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A Two Stage Model of Assignment and Market

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Abstract. This paper studies a two stage economy where the non-monetary assignments of indivisible objects are followed by market transactions. In this economy, there are finitely many players and finitely many types of indivisible objects and one divisible good called money. Every player demands at most one object besides money. The first stage is governed by a non-monetary assignment mechanism, while the second stage is governed by the market. We impose the obtainability condition on the first stage mechanism, which requires that each player has an option to obtain any unassigned object. This condition is satisfied by a broad class of mechanisms, including the Boston mechanism and deferred acceptance algorithm. We define an equilibrium concept called perfect market equilibrium (PME) and its refined concept. We then analyze three classes of situations, the case with abundant money, the case where some players (e.g., firms) cannot obtain objects (e.g., degree) in the first stage, waiting for some other players (e.g., students) obtain them and trade the objects with them in the future, and the third case with no money. We set forth some sufficient conditions under which existence and efficiency are guaranteed and compare the three situations in terms of these conditions.

Keywords: two stage economy, assignment mechanism, market, indivisible object, perfect market equilibrium

JEL Classification Numbers: C78, D41, D47, D51

"Laura had always been a pioneer girl rather than a farmer's daughter, always moving on to new places before the fields grew large." –Laura Ingalls Wilder, "First four years"[1]

1. Introduction

This paper studies a two stage model where the non-monetary assignments of indivisible objects are followed by market transactions. This model is related to the following couple of situations. First, consider a problem of college admission where students select a college to be admitted. They do so strategically, taking into account their future job prospects, rather than truthfully expressing their intrinsic preferences such as their love for campus. Next, consider the following historical case of the United States: after Homestead Act was

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[1]See Wilder (1973).

enacted in 1862, pioneers of the great prairie obtained a piece of land (160 acres) in return for living there and cultivating the land. After acquisition, the lands were freely traded in the market. Also, imagine several firms that strategically develop new technology to obtain patents. The intellectual property rights are assigned to firms on the first-come-first-served basis. After the acquisition of the rights, they can sell their patents to other firms or keep them and commercialize the invented technology in the market. The fourth example is a situation in which office spaces in a newly constructed building are alloted to faculty members. In the first stage, the department assigns rooms to the members based on some predetermined rule. Then, the members are allowed to exchange their rooms afterwards.

A common observation in the above cases is that there are two stages, an assignment stage and a market stage, and that assignment through a formal or informal non-price mechanism of the first stage affects and is affected by what people obtain in the subsequent market, and therefore, players therein would choose objects to obtain in the assignment stage in a strategic manner, taking into account their prospects in the market stage. This observation raises a number of questions: under what conditions does equilibrium exist? when does the first stage assignment matter in terms of efficiency? what do we miss if the assignment stage and the market stage are separately analyzed?

In order to examine these questions, we construct a two-stage model. In this model, there are finitely many players and finitely many types of indivisible objects and one divisible good called money. Every player demands at most one object besides money. Also, players have different priorities at each object type in the first stage. Each object has a limited amount of capacity, called quota. Each player has a quasi-linear utility function.

The first stage is governed by a non-monetary assignment mechanism. A mechanism is a pair of the set of strategy profiles and an assignment rule. Given a mechanism, players simultaneously choose strategies to obtain one unit of some object. Then, the mechanism chooses an allocation based on the selected strategy profile and priorities. That is if the number of the players who choose a certain object type exceeds its quota, then the players with top priority will obtain the objects up to the quota; otherwise, the objects are alloted to all the players who choose it. In the analysis, we do not assume any specific mechanism, but put conditions that mechanisms should satisfy. The conditions we require are so broad that in most of the analysis, both the Boston mechanism and the deferred acceptance algorithm satisfy them. Throughout this paper, we assume that the mechanism satisfies a condition called *obtainability*, which requires that every player has a strategy to obtain an object if there is an unassigned object that is available to the player under some strategy profile.

The second stage is governed by the market. The players are endowed with the objects assigned in the first stage as well as money. The priority no longer matters in the second stage. The players can trade objects as a price taker. Each player's payoff is determined by the indivisible good and the money held at the end of the second stage. In particular, what they obtain in the first stage matter only to the extent that it affects the final allocation in the second stage.

We introduce an equilibrium concept called perfect market equilibrium (PME) to analyze these situations. PME requires that a market equilibrium be realized in each market of the second stage, and that each player selects an optimal strategy in the mechanism, taking into consideration what will happen in the second stage. In order to capture players' incentives in the first stage, we define an induced game, where the payoff of each strategy profile is defined by the corresponding market equilibrium outcome. PME is a Nash equilibrium in this induced game.

We also consider a refined concept of PME, Permutation independent PME (PIPME). It requires that the price profiles of the second stage are the same between two initial endowment profiles as long as their total endowments are the same. This concept reflects the idea of anonymity and (partial) price-taking behavior. It reflects, in addition to perfection, the idea of anonymity as changes in object holders would not change the price system as long as the total endowments are unchanged. It also reflects the idea of price taking behavior in the sense that even if one changes his/her strategy in the assignment stage, it would not affect the second stage price system as long as the total endowments are unchanged. Indeed, in many mechanisms, including the Boston mechanism and the deferred acceptance algorithm, if there are a sufficient number of players who take undominated strategies, a unilateral deviation would not affect the total amount of objects available for the second stage market. Our interest resides in the allocation of indivisible objects and money in PME as well as PIPME.

In the analysis, two criteria are used to evaluate the allocation in PME: one is Pareto optimality, and the other is efficiency. If an allocation is Pareto optimal, then there is no allocation where all the players weakly prefer and at least one player strictly prefers to this allocation. An efficient allocation maximizes the social welfare, which is equal to the sum of the players' utility values. We also introduce ω-optimality and ω-efficiency given an endowment ω of the second stage, which correspond to optimality and efficiency, respectively. Given ω, an ω-optimal (resp. ω-efficient) allocation is not necessarily an optimal (resp. efficient) allocation, especially when there are some unassigned objects in the first stage.

With this two-stage model and solution concepts, we analyze three types of situations. The first type of situation, analyzed in Section 3 is the one in which players have abundant money. This situation corresponds to an assignment stage followed by monetary transactions.

The existence of market equilibrium in the second stage is guaranteed as shown by Gale (1984). Therefore, PME always exists in this case. We show the efficiency and the uniqueness of PME object allocation provided that every object is *scarce*, or there is a sufficient demand for every object. Its proof is an application of the first fundamental theorem of welfare economics.[2] The assumption of scarcity, however, is essential to the results. We discuss an example where efficiency and uniqueness do not hold due to the lack of scarcity. If the scarcity assumption is violated, an $\omega-$ efficient allocation in the market may not coincide with the unique efficient allocation in the two-stage economy.

The second type of situation is analyzed in Section 4. It is the one in which the buyers and the sellers of the objects in the second stage market are inherently separated. In other words, the players who obtain indivisible objects in the first stage will turn to be their sellers in the subsequent market. Students choose a college to be admitted, while firms

[2]See, e.g., Mas-Colell, Whinston, Green, et al. (1995).

hire a student based on the college he/she graduates from. No student hopes to stay in the college forever, and all of them enter the labor market as a seller after their graduation, trading their labor backed by their degrees with firms that evaluate the skills attached to the college degrees. Students strategically choose a college, taking into account their future job prospects.

In this application, we show that PME always exists as market equilibrium does provided the scarcity of objects for both students and firms. Also, we show that PME achieves an efficient and unique object allocation. In addition to this, we show that there is a strict order of positive prices on object types in the second stage. This implies that students preferences happen to be same as sorted by strictly positive prices because students does not feel any value on object holding. Then, the priority structure determines which player can owns what objects in the first stage. In the second stage, all the students with an object in the first stage can sell his/her owning to firms. And, object allocation in the market becomes efficient and unique because firms' monetary endowment assures that objects are exchanged from a student to a firm who values it.

The third type of situation, analyzed in Section 5, is an assignment stage followed by exchanges where there is no money, or monetary transactions are considered inappropriate.

The existence of market equilibrium in the second stage is not guaranteed unless various conditions are met. For example, if the quota of some object exceeds one, market equilibrium may not exist. This also implies that the existence of PME is not guaranteed unless the quota of each object type is limited to one. On the other hand, Pareto optimality of PME does not require even scarcity on condition that it exists. Again, the proof for optimality is an application of that of the first fundamental theorem of welfare economics.

To examine the relationships between the perfection of players and the first stage mechanism, we relate PME to stability. An allocation is said to be stable if every player prefers his/her assignment to any object that is held by another player whose priority is lower than the player in question and to any unassigned object. We introduce another equilibrium concept, called stable market equilibrium (SME) for the analysis with no money. It imposes stability on the market equilibrium object allocation. SME, unlike PME, considers neither the incentive to deviate in the first stage nor off-the-equilibrium outcomes, and therefore, it is much easier to construct SME than PME. We show that if there is no priority cycle defined by Ergin (2002), then together with some other conditions, SME exists, and there exists a PME of which object allocation is the same as that of SME.

Since we analyze a two-stage model that consists of non-monetary assignments in the first stage and market transactions in the second, our analysis is based on a variety of existing literature even if we limit our attention to the papers that are directly related to the present one.

The present model closely follows the literature on assignment problems. The college admissions problem is adopted from Gale and Shapley (1962) and Roth and Sotomayor (1989). Sotomayor (2008) formulates a game form and define a Nash equilibrium to analyze stable matching mechanisms. What is new in the present paper, other than considering a two stage economy, is that we set forth the condition of obtainability, which requires that anyone can obtain any unassigned object as long as it is physically available. This condition is not so stringent that the set of mechanisms satisfying it includes serial dictatorship, the Boston mechanism, and the deferred acceptance algorithm as examples.

In the sense that similar results hold as long as the mechanism satisfies obtainability, the second stage market will minimize the difference between these mechanisms, correcting any efficiency loss through market transactions.[3]

Ergin (2002) shows that no cycle of priority, or acyclicity, is equivalent to Pareto optimality of the outcome in the deferred acceptance mechanism. This condition of acyclicity turns out to play a critical role in relating stable allocations to PME allocations.[4]

This paper is based upon some results in the existing literature on markets with indivisible goods. Shapley and Scarf (1974) shows non-emptiness of core and existence of competitive equilibrium when there is no money. We use their result directly in proving the existence of market equilibrium in the case of no money. Kaneko (1982) shows non-emptiness of core under no-transferable utility. Wako (1984) shows strong core is inside the set of competitive equilibrium and conditions under which strong core exists. Quinzii (1984) shows the existence of competitive equilibrium in an economy with indivisible goods and money. We use this result directly in stating the existence result of market equilibrium in the case of abundant money.

We consider the case of abundant money, which requires that everyone has a sufficient amount of money that matches his/her highest willingness-to-pay, and the case of no money. If money holdings are in between, i.e., some player has a positive amount of money but not as much as his/her reservation value for some goods, then we face difficulty in the proof of existence. Indeed, Gale (1984) shows that competitive equilibrium exists if, among others, demand correspondence has a closed graph. But, in the in-between case, the demand correspondence for some goods does not become closed in general, and the existence proof fails to work.[5]

If we view the second stage endowment as the assignment of property rights, then the analysis of the present paper is related to Coase's theorem (see Coase (1960)). In the present context, the theorem implies that irrespective of the assignment of property right, the market will lead to an efficient allocation. Papers related to Coase's theorem are abundant. In the present context, it is worth mentioning Demsetz (1964), which states that under smooth markets, zero pricing of scarce good does not lead to inefficiency, and Jehiel and Moldovanu (1999), which considers assignment with resale and shows that the assignment of property right is irrelevant if there are resale processes. In the present paper, if money is abundant and the scarcity condition holds, then the situation becomes a special case of Jehiel and Moldovanu (1999). In other cases, however, their presumption does not hold, and the result may not hold in general.

The rest of the paper is organized as follows. Section 2 presents a model and solution concepts as well as some preliminary results. Section 3 studies situations with abundant money. Section 4 studies a situation where the population is divided into two groups,

[3]Experimental studies may be needed to discern these mechanisms. If one confines attention on the first assignment stage, one may refer to Chen and Sönmez (2006) that compares three mechanisms, the Boston mechanism, the deferred acceptance algorithm, and top trading cycles, in an experiment.

[4]See also Kojima and Manea (2010), which takes axiomatic approaches on deferred acceptance mechanisms.

[5]To be precise, the demand correspondence does not have a closed graph if one has no money, either. However, in this particular case, we have the existence result due to Shapley and Scarf (1974) and others as we have seen.

students and firms. Section 5 studies situations with no money. Some proofs and the definitions of some mechanisms are relegated to appendices.

2. Model

We consider a two stage economy. In the first stage, players play a game to obtain objects, while in the second stage, the market opens to allocate objects and money, if any. The object allocation in the first stage is governed by a mechanism, which may or may not be a formal one, depending on applications. On the other hand, an object allocation in the second stage is determined through a pure exchange economy based on the profile of the initial endowments, objects and money. In this model, the initial object endowment profile in the second stage is the outcome of the first stage.

2.1. **Preliminaries.** N is a finite set of players. O is a finite set of objects. There is a null object, denoted ϕ. We may call ϕ an object and any a in O a tangible object whenever convenient. Let $\bar{O} = O \cup \{\phi\}$. The objects are indivisible, and each agent demands at most one unit of the object. For any $a \in O$, a has a quota $q^a \in \{1, 2, \ldots\}$. Also, let q^ϕ be any number satisfying $q^\phi > |N|$. A quota profile is denoted by $q = (q^a)_{a \in \bar{O}}$. Given a vector $\mu = (\mu_i)_{i \in N} \in \bar{O}^N$ and $a \in \bar{O}$, let $\mu^a = \{i \in N | \mu_i = a\}$ be the set of the players who hold a. An object allocation is μ that satisfies $|\mu^a| \leq q^a$ for all $a \in O$. $\mathcal{A}^+ = \{\mu \in \bar{O}^N \mid \forall a \in \bar{O}\ |\mu^a| \leq q^a\}$ is the set of all the object allocations.

In this economy, the players may have money as endowment. Let $\bar{m}_i \in \mathbb{R}_+$ be the monetary endowment of player $i \in N$. We write $\bar{m} = (\bar{m}_i)_{i \in N}$. An *allocation* is given by $x = (\mu, m) \in X \equiv \mathcal{A}^+ \times \mathbb{R}^N_+$ with $\sum_{i \in N} m_i \leq \sum_{i \in N} \bar{m}_i$. Note that there is a liquidity constraint, i.e., no player can borrow money.

For every $i \in N$, R_i is a preference relation over $\bar{O} \times \mathbb{R}_+$, the set of pairs of objects and money. $R = (R_i)_{i \in N}$ is a preference profile of the players. Let P_i denote i's strict preference over $\bar{O} \times \mathbb{R}_+$, i.e., for all x and x' in X, xP_ix' if xR_ix' and not $x'R_ix$. We write $P = (P_i)_{i \in N}$. Also, I_i ($i \in N$) is the indifference relation induced by R_i, i.e., for all x and x', xI_ix' if xR_ix' and $x'R_ix$.

We assume that the preference relation R_i ($i \in N$) is rational, i.e., complete and transitive. Let $\mathcal{R}$ (resp. $\mathcal{P}$) be the set of (resp. strict) preferences.

We measure i's value $v_i(a)$ of object a by money. Let $v_i(a)$ ($i \in N$, $a \in O$) be a number such that $(\phi, v_i(a))$ and $(a, 0)$ are indifferent, i.e., $(\phi, v_i(a))I_i(a, 0)$. We assume such a number exists. We assume that there is no income effect, i.e.,

$$\forall i \in N \forall a \in O \forall m_i \geq 0 \forall r \in \mathbb{R}\ (\phi, m_i)R_i(a, 0) \Rightarrow (\phi, m_i + r)R_i(a, r).$$

We also assume that $(\mu_i, m_i)P_i(\mu_i, m'_i)$ if and only if $m_i > m'_i$. Therefore, we can represent player i's preference by a quasi-linear utility function, i.e.,

$$u_i(a, m'_i) = v_i(a) + m'_i.$$

We assume that these values are generic unless otherwise mentioned. In particular, we assume that for all $N', N'' \subset N$, and all allocations $\mu', \mu'' \in \mathcal{A}^+$,

$$\sum_{i \in N'} v_i(\mu'_i) \neq \sum_{j \in N''} v_j(\mu''_j)$$

holds.

We use two criteria to evaluate allocations in terms of utility. One is Pareto criterion and the other is social welfare. Consider two allocations x and x' in X. We say that x *Pareto dominates* x' if for all $i \in N$, $x_i R_i x'_i$ holds, and for some $j \in N$, $x_j P_j x'_j$ holds. An allocation x is *Pareto optimal* if there is no allocation that Pareto dominates x.

The second criterion is social welfare. For each allocation (μ, m), a *social welfare* is given by $W(\mu) = \sum_{i \in N} v_i(\mu_i)$. We say that (μ, m) is *efficient* if $\mu \in \arg\max_{\mu' \in \mathcal{A}^+} W(\mu')$ holds.

For every $a \in O$, $\succeq_a$ is a total order over N at $a \in O$, i.e., it is complete, transitive, and anti-symmetry. This binary relation $\succeq_a$ induces $\succ_a$ for all $a \in O$: for all $i, j \in N$, $i \succ_a j$ holds if and only if $i \succeq_a j$ but not $j \succeq_a i$. It defines the order of players' priority at object a, i.e., $i \succ_a j$ means that i has higher priority than j at a. Let $\succ = (\succ_a)_{a \in O}$ be the priority profile at all the objects. $\mathcal{S}$ is a set of all the priority profiles. Given the set N of players, an economy $\mathcal{E}$ is denoted by $\mathcal{E} = \langle R, \succ, q, \bar{m} \rangle$.

2.2. **The first stage: Assignment.** In the first stage, the players obtain objects based on priority through a mechanism. A mechanism is a pair

$$M = \langle \Sigma, \lambda \rangle,$$

where $\Sigma = (\Sigma_i)_{i \in N}$ is the set of strategy profiles with Σ_i being the set of i's strategies and $\lambda : \Sigma \to \mathcal{A}^+$ is an outcome function. Given an economy $\mathcal{E} = \langle R, \succ, q, \bar{m} \rangle$, both Σ and λ may reflect $\succ$ and q, but neither R nor $\bar{m}$. That is, if the number of the players who choose a certain object type exceeds its quota, then the players with top priority will obtain the objects up to the quota; otherwise, the objects are alloted to all the players who choose it. Given a mechanism M, for each $i \in N$, $A_i \subset \bar{O}$ is the set of available object types for player i. An object type a is in A_i if $\lambda_i(\sigma) = a$ holds for some $\sigma \in \Sigma$. Assume that for all i in N, ϕ is in A_i. Let $\mathcal{A}$ be a subset of $\mathcal{A}^+$ such that for all $\mu \in \mathcal{A}$ and all $i \in N$, $\mu_i \in A_i$ holds. We may write $\lambda : \Sigma \to \mathcal{A}$ in the sequel.

Next, we define obtainability, which roughly states that if there is an unassigned object, each player has an option of obtaining it.

Condition 2.1. *A mechanism $M = \langle \Sigma, \lambda \rangle$ satisfies* obtainability, i.e.,

> **(Obtainability):** For all $\sigma \in \Sigma$, if $|\lambda^a(\sigma)| < q^a$ holds for some $a \in O$, then for all $i \in N$ with a in A_i, there exists $\hat{\sigma}_i \in \Sigma_i$ that satisfies $\lambda(\hat{\sigma}_i, \sigma_{-i}) = a$.

This condition is satisfied by various mechanisms, including serial dictatorship, the first-come-first-served rule (the Boston mechanism), and the deferred acceptance algorithm.

2.3. **The second stage: Market.** The players participate in the market in the second stage. To begin with, several related concepts are defined given the initial object allocation ω, which is the outcome of the first stage.

Given an initial object allocation $\omega \in \mathcal{A}$ of the second stage, an allocation $x = (\mu, m) \in X$ is *ω-feasible* if for all $a \in O$, $|\mu^a| \leq |\omega^a|$ holds. $\mathcal{A}^\omega$ denotes the set of ω-feasible allocations. Also, $O^\omega = \{a \in O \| \omega^a| > 0\}$ is the set of available objects, and $\bar{O}^\omega = O^\omega \cup \{\phi\}$.

Next, given $\omega \in \mathcal{A}$ and $a \in O$, let $|\omega^a|$ be a *total endowment* of object a in the second stage. We write a total endowment profile, or simply total endowment, $|\omega| = (|\omega^a|)_{a \in O}$. The initial endowment ω of the second stage is said to be *exhaustive* if every object is assigned to some player, i.e., $|\omega| = q$ holds. Also, an object allocation μ is *ω-exhaustive* if $|\mu| = |\omega|$ holds where $|\mu|$ is similarly defined as $|\omega|$.

Note that the quantity restriction is only on the objects in O, i.e., not on ϕ. Next, ω-Pareto optimality and ω-efficiency are defined.

Definition 2.1. *Given an initial object allocation ω of the second stage, an allocation x is* ω-Pareto optimal (ω-optimal) *if there does not exist an ω-feasible allocation x' that Pareto dominates x. Also, an allocation (μ, m) is* ω-efficient *if there does not exist an ω-feasible allocation (μ', m') such that $W(\mu') > W(\mu)$.*

The following lemma states the relationship between ω-optimality (resp. ω-efficiency) and Pareto optimality (resp. efficiency). It is a direct consequence of the respective definitions.

Lemma 2.1. *If ω is exhaustive, then an ω-optimal (resp. ω-efficient) allocation is also Pareto optimal (resp. efficient).*

Proof.
Suppose that $\omega \in \mathcal{A}$ is exhaustive. Then $\mathcal{A}^\omega = \mathcal{A}^+$ holds. Thus, the definition of ω-optimality (resp. ω-efficiency) becomes identical to that of Pareto optimality (resp. efficiency). □

The concept we use for the second stage is market equilibrium. The second stage market is given by $\langle \bar{m}, R, \omega \rangle$.

Definition 2.2. *Given a triple $\langle \bar{m}, R, \omega \rangle$, $(p, x) = (p, \mu, m) \in \mathbb{R}_+^{\bar{O}^\omega} \times \mathcal{A}^\omega \times \mathbb{R}_+^N$ is a* market equilibrium *under $\langle \bar{m}, R, \omega \rangle$ (or simply under ω if there is no confusion) if x is ω-feasible, $p_\phi = 0$, and*

(1) $\forall i \in N \ p_{\mu_i} + m_i = p_{\omega_i} + \bar{m}_i$,
(2) $\forall i \in N \ \forall a \in \bar{O}^\omega [\bar{m}_i + p_{\omega_i} \geq p_a \Rightarrow (\mu_i, m_i) R_i (a, \bar{m}_i + p_{\omega_i} - p_a)]$,
(3) $\forall a \in O^\omega \ [|\mu^a| < |\omega^a| \Rightarrow p_a = 0]$.

Note that Definition 2.2, especially $p_\phi = 0$ and (3), together with ω-feasibility implies that the objects in O are free disposal.

2.4. **The two stage economy and perfect market equilbirium.** We combine the two stages, considering an economy $\mathcal{E} = \langle R, \succ, q, \bar{m} \rangle$. First, we introduce an induced game.

Definition 2.3. *Given a mechanism $M = \langle \Sigma, \lambda \rangle$ and a profile $(p(\omega), x(\omega))_{\omega \in \mathcal{A}}$, player i's* induced payoff is

$$\tilde{u}_i(\sigma) = u_i(x(\lambda(\sigma))).$$

Given a mechanism $M = \langle \Sigma, \lambda \rangle$ and a profile $(p(\omega), x(\omega))_{\omega \in \mathcal{A}}$, an induced game Γ is a profile $\langle N, \Sigma, (\tilde{u}_i)_{i \in N} \rangle$.

Given an induced game Γ, we naturally extend the strategy space to include mixed strategies. A mixed strategy ρ_i of player $i \in N$ is a probability distribution over Σ_i, i.e.,

$$\rho_i \in \Delta(\Sigma_i) \equiv \Big\{ \rho_i : \Sigma_i \to [0,1] \mid \sum_{\sigma_i \in \Sigma_i} \rho_i(\sigma_i) = 1 \Big\}.$$

We allow mixed strategies in the definition of equilibrium. To do so, let us define the expected payoff under a mixed strategy profile ρ and a market equilibrium profile $(\mu(\omega), m(\omega))_{\omega\in\mathcal{A}}$ as follows:

$$\mathbf{E}\left[\tilde{u}_i(\cdot)|\rho\right] = \sum_{\sigma\in\Sigma}\rho(\sigma)\left[v_i(\mu_i(\lambda(\sigma))) + m_i(\lambda(\sigma))\right],$$

where $\rho(\sigma) = \Pi_{i\in N}\rho_i(\sigma_i)$ is the product of $\rho_i(\sigma_i)$ across the players.

Now, we present an equilibrium concept that reflects the idea of perfection.

Definition 2.4. *Given a mechanism* $M = \langle\Sigma, \lambda\rangle$ *and a two stage economy* $\mathcal{E} = \langle R, \succ, q, \bar{m}\rangle$, $(\rho, (p(\omega), x(\omega))_{\omega\in\mathcal{A}})$ *is a* perfect market equilibrium (PME) *if*

(1) *for all* $\omega \in \mathcal{A}$, $(p(\omega), x(\omega))$ *is a market equilibrium under* ω*;*
(2) ρ *is a Nash equilibrium of the induced game* Γ, *i.e.,*

$$\mathbf{E}\left[\tilde{u}_i(\cdot)|\rho\right] \geq \mathbf{E}\left[\tilde{u}_i(\cdot)|(\rho'_i, \rho_{-i})\right].$$

We also consider a refined concept of PME. The following concept of permutation independent PME requires that if the total endowments of the second stage are the same between the two outcomes of the first stage, then the equilibrium price vectors are the same. This reflects the idea of anonymity, i.e., changes in object holders would not change the price system as long as the total endowments are unchanged.[6]

Definition 2.5. $(\rho, (p(\omega), x(\omega))_{\omega\in\mathcal{A}})$ *is a* permutation independent perfect market equilibrium (PIPME) *if it is a PME, and for all* $\omega, \hat{\omega} \in \mathcal{A}$,

(PI): $|\omega| = |\hat{\omega}| \;\Rightarrow\; p(\omega) = p(\hat{\omega})$.

Let $Q = \sum_{a\in O} q^a$. We sometimes assume in the sequel that objects are scarce, which turns out to be essential for some of the subsequent results.

Condition 2.2.
Every object in O *is* scarce *if*

(Scarcity): *for all* $a \in O$, $|\{i \in N | v_i(a) > 0\}| > 2Q$ *holds.*

Lemma 2.2. *Assume (Scarcity). Then, we have the following:*

(1) μ *is exhaustive if an allocation* (μ, m) *is Pareto optimal;*
(2) *given* $\omega \in \mathcal{A}$, μ *is* ω*-exhaustive if an allocation* (μ, m) *is* ω*-optimal;*
(3) *given* $\omega \in \mathcal{A}$, $p_a > 0$ *holds for all* $a \in O$, *and* μ *is* ω*-exhaustive if* (p, μ, m) *is a market equilibrium under* ω.

Proof.
Assume (Scarcity) in Assumption 2.2 throughout the proof. Suppose an allocation (μ, m) is Pareto optimal. And, suppose the contrary. Then, there exists $a \in O$, $|\mu^a| < q^a$. The scarcity implies that there exists $i \in \mu^\phi$ s.t. $(a, m_i)P_i(\phi, m_i)$. We can construct

[6] This definition also reflects the idea of price taking behavior in the sense that even if one changes his/her strategy in the assignment stage, it would not affect the second stage price system as long as the total endowments are unchanged. Indeed, in many mechanisms, including the Boston mechanism and the deferred acceptance algorithm, if there are a sufficient number of players who take undominated strategies, a unilateral deviation would not affect the total amount of objects available for the second stage market.

an allocation (ν, m) where $\nu_i = a$ and $\nu_j = \mu_j$ for j other than i. Then, (ν, m) Pareto dominates (μ, m). This is a contradiction to Pareto optimality.

The second claim is verified in the same manner as the first.

As for (3), suppose the contrary, i.e., there exists a market equilibrium (p, μ, m) under ω such that for some a in O $p_a = 0$ holds. Take such object a. Since the object a is scarce, $|\{i \in N | v_i(a) > 0\}| > 2Q$ holds. This implies that there exists at least one j in μ^ϕ s.t. $v_j(a) > 0$ and $m_j = \bar{m}_j$. Therefore, $\mu_j = \phi$ and m_j does not maximize j's utility. This is a contradiction to that (p, μ, m) is a market equilibrium under ω. Once $p_a > 0$ is proven, $|\mu^a| = |\omega^a|$ immediately follows from the definition of market equilibrium (excess supply of some object implies that its price is zero). □

We now consider the incentives in the first stage and claim the following.

Lemma 2.3. *Assume (Scarcity) and (Obtainability). If $(\sigma, (p(\omega), x(\omega))_{\omega \in \mathcal{A}})$ is a PME in pure strategy, then $\lambda(\sigma)$ is exhaustive.*

Proof.
Assume (Scarcity) and (Obtainability). Let $(\sigma, (p(\omega), x(\omega))_{\omega \in \mathcal{A}}) = (\sigma, (p(\omega), \mu(\omega), m(\omega))_{\omega \in \mathcal{A}})$ be a PME. Suppose the contrary, i.e., that $|\lambda^a(\sigma)| < q^a$ holds for some $a \in O$. Take such an a. Then, (Scarcity) implies that there exists $i \in N$ such that $\lambda_i(\sigma) = \phi$, $\mu_i(\lambda(\sigma)) = \phi$, and $v_i(a) > 0$. Take such an i. Note that this player i is the one who obtains an object in O in neither stage. (Obtainability) implies that there exists $\hat{\sigma}_i \in \Sigma_i$ such that $\lambda(\hat{\sigma}_i, \sigma_{-i}) = a$ holds. Take such a $\hat{\sigma}_i$. Then $v_i(a) > 0$ implies $(\lambda_i(\hat{\sigma}_i, \sigma_{-i}), \bar{m}_i) P_i (\mu_i(\lambda(\sigma)), \bar{m}_i)$. In the second stage, we have $x_i(\lambda(\hat{\sigma}_i, \sigma_{-i})) R_i (\lambda_i(\hat{\sigma}_i, \sigma_{-i}), \bar{m}_i)$. Hence,

$$x_i(\lambda(\hat{\sigma}_i, \sigma_{-i}))\ P_i\ (\mu_i(\lambda(\sigma)), \bar{m}_i),$$

i.e., player i has an incentive to deviate and obtain a. This is a contradiction to that $(\sigma, (p(\omega), x(\omega))_{\omega \in \mathcal{A}})$ is a PME. □

3. Perfect Market Equilibrium under Abundant Money

In this section, we assume that for all $i \in N$ $A_i = \bar{O}$ holds.

We assume that everyone has a sufficient amount of money. Formally, we have the following.

Condition 3.1.
Money is abundant *(for all the players), i.e.,*

(Abundance): *for all $i \in N$, $\bar{m}_i \geq \max_{a \in \bar{O}} v_i(a)$ holds.*

If money is abundant, then under the assumption of no income effect, utility becomes transferable, and the definition of Pareto optimality (resp. ω-optimality) is reduced to that of efficiency (resp. ω-efficiency).

3.1. Existence and efficiency.

First, we have the existence result of market equilibrium in the second stage due to Quinzii (1984).

Claim 3.1. *(Quinzii (1984)) Assume (Abundance) in Assumption 3.1. Then for all $\omega \in \mathcal{A}$, there exists at least one market equilibrium under ω.*

If there exists a market equilibrium under every ω, then by assigning a market equilibrium allocation under each ω, we can construct a game for the first stage. Then PME exists in the mixed strategy profile space since the existence of PME is reduced to the existence of Nash equilibrium. Thus, the following result is stated without proof.

Theorem 3.2. *Assume (Abundance). Then, there exists at least one PME.*

The abundance of money (Abundance) is crucial to the existence result. Indeed, as we shall see in Section 5, if money is not abundant, then market equilibrium may not exist under ω, and therefore, PME may not exist, either.

If money is abundant, the quasi-linearity of the utility functions implies that the demand correspondence is independent of the initial allocation $(\omega, \bar{m})$ of the second stage, i.e., the object choice μ_i of player i is given by

$$\mu_i \in \arg\max_{a \in \bar{O}^\omega} \{v_i(a) - p_a\}. \tag{3.1}$$

Corollary 3.3. *Assume (Abundance). Suppose that $(\rho, (p(\omega), x(\omega))_{\omega \in \mathcal{A}})$ is a PME. Then there exists at least one PIPME $(\rho^*, (p^*(\omega), x^*(\omega))_{\omega \in \mathcal{A}})$.*

To prove this corollary, let us define the following. Given $\omega \in \mathcal{A}$, let

$$\Omega_\omega = \{\hat{\omega} \in \mathcal{A} \mid |\hat{\omega}| = |\omega|\}.$$

It is verified that these sets form equivalence classes. Let $\Omega = \{\Omega^1, \ldots, \Omega^L\}$ be a partition of $\mathcal{A}$, i.e., $\Omega^\ell \cap \Omega^{\ell'} = \emptyset$ for $\ell \neq \ell'$ and $\cup_{i=1}^{L} \Omega^\ell = \mathcal{A}$.

Proof.
Assume (Abundance). Suppose that $(\rho, (p(\omega), \mu(\omega), m(\omega))_{\omega \in \mathcal{A}})$ is PME. We construct a PIPME $(\rho^*, (p^*(\omega), \mu^*(\omega), m^*(\omega))_{\omega \in \mathcal{A}})$ as follows. Consider $\Omega = \{\Omega^1, \ldots, \Omega^L\}$. For each $\ell = 1, \ldots, L$, take an $\hat{\omega}^\ell \in \Omega^\ell$ in an arbitrary manner. Then for each $\ell = 1, \ldots, L$ and each $\omega \in \Omega^\ell$, let

$$\begin{aligned} p^*(\omega) &= p(\hat{\omega}^\ell), \\ \mu^*(\omega) &= \mu(\hat{\omega}^\ell), \\ m_i^* &= \bar{m}_i - p^*_{\mu_i^*(\omega)} + p^*_{\omega_i}, \ i \in N. \end{aligned}$$

Given the price $p^*(\omega)$, for each player i, the optimal object is $\mu_i(\hat{\omega}^\ell)$ since the demand correspondence does not depend on the initial endowment by Equation (3.1). Also, m_i^* is determined by player i's budget constraint. Thus, $(p^*(\omega), \mu^*(\omega), m^*(\omega))$ is a market equilibrium under ω. This completes the construction of the second stage equilibrium profile $(p^*(\omega), x^*(\omega))_{\omega \in \mathcal{A}}$ with (PI). Since the first stage strategy profile ρ^* is simply a Nash equilibrium of the induced game. This completes the proof.

□

The next lemma states that every market equilibrium is ω-efficient, and therefore, that the final object allocations in two markets with the same total initial endowment are the same even if the initial endowment profiles are different.

Lemma 3.4. *Assume (Abundance). Given any $\omega \in \mathcal{A}$, if (p, μ, m) is a market equilibrium under ω, then (μ, m) is ω-efficient. For any $\omega, \hat{\omega} \in \mathcal{A}$, if $|\omega| = |\hat{\omega}|$ holds, then, for any* *market equilibria* $(p(\omega), \mu(\omega), m(\omega))$ *under* ω *and* $(p(\hat{\omega}), \eta(\hat{\omega}), m(\hat{\omega}))$ *under* $\hat{\omega}$, $\mu(\omega) = \eta(\hat{\omega})$ *holds.*

Proof.
Assume (Abundance). First, we show ω-efficiency. Suppose the contrary, i.e., that there exists $\eta \in \mathcal{A}^\omega$ such that $W(\eta) > W(\mu)$ holds.

For every player i, (3.1) implies

$$v_i(\mu_i) - p_{\mu_i} \geq v_i(\eta_i) - p_{\eta_i}. \tag{3.2}$$

From (3.2), we have

$$p_{\eta_i} - p_{\mu_i} \geq v_i(\eta_i) - v_i(\mu_i) \tag{3.3}$$

for all $i \in N$.

By taking the summation of the both sides across $i \in N$, (3.3) implies

$$\sum_{i \in N} [p_{\eta_i} - p_{\mu_i}] \geq W(\eta) - W(\mu) > 0. \tag{3.4}$$

Therefore, we have

$$\sum_{i \in N} p_{\eta_i} > \sum_{i \in N} p_{\mu_i}. \tag{3.5}$$

Rewriting the above inequality, we have

$$\sum_{a \in O} |\eta^a| p_a > \sum_{a \in O} |\mu^a| p_a. \tag{3.6}$$

This implies that there exists an object $a \in O$ such that $|\eta^a| > |\mu^a|$ and $p_a > 0$ hold. However, $|\mu^a| < |\eta^a| \leq |\omega^a|$ implies $p_a = 0$ by the equilibrium condition. This is a contradiction.

Next, suppose that (p, μ, m) and $(\hat{p}, \hat{\mu}, \hat{m})$ are market equilibria under ω and $\hat{\omega}$, respectively, such that $|\omega| = |\hat{\omega}|$ holds. Suppose the contrary, i.e., that $\mu \neq \hat{\mu}$ holds. By the genericity of v's, we have $W(\mu) \neq W(\hat{\mu})$. This is a contradiction to what we have proven above.

□

Once we have proven the above lemmata, it is relatively straightforward to show the efficiency and uniqueness (with respect to the object allocation) of PME provided that the scarcity condition holds.

Theorem 3.5. *Assume (Obtainability), (Scarcity), and (Abundance). If* $(\sigma, (p(\omega), \mu(\omega), m(\omega))_{\omega \in \mathcal{A}})$ *is PME, then* $x(\lambda(\sigma))$ *is efficient, and therefore,* $\mu(\lambda(\sigma))$ *is unique.*

Proof.
Assume (Obtainability), (Scarcity), and (Abundance). Suppose that $(\sigma,\ (p(\omega),\ \mu(\omega),\ m(\omega))_{\omega \in \mathcal{A}})$ is a PME. Then from Lemma 3.4, $\mu(\lambda(\sigma))$ is ω-efficient with $\omega = \lambda(\sigma)$. Lemma 2.3 implies $|\lambda^a(\sigma)| = q^a$ holds for all $a \in O$. Then Lemma 2.1 implies that $(\mu(\lambda(\sigma)), m(\lambda(\sigma)))$ is efficient.

The uniqueness follows due to the genericity assumption. □

3.2. **Violation of scarcity may cause inefficiency.** Scarcity plays a critical role in some results of Subsection 3.1. If the scarcity condition (Scarcity) is violated, neither efficiency nor uniqueness of object allocation is guaranteed. To see this, we consider the following example.

i	A	B
$v_i(x)$	100	80
$v_i(y)$	40	60
$v_i(z)$	50	-10

TABLE 3.1. Values

$$B \succ_a A, \;\; a = x, y, z$$

TABLE 3.2. Priority

Let $N = \{A, B\}$ and $O = \{x, y, z\}$. Also, let the values and the priority be given by Tables 3.1 and 3.2, respectively. For example, we have $v_A(x) = 100$ and $B \succ_x A$. In this economy, there is a PME where efficiency is not attained. Note that

$$\arg\max_{\mu \in \mathcal{A}^+} [v_A(\mu_A) + v_B(\mu_B)] \;\; = \{(x, y)\}$$

holds, i.e., efficiency is attained if and only if A otains x, and B obtains y provided that no money is wasted. We would like to construct a PME where (z, x) is the object allocation obtained on the equilibrium path so that efficiency is not attained.

	on-path	A's dev	B's dev
ω	(z, x)	(y, x)	(z, y)
p_x	70	100	–
p_y	–	40	50
p_z	10	–	40

TABLE 3.3. Prices

	on-path	A's dev	B's dev
ω	(z, x)	(y, x)	(z, y)
μ	(z, x)	(x, y)	(z, y)
A's gain	50	40	60
B's gain	80	120	50
$W(\mu)$	130	160	110

TABLE 3.4. Utility gains

Suppose that the price systems (p_x, p_y, p_z) under (y, x) and (z, x) are given in Table 3.3. Under this price system profile, $\omega = (\omega_A, \omega_B) = (z, x)$ is the equilibrium outcome of the first stage as well as the second. Note that y is not available in this subgame, which is critical for this example. In order to check the incentive of A, consider $\omega' = (y, x)$. This is the most stringent constraint. The utility gains after obtaining the respective objects on the path as well as off the paths that can be reached by a unilateral deviation are shown in Table 3.4. For example, if $\omega = (z, x)$ holds, then $\mu = (z, x)$ is realized with the utility gain of player A being $v_A(z) = 50$, while if $\omega' = (y, x)$ holds, then $\mu = (x, y)$ is realized with the gain of A being $v_A(x) - p_x + p_y = 40$. Player B's incentive not to deviate to y in the first stage is similarly checked by using the same tables (Tables 3.3 and 3.4).

There is another PME with a different outcome. Table 3.5 shows the price systems (p_x, p_y, p_z) under (y, x), (z, x), and (y, z), which are ω on the equilibrium path, the one attained by A's unilateral deviation, and the one attained by B's unilateral deviaiton,

	on-path	A's dev	B's dev
ω	(y,x)	(z,x)	(y,z)
p_x	90	90	–
p_y	70	–	30
p_z	–	40	30

Table 3.5. Prices

	on-path	A's dev	B's dev
ω	(y,x)	(z,x)	(y,z)
μ	(x,y)	(z,x)	(z,y)
A's gain	80	50	50
B's gain	80	80	60
$W(\mu)$	160	130	110

Table 3.6. Utility gains

respectively. Under this price system profile, $\omega = (y, x)$ is the equilibrium outcome of the first stage, and $\mu = (x, y)$ is the object allocation of the second stage, and so on, as shown in Table 3.6. It is verified that no player has an incentive to make a unilateral deviation. Under $\omega = (y, x)$, the players trade their holdings so that the final object allocation becomes (x, y).

Thus, this example shows that if scarcity does not hold, then neither efficiency nor uniquenss is guaranteed in some PME. Also, by reducing one unit of object z, the welfare of the economy in some PME is strictly increased.

4. College Admission and Labor Market

We consider a decentralized labor market after college admission. There are two sets of players. N_s is the set of students. N_f is the set of firms. We have $N = N_s \cup N_f$ and $N_s \cap N_f = \emptyset$. College degrees are objects. A firm can demand a degree only if it is owned by some student.[7] Every student selects a college (including not going to college, corresponding to ϕ), taking into account the future job prospect. For all $i \in N_s$, i's action set in the first stage is $A_i = \bar{O}$. On the other hand, for all $i \in N_f$, i's action set in the first stage is $A_i = \{\phi\}$. $A = \times_{i \in N} A_i$ is the set of action profiles. Note that those who choose colleges in the first stage are the students.

We assume the following.

Condition 4.1.

> **(Zero) for N_s:** *$v_i(a) = 0$ holds for all $i \in N_s$ and all $a \in \bar{O}$,*
> **(No money) for N_s:** *$\bar{m}_i = 0$ holds for all $i \in N_s$,*
> **(Abundance) for N_f:** *$\bar{m}_i > \max_{a \in O} v_i(a)$ holds for all $i \in N_f$.*

In the presence of (Zero) for N_s, we assume genericity only for N_f. Assumptions (Zero) for N_s implies that the firms, not the students, intrinsically demand the college degrees. Assumption (Abundance) for N_f implies that the firms have a sufficient amount of money to pay the wages up to the marginal productivity of the students.

Given $N_0 \subset N$, let $\mathcal{A}_{N_0} = \{\omega \in \mathcal{A} | i \in N \setminus N_0 \to \omega_i = \phi\}$.

Lemma 4.1. *Assume (Zero) and (No money) for N_s, and (Abundance) for N_f. Given $\omega \in \mathcal{A}$, there exists at least one market equilibrium under $\langle \bar{m}, R, \omega \rangle$.*

Proof. See Appendix A.1. $\square$

[7] We do not consider signaling effects here.

Then PME exists in the mixed strategy since the existence of PME is reduced to the existence of subgame perfect equilibrium. The following result is stated below without proof.

Theorem 4.2. *Assume (Zero) and (No money) for N_s, and (Abundance) for N_f. Then there exists at least one PME.*

We have the following corollary that corresponds to Corollary 3.3.

Corollary 4.3. *Assume (Zero) and (No money) for N_s, and (Abundance) for N_f. Suppose that $(\rho, (p(\omega), x(\omega))_{\omega \in \mathcal{A}})$ is a PME. Then there exists at least one PIPME $(\rho^*, (p^*(\omega), x^*(\omega))_{\omega \in \mathcal{A}})$.*

The proof of this corollary is essentially the same as that of Corollary 3.3. Nonetheless, let us lay it out here as details are different between the two.

Proof.
Assume (Zero) and (No money) for N_s, and (Abundance) for N_f. Suppose that $(\rho, (p(\omega), \mu(\omega), m(\omega))_{\omega \in \mathcal{A}})$ is PME. We construct a PIPME $(\rho^*, (p^*(\omega), \mu^*(\omega), m^*(\omega))_{\omega \in \mathcal{A}})$ as follows. Let $\Omega = \{\Omega^1, \ldots, \Omega^L\}$ be the partition of $\mathcal{A}$ as defined in the proof of Corollary 3.3. For each $\ell = 1, \ldots, L$, take an $\hat{\omega}^\ell \in \Omega^\ell$ in an arbitrary manner. Then for each $\ell = 1, \ldots, L$ and each $\omega \in \Omega^\ell$, let

$$\begin{aligned} p^*(\omega) &= p(\hat{\omega}^\ell), \\ \mu^*(\omega) &= \mu(\hat{\omega}^\ell), \\ m_i^* &= \bar{m}_i - p^*_{\mu_i^*(\omega)} + p^*_{\omega_i}, \ i \in N. \end{aligned}$$

Given the price $p^*(\omega)$, for each player $i \in N_f$, the optimal object is $\mu_i(\hat{\omega}^\ell)$ since i's demand correspondence does not depend on the initial endowment. For $i \in N_s$, since i's demand correspondence does not depend on the initial endowment by (Zero), either. Also, m_i^* is determined by player i's budget constraint for $i \in N$.

Thus, $(p^*(\omega), \mu^*(\omega), m^*(\omega))$ is a market equilibrium under ω. This completes the construction of the second stage equilibrium profile $(p^*(\omega), x^*(\omega))_{\omega \in \mathcal{A}}$ with (PI). Since the first stage strategy profile ρ^* is simply a Nash equilibrium of the induced game. This completes the proof.

□

In order to state some of the subsequent results, we need to modify the definition of scarcity.

Condition 4.2.

(Scarcity'): $|N_s| > Q$ *and* $|\{i \in N_f |\ v_i(a) > 0\}| > Q$ *$(a \in O)$ hold.*

The next lemma shows that under (Scarcity'), among others, given $\omega \in \mathcal{A}$, every object in O^ω has a positive equilibrium price, that all the objects in O^ω are allocated to the firms, and that there is no remaining object.

Lemma 4.4. *Assume (Zero) and (No money) for N_s, (Abundance) for N_f, and (Scarcity'). Given $\omega \in \mathcal{A}$, if (p, μ, m) is a market equilibrium under $\langle \bar{m}, R, \omega \rangle$, then μ is ω-exhaustive, and $p_a > 0$ and $\mu^a \subset N_f$ hold for all a in O^ω.*

Proof.
Assume (Zero) and (No money) for N_s, (Abundance) for N_f, and (Scarcity'). Take $\omega \in \mathcal{A}$ as given. Suppose that (p, μ, m) is a market equilibrium under $\langle \bar{m}, R, \omega \rangle$. Suppose not, i.e., for some a in O^ω, $p_a = 0$ holds in the market equilibrium. Take such an object a. Due to (Scarcity'), $|\{i \in N_f | v_i(a) > 0\}| > Q$ holds. This implies that there exists at least one j in $\mu^\phi \cap N_f$ such that $v_j(a) > 0$ and $m_j = \bar{m}_j$, which implies $(a, m_j)\ P_j\ (\phi, \bar{m}_j)$. Therefore, (ϕ, m_j) does not maximize j's utility. This is a contradiction, i.e., $p_a > 0$ holds for all $a \in O^\omega$.

Then, the positive prices of all tangible objects in O and Definition 2.2 imply that μ is ω-exhaustive.

Repeating the same argument as in the proof of Lemma 4.1, we verify that the positive price of an object a in turn implies $\mu^a \subset N_f$ for all $a \in O^\omega$.

□

The next lemma shows the ω-efficiency of a market equilibrium under ω and compares the outcomes across different initial endowments of the second stage.

Lemma 4.5. *Assume (Zero) and (No money) for N_s, (Abundance) for N_f, and (Scarcity'). Given $\omega \in \mathcal{A}$, suppose that (p, x) is a market equilibrium under $\langle \bar{m}, R, \omega \rangle$. Then, x is ω-efficient. Also, suppose that (p, μ, m) is a market equilibrium under $\langle \bar{m}, R, \omega \rangle$, and that $(\hat{p}, \hat{\mu}, \hat{m})$ is a market equilibrium under $\langle \bar{m}, R, \hat{\omega} \rangle$, where ω and $\hat{\omega}$ are in $\mathcal{A}$ such that $|\omega| = |\hat{\omega}|$ holds. Then, $\mu = \hat{\mu}$ holds. In addition, there exists m' such that (p, μ, m') is a market equilibrium under $\langle \bar{m}, R, \hat{\omega} \rangle$.*

Proof. See Appendix A.2. □

Theorem 4.6. *Assume (Zero) and (No money) for N_s, (Abundance) for N_f, (Obtainability) and (Scarcity'). Suppose also that $(\sigma, (p(\omega), \mu(\omega), m(\omega))_{\omega \in \mathcal{A}})$ is a PME. Then $(\mu(\lambda(\sigma)), m(\lambda(\sigma)))$ is efficient, and therefore, $\mu(\lambda(\sigma))$ is unique.*

Since the proof is similar to that of Lemma 3.4, we relegate it to the appendix.

Proof. See Appendix A.3. □

In the following examples, we show that if (Zero) is violated, i.e., if students with no money have a positive value for some object, then PME may not exist, and that even if it exists, a PME object allocation may not be efficient.

Example 4.1.

	A	B	C
x	10	20	20
y	20	10	10

TABLE 4.1. Value for N_s

	D	E	F
x	1	1	2
y	2	2	1

TABLE 4.2. Value for N_f

Let the values for N_s and N_f be given in Tables 4.1 and 4.2, respectively. Suppose

$$\omega = (x, y, y, \phi, \phi, \phi).$$

Then we have no market equilibrium in the second stage under ω. To begin with, we have $p_x \leq 20$ and $p_y \leq 10$. For if not, there would be excess supply with a positive price. Also, we have $p_x, p_y \geq 2$; if not, there will be excess demand for the object of which price is below 2. Consider two cases. First, suppose $2 \leq p_x \leq p_y \leq 10$. Then both B and C can afford x, and therefore, the demand for x is at least two, which leads to excess demand as there is only one unit of object x. Second, suppose $p_x > p_y \geq 2$. Then no player demands x, which leads to excess supply for x, supplied by A, with a positive price. Thus, no market equilibrium exists.

Example 4.2. *Let the values for N_s, N_f, and priority are given by Tables 4.3, 4.4, and 4.5, respectively. Also, let the initial endowment profile of money be given by $m = (0, 0, \bar{m}_C, \bar{m}_D)$.*

In this economy, the efficient object allocation is $\mu = (\phi, x, \phi, \phi)$. However, B cannot obtain x in PME. Actually, p_x will be in $[25, 30]$ when the second stage becomes a market equilibrium under an initial endowment (x, ϕ, ϕ, ϕ). Then, A's utility becomes at least 25 after having x in the first stage. If A does not get x in the first stage, A's utility becomes 0 regardless of the value of p_x. Therefore, getting x is strictly better for A in the first stage. A can always get x because A has the highest priority. Hence, B will never obtain x in a PME allocation.

	A	B
x	15	90

TABLE 4.3.
Values for N_s

	C	D
x	30	25

TABLE 4.4.
Values for N_f

$A \succ_x B$

TABLE 4.5.
Priority for N_s

5. PERFECT MARKET EQUILIBRIUM UNDER NO MONEY

In this section, we assume that $A_i = \bar{O}$ holds for all $i \in N$. We also assume that nobody has money, i.e., for all $i \in N$, $\bar{m}_i = 0$. We write, for all $\mu, \eta \in \mathcal{A}^+$, $\mu_i R_i^0 \eta_i$ whenever $(\mu_i, 0) R_i (\eta_i, 0)$ holds. P_i^0 and I_i^0 are similarly defined by using P_i and I_i, respectively. Also, we assume that all goods are valuable for all the players, i.e., for all i in N, for all $a \in O$ $v_i(a) > 0$ holds. In this section, we often need a restriction on quotas. We assume that the quota of each object in O is one.

For convenience, we summarize some of the assumptions in the following.

Condition 5.1.

(No money): *for all $i \in N$,$\bar{m}_i = 0$,*
(+Value): *for all i in N and for all $a \in O$, $v_i(a) > 0$,*
(Quota1): *for all $a \in O$, $|q^a| = 1$.*

If there is no money, the definition of Pareto optimality is reduced to the following. We say that an object allocation μ *Pareto dominates* another object allocation η if for all $i \in N$, $\mu_i R_i^0 \eta_i$ holds, and for some $j \in N$, $\mu_j P_j^0 \eta_j$. An object allocation μ is *Pareto* *optimal* if there is no object allocation that Pareto dominates μ. Accordingly, ω-Pareto optimal allocation is defined.

5.1. **Existence and optimality.** The condition for the existence of market equilibrium in the second stage is non-trivial in the case of no money. Shapley and Scarf (1974) essentially showed that for any initial endowment, a market equilibrium exists if the quota of each object is one.

Lemma 5.1. *Assume (No money), (+Value), and (Quota1) in Assumption 5.1. Then for all $\omega \in \mathcal{A}$, market equilibrium (p, μ, m) exists under ω.*

Proof.
Assume (No money), (+Value), and (Quota1). Shapley and Scarf (1974) shows that there is a sequence of top trading cycles $S_1, \dots, S_L$ where $S_1 \neq \emptyset$ is a top trading cycle in N, $S_{\ell+1} \neq \emptyset$ is a top trading cycle in $N \setminus \cup_{\ell'=1}^{\ell} S_{\ell'}$ ($\ell = 1, \dots, L-1$), and $\cup_{\ell=1}^{L} S_\ell = N$ (see Appendix B.3 for the definition of top trading cycles). Next, Shapley and Scarf (1974) attaches, in the present notation, a price p_ℓ to each good held by a player in S^ℓ ($\ell = 1, \dots, L-1$) in such a way that we have

$$p^1 > \cdots > p^L > 0.$$

Then, the price system defined above constitutes a competitive price system.

We let p_1, the highest price, not exceed $\min_{i \in N} \min_{a,b \in \bar{O},\ a \neq b} [v_i(a) - v_i(b)]$, which is positive due to genericity. Then, no player has an incentive to deviate in the second stage under ω. □

Using this claim and the existence result of subgame perfect equilibrium for a finite game, we have the existence result for PME, which is stated without proof.

Theorem 5.2. *Assume (No money), (+Value), and (Quota1). Then, there exists at least one PME.*

Next, we show that a market equilibrium allocation is "optimal" given the initial endowment.

Lemma 5.3. *Assume (No money). Suppose that given ω, μ is a market equilibrium object allocation. Then, μ is ω-optimal.*

Since the proof of this lemma is an application of the standard proof of the first fundamental theorem of welfare economics, we relegate it to the appendix.

Proof. See Appendix A.4. □

The following proposition holds even if there is no scarcity of objects.

Theorem 5.4. *Assume (Obtainability) and (No money). If $(\sigma, (p(\omega), \mu(\omega), 0)_{\omega \in \mathcal{A}})$ is PME, then $\mu(\lambda(\sigma))$ is Pareto optimal.*

Proof.
Assume (Obtainability) and (No money). Let $\mu \in \mathcal{A}^+$ be a PME allocation, and let p be

the price profile in this PME. Suppose the contrary, i.e., that there exists $\eta \in \mathcal{A}^+$ that Pareto dominates μ. Partition N into N_e and N_d where we have

$$\eta_i I_i^0 \mu_i \quad \text{if} \quad i \in N_e,$$
$$\eta_i P_i^0 \mu_i \quad \text{if} \quad i \in N_d.$$

Note $N_d \neq \emptyset$. Since there is no indifferent object other than itself, $\eta_i = \mu_i$ holds for all $i \in N_e$. Take any $i_0 \in N_d$. Player i_0 would have obtained η_{i_0} if it were available in either stage. In the first stage, therefore, it must be the case that another player in N_d who obtained η_{i_0} under μ; otherwise, player i_0 could have obtained it directly in the first stage. Also, player i_0 could have obtained it in the second stage if $p_{\eta_{i_0}} \leq p_{\mu_{i_0}}$. But, repeating the same proof as the one in Lemma 5.3, we prove this would lead to a contradiction.

□

5.2. **Quotas and values.**

Existence is not guaranteed if the quota exceeds one for some object type as the next example shows.

Example 5.1.

	A	B	C
x	10	20	20
y	20	10	10

TABLE 5.1. Value

Let the values of this economy be given in Table 5.1. Suppose

$$\omega = (x, y, y).$$

Then we have no market equilibrium in the second stage under ω. To begin with, we have $p_x \leq 20$ and $p_y \leq 10$. For if not, there would be excess supply with a positive price. Consider two cases. First, suppose $p_x \leq p_y \leq 10$. Then both B and C can afford x, and therefore, the demand for x is at least two, which leads to excess demand as there is only one unit of object x. Second, suppose $p_x > p_y$. Then no player demands x, which leads to excess supply for x with a positive price. Thus, no market equilibrium exists.

Also, existence is not guaranteed if (+Value) in Assumption 5.1 is violated.

Example 5.2.

	A	B	C
x	20	-10	20
y	10	-20	10
ϕ	0	0	0

TABLE 5.2. Value

Let the values of this economy be given in Table 5.2. Suppose

$$\omega = (\phi, x, y).$$

Then we have no market equilibrium in the second stage under ω. Suppose the contrary, i.e., that p is a market equilibrium price. First, we have $p_\phi = 0$. Next, we would like to show $p_x = 0$. Suppose not, i.e., $p_x > 0$. Then there must be a positive demand for x, which occurs only if $p_y \geq p_x > 0$ since C must demand x. This implies that there is no demand for y since A has neither money nor object with a positive price. This is a contradiction. Thus, $p_x = 0$ holds. But, this would induce the excess demand for x. Hence, no market equilibrium exists.

5.3. **Stability and Market Equilibrium.** We define the concept of stable market equilibrium (SME), which is a market equilibrium of which object allocation is stable. This subsection studies the relationship between SME and PME. SME requires that the object allocation of a market equilibrium should be stable. It considers neither the incentive in the first stage nor off-the-path market equilibria of the second stage. Therefore, while it is easy to verify some allocation is an SME allocation, it is not clear if the players really follow this equilibrium. On the other hand, PME takes into account all the incentives, both on and off-the equilibrium path, and in general, it is hard to characterize.

The stability of object allocations is also defined in the standard manner.

Definition 5.1. *An object allocation $\mu \in \mathcal{A}$ is* stable *if*

- $\forall i \in N, \forall j \in N\ [\mu_j \in O \wedge i \succ_{\mu_j} j \Rightarrow \mu_i R_i^0 \mu_j]$,
- $\forall a \in \bar{O}\ \forall i \in N\ [|\mu^a| < q^a \Rightarrow \mu_i R_i^0 a]$.

Lemma 5.5. *Assume (Scarcity). For all $a \in O$ $|\mu^a| = q^a$ holds if an object allocation μ is stable.*

Proof.
Assume (Scarcity). Suppose an object allocation μ is stable. Suppose not, i.e., there exists $a \in O$ $|\mu_a| < q^a$. The scarcity implies that there exists $i \in \mu^\phi$ s.t. $v_i(a) > 0$. However, this is a contradiction to that μ is stable. $\square$

Definition 5.2. *Given R and $\succ$, $(p, \mu, 0)$ is a* stable market equilibrium *(SME) if*

- *$(p, \mu, 0)$ is a market equilibrium under μ itself;*
- *μ is stable.*

In order to further study the solution concepts, we introduce the concept of priority cycle as stated in Ergin (2002)[8].

Definition 5.3. *Let $\succ$ be a priority structure and q be a quota profile.* A priority cycle *is constituted of distinct $a, b \in O$ and $i, j, k \in N$ such that the following is satisfied:*
(C) Cycle condition: $i \succ_a j \succ_a k \succ_b i$.

[8]The definition of the cycle and acyclicity are different from that of Ergin (2002) in that Ergin (2002) includes the condition on scarcity in the definition as well

Definition 5.4. *Let $\succ$ be a priority structure and q be a vector of quotas. A* generalized cycle of priority *is constituted of distinct $a_1, a_2, \ldots, a_n \in O$ and $i, k_1, \ldots, k_n \in N$ such that the following are satisfied:*
(C') Cycle condition: $k_1 \succ_{a_1} i \succ_{a_1} k_n \succ_{a_n} k_{n-1} \succ_{a_{n-1}} k_{n-2} \ldots k_2 \succ_{a_2} k_1$.

If $\succ$ has a generalized cycle, then it also has a cycle. However, this assertion can be shown in the same way as in Ergin (2002). If the priority structure is not cyclical, it is called *acyclical*. The following proposition states the existence of SME.

Proposition 5.6. *Assume (Scarcity), (No money), (+Value), and (Quota1). SME exists if the priority structure is acyclical.*

Proof. See Appendix A.5. □

Next, we have the following proposition, stating that an SME object allocation is Pareto optimal.

Proposition 5.7. *Assume (Scarcity) and (No money). Given SME object allocation μ, μ is Pareto optimal.*

Proof.
Assume (Scarcity) and (No money). From Lemma 5.3, for all $\omega \in \mathcal{A}$, μ is ω-optimal. From Lemma 5.5, for all $a \in O$, $|\mu^a| = q^a$ holds. Thus, μ is Pareto optimal. □

Condition 5.2. *A mechanism $M = \langle \Sigma, \lambda \rangle$ is a* generalized first-come-first-served mechanism *if the following condition is satisfied.*

(G): *For all $\mu \in \mathcal{A}^+$, there exists $\hat{\sigma} \in \Sigma$ that satisfies the following properties*:

(1) $\lambda(\hat{\sigma}) = \mu$;
(2) *given $i \in N$ and $\sigma'_i \in \Sigma_i$, denote $\eta = \lambda(\sigma'_i, \hat{\sigma}_{-i})$; then either one of the following two cases holds*:
  - $\eta = \mu$;
  - *$\eta_i \neq \mu_i$, and $\eta_k = \mu_k$ holds for all tangible object holder $k \in \cup_{a \in O} \mu^a$ except for i him/herself and the player at η_i who is lower in priority than i and any other player who was at η_i under μ.*

Given $i \in N$ and $\mu_i \in \bar{O}$, we call such a strategy, often denoted by $\hat{\sigma}_i$ a Go-and-Get (GG) *strategy of player i for μ_i.*

This condition is satisfied, among others, by the first-come-first-served rule (the Boston mechanism)[9]. It states that other than a deviator i, the only player j whose first stage outcome is affected by i's deviation is the one who is *directly* pushed out by i. The object allocation for the other players is independent of such a deviation.

We now present the following result.

[9] See Appendix B.1 for its definition.

Proposition 5.8. *Assume (Obtainability), (G), (Scarcity), (No money), (+Value), and (Quota1). Assume also that there is no cycle of priority. Then, if $(p, \mu, 0)$ is an SME, there exists a PME $(\hat{\rho}, (\hat{p}(\omega), \hat{\mu}(\omega), 0)_{\omega \in \mathcal{A}})$ such that $\hat{\mu}(\lambda(\hat{\sigma})) = \mu$ holds for all $\hat{\sigma}$ with $\hat{\rho}(\hat{\sigma}) > 0$.*

Proof. See Appendix A.6. □

Condition 5.3.

> **(DA):** $M = \langle \Sigma, \lambda \rangle$ *satisfies the definition of the deferred acceptance algorithm in Appendix B.2.*

Proposition 5.9. *Assume (Obtainability), (DA), (Scarcity), (No money), (+Value), and (Quota1). Assume also that there is no cycle of priority. Then, if $(p, \mu, 0)$ is an SME, there exists a PME $(\hat{\rho}, (\hat{p}(\omega), \hat{\mu}(\omega), 0)_{\omega \in \mathcal{A}})$ such that $\hat{\mu}(\lambda(\hat{\sigma})) = \mu$ holds for all $\hat{\sigma}$ with $\hat{\rho}(\hat{\sigma}) > 0$.*

Proof. See Appendix A.7. □

The following is an example where an SME object allocation is not a PME object allocation if there is a priority cycle.

Example 5.3. *Assume (Obtainability),(Scarcity), (No money), (+Value), and (Quota1). Assume (G) or (DA). Players' object values are shown in Table 5.3. There is a priority cycle:*

$$B \succ_x C \succ_x A \succ_y B.$$

We show that $\mu = (y, x, z)$ is a SME object allocation, but not a PME object allocation.

	A	B	C
x	30	20	10
y	20	10	20
z	10	30	30

Table 5.3. Values

$$B \succ_x C \succ_x A$$
$$A \succ_y B \succ_y C$$
$$A \succ_z C \succ_z B$$

Table 5.4. Priority

We can construct an SME, where $p_z \geq p_x > p_y$ and $\mu = (\mu_A, \mu_B, \mu_C) = (y, x, z)$ hold. Note that this is a Pareto optimal allocation.

There is, however, no PME of which object allocation is $\mu = (y, x, z)$ on the equilibrium path. To see this, suppose the contrary, i.e., that there is a PME $(\rho, (p(\omega), \mu(\omega), m(\omega))_{\omega \in \mathcal{A}})$ of which object allocation is $\mu = (y, x, z)$ on the equilibrium path with a positive probability.

ω	$p(\omega)$	$\mu(\omega)$
(x, y, z)	$[p_x \geq p_z > p_y] \vee [p_z \geq p_x > p_y]$	(x, y, z)
(x, z, y)	$[p_x \geq p_z > p_y] \vee p_z \geq p_x > p_y]$	(x, z, y)
(y, x, z)	$p_z \geq p_x > p_y$	(y, x, z)
(y, z, x)	$p_z > p_x = p_y$	(x, z, y)
(z, x, y)	$p_x = p_z > p_y$	(x, z, y)
(z, y, x)	$p_x = p_z > p_y$	(x, y, z)

TABLE 5.5. market equilibria under ω

Table 5.5 shows market equilibria under exhaustive endowments. For example, if $\omega = (x, y, z)$*, then there exists a unique object allocation* $\mu(\omega) = (x, y, z)$ *that is a market equilibrium under* ω *with the price condition* $[p_x \geq p_z > p_y] \vee [p_z \geq p_x > p_y]$ *as given in the column of* $p(\omega)$*. Due to Lemma 2.3, we do not have to consider* ω *such that* $|\omega^a| < q^a$ *for some* $a \in O$.

Table 5.5 shows that the SME object allocation $\mu = (y, x, z)$ *is achieved when the initial object endowment is* (y, x, z) *itself, and that the price vector satisfies* $p_z \geq p_x > p_y$.

We examine that (y, x, z) *is not achieved as the initial object endowment neither under (G) nor (DA).*

Suppose (G). We consider that $c^1 = (y, x, z)$ *is the players' first choices in the message submitted to the mechanism. Then, we check whether this* c^1 *constitutes a PME strategy* σ *or not. Due to Lemma 2.3, it suffices to check the incentive to deviate from* $c^1 = (y, x, z)$*. It is verified that player A has an incentive to deviate from* $c^1_A = y$ *in the first stage. Indeed, consider the case that A changes to* $\hat{c}^1_A = z$ *where A is ranked the highest in terms of priority. Since C's second choice must be* y *due to the definition of PME, and the initial endowment becomes* (z, x, y)*. If* (z, x, y) *is an initial endowment, the unique market equilibrium object allocation becomes* (x, z, y) *according to Table 5.5. Therefore, A can be better off by the deviation. This is a contradiction to that* $c^1 = (y, x, z)$ *constitutes a PME strategy* σ*. It is verified that the best-response profile of the first-stage is either* (z, x, y) *or* (x, z, y)*. Therefore, for any* ρ *of PME, there is no* σ *which consists of* $c^1 = (y, x, z)$ *as a first round action profile in the support of* ρ.

Next, suppose (DA). Under the truth-telling strategy, the initial object allocation is (y, x, z)*. However, this truth-telling strategy itself is not PME strategy because A has an incentive to deviate and obtain* z*. Repeating almost the same argument above, PME object allocation is* (x, z, y)*. Therefore, SME object allocation is not achieved as any PME object allocation including the truth-telling strategy.*

The assumption of no priority cycle is crucial in Theorem 5.8 and Theorem 5.9. This example has a priority cycle, $A \succ_z C \succ_z B \succ_x A$*, and the theorems do not hold. The PME allocation in this example is* (x, z, y)*. Note that the object* x *(resp.* z*) is the most favorite object for A (resp. B). Since B has the lowest priority at the object* z*, B cannot obtain it by himself. However, A can obtain it for B in the first stage and keep it from C who also likes* z *the best. As the cycle exists, B has higher priority than A at the object* x*. Then, B obtains* x *for A in order to exchange it with* z*. Therefore, the cyclicity enables both A and B to obtain what they like most in the second stage.*

A PME object allocation in the two stage economy may be different from an SME object allocation if the assumptions of Theorem 5.8 and Theorem 5.9 are violated. The concept of SME does not consider the incentive of the first stage. Therefore, if we separately analyze each stage of the two stage economy, the outcome would be different from that attained in the analysis of the two stage economy based on perfection.

Appendix A. Proofs

A.1. **Proof of Lemma 4.1.**

Proof. Assume (Zero) and (No money) for N_s, and (Abundance) for N_f. Take $\omega \in \mathcal{A}$ as given. Consider $\langle \bar{m}, R, \omega \rangle$ in the second stage. We construct an auxiliary economy in which the players in N_s also have a sufficient amount of money, i.e., (Abundance) holds instead of (No money) for N_s. Let $\bar{m}'_i$ be the initial money holdings of $i \in N_s$, and let $\bar{m}' = \big((\bar{m}_i)_{i \in N_f}, (\bar{m}'_i)_{i \in N_s}\big)$. Then, for any $\omega \in \mathcal{A}$, a market equilibrium under $\langle \bar{m}', R, \omega \rangle$ exists from Claim 3.1. Take one of such market equilibria, denoted by (p, μ, m). Partition $\bar{O}$ into $\bar{O}_+$ and $\bar{O}_0$ where we have

$$\begin{aligned} a \in \bar{O}_+ \quad &\text{if} \quad p_a > 0, \\ a \in \bar{O}_0 \quad &\text{if} \quad p_a = 0. \end{aligned}$$

$\bar{O}_+$ may be empty, while $\bar{O}_0$ is always nonempty as $p_\phi = 0$ always holds.

Suppose that $\bar{O}_+$ is not empty in (p, μ, m). Take any object $a \in \bar{O}_+$. Then, from (3) of Definition 2.2, this implies that $|\mu^a| = |\omega^a|$ holds for object a. We also have $\mu^a \subset N_f$. For if not, there exists $j \in N_s$ with $j \in \mu^a$. (Zero) implies

$$v_j(a) - p_a = 0 - p_a < 0 = v_j(\phi) - p_\phi,$$

which violates the optimization condition of player j.

Next, take any player $i \in N_f$. Then, in the equilibrium, $(\mu_i, \bar{m}_i - p_{\mu_i}) R_i (b, \bar{m}_i - p_b)$ holds for all $b \in \bar{O}$. Note that $m_i = \bar{m}_i - p_{\mu_i}$ holds.

Let $m^* = \big((m_i)_{i \in N_f}, (m_i - \bar{m}'_i)_{i \in N_s}\big)$. Next, we would like to show (p, μ, m^*) is a market equilibrium under the original economy $\langle \bar{m}, R, \omega \rangle$.

Since neither the price nor the initial endowments of the players in N_f is altered, it suffices to check the incentive of N_s. Since the above argument implies that $\mu_j \in \bar{O}_0$ holds for all j in N_s, the reduction of the money endowment for N_s does not affect their incentive in the market even if there is an object with a positive price. Thus, (p, μ, m^*) is a market equilibrium under $\langle \bar{m}, R, \omega \rangle$.

If $\bar{O}_+$ is empty, then also $\mu_i \in \bar{O}_0$ holds for all i in N_s, and the money has no impact on the incentives of the members in N_s, either.

□

A.2. **Proof of Lemma 4.5.** This proof is similar to that of Lemma 3.4.

Proof.
Assume (Zero) and (No money) for N_s, (Abundance) for N_f, and (Scarcity'). Suppose that (p, μ, m) is a market equilibrium under $\langle \bar{m}, R, \omega \rangle$. First, we show ω-efficiency. Suppose the contrary, i.e., that there exists $\gamma \in \mathcal{A}^\omega \cap \mathcal{A}_{N_f}$ such that $W(\gamma) > W(\mu)$ holds. For every player i, the optimization for $i \in N_f$ implies

$$v_i(\mu_i) - p_{\mu_i} \geq v_i(\gamma_i) - p_{\gamma_i}. \tag{A.1}$$

Rewriting (A.1), we have

$$p_{\gamma_i} - p_{\mu_i} \geq v_i(\gamma_i) - v_i(\mu_i) \tag{A.2}$$

for all $i \in N_f$. By taking the summation of the both sides across $i \in N_f$, (A.2) implies

$$\sum_{i \in N_f} [p_{\gamma_i} - p_{\mu_i}] \geq W(\gamma) - W(\mu) > 0. \tag{A.3}$$

Therefore, we have

$$\sum_{i \in N_f} p_{\gamma_i} > \sum_{i \in N_f} p_{\mu_i}. \tag{A.4}$$

Since $\mu^a \subset N_f$ holds for all $a \in O^\omega$ from Lemma 4.4, we can sum both sides across N instead of N_f, i.e.,

$$\sum_{i \in N} p_{\gamma_i} > \sum_{i \in N} p_{\mu_i}. \tag{A.5}$$

Rewriting the above inequality, we have

$$\sum_{a \in O} |\gamma^a| p_a > \sum_{a \in O} |\mu^a| p_a. \tag{A.6}$$

This implies that there exists an object $a \in O$ such that $|\gamma^a| > |\mu^a| = |\omega^a|$ and $p_a > 0$ hold from Lemma 4.4. However, $|\gamma^a| > |\omega^a|$ is a contradiction to that γ is ω-feasible.

Next, we will show there exists m' such that (p, μ, m') is a market equilibrium under $\langle \bar{m}, R, \hat{\omega} \rangle$. Due to quasi-linearity of the utility functions, for all $i \in N_f$, $\mu_i \in \text{argmax}_{a \in \bar{O}}[v_i(a) - \bar{m}_i]$ holds even if $\hat{\omega}$ is the initial object endowment. Also, for all $j \in N_s$, if $\hat{\omega}_j \in O$, j sells his/her object from Lemma 4.4. Let $m'_i = \bar{m}_i - p_{\mu_i}$ for i in N_f and $m'_j = p_{\hat{\omega}_j}$ for $j \in N_s$. Then, since (p, μ, m) is a market equilibrium under $\langle \bar{m}, R, \omega \rangle$, the construction of m' implies that for all $i \in N_f$

$$(\mu_i, m'_i) R_i (a', \bar{m}_i - p_{a'})$$

holds for every $a' \in O$. Also, for all j in N_s with $\hat{\omega}_j = a$ $(a \in O)$

$$(\phi, m'_j) P_j (a', p_a - p_{a'})$$

holds for every $a' \in O$ with $a' \neq a$. Also, for all j in N_s with $\hat{\omega}_j = \phi$, $\mu_j = \phi$ and $m'_j = 0$ holds. We also have $p_a > 0$ and $|\mu^a| = |\hat{\omega}^a|$ for all $a \in O$. Thus, (p, μ, m') is a market equilibrium under $\langle \bar{m}, R, \hat{\omega} \rangle$. □

A.3. **Proof of Theorem 4.6.** Assume (Zero) and (No money) for N_s, (Abundance) for N_f, (Obtainability) and (Scarcity'). Suppose that the profile $(\sigma, (p(\omega), \mu(\omega), m(\omega))_{\omega \in \mathcal{A}})$ is PME. In the following, note that $\lambda(\sigma) \in \mathcal{A}$ and $\mu(\lambda(\sigma)) \in \mathcal{A}_{N_f}$ hold for all $\sigma \in \Sigma$ (see Lemma 4.1).

To prove the theorem, it is useful to have the following lemma, which states that every object is assigned to some player.

Lemma A.1. *Assume (Zero) and (No money) for N_s, (Abundance) for N_f, (Obtainability) and (Scarcity'). Suppose that the profile $(\sigma, (p(\omega), \mu(\omega), m(\omega))_{\omega \in \mathcal{A}})$ is PME. Then, $|\lambda^a(\sigma)| = q^a$ holds for every $a \in O$.*

Proof.
Assume (Zero) and (No money) for N_s, (Abundance) for N_f, (Obtainability) and (Scarcity'). Suppose $(\sigma, (p(\omega), \mu(\omega), m(\omega))_{\omega \in \mathcal{A}})$ is PME. Suppose the contrary, i.e., that $|\lambda^a(\sigma)| < q^a$ holds for some $a \in O$. Take such an a.

(Scarcity') together with Lemma 4.4 implies that there exists $i \in N_s$ with $\lambda_i(\sigma) = \phi$ and $\mu_i(\lambda(\sigma)) = \phi$. Take such an i. Note that this player i is the one who obtains any object in O in neither stage. (Obtainability) implies that there exists $\hat{\sigma}_i \in \Sigma_i$ such that $\lambda(\hat{\sigma}_i, \sigma_{-i}) = a$ holds. Take such a strategy $\hat{\sigma}_i$. Then, i can sell object a at a strictly positive price $p_a > 0$ by Lemma 4.4.

Thus, player i has an incentive to deviate and obtain a. This is a contradiction to that $(\sigma, (p(\omega), x(\omega))_{\omega \in \mathcal{A}})$ is a PME.

□

Now, we return to the proof of the theorem. From Lemma 4.5, $\mu(\lambda(\sigma))$ is ω-efficient. Also, Lemmata 4.4 and A.1 imply

$$|\mu^a(\lambda(\sigma))| = |\lambda^a(\sigma)| = q^a$$

holds for all $a \in O$. Then Lemma 2.1 implies that $\mu^a(\lambda(\sigma))$ is efficient. The uniqueness follows due to the genericity assumption.

□

A.4. **Proof of Lemma 5.3.**

Proof.
Assume (No money) in Assumption 5.1. Since μ is a market equilibrium object allocation, there exists $p \in \mathbb{R}^O_+$ with $(p, \mu, 0)$ being a market equilibrium under ω. Suppose the contrary that there exists $\eta \in \mathcal{A}^\omega$ such that η Pareto dominates μ. Partition N into N_e and N_d where we have

$$\begin{aligned} \eta_i I_i^0 \mu_i \quad &\text{if} \quad i \in N_e, \\ \eta_i P_i^0 \mu_i \quad &\text{if} \quad i \in N_d. \end{aligned}$$

Note $N_d \neq \emptyset$. Since there is no indifferent object other than itself, $\eta_i = \mu_i$ holds for all $i \in N_e$. Take any $i_0 \in N_d$. Player i_0 could have obtained η_{i_0} in the second stage if $p_{\eta_{i_0}} \leq p_{\mu_{i_0}}$. Therefore, we must have

$$p_{\eta_{i_0}} > p_{\mu_{i_0}}.$$

There exists $i_1 \in N_d$ who obtained η_{i_0} under μ, i.e., $\mu_{i_1} = \eta_{i_0}$. Thus, we have

$$p_{\mu_{i_1}} > p_{\mu_{i_0}}.$$

We repeat the same procedure to construct a sequence $(i_0, i_1, i_2, \dots)$ with

$$p_{\mu_{i_{k+1}}} > p_{\mu_{i_k}}, \quad k = 0, 1, 2, \dots, \tag{A.1}$$

until the same player reappear along the sequence, i.e., $i_K = i_L$ for some $K < L$. Adding (A.1) from $k = K$ to $k = L - 1$, we obtain

$$\sum_{k=K}^{L-1} p_{\mu_{i_{k+1}}} > \sum_{k=K}^{L-1} p_{\mu_{i_k}}. \tag{A.2}$$

The both sides are the same since $i_L = i_K$ holds. This is a contradiction.

□

A.5. **Proof of Proposition 5.6.** Before proving the proposition, we state and prove the following lemma.

Lemma A.2. *Assume (Scarcity) in Assumption 2.2 and (No money), (+Value), and (Quota1) in Assumption 5.1. If $\mu \in \mathcal{A}^+$ is Pareto optimal, all the players in the trading cycle mechanism with an initial object μ are in a trading cycle as a singleton.*

Proof.
Assume (Scarcity) in Assumption 2.2 and (No money), (+Value), and (Quota1) in Assumption 5.1. Suppose not, i.e., there is a trading cycle $(i_1, \ldots, i_K)$ with $K > 1$. Then, $\mu_{i_{k+1}} P^0_{i_k} \mu_{i_k}$ and $\mu_{i_1} P^0_{i_K} \mu_{i_K}$ hold. This implies that we can construct $\gamma \in \mathcal{A}^+$ such that for all $j \in N \setminus \{i_1, \ldots, i_K\}$, $\gamma_j = \mu_j$, for all $k = 1, \ldots, K-1$, $\gamma_{i_k} = \mu_{i_{k+1}}$ and $\gamma_{i_K} = \mu_{i_1}$. Then, γ Pareto dominates μ. This is a contradiction to the assumption. □

Now, the proof of the proposition is provided. Assume (Scarcity) in Assumption 2.2 and (No money), (+Value), and (Quota1) in Assumption 5.1. Also, suppose that the priority structure is acyclical. Then, Ergin (2002) shows that the DA object allocation μ becomes a stable and Pareto optimal allocation.

First, we assign a price to each object a using the trading cycle mechanism with an initial object μ (see Appendix B.3 for the definition of the trading cycle mechanism). Let $(p_a)_{a \in O}$ be such a constructed price where p_1, the highest price, not exceed

$$\min_{i \in N} \min_{a,b \in \bar{O},\ a \neq b} [v_i(a) - v_i(b)],$$

which is positive due to genericity. Then, for all $i \in N$, for all $a \in O$ if $p_{\mu_i} \geq p_a$ holds with $a \neq \mu_i$, then $\mu_i P^0_i a$ holds. Also, no player has an incentive to deviate in the second stage under μ. Since Lemma A.2 implies that every player forms a trading cycle as a singleton, the object allocation after the trading cycle mechanism with an initial object μ is μ. Therefore, $(p, \mu, 0)$ is a market equilibrium under μ itself.

□

A.6. **Proof of Proposition 5.8.**

Proof.
Assume (Obtainability), (G), (Scarcity), (No money), (+Value), and (Quota1). Suppose that $(p, \mu, 0)$ is an SME. We construct a first stage equilibrium strategy $\hat{\rho}$ in the following manner.

First, we construct a profile $(\hat{p}(\omega), \hat{\mu}(\omega), 0)_{\omega \in \mathcal{A}}$ of the second stage equilibrium outcomes. By the definition of SME, $(p, \mu, 0)$ is a market equilibrium under μ itself as an initial object allocation of the second stage. Let $(\hat{p}(\mu), \hat{\mu}(\mu), 0) = (p, \mu, 0)$. Note that $|\mu^a| = q^a$ holds for all a in O from 5.5. For each $\omega \in \mathcal{A}$ with $\omega \neq \mu$, there exists at least one ME from 5.1. For such ω, let $(\hat{p}(\omega), \hat{\mu}(\omega), 0)$ be a market equilibrium of the second stage under ω. Note that $|\mu^a| = \omega^a$ for all a in O holds by Lemma 2.2.

Given a constructed profile of $(\hat{p}(\omega), \hat{\mu}(\omega), 0)_{\omega \in \mathcal{A}}$ and a given mechanism M, we define an induced payoff and game $\Gamma = \langle N, \Sigma, (\tilde{u}_i)_{i \in N} \rangle$. Let $\hat{\sigma}$ be a GG strategy profile for μ. Its existence is guaranteed by the condition (G). By the construction of $(\hat{p}(\omega), \hat{\mu}(\omega), 0)_{\omega \in \mathcal{A}}$, $\hat{\mu}(\lambda(\hat{\sigma})) = \mu$ holds.

We consider $\hat{\rho}$ that puts probability one on this $\hat{\sigma}$. Then, the outcome of the first stage is μ with probability one under $\hat{\rho}$.

We want to show that $(\hat{\rho}, (\hat{p}(\omega), \hat{\mu}(\omega), 0)_{\omega \in \mathcal{A}})$ is PME. Suppose not, i.e., that there exists $i \in N$ with $\rho'_i \neq \hat{\rho}_i$ satisfying

$$\mathbf{E}\left[u_i(\cdot)|(\rho'_i, \hat{\rho}_{-i})\right] > \mathbf{E}\left[u_i(\cdot)|\hat{\rho}\right]. \tag{A.1}$$

Fix this player i throughout the proof.

Note that the allocation of the second stage changes only if someone changes the action in the first stage since all the objects allocation in the second stage depends on the initial object allocation of the second stage, i.e. $\lambda(\sigma)$ induced by $\hat{\rho}$.

Note also that if one has an incentive to deviate by using a mixed strategy, the player has an incentive to do so by using some pure strategy as well. Assume, therefore, that ρ'_i puts probability one on $\sigma'_i \neq \hat{\sigma}_i$ in the first round.

Inequality (A.1) holds only if player i obtains $b \neq \mu_i$ with $bP^0_i\mu_i$ in the second stage with a positive probability.

Let us write $\hat{\omega} = \lambda(\sigma'_i, \hat{\sigma}_{-i})$. Note that $|\hat{\omega}^a| = q^a$ holds for all a in O. Player i, who obtains $\hat{\omega}_i$ from player, say, $j \neq i$ by deviation, is willing to trade: otherwise, player i would not have deviated in the first place due to the stability of μ and the condition (G). Since there is no money, the only way to trade is through a trading cycle, $(k_0, k_1, \ldots, k_{\bar{n}})$ with $k_0 = k_{\bar{n}} = i$ such that

$$\hat{\omega}_{k_{n+1}} \ P^0_{k_n} \ \hat{\omega}_{k_n}$$

holds for $n = 0, 1, \cdots, \bar{n} - 1$.

Note that every player ℓ except for i and j holds same object μ_ℓ under $\hat{\omega}$. Also, this implies that that j holds μ_i or ϕ under $\hat{\omega}$.

Let us divide the analysis into two cases. First, suppose that the above cycle does not contain μ_i.

Each player in the cycle could not obtain what he/she likes due to priority. Indeed, the condition (G) together with μ being an SME implies that for all $n = 0, 1, 2, \cdots, \bar{n} - 1$,

$$k_{n+1} \succ_{\hat{\omega}_{k_{n+1}}} k_n.$$

Consider $k_{\bar{n}-1}$. This player was willing to obtain $\hat{\omega}_i = \mu_j$ in the first stage under μ as well as under $\hat{\omega}$, but could not. Note $\hat{\omega}_{k_{\bar{n}-1}} = \mu_{k_{\bar{n}-1}}$ due to the condition (G). Therefore, we have

$$j \ \succ_{\mu_j} k_{\bar{n}-1}.$$

Also, the fact that player i deprived j of μ_j implies

$$i \succ_{\mu_j} j,$$

since $\hat{\sigma}$ is a GG strategy profile for μ. Thus, we have a priority cycle:

$$i \succ_{\mu_j} j \succ_{\mu_j} k_{\bar{n}-1} \succ_{\mu_{\bar{n}-1}} k_{\bar{n}-2} \succ_{\mu_{\bar{n}-2}} \ \cdots \ \succ_{\mu_{k_2}} k_1 \succ_{\mu_{k_1}} i,$$

which contradicts to acyclicity.

Second, suppose that the trading cycle contains μ_i. Similarly, we have

$$k_{n+1} \succ_{\hat{\omega}_{k_{n+1}}} k_n$$

for all $n = 0, 1, 2, \cdots, \bar{n}-1$. Now, there is a player, say, k_ℓ in the cycle, who likes to obtain μ_i but cannot in the first stage under μ, i.e.,

$$\mu_i P^0_{k_\ell} \mu_{k_\ell}, \text{ and } i \succ_{\mu_i} k_\ell.$$

Consider another trading cycle $i = k_0, k_1, \ldots, k_\ell, i$ under μ rather than $\hat{\omega}$. The resulting outcome Pareto dominates μ, which is a contradiction.

□

A.7. **Proof of Proposition 5.9.**

Proof.
Assume (Obtainability),(DA), (Scarcity), (No money), (+Value), and (Quota1). Suppose that $(p, \mu, 0)$ is an SME. We construct a first stage equilibrium strategy $\hat{\rho}$ in the following manner.

First, we construct a profile $(\hat{p}(\omega), \hat{\mu}(\omega), 0)_{\omega\in\mathcal{A}}$ of the second stage equilibrium outcomes. By the definition of SME, $(p, \mu, 0)$ is a market equilibrium under μ itself as an initial object allocation of the second stage. Let $(\hat{p}(\mu), \hat{\mu}(\mu), 0) = (p, \mu, 0)$. Note that $|\mu^a| = q^a$ holds for all a in O from Lemma 5.5. For each $\omega \in \mathcal{A}$ with $\omega \neq \mu$, there exists at least one ME from Lemma 5.1. For such ω, let $(\hat{p}(\omega), \hat{\mu}(\omega), 0)$ be a market equilibrium of the second stage under ω. Note that $\hat{\mu}(\omega)$ is ω-exhaustive by Lemma 2.2.

Given the constructed profile $(\hat{p}(\omega), \hat{\mu}(\omega), 0)_{\omega\in\mathcal{A}}$ of the second stage equilibrium outcomes and a given DA mechanism M, we consider the induced game $\Gamma = \langle N, \Sigma, (\tilde{u}_i)_{i\in N}\rangle$. Let $\hat{\sigma} = \zeta^*$ be the profile of the truth-telling strategies as defined in Appendix B.2. Then, $\lambda(\hat{\sigma}) = \mu$ holds by Proposition 5.7 and B.2. By the construction of $(\hat{p}(\omega), \hat{\mu}(\omega), 0)_{\omega\in\mathcal{A}}$, $\hat{\mu}(\lambda(\hat{\sigma})) = \mu$ holds.

We consider $\hat{\rho}$ that puts probability one on this $\hat{\sigma}$. Then, the outcome of the first stage is μ with probability one under $\hat{\rho}$.

We want to show that $(\hat{\rho}, (\hat{p}(\omega), \hat{\mu}(\omega), 0)_{\omega\in\mathcal{A}})$ is PME. Suppose not, i.e., that there exists $i \in N$ with $\rho'_i \neq \hat{\rho}_i$ satisfying

$$\mathbf{E}\left[u_i(\cdot)|(\rho'_i, \hat{\rho}_{-i})\right] > \mathbf{E}\left[u_i(\cdot)|\hat{\rho}\right]. \tag{A.2}$$

Fix this player i throughout the proof.

Note that the allocation of the second stage changes only if someone changes the action in the first stage since all the objects allocation in the second stage depends on the initial object allocation of the second stage, i.e. $\lambda(\hat{\sigma})$ induced by $\hat{\rho}$.

Note also that if one has an incentive to deviate by using a mixed strategy, the player has an incentive to do so by using some pure strategy as well. Assume, therefore, that ρ'_i puts probability one on $\sigma'_i \neq \hat{\sigma}_i$ in the first round.

Inequality (A.2) holds only if player i obtains an object that is preferred to μ_i in the second stage with a positive probability.

Let us write $\hat{\omega} = \lambda(\sigma'_i, \hat{\sigma}_{-i})$. Note that $|\hat{\omega}| = q$ holds. Player i, who obtains $\hat{\omega}_i$ from player, say, $j \neq i$ by deviation, is willing to trade: otherwise, player i would not have deviated in the first place due to the strategy-proofness of DA as shown by Dubins and Freedman (1981). Since there is no money, the only way to trade is through a trading cycle, $(k_0, k_1, \ldots, k_{\bar{n}})$ with $k_0 = k_{\bar{n}} = i$ such that

$$\hat{\omega}_{k_{n+1}} \; P^0_{k_n} \; \hat{\omega}_{k_n}$$

holds for $n = 0, 1, \cdots, \bar{n}-1$. Also, let $O^e = \{\hat{\omega}_{k_1}, \hat{\omega}_{k_2}, \ldots, \hat{\omega}_{k_{\bar{n}}}\}$ be the set of object types exchanged by the players in this trade cycle.

Note that $\hat{\omega}$ satisfies the following properties due to DA's stability and strategy proofness.

$$\forall k \in N \setminus \{i\} \;\; \forall l \in N \quad [\hat{\omega}_l \in O \wedge k \succ_{\hat{\omega}_l} l \Rightarrow \hat{\omega}_k R_k^0 \hat{\omega}_l], \tag{A.3}$$

$$\forall k \in N \qquad [\hat{\omega}_k P_i^0 \lambda_i(\hat{\sigma}) \Rightarrow k \succ_{\hat{\omega}_k} i]. \tag{A.4}$$

Now, we consider an auxiliary situation by altering the order of moves in DA. Note that DA object allocation is not affected by the order of moves as discussed in Dubins and Freedman (1981)[10]. First, we run DA algorithm without i. After this algorithm is tentatively terminated, we put in player i in the algorithm and continue it until it stops. Let t^* be the step right after the algorithm is tentatively terminated, i.e., at step t^*, i is put in the algorithm. Also, let η be a profile of players' object holdings except i when the algorithm is tentatively terminated. Note that (Scarcity) implies

$$|\eta| = q. \tag{A.5}$$

Suppose that i's strategy is σ_i', and that all of the other players take the truth-telling strategies. Then, the stability of DA implies

$$\forall k \in N \setminus \{i\} \;\; \forall l \in N \setminus \{i\} \quad [\eta_l \in O \wedge k \succ_{\eta_l} l \Rightarrow \eta_k R_k^0 \eta_l]. \tag{A.6}$$

We consider the steps after t^*. Suppose that i obtains $\hat{\omega}_i$ at step $\tau_1 \geq t^*$ under σ_i' for the first time. Then, we have the following result, which is stated as a lemma inside this proof.

Lemma A.3. *If there is no cycle of priority, then we have*

$$\forall t = t^*, \ldots, \tau_1 - 1 \; [i \notin \cup_{a \in O} \Phi^{a,t}]. \tag{A.7}$$

Proof. Suppose not, i.e., that there exist $t = t^*, \ldots, \tau_1 - 1$ and $a \neq \hat{\omega}_i$ such that i is in $\Phi^{a,t}$ for the first time (note that the case of $a = \hat{\omega}_i$ has already been taken care of by the definition of τ_1). Equation (A.5) implies that there exists a player j' in $\Phi^{a,t-1}$ who is rejected at step t, i.e., $j' \in N \setminus \Phi^{a,t}$. Moreover, $N \setminus \cup_{b \in \bar{O}} \Phi^{b,t} = \{j'\}$ because all of the other players who are not assigned any object in O have gone to ϕ. Since i obtains $\hat{\omega}_i$ at step τ_1, there exists a step $t_{\bar{\kappa}}$ with $t < t_{\bar{\kappa}} < \tau_1$ such that i is rejected at a, i.e., $i \in N \setminus \Phi^{a,t_{\bar{\kappa}}}$. Let (a', i', t') be a rejection triple that describes a situation in which $i' \in N$ is at $a' \in O$ in step $t'-1$ and rejected at a' in step t'. Then, we have a chain of rejection triples,

$$(a, j', t) = (a_1, j_1, t_1), (a_2, j_2, t_2), \ldots, (a_{\bar{\kappa}}, j_{\bar{\kappa}}, t_{\bar{\kappa}}) = (a, i, t'),$$

where j_κ is rejected at a_κ as $j_{\kappa-1}$ chooses a_κ in step t_κ ($\kappa = 2, \ldots, \bar{\kappa}-1$). Suppose that all the objects except for a in the rejection chain are distinct. Then, this chain of rejection triples constitute a cycle of priority,

$$j_{\bar{\kappa}-1} \succ_a i \succ_a j_1 \succ_{a_2} j_2 \succ_{a_3} \cdots \succ_{a_{\bar{\kappa}-1}} j_{\bar{\kappa}-1}.$$

This is a contradiction.

[10]DA algorithm discussed in this auxiliary situation is essentially the same as the one defined in Dubins and Freedman (1981).

Next, suppose that all the objects except for a in the rejection chain are not distinct. Then, we can also find a shorter cycle of priority than before by the same argument. This completes the proof of Lemma A.3. □

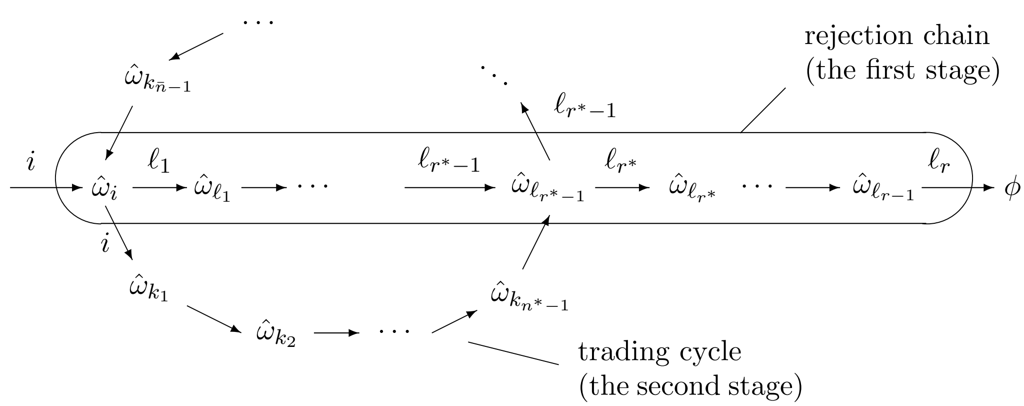

Figure A.1. A Trading Cycle and a Rejection Chain with Intersection

Let us continue the proof of the proposition. By Lemma A.3, given σ_i', $\hat{\omega}_i$ is the object where i is accepted for the first time after t^*. This implies that $\Phi^{a,\tau_1} = \eta^a$ for all objects a in $\bar{O}$ except for $\hat{\omega}_i$. When i comes to $\hat{\omega}_i$ at the step τ_0, there must be a player, say ℓ_1, in $\hat{\omega}_i$ because $\eta^a = \Phi^{a,t^*-1}$ $(a \in \bar{O})$ holds. And, ℓ_0 is rejected when i comes to $\hat{\omega}_i$ at step τ_1. A rejection triple $(\hat{\omega}_i, \ell_1, \tau_1)$ denotes the situation in which ℓ_1 is at $\hat{\omega}_i$ in step $\tau_1 - 1$ and rejected at $\hat{\omega}_{\ell_0}$ in step τ_1.

Note that after step t^*, DA ends at step τ_{r+1} when player ℓ_r, goes to ϕ after rejected from an object $\hat{\omega}_{\ell_{r-1}} \in O$. Also, after ℓ_1 is rejected from $\hat{\omega}_i$ at step τ_1, only one player goes to a new object at every step till τ_{r+1} under the assumptions.

Then, there is a chain of rejection triples $(\hat{\omega}_{\ell_0}, \ell_1, \tau_1)$, $\ldots$, $(\hat{\omega}_{\ell_{r-1}}, \ell_r, \tau_r)$, where $\ell_0 = i$ and for each $r' = 1, \ldots, r$, $\ell_{r'}$ is rejected at $\hat{\omega}_{r'-1}$ at $\tau_{r'}$. Note that $\hat{\omega}_{r'}$ is the object in O that is obtained by $\ell_{r'}$ in the first stage under $(\sigma_i', \hat{\sigma}_{-i})$. Note also that all the objects in this chain are distinct; otherwise, there is a cycle of priority. This can be shown by the same procedure as in the proof of Lemma A.3.

There exists an object in O^e that appears in the rejection chain since at least $\hat{\omega}_i$ is in O^e. Therefore, at least one player, either player i or the one who is rejected after step τ_1, goes to an object in O^e. Let $r^* = 1, \ldots, r$ be the greatest number among r's such that $\hat{\omega}_{\ell_{r'-1}}$ is in O^e, and $(\hat{\omega}_{\ell_{r'-1}}, \ell_{r'}, \tau_{r'})$ is in the rejection chain. Note that the last player ℓ_r in the chain goes to ϕ, which is not in O^e. Therefore, ℓ_{r^*} must go to some object not in O^e. This implies that ℓ_{r^*} is not in the trading cycle. Let n^* be a number such that $k_{n^*} = \ell_{r^*-1}$.

Note that $\eta_{\ell_{r^*}} = \hat{\omega}_{k_{n^*}}$. Also, a player k_{n^*-1} who is in the trading-cycle and will obtain $\hat{\omega}_{k_{n^*}}$ after the exchange satisfies that $\hat{\omega}_{k_{n^*}} P^0_{k_{n^*-1}} \eta_{k_{n^*-1}}$: otherwise k_{n^*-1} has not been rejected from η by step t^* under the truth-telling strategy, and this leads to a contradiction.

Then, equation (A.6) implies $\ell_{r^*} \succ_{\hat{\omega}_{k_{n^*}}} k_{n^*-1}$. Also, equation (A.3) implies that $k_{n^*} \succ_{\hat{\omega}_{k_{n^*}}} g$ holds.

Therefore, we can find a cycle of priority consisting of players in the trading cycle and ℓ_{r^*} rejected from $\hat{\omega}_{k_{n^*}}$ (note here that k_{n^*} is identical with ℓ_{r^*-1}; see Figure A.1),

$$k_{n^*} \succ_{\hat{\omega}_{k_{n^*}}} \ell_{r^*} \succ_{\hat{\omega}_{k_{n^*}}} k_{n^*-1} \cdots \succ_{\hat{\omega}_{k_1}} i \succ_{\hat{\omega}_i} k_{\bar{n}-1} \ldots k_{n^*+1} \succ_{\hat{\omega}_{k_{n^*+1}}} k_{n^*}.$$

Hence, this is a contradiction.

□

Appendix B. Mechanisms and Trading Cycles

B.1. **First-come-first-served rule (Boston mechanism).**

A *First-come-first-served rule (FCFS)* with priority, often called the Boston mechanism, is defined here as a game in extensive form. In the first round, each player $i \in N$ simultaneously chooses an object $c_i^1 \in A_i$. Each object in O is assigned to the players who choose it based on priority. In the tth round ($t = 2, 3, \ldots$), the remaining players, who have not been assigned to any object in O at the beginning of the t th round, simultaneously choose an object. A strategy of player $i \in N$ is the profile that consists of c_i^1 in the first round and c_i^t in each proper subgame of the tth round in which player i is a remaining player.

Formally, we define a function $\bar{\lambda}$ of players' actions in the following manner and then define the outcome function λ as a function of strategy profiles after that.

FCFS rule: Let $q^{a0} = q^a$, $\bar{\lambda}^{0a} = \emptyset$, $\bar{O}^0 = \bar{O}$ and $N^0 = N$ as initial values. Construct $(\bar{\lambda}^{ta}(\cdot))_{a \in \bar{O}}$ ($t = 1, 2, \ldots$) as follows.
t-th round ($t = 1, 2, \ldots$). Given a sequence of the first $(t-1)$ rounds choice profiles $h^t = (c^1, \ldots, c^{t-1})$, which we call a *history* at $t = 2, 3, \ldots$, where $c^\tau = (c_i^\tau)_{i \in N^{\tau-1}} \in \times_{i \in N^{\tau-1}} A_i \cap \bar{O}^{\tau-1}$ ($\tau = 1, \ldots, t$), and h^1 is the *null history*, and given $\bar{\lambda}^{t-1,a} = \bar{\lambda}^{t-1,a}(h^t)$ ($a \in \bar{O}$), let

$$\begin{aligned} q^{at} &= \max\{q^{a,t-1} - |\bar{\lambda}^{t-1,a}|, 0\}, \quad \text{(remaining quota)}, \\ \bar{O}^t &= \bar{O}^{t-1} \setminus \{a' \in \bar{O} | q^{a't} = 0\}, \quad \text{(remaining objects)}, \\ N^t &= N^{t-1} \setminus [\cup_{a \in \bar{O}} \bar{\lambda}^{t-1,a}], \quad \text{(remaining players)}. \end{aligned}$$

Suppose now that c^t is a choice profile by the players in N^t. Let $h^{t+1} = h^t \circ c^t$ be a concatenation. For each $a \in \bar{O}$, construct $\bar{\lambda}^{ta} = \bar{\lambda}^{ta}(h^t \circ c^t)$ in the following manner:

(1) $\bar{\lambda}^{ta} \subset N^a(c^t) \equiv \{j \in N^t | c_j^t = a\}$, (only applicants to a may be assigned to it),
(2) if the number of applicants to a exceeds its quota, i.e., $|N^a(c^t)| > q^{at}$, then the following two hold:
   (a) $|\bar{\lambda}^{ta}| = q^{at}$ (all the units of a are assigned);
   (b) $\forall i \in \bar{\lambda}^{ta}\ \forall j \in [N^a(c^t) \setminus \bar{\lambda}^{ta}]\ [i \succ_a j]$ (the assignment is based on priority in the same round);

(3) if the number of applicants to a does not exceed its quota, i.e., $|N^a(c^t)| \leq q^{at}$, then $\bar{\lambda}^{ta} = N^a(c^t)$ (all the applicants to a are accomodated).

For all $i \in N$ and all $a \in \bar{O}$, let $\bar{\lambda}_i(c^1, \ldots, c^t) = a$ if $i \in \cup_{\tau=1}^{t} \bar{\lambda}^{\tau a}$ holds.

Continue this process until N^t stops changing, i.e., until $t = T$ where $N^T = N^{T-1}$. If not, the game moves to the $(t+1)$ th round.

Call $h^{T+1} = (c^1, \ldots, c^T)$ the *complete history*.

If player $i \in N$ is not assigned to any object in $\bar{O}$ until T, then let $\bar{\lambda}_i(h^{T+1}) = \phi$. This completes the construction of $\bar{\lambda}$.

A strategy σ_i of a remaining player $i \in N^{t-1}$ in the tth round ($t = 1, 2, \ldots$) is a function from the set of histories into $A_i \cap \bar{O}^{t-1}$, i.e., given a history h^t in the t-th round, $\sigma_i(h^t) \in A_i \cap \bar{O}^{t-1}$ (only available objects can be chosen).

Based on the above construction of $\bar{\lambda}$ as a function of histories, we define the outcome function λ as a function of strategy profiles. Given a strategy profile $\sigma \in \Sigma$ and a history h^t for some $t = 0, 1, 2, \ldots$, $c^t = (\sigma_i(h^t))_{i \in N^{t-1}}$. This way, we induce the complete history h^{T+1} (the actual value of T depends on strategy profiles). Let $\lambda_i(\sigma) = \bar{\lambda}_i(h^T)$ if h^{T+1} is the complete history induced by σ. Let $\sigma|_{h^t}$ be the induced strategy profile after history h^t, and $\lambda_i(\sigma|h^t)$ be the eventual outcome of player $i \in N$ induced by σ after history h^t.

A couple of remarks are in order. First, one cannot choose an object that is no longer available, i.e., an object $a \in O$ with $q^{at} = 0$. Second, if a player chooses the null object in the tth round, then he/she "obtains" ϕ immediately and is not entitled to any other object in later rounds.

B.2. **Deferred acceptance algorithm.**
We define an allocation rule derived from the Deferred Acceptance algorithm. For each $i \in N$, let $\zeta_i = (\zeta_i^1, \ldots, \zeta_i^{|\bar{O}|})$ be an order of object types submitted by player i, and let ζ_i^* be the order of objects induced by R_i, or the truth-telling strategy of player i, i.e., for all $k, \ell = 1, \ldots, |\bar{O}|$, $k < \ell$ implies $\zeta_i^{*k} R_i^0 \zeta_i^{*\ell}$. Then, let Σ_i be the set of the orders of object types of player i. Also, let $\zeta = (\zeta_i)_{i \in N}$ and $\Sigma = \times_{i \in N} \Sigma_i$.

A *non-monetary allocation rule* is a function $\varphi : \Sigma \times \mathcal{S} \to \mathcal{A}$. Given $\zeta \in \Sigma$ and $\succ \in \mathcal{S}$, player $i \in N$ is assigned to object $\varphi_i(\zeta, \succ)$.

Let us now specify φ so that it reflects the Deferred Acceptance algorithm. Given $N' \subset N$ and $a \in \bar{O}$, let $\psi^a(N')$ be a subset of N' defined as follows: (i) if $|N'| \leq q^a$, then $\psi^a(N') = N'$ holds; (ii) if $|N'| > q^a$, then $|\psi^a(N')| = q^a$, and for all $i \in \psi^a(N')$ and all $j \in N' \setminus \psi^a(N')$, we have $i \succ_a j$. For all $N' \subset N$, $a \in \bar{O}$, and all $\nu \in \bar{O}^N$, let $I^a(\nu, N') = \{i \in N' | \nu_i = a\}$.

Given $\zeta \in \Sigma$ and $\succ$, we consider the following steps.

Step 1:

$$
\begin{aligned}
N^1 &= N, \\
n_i^1 &= 1, \ (i \in N) \\
\nu^1 &= (\zeta_i^{n_i^1})_{i \in N}, \\
\Phi^{a,1} &= \psi^a(I^a(\nu^1, N^1)) \ (a \in \bar{O}).
\end{aligned}
$$

Step t ($t = 2, 3, \ldots$):

$$\begin{aligned} N^t &= N \setminus (\cup_{a \in O} \Phi^{a,t-1}), \\ n_i^t &= \begin{cases} n_i^{t-1} + 1 & \text{if } i \in N^t, \nu_i^{t-1} \neq \phi, \text{ and } n_i^{t-1} < |\bar{O}|, \\ n_i^{t-1} & \text{otherwise, } (i \in N) \end{cases} \\ \nu^t &= (\zeta_i^{n_i^t})_{i \in N}, \\ \Phi^{a,t} &= \psi^a \left(I^a(\nu^t, N^t) \cup \Phi^{a,t-1} \right) \; (a \in \bar{O}). \end{aligned}$$

Continue this process until everyone is assigned to some object in $\bar{O}$, i.e., $\cup_{a \in \bar{O}} \Phi^{a,t} = N$. Then, let $\Phi^a = \Phi^{a,t}$ for each $a \in \bar{O}$.

Given ζ and $\succ$, an allocation rule φ is said to be the *Deferred Acceptance(DA) rule* if for all $i \in N$ and all $a \in \bar{O}$, $\varphi_i(\zeta, \succ) = a$ if and only if $i \in \Phi^a$.

Lemma B.1. *Assume (Scarcity) and (No money). Let $\eta = \varphi(\zeta^*, \succ)$. Suppose also that $\mu \neq \eta$ is stable. Then, μ is Pareto dominated by η.*

Proof.
Assume (Scarcity) and (No money). Suppose not, i.e., that a stable object allocation $\mu \neq \eta = \varphi(\zeta^*, \succ)$ is not Pareto dominated by η. Then under genericity, there exists $k_1 \in N$ such that $\mu_{k_1} P^0_{k_1} \eta_{k_1}$ holds. Take such a player k_1. Let $a = \mu_{k_1}$. In DA algorithm with ζ^*, in some step t_1, k_1 is rejected at a, i.e., $\nu_{k_1}^{t_1} = a$ and $k_1 \notin \Phi^{a,t_1}$. Therefore, we have $j \succ_a k_1$ for all $j \in \Phi^{a,t_1}$. Then, the stability of η implies $\forall j \in \Phi^{a,t_1}$ $\mu_j R^0_j a$ holds. By Lemma 5.5, the scarcity and stability of μ and η imply $|\mu^a| = |\eta^a| = q^a$ holds. Then, $\exists k_2 \in \Phi^{a,t_1}$ $\mu_{k_2} \neq a$. Note that $\mu_{k_2} P^0_{k_2} a$ holds; otherwise, μ is not stable because k_2 can deprive k_1 of a. Then, for such k_2, there exists a step t_2 s.t. $t_2 < t_1$ and k_2 is rejected at μ_{k_2}.

In this way, we can construct a sequence of players $\{k_n\}_{n=1}^{\infty}$ such that for each $n = 3, 4, \ldots$, there exists a step t_n such that $t_n < t_{n-1}$ and k_n is rejected at μ_{k_n}. This is a contradiction since there are finitely many steps in DA algorithm. □

The following proposition is a direct consequence of the above lemmata, which is stated without a proof.

Proposition B.2. *Assume (Scarcity) in Assumption 2.2 and (No money) in Assumption 5.1. Suppose μ is stable and Pareto optimal. Also, suppose η is a DA object allocation. Then, $\mu = \eta$.*

B.3. **Top trading cycles.** We define the top trading cycles due to Shapley and Scarf (1974) in this appendix.[11]

Definition B.1. *Assume (No money), (+Value), and (Quota1). Consider an object allocation $\mu \in \mathcal{A}^+$.*

The following is the top trading cycles *with an initial object μ.*

Given $N' \subset N$ and $\mu \in \mathcal{A}^+$, we define a trading cycle *among N' under μ as a nonempty subset S of N', whose $K-1$ members can be indexed in a cyclic order: $S =$*

[11] See also Kesten (2006) and Piccione and Rubinstein (2007).

$\{i_1, i_2, \ldots, i_{K-1}\}$ *with* $i_K = i_1$, *in such a way that each trader* i_k *(*$k = 1, \ldots, K-1$*) weakly prefers* $\mu_{i_{k+1}}$ *to* μ_j *for all* $j \in N'$.

We then define the following algorithm.

Step0: Assign $\mu \in \mathcal{A}^+$ *to the players in* N. *Let* $N_1 = N$. *Let* p_0 *be any positive number.*

Step $t(t \geq 1)$*: There is at least one trading cycle among* N_t *under* μ. *Take one of them and denote it* S_t, *which may be a singleton. Let* $N_{t+1} = N_t \setminus S_t$. *Let the price of all the objects held by the players in* S_t *be* p_t *satisfying* $p_t < p_{t-1}$.

Stop when N_{t+1}. *Otherwise, go to Step* $t + 1$.

Note that the above algorithm is terminated in a finite number of steps since at least one player is removed from the mechanism in each step. Since for all i in N, for all a in O, $v_i(a) > 0$ holds, ϕ is never chosen by any player until all the objects in O are removed.

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