is, the correspondence between the optimal room charge and the given room charge of the counterpart. Starting from the first optimization of Hotel A, optimal room charge JPY 42,000 of Hotel A in response to the initial room charge JPY 42,292 of Hotel B, after the iterative optimizations by turns, the room charges settle at the crossing point of the two graphs, JPY 10,500 and JPY 6,500 for Hotels A & B, respectively.

Figure 5: Optimal room charges of Hotels A & B in response to the counterpart's room charge

These equilibrium results imply that the revenues of the hotels may significantly decrease once they are in a price competition repetitively optimizing their overbooking levels and room charges to maximize their expected sales less the oversale cost. Thus, it is important for hotels to keep this possibility in mind when they implement this model to determine their optimal overbooking strategies and room charges.

4 Conclusion

In this study, we have proposed a new quantitative overbooking model to calculate the optimal overbooking level and room charge that maximizes the expected sales less the oversale cost in an online booking system. The revenue management that takes into account the overbooking by hotels in online booking systems, is particularly important for hotels offering their rooms through the systems, because of the increasing cancellations at the last minute and no-shows, which are attributable to the convenience of the booking and cause serious damage to the management of the hotels. Moreover, the model takes into account the effect of the room charges of the rival hotels on the choice probability of the target hotel. Furthermore, we consider the equilibrium room charges of the hotels by this optimization.

We have presented the numerical examples of the optimal setting of the overbooking level and the room charge by using actual online booking data of the major two luxury hotels in Shinjuku area in Tokyo, which were crawled from a Japanese booking website. In particular, we have obtained concrete optimal overbooking levels and room charges along with the maximized expected sales less the oversale cost in the four different cases of the cancellation rate and oversale cost per room.

Not only the results represent the features naturally expected for the overbooking model: when the oversale cost per room is low, it is better to overbook in order to compensate the decrease in revenue due to the no-shows and cancellations at the last minute; when the oversale cost per room is high, excessive overbooking ends up with a decrease in the expected sales less the oversale cost due to the high oversale cost, but also indicate concrete quantitative levels, which cannot be obtained by the qualitative observation. For instance, when the oversale cost per room is high and the cancellation rate is low, even minor degree of the overbooking can lead to the decrease in the expected sales less the oversale cost. Moreover, when the oversale cost per room and the cancellation rate are both high, the result exhibits the explicit trade-off between the compensation for the sales loss by the overbooking and the oversale cost, which is useful to determine the overbooking level.

In the numerical example of the equilibrium room charges between the two hotels, we have calculated the transition of the room charges when the hotels iteratively optimize them in turns. We have found that the equilibrium prices settle at significantly low levels compared to the original room charges and accordingly the expected sales less the oversale cost drop.

Hotels can make use of the model to develop their overbooking strategies and to set the room charges in online booking systems, by taking into account the rival hotels' room charges along with their empirical data on the cancellation rate and the oversale cost, particularly for peak seasons. It is also important for hotels to keep in mind that once a price competition occurs, it may dramatically decrease their revenues.

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A Proof of Theorem 1

Let

$$f(x^{(i)}) = \mathbf{E}\left[\min\left(L_i, (1-r)\min(R_T^{(i)}, L_i^{ob})\right) x^{(i)} - \max\left((1-r)\min(R_T^{(i)}, L_i^{ob}) - L_i, 0\right) c_i\right]. \tag{23}$$

First, we show that for a fixed $L_i^{ob} \geq L_i$, the maximum point with respect to $x^{(i)} \in (0, \infty)$ exists. To show this, it suffices to prove the followings.

1.

$$\lim_{x^{(i)} \to 0} f(x^{(i)}) \leq 0, \tag{24}$$

2.

$$\lim_{x^{(i)}\to+\infty} f(x^{(i)}) = 0, \tag{25}$$

3. there exists $\bar{x}^{(i)} \in (0,\infty)$ such that

$$f(\bar{x}^{(i)}) > 0, \tag{26}$$

4. $f(x^{(i)})$ is a continuous function with respect to $x^{(i)}$.

Let $\gamma^* = (1,1,\ldots,1) \in \Gamma$ and $\gamma^\# \in \Gamma$ be

$$\gamma_j^\# = \begin{cases} 1, & j = i, J \\ 0, & \text{otherwise,} \end{cases} \qquad j = 1,\cdots,J.$$

1. $\lim_{x^{(i)}\to 0} f(x^{(i)}) \le 0$

Noting that $\lim_{x^{(i)}\to 0} f(x^{(i)}) = -c_i \lim_{x^{(i)}\to 0} \mathbf{E}\left[\max\left((1-r)\min(R_T^{(i)}, L_i^{ob}) - L_i, 0\right)\right]$, we have $\lim_{x^{(i)}\to 0} f(x^{(i)}) \le 0$, since $-c_i\mathbf{E}\left[\max\left((1-r)\min(R_T^{(i)}, L_i^{ob}) - L_i, 0\right)\right]$ is decreasing as $x^{(i)}$ decreases to 0 and satisfies

$$0 \ge -c_i\mathbf{E}\left[\max\left((1-r)\min(R_T^{(i)}, L_i^{ob}) - L_i, 0\right)\right] \ge -c_i\mathbf{E}\left[\max\left((1-r)\min(N_T, L_i^{ob}) - L_i, 0\right)\right]. \tag{27}$$

2. $\lim_{x^{(i)}\to+\infty} f(x^{(i)}) = 0$

For the second term of $f(x^{(i)})$ in (23), since

$$0 \le \max\left((1-r)\min(R_T^{(i)}, L_i^{ob}) - L_i, 0\right) \le R_T^{(i)}, \tag{28}$$

we have

$$
\begin{aligned}
0 &\leq \mathbf{E}\left[\max((1-r)\min(R_T^{(i)}, L_i^{ob}) - L_i, 0)\right] \\
&\leq \mathbf{E}[R_T^{(i)}] \\
&= \sum_{k=0}^{\infty} \mathbf{E}[R_T^{(i)} | N_T = k] \frac{(\lambda T)^k}{k!} e^{-\lambda T} \\
&\leq \sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} m \binom{k}{m} p_i^{(\gamma\#)m} (1 - p_i^{(\gamma\#)})^{k-m} \right) \frac{(\lambda T)^k}{k!} e^{-\lambda T} \\
&= e^{-\lambda T} \sum_{m=0}^{\infty} m p_i^{(\gamma\#)m} \left(\sum_{k=m}^{\infty} \binom{k}{m} (1 - p_i^{(\gamma\#)})^{k-m} \frac{(\lambda T)^k}{k!} \right) \\
&= e^{-\lambda T} \sum_{m=0}^{\infty} m p_i^{(\gamma\#)m} \left(\sum_{k'=0}^{\infty} \binom{k'+m}{m} (1 - p_i^{(\gamma\#)})^{k'} \frac{(\lambda T)^{k'+m}}{(k'+m)!} \right) \\
&= e^{-\lambda T} \sum_{m=0}^{\infty} m p_i^{(\gamma\#)m} \frac{(\lambda T)^m}{m!} \left(\sum_{k'=0}^{\infty} \frac{(\lambda T(1 - p_i^{(\gamma\#)}))^{k'}}{k'!} \right) \\
&= e^{-\lambda T} e^{\lambda T(1 - p_i^{(\gamma\#)})} \sum_{m=0}^{\infty} m p_i^{(\gamma\#)m} \frac{(\lambda T)^m}{m!} \\
&= \lambda T p_i^{(\gamma\#)} e^{-\lambda T p_i^{(\gamma\#)}} \sum_{m'=0}^{\infty} \frac{(\lambda T p_i^{(\gamma\#)})^{m'}}{m'!} \\
&= \lambda T p_i^{(\gamma\#)}.
\end{aligned} \tag{29}
$$

Hence,

$$
\lim_{x^{(i)} \to +\infty} \mathbf{E}\left[\max\left((1-r)\min(R_T^{(i)}, L_i^{ob}) - L_i, 0\right)\right] = 0. \tag{30}
$$

Similarly, for the first term of $f(x^{(i)})$ in (23), since

$$
0 \leq \mathbf{E}\left[\min\left(L_i, (1-r)\min(R_T^{(i)}, L_i^{ob})\right) x^{(i)}\right] \leq \mathbf{E}\left[R_T^{(i)} x^{(i)}\right] \tag{31}
$$

and

$$
\lim_{x^{(i)} \to \infty} \mathbf{E}\left[R_T^{(i)} x^{(i)}\right] \leq \lim_{x^{(i)} \to \infty} \lambda T x^{(i)} p_i^{(\gamma\#)} = 0, \tag{32}
$$

where the last equation will be shown later in the case of the mixed logit model (3), we have

$$
\lim_{x^{(i)} \to \infty} \mathbf{E}\left[\min\left(L_i, (1-r)\min(R_T^{(i)}, L_i^{ob})\right) x^{(i)}\right] = 0. \tag{33}
$$

Hence

$$
\lim_{x^{(i)} \to +\infty} f(x^{(i)}) = 0. \tag{34}
$$

To show (32) in the case of the mixed logit model in (3), noting that

$$p_i^{(\gamma\#)} x^{(i)} = \int_0^\infty x^{(i)} \frac{e^{-\beta x^{(i)}+\alpha_i}}{\sum_{j=1}^J e^{-\beta x^{(j)}+\alpha_j}} h(\beta) d\beta, \tag{35}$$

and for $\beta > 0$, setting

$$\begin{aligned} x^{(*)} &= \max_{j \neq i} x^{(j)}, \\ C &\equiv \sum_{j \neq i} e^{(\alpha_j - \alpha_i)} > 0, \end{aligned} \tag{36}$$

we have

$$\begin{aligned} \frac{x^{(i)}}{1 + \sum_{j \neq i} e^{-\beta(x^{(j)} - x^{(i)}) + (\alpha_j - \alpha_i)}} &\leq \frac{x^{(i)}}{1 + \sum_{j \neq i} e^{\beta(x^{(i)} - x^{(*)}) + (\alpha_j - \alpha_i)}} \\ &= \frac{x^{(i)}}{1 + \left(\sum_{j \neq i} e^{(\alpha_j - \alpha_i)}\right) e^{\beta(x^{(i)} - x^{(*)})}} \\ &= \frac{x^{(i)}}{1 + C e^{\beta(x^{(i)} - x^{(*)})}} \\ &= \frac{x^{(i)} - x^{(*)}}{1 + C e^{\beta(x^{(i)} - x^{(*)})}} + \frac{x^{(*)}}{1 + C e^{\beta(x^{(i)} - x^{(*)})}} \\ &\leq \frac{x^{(i)} - x^{(*)}}{C e^{\beta(x^{(i)} - x^{(*)})}} + x^{(*)} \\ &\leq \frac{1}{Ce\beta} + x^{(*)}. \end{aligned} \tag{37}$$

With the assumption that

$$\int_0^\infty \frac{1}{\beta} h(\beta) d\beta < \infty, \tag{38}$$

we have

$$\lim_{x^{(i)} \to \infty} p_i^{(\gamma\#)} x^{(i)} = 0 \tag{39}$$

by the dominated convergence theorem.

3. The existence of $\bar{x}^{(i)} \in (0, \infty)$ such that $f(\bar{x}^{(i)}) > 0$

To prove (26), it suffices to show that

$$\lim_{x^{(i)} \to \infty} \frac{\mathbf{E}\left[\max\left((1-r)\min(R_T^{(i)}, L_i^{ob}) - L_i, 0\right) c_i\right]}{\mathbf{E}\left[\min\left(L_i, (1-r)\min(R_T^{(i)}, L_i^{ob})\right) x^{(i)}\right]} = 0.$$

This follows from the fact that

$$\begin{aligned}\frac{\mathbf{E}\left[\max\left((1-r)\min(R_T^{(i)},L_i^{ob})-L_i,0\right)c_i\right]}{\mathbf{E}\left[\min\left(L_i,(1-r)\min(R_T^{(i)},L_i^{ob})\right)x^{(i)}\right]} &\le \frac{c_i\mathbf{E}\left[R_T^{(i)}\right]}{x^{(i)}(1-\bar{r})\mathbf{P}\left(R_T^{(i)}=1\right)}\\ &\le \frac{c_i\lambda T p_i^{(\gamma^{\#})}}{x^{(i)}(1-\bar{r})\frac{\lambda T p_i^{(\gamma^*)}}{e^{\lambda T p_i^{(\gamma^*)}}}}\\ &\le \frac{c_i e^{\lambda T}p_i^{(\gamma^{\#})}}{(1-\bar{r})x^{(i)}p_i^{(\gamma^*)}} \to 0 \ (x^{(i)}\to\infty). \end{aligned} \tag{40}$$

Here, we used

$$\begin{aligned}\mathbf{P}\left(R_T^{(i)}=1\right) &= \sum_{k=0}^{\infty}\mathbf{P}\left(R_T^{(i)}=1|N_T=k\right)\mathbf{P}(N_T=k)\\ &\ge \sum_{k=0}^{\infty} kp_i^{(\gamma^*)}\left(1-p_i^{(\gamma^*)}\right)^{k-1}\frac{e^{-\lambda T}}{k!}(\lambda T)^k\\ &= \lambda T p_i^{(\gamma^*)}e^{-\lambda T}\sum_{k'=0}^{\infty}\left(1-p_i^{(\gamma^*)}\right)^{k'}\frac{(\lambda T)^{k'}}{k'!}\\ &= \lambda T p_i^{(\gamma^*)}e^{-\lambda T}e^{\lambda T\left(1-p_i^{(\gamma^*)}\right)}\\ &= \lambda T p_i^{(\gamma^*)}e^{-\lambda T p_i^{(\gamma^*)}}, \end{aligned} \tag{41}$$

and

$$\lim_{x^{(i)}\to\infty}\frac{p_i^{(\gamma^{\#})}}{x^{(i)}p_i^{(\gamma^*)}}=0, \tag{42}$$

which is shown particularly in the case of the mixed logit model in (3) as follows.

For $\bar{\beta}>0$,

$$\begin{aligned}\frac{p_i^{(\gamma^{\#})}}{x^{(i)}p_i^{(\gamma^*)}} &= \frac{\int_0^\infty \frac{1}{1+e^{-\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}}h(\beta)d\beta}{x^{(i)}\int_0^\infty \frac{1}{1+\sum_{j\neq i}e^{-\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}}h(\beta)d\beta}\\ &\le \frac{\int_0^{\bar{\beta}} \frac{1}{1+e^{-\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}}h(\beta)d\beta}{x^{(i)}\int_0^{\bar{\beta}} \frac{1}{1+\sum_{j\neq i}e^{-\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}}h(\beta)d\beta} + \frac{\int_{\bar{\beta}}^\infty \frac{1}{1+e^{-\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}}h(\beta)d\beta}{x^{(i)}\int_0^{\bar{\beta}} \frac{1}{1+\sum_{j\neq i}e^{-\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}}h(\beta)d\beta}. \end{aligned} \tag{43}$$

For the first term in the right hand side of (43), for $0 \leq \beta \leq \bar{\beta}$,

$$
\begin{aligned}
&\frac{1}{1+e^{-\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}} \\
&= \frac{1}{1+\sum_{j\neq i} e^{-\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}} \frac{1+\sum_{j\neq i} e^{-\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}}{1+e^{-\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}} \\
&= \frac{1}{1+\sum_{j\neq i} e^{-\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}} \left(1+\frac{\sum_{j\neq i,J} e^{-\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}}{1+e^{-\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}}\right) \\
&= \frac{1}{1+\sum_{j\neq i} e^{-\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}} \left(1+\frac{e^{-\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}}{1+e^{-\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}} \sum_{j\neq i,J} e^{-\beta(x^{(j)}-x^{(J)})+(\alpha_j-\alpha_J)}\right) \\
&\leq \frac{1}{1+\sum_{j\neq i} e^{-\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}} \left(1+\left(\sum_{j\neq i,J} e^{(\alpha_j-\alpha_J)}\right) \max_{j\neq i,J,0\leq\beta\leq\bar{\beta}} e^{-\beta(x^{(j)}-x^{(J)})}\right). \qquad (44)
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\int_0^{\bar{\beta}} \frac{1}{1+e^{-\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}} h(\beta)d\beta \\
&\leq \left(\int_0^{\bar{\beta}} \frac{1}{1+\sum_{j\neq i} e^{-\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}} h(\beta)d\beta\right) \left(1+\left(\sum_{j\neq i,J} e^{(\alpha_j-\alpha_J)}\right) \max_{j\neq i,J,0\leq\beta\leq\bar{\beta}} e^{-\beta(x^{(j)}-x^{(J)})}\right) \\
&\qquad (45)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\int_0^{\bar{\beta}} \frac{1}{1+e^{-\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}} h(\beta)d\beta}{x^{(i)} \int_0^{\bar{\beta}} \frac{1}{1+\sum_{j\neq i} e^{-\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}} h(\beta)d\beta} &\leq \frac{1}{x^{(i)}} \left(1+\left(\sum_{j\neq i,J} e^{(\alpha_j-\alpha_J)}\right) \max_{j\neq i,J,0\leq\beta\leq\bar{\beta}} e^{-\beta(x^{(j)}-x^{(J)})}\right) \\
&\to 0 \ (x^{(i)} \to \infty). \qquad (46)
\end{aligned}
$$

For the second term in the right hand side of (43), for $\beta \geq \bar{\beta}$ and $x^{(i)} > \max_{j\neq i} x^{(j)}$, since

$$
\frac{1}{1+e^{-\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}} \leq \frac{1}{1+e^{-\bar{\beta}(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}}, \qquad (47)
$$

we have

$$
\int_{\bar{\beta}}^{\infty} \frac{1}{1+e^{-\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}} h(\beta)d\beta \leq \frac{1}{1+e^{-\bar{\beta}(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}}. \qquad (48)
$$

Hence,

$$\frac{\int_{\bar\beta}^{\infty}\frac{1}{1+e^{-\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}}h(\beta)d\beta}{x^{(i)}\int_0^{\bar\beta}\frac{1}{1+\sum_{j\neq i}e^{-\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}}h(\beta)d\beta} \le \frac{1}{x^{(i)}\int_0^{\bar\beta}\frac{1+e^{-\bar\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}}{1+\sum_{j\neq i}e^{-\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}}h(\beta)d\beta}$$
$$\le \frac{1}{x^{(i)}\int_0^{\bar\beta}\frac{1+e^{-\bar\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}}{1+\sum_{j\neq i}e^{-\bar\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}}h(\beta)d\beta}$$
$$= \frac{1}{x^{(i)}\left(\int_0^{\bar\beta}h(\beta)d\beta\right)\frac{1+e^{-\bar\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}}{1+\sum_{j\neq i}e^{-\bar\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}}}. \tag{49}$$

Since

$$\lim_{x^{(i)}\to\infty}\frac{1+e^{-\bar\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}}{1+\sum_{j\neq i}e^{-\bar\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}} = \frac{e^{-\bar\beta x^{(J)}+(\alpha_J-\alpha_i)}}{\sum_{j\neq i}e^{-\bar\beta x^{(j)}+(\alpha_j-\alpha_i)}}, \tag{50}$$

we have

$$\frac{\int_{\bar\beta}^{\infty}\frac{1}{1+e^{-\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}}h(\beta)d\beta}{x^{(i)}\int_0^{\bar\beta}\frac{1}{1+\sum_{j\neq i}e^{-\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}}h(\beta)d\beta} \le \frac{1}{x^{(i)}\left(\int_0^{\bar\beta}h(\beta)d\beta\right)\frac{1+e^{-\bar\beta(x^{(J)}-x^{(i)})+(\alpha_J-\alpha_i)}}{1+\sum_{j\neq i}e^{-\bar\beta(x^{(j)}-x^{(i)})+(\alpha_j-\alpha_i)}}}$$
$$\to 0 \ (x^{(i)}\to\infty). \tag{51}$$

Therefore,

$$\lim_{x^{(i)}\to\infty}\frac{p_i^{(\gamma^{\#})}}{x^{(i)}p_i^{(\gamma^*)}} = 0. \tag{52}$$

4. The continuity of $f(x^{(i)})$

We first show the continuity of the first term of $f(x^{(i)})$ in (23).
Set

$$g(x^{(i)}) = \mathbf{E}\left[\min(L_i,(1-r)\min(R_T^{(i)},L_i^{ob}))\right], \tag{53}$$

$$g_N(x^{(i)}) = \sum_{k=0}^{N}\min(L_i,(1-r)\min(k,L_i^{ob}))\mathbf{P}(R_T^{(i)}=k). \tag{54}$$

Then,

$$\lim_{N\to\infty}\sup_{x^{(i)}\in(0,\infty)}|g_N(x^{(i)})-g(x^{(i)})| = 0, \tag{55}$$

since

$$\sup_{x^{(i)}\in(0,\infty)}|g_N(x^{(i)})-g(x^{(i)})| \le \sup_{x^{(i)}\in(0,\infty)}\sum_{k=N+1}^{\infty}k\mathbf{P}(R_T^{(i)}=k)$$
$$= \sup_{x^{(i)}\in(0,\infty)}\mathbf{E}\left[R_T^{(i)}1_{\{R_T^{(i)}\ge N+1\}}\right]$$
$$\le \mathbf{E}\left[N_T 1_{\{N_T\ge N+1\}}\right]\to 0,\ (N\to\infty). \tag{56}$$

Hence, $g(x^{(i)})$ is a continuous function, since the continuous function $g_N(x^{(i)})$ converges to $g(x^{(i)})$ uniformly in $x^{(i)} \in (0, \infty)$, and the continuity of $x^{(i)}g(x^{(i)})$ follows.

In the same manner, it follows that the second term of $f(x^{(i)})$ in (23),

$$\mathbf{E}\left[-\max((1-r)\min(R_T^{(i)}, L_i^{ob}) - L_i, 0)c_i\right] \tag{57}$$

is a continuous function with respect to $x^{(i)}$. Hence, $f(x^{(i)})$ is continuous on $(0, \infty)$.

Finally, for fixed $x^{(i)} \in (0, \infty)$,

$$\mathbf{E}\left[\min\left(L_i, (1-r)\min(R_T^{(i)}, L_i^{ob})\right)x^{(i)} - \max\left((1-r)\min(R_T^{(i)}, L_i^{ob}) - L_i, 0\right)c_i\right] \tag{58}$$

is strictly decreasing for $L_i^{ob} \geq \frac{L_i}{1-\bar{r}}$.

In fact, for $L_i^{ob} \geq \frac{L_i}{1-\bar{r}}$, if L_i^{ob} increases from $\bar{L}_i^{ob}$ to $\bar{L}_i^{ob}+1$, only the probabilities of $R_T^{(i)} = \bar{L}_i^{ob}$ and $R_T^{(i)} = \bar{L}_i^{ob}+1$ change. That is, for selection orders τ that contain $\bar{L}_i^{ob}+1$ of i, the value of $R_T^{(i)}$ changes from $\bar{L}_i^{ob}$ to $\bar{L}_i^{ob}+1$, and the probability for such selection orders changes from 0 to some positive value. For the first term $\mathbf{E}\left[\min\left(L_i, (1-r)\min(R_T^{(i)}, L_i^{ob})\right)x^{(i)}\right]$, since $\min\left(L_i, (1-r)\min(R_T^{(i)}, L_i^{ob})\right)x^{(i)} = L_i x^{(i)}$, it is unchanged. For the second term $\mathbf{E}\left[-\max\left((1-r)\min(R_T^{(i)}, L_i^{ob}) - L_i, 0\right)c_i\right]$, since only for such selection orders τ, $-\max\left((1-r)\min(R_T^{(i)}, L_i^{ob}) - L_i, 0\right)c_i$ decreases from
$-\left((1-r)\bar{L}_i^{ob} - L_i\right)c_i$ to $-\left((1-r)(\bar{L}_i^{ob}+1) - L_i\right)c_i$, and the probability changes from 0 to some positive value, it is strictly decreasing, and hence (58) is strictly decreasing for $L_i^{ob} \geq \frac{L_i}{1-\bar{r}}$. Hence, (58) has a maximum at some $L_i^{ob} \in [L_i, \infty)$.

Therefore,

$$\max_{L_i^{ob} \geq L_i} \max_{x^{(i)} \in (0,\infty)} \mathbf{E}\left[\min\left(L_i, (1-r)\min(R_T^{(i)}, L_i^{ob})\right)x^{(i)} - \max\left((1-r)\min(R_T^{(i)}, L_i^{ob}) - L_i, 0\right)c_i\right]. \tag{59}$$

is attained at some $(x_*^{(i)}, L_{i*}^{ob})$ in $(0, \infty) \times [L_i, \infty)$, which is also a maximum point for the objective function in (16). $\square$

B Proof of Theorem 2

Let

$$f(x_1) = x_1 p_1 = \frac{x_1}{1 + e^{\beta(x_1 - x_2) + \alpha_2 - \alpha_1}}, \tag{60}$$

$$g(x_2) = x_2 p_2 = \frac{x_2}{1 + e^{\beta(x_2 - x_1) + \alpha_1 - \alpha_2}}. \tag{61}$$

Since $\lim_{x_1 \to 0} f(x_1) = 0, \lim_{x_1 \to \infty} f(x_1) = 0$ and $f(x_1) \geq 0$ on $(0, \infty)$, $f(x_1)$ attains its maximum at some x_1 that satisfies $f'(x_1) = 0$.

Noting that

$$f'(x_1) = \frac{1 + e^{\beta(x_1 - x_2) + \alpha_2 - \alpha_1} - \beta x_1 e^{\beta(x_1 - x_2) + \alpha_2 - \alpha_1}}{\left(1 + e^{\beta(x_1 - x_2) + \alpha_2 - \alpha_1}\right)^2}, \tag{62}$$

we have

$$e^{\beta x_2} = (\beta x_1 - 1)e^{\beta x_1 + \alpha_2 - \alpha_1}, \tag{63}$$

and

$$\begin{aligned} x_2 &= \frac{1}{\beta}\log(\beta x_1 - 1) + x_1 + \frac{\alpha_2 - \alpha_1}{\beta} \\ &= h(x_1). \end{aligned} \tag{64}$$

Since $h : (\frac{1}{\beta}, \infty) \to (-\infty, \infty)$ is strictly increasing and hence bijective, there exists $h^{-1} : (-\infty, \infty) \to (\frac{1}{\beta}, \infty)$ such that

$$x_1 = h^{-1}(x_2). \tag{65}$$

Similarly, $g(x_2)$ attains its maximum at x_2 satisfying

$$e^{\beta x_1} = (\beta x_2 - 1)e^{\beta x_2 + \alpha_1 - \alpha_2} \tag{66}$$

or equivalently

$$\begin{aligned} x_1 &= \frac{1}{\beta}\log(\beta x_2 - 1) + x_2 + \frac{\alpha_1 - \alpha_2}{\beta} \\ &= i(x_2), \end{aligned} \tag{67}$$

and $i : (\frac{1}{\beta}, \infty) \to (-\infty, \infty)$ has the inverse $i^{-1} : (-\infty, \infty) \to (\frac{1}{\beta}, \infty)$ such that

$$x_2 = i^{-1}(x_1). \tag{68}$$

The equilibrium point is a solution of the simultaneous equations,

$$e^{\beta x_2} = (\beta x_1 - 1)e^{\beta x_1 + \alpha_2 - \alpha_1} \tag{69}$$

$$e^{\beta x_1} = (\beta x_2 - 1)e^{\beta x_2 + \alpha_1 - \alpha_2}. \tag{70}$$

Substituting (69) for (70), we have

$$1 = (\beta x_2 - 1)(\beta x_1 - 1), \tag{71}$$

and

$$1 = \left(\log(\beta x_1 - 1) + \beta x_1 + \alpha_2 - \alpha_1 - 1\right)(\beta x_1 - 1). \tag{72}$$

Setting $\beta x_1 - 1 = v$ and $\beta x_2 - 1 = w$, we have $v, w > 0$,

$$w = \frac{1}{v}, \tag{73}$$

and

$$1 = \left(\log v + v + \alpha_2 - \alpha_1\right) v. \tag{74}$$

Since the right hand side of (74) is strictly increasing, the simultaneous equations (69) and (70) have a unique solution $(x_*^{(1)}, x_*^{(2)}) \in (\frac{1}{\beta}, \infty)^2$.

Finally, we show that for any $(x_0^{(1)}, x_0^{(2)}) \in [0, \infty)^2$ and $F : [0, \infty)^2 \to [0, \infty)^2$ such that $F(x_1, x_2) = (h^{-1}(x_2), i^{-1}(x_1))$, $\{(x_n^{(1)}, x_n^{(2)})\}_{n \in \mathbf{N}}$ defined by $(x_{n+1}^{(1)}, x_{n+1}^{(2)}) = F(x_n^{(1)}, x_n^{(2)})$ converges to $(x_*^{(1)}, x_*^{(2)})$ as $n \to \infty$.

Since for $h : (\frac{1}{\beta}, \infty) \to (-\infty, \infty)$,

$$\begin{aligned}\frac{dh(x_1)}{dx_1} &= \frac{\beta}{\beta(\beta x_1 - 1)} + 1 \\ &= \frac{\beta x_1}{\beta x_1 - 1} \neq 0,\end{aligned} \tag{75}$$

we have for $h^{-1} : (-\infty, \infty) \to (\frac{1}{\beta}, \infty)$,

$$\frac{dh^{-1}(x_2)}{dx_2} = \frac{\beta x_1 - 1}{\beta x_1} = \frac{\beta h^{-1}(x_2) - 1}{\beta h^{-1}(x_2)}, \tag{76}$$

and

$$0 < \frac{dh^{-1}(x_2)}{dx_2} < 1. \tag{77}$$

Consider a ball $B_R = \{(x_1, x_2) | |(x_1, x_2) - (x_*^{(1)}, x_*^{(2)})| \leq R\}$ with the radius $R > 0$ and the center $(x_*^{(1)}, x_*^{(2)})$. Here, we take $R > 0$ so that $(x_0^{(1)}, x_0^{(2)}) \in B_R$. We show that F restricted on B_R is a contraction map from $B_R \cap [0, \infty)^2$ to $B_R \cap [0, \infty)^2$.

For any $(x_1, x_2) \in B_R \cap [0, \infty)^2$, $F(x_1, x_2) = (h^{-1}(x_2), i^{-1}(x_1))$ and

$$\begin{aligned}|h^{-1}(x_2) - x_*^{(1)}| &= |h^{-1}(x_2) - h^{-1}(x_*^{(2)})| \\ &\leq \left| \int_{x_*^{(2)}}^{x_2} \frac{dh^{-1}(x_2)}{dx_2} dx_2 \right| \\ &\leq |x_2 - x_*^{(2)}|.\end{aligned} \tag{78}$$

Similarly,

$$\begin{aligned}|i^{-1}(x_1) - x_*^{(2)}| &= |i^{-1}(x_1) - i^{-1}(x_*^{(1)})| \\ &\leq \left| \int_{x_*^{(1)}}^{x_1} \frac{di^{-1}(x_1)}{dx_1} dx_1 \right| \\ &\leq |x_1 - x_*^{(1)}|.\end{aligned} \tag{79}$$

Hence, $F(x_1, x_2) \in B_R \cap [0, \infty)^2$.

Next, noting that $|(x_1, x_2)| \leq R + |(x_*^{(1)}, x_*^{(2)})|$ for all $(x_1, x_2) \in B_R \cap [0, \infty)^2$, from (76), we have

$$0 < \frac{dh^{-1}(x_2)}{dx_2} < 1 - \frac{1}{\beta(R + |(x_*^{(1)}, x_*^{(2)})|)}, \tag{80}$$

and similarly

$$0 < \frac{di^{-1}(x_1)}{dx_1} < 1 - \frac{1}{\beta(R + |(x_*^{(1)}, x_*^{(2)})|)}. \tag{81}$$

Hence, taking $0 < r_R = 1 - \frac{1}{\beta(R+|(x_*^{(1)},x_*^{(2)})|)} < 1$, we have $|F(x_1, x_2) - F(\bar{x}_1, \bar{x}_2)| \leq r_R|(x_1, x_2) - (\bar{x}_1, \bar{x}_2)|, \forall (x_1, x_2), (\bar{x}_1, \bar{x}_2) \in B_R \cap [0, \infty)^2$, which implies that F is a contraction map from $B_R \cap [0, \infty)^2$ to $B_R \cap [0, \infty)^2$.

For $(x_0^{(1)}, x_0^{(2)}) \in B_R \cap [0, \infty)^2$, define $\{(x_n^{(1)}, x_n^{(2)})\}_{n \in \mathbf{N}}$ by

$$(x_{n+1}^{(1)}, x_{n+1}^{(2)}) = F(x_n^{(1)}, x_n^{(2)}). \tag{82}$$

Since F is a contraction map, there exists $(x_\infty^{(1)}, x_\infty^{(2)}) \in B_R \cap [0, \infty)^2$ such that

$$\lim_{n \to \infty} (x_n^{(1)}, x_n^{(2)}) = (x_\infty^{(1)}, x_\infty^{(2)}). \tag{83}$$

Noting that F is a continuous map, taking the limit $n \to \infty$ on

$$(x_{n+1}^{(1)}, x_{n+1}^{(2)}) = F(x_n^{(1)}, x_n^{(2)}), \tag{84}$$

we have

$$(x_\infty^{(1)}, x_\infty^{(2)}) = F(x_\infty^{(1)}, x_\infty^{(2)}), \tag{85}$$

which implies that $(x_\infty^{(1)}, x_\infty^{(2)})$ is the equilibrium point, and therefore $(x_\infty^{(1)} x_\infty^{(2)}) = (x_*^{(1)}, x_*^{(2)})$.
$\square$