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Dynamic Implementation, Verification, and Detection

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Abstract

We investigate implementation of social choice functions, where we impose severe restrictions on mechanisms, such as boundedness, permitting only tiny transfers, and uniqueness of an iteratively undominated strategy profile in the ex-post term. We assume that there exists some partial information about the state that is verifiable. We consider the dynamic aspect of information acquisition, where players share information, but the timing of receiving information is different across players. By using this aspect, the central planner designs a dynamic, not a static, mechanism, in which each player announces what he (or she) knows about the state at multiple stages with sufficient intervals. By demonstrating a sufficient condition on the state and on the dynamic aspect, namely full detection, we show that a wide variety of social choice functions are uniquely implementable even if the range of players' lies that the verified information can directly detect is quite narrow. With full detection, we can detect all possible lies, not by the verified information alone, but by processing a chain of detection triggered by this information. This paper does not assume either expected utility or quasi-linearity.

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1. Introduction

This paper investigates unique implementation of a social choice function (SCF), where the central planner attempts to achieve the allocation implied by the SCF that is contingent on the state. The central planner, however, cannot observe the state before determining an allocation. Hence, the central planner designs a mechanism to induce informed players to reveal their knowledge about the state. In this case, the mechanism must incentivize these players to make truthful announcements as *unique* equilibrium behavior[3]. The requirement of uniqueness is a quite substantial restriction in the implementation problem. The basic problem is therefore to clarify whether the central planner can design such an effective mechanism.[4]

This paper has two main departures from the previous works on implementation. First, this paper permits some partial information about the state to become *verifiable* after the central planner determines an allocation. For example, by conducting a follow-up survey, the central planner can obtain a resultant consequence of his (or her) allocation decision, which is verifiable and includes a partial information about the state. The central planner can utilize this verified information as a clue to detecting players' lying.

Second, this paper seriously, and carefully, considers a *dynamic* aspect of players' information acquisition. We assume that each player receives information about the state not all at once but sequentially. The central planner requires each player to announce what he (or she) knows at the early stage, i.e., at the first stage. He also requires this player to announce what he knows at the later stage, i.e., at the second stage, where he is more informed than at the first stage. Hence, the central planner requires each player to announce at multiple stages with sufficient intervals.

By making the monetary transfers contingent on the verified information as well as their announcements, the central planner attempts to design not a static mechanism, but a *dynamic mechanism*, which can effectively penalize any detected liar, making players willing to tell the truth. The purpose of this paper is to clarify the extent to which a wider

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[3] This paper does not investigate direct mechanisms, but indirect mechanisms, in which each player is required to announce what he (or she) knows many times.

[4] For the surveys on implementation theory, see Moore (1992), Palfrey (1992), Osborne and Rubinstein (1994, Chapter 10), Jackson (2001), and Maskin and Sjöström (2002), for instance.

variety of SCFs are uniquely implementable with partial verification and multi-stage revelation than without them.

This paper makes numerous severe restrictions on mechanism design. First, we select *iterative dominance* as the equilibrium concept, which is defined as the set of all strategy profiles that survive through the iterative removal of strategies that are dominated in the ex-post term, i.e., weakly dominated at every state, and strictly dominated at some states. This is the set of all strategy profiles that survive through the iterative removal of strategies that are strictly dominated irrespective of the specification of the full-support prior distribution. We then require the uniqueness of an iteratively undominated strategy profile. Since this iterative dominance notion is a very weak equilibrium concept, our uniqueness requirement should be a very severe restriction.

Second, we require a mechanism to be *detail-free* in terms of prior distribution on the state space. Since our definition of iterative dominance is on the ex-post term, the mechanism must be made independent of the specification of prior distribution.

Third, we require a mechanism to be *bounded* in that it is not incorporated with any construction that has no equilibrium, such as the integer game. Because of this boundedness requirement, we focus on a class of mechanisms in which the message space is finite for each player.

Fourth, we permit only *tiny* transfers because of players' limited liability. To be precise, we require any transfer to be close to zero off the equilibrium path and no transfers on the equilibrium path.

The above requirements will make solving our implementation problem challenging. This paper demonstrates a sufficient condition on the state space and the dynamic aspect of information acquisition, under which an SCF is uniquely implementable in iterative dominance with partial verification, where we design a mechanism that is dynamic, bounded, and detail-free, and utilizes only tiny transfers.

To design an effective bounded mechanism with tiny transfers, we will apply the basic concept of mechanism design that originates in Abreu and Matsushima (1992a, 1992b, 1994), where the central planner requires each player to make multiple announcements at once, randomly selects one announcement profile from their announcements, and fines the first deviants from some *reference*. Once we can establish the reference truthfully, the mechanism design á la Abreu-Matsushima properly motivates all players to make truthful announcements, successfully implementing the SCF. The remaining problem is therefore to clarify the manner in which we can establish such a truthful reference.

This remaining problem appears difficult to solve, because the relevance of the verified information to the state is limited and the prior distribution is unspecified in our model. In fact, the range of players' lies that the verified information can directly detect is quite narrow, and we cannot even apply any device of a proper scoring rule to incentivize players to truthfully reveal the distributions implied by their private information.

This paper will overcome this difficulty by considering the dynamic aspect of players' information acquisition as follows. The main part of this paper assumes complete information in that there exist three or more players and the state is common knowledge among them immediately before the central planner determines the allocation. This assumption automatically guarantees incentive compatibility by using direct mechanisms such as majority rules, where truth-telling is a Nash equilibrium. However, in such mechanisms, it is also a Nash equilibrium for all players to tell the same lie, violating uniqueness. To overcome the difficulty of uniqueness, we design a multi-stage dynamic mechanism where we assume that the timing of receiving information is different across players, and the central planner can require a player to announce his private information, i.e., partial information about the state, before all players observe this information.

We introduce a concept concerning this dynamic aspect, namely "chain of detection", as follows. The verified information detects a limited, but non-empty, class of some players' lies at the first stage, which motivates these players to reveal their respective aspects of the state truthfully at the first stage. This truthful revelation along with the verified information detects another class of lies at the first stage, which motivates the relevant players to reveal other aspects of the state truthfully at the first stage; and so on.

This paper demonstrates a condition on the state space and the dynamic aspect of information acquisition, namely *full detection*, implying that there exists such a chain of detection, through which we can iteratively detect all possible lies at the first stage. Hence, with full detection, truth-telling is the only announcement for each player that survives through the iterative removal of detected lies at the first stage.

By penalizing detected liars in an appropriate manner, with full detection we can make all players reveal their private information truthfully at the first stage. Hence, we can establish the truthful reference as the combination of the verified information and their truthfully announced private information.

Based on these observations, we show as the main theorem of this paper that full detection is a sufficient condition, under which an SCF is uniquely implementable in iterative dominance with partial verification, where the designed mechanism is dynamic, bounded, detail-free, and permits only tiny transfers.

Full detection appears to be an involved condition. In fact, to detect a player's lie, we must find out a state at which the other players never announce any message profile that they may possibly announce provided his lie is true. However, by demonstrating a tractable sufficient condition for full detection, we can show that despite this complexity, full detection covers a wide range of state space formulations. This contrasts with the case without verification, where any non-trivial deterministic SCF is never implementable in the exact term.

To detect all possible lies, the central planner designs a dynamic mechanism, not a static mechanism, in which each player is required to make announcements twice at two distinct stages, i.e., at the first stage and at the second stage. The central planner regards their first announcements along with the verified information as the reference, while he utilizes only their second announcements for the determination of allocation.

At the first stage, each player is informed of his private information that the central planner asks him to reveal, but he is less informed than at the second stage. By requiring players to make announcements when they are less informed, the central planner can prevent them from finding a means of escape from detection, making the truthful reference easier to be established.

Let us consider an example with $n \geq 2$ players, where a state is described by $\omega = (\omega_0, \omega_1, ..., \omega_n)$. Assume that ω_i is either 1, 2, or 3 for each $i \in \{0,1,...,n\}$, and ω_0 is ex-post verifiable. The state space Ω is defined as a subset of $\{1,2,3\}^{n+1}$. At the first stage, each player $i \in \{0,1,...,n\}$ observes ω_i as his private information, and the central planner requires him to announce it.

If $\Omega = \{1,2,3\}^{n+1}$, i.e., there exists no state that can be ignored, it is impossible to detect any lie. Let us suppose that Ω is a *proper* subset of $\{1,2,3\}^{n+1}$, and that each player's observation is always *different* from his neighbor's observation, i.e., $\omega \in \{1,2,3\}^{n+1}$ belongs to Ω if and only if

$$\omega_i \neq \omega_{i-1} \text{ for all } i \in \{1,...,n\}.$$

Because of this proper-subset nature, the verified information ω_0 can directly detect any lie about ω_1, because player 1 cannot exclude the possibility that his lie $\omega_1' \neq \omega_1$ is equivalent to ω_0, i.e., $\omega_1' = \omega_0$. This motivates player 1 to tell the truth about ω_1. In the same manner, the truthful announcement about ω_1 can detect any lie about ω_2. This motivates player 2 to tell the truth about ω_2. Recursively, any player $i \in \{1,...,n\}$ is well motivated to tell the truth about ω_i, implying full detection.

In the process of such iterative removal of detected lies, it is important to assume that each player i is not informed of ω_{i-1} at the first stage. Otherwise, he can find a way to escape from detection by announcing $\tilde{\omega}_i \notin \{\omega_i, \omega_{i-1}\}$. Hence, it is crucial for the central planner to require each player i to announce about ω_i before he observes ω_{i-1}.

This paper further investigates the case in which full detection does not hold, i.e., the case of *partial detection*. By replacing uniqueness of strategy by uniqueness of outcome, we define *full* implementation in iterative dominance. We define a concept of measurability of an SCF with respect to partial detection. We then show that with partial detection, this measurability is sufficient for the SCF to be fully implementable in iterative dominance.

Throughout this paper, we mostly assume complete information at the second stage. Without any substantial modification, however, we can replace this complete information by a class of *incomplete* information, where each player can observe not all players' private information, but some players' private information, at the second stage. By requiring a version of ex-post incentive compatibility, we show that the same arguments hold even in this class of incomplete information environments.

If players share information about the state, i.e., if players' observations overlap each other, the requirement of incentive compatibility is less restrictive than otherwise.

However, the requirement of uniqueness is more restrictive, because players are given more room to coordinate undetected lies.

This paper shows that we can overcome this tradeoff between incentive compatibility and uniqueness by incorporating both the revelation at the early stage, where players are less informed than at the later stage, and the revelation at the later stage into dynamic mechanism design. In this manner, we can effectively separate the issue of incentive compatibility, which is relevant to the informational structure at the second stage, from the issue of uniqueness, i.e., the establishment of reference at the first stage. This finding is in contrast with the previous literature of robust implementation such as Bergemann and Morris (2009), which generally used only static mechanisms[5].

We should refer to another departure from the previous works; we do not assume either expected utility or quasi-linearity. This paper makes only basic assumptions on preferences such that each player's utility function is continuous in lottery over allocations, and is continuous and increasing in monetary transfer.

The organization of this paper is as follows. Section 2 explains related literature. Section 3 shows the basic model. Section 4 investigates the case of full verification. Section 5 investigates the case of partial verification. Section 6 introduces the concepts of detection and full detection, and demonstrates the main theorem of this paper. Section 7 investigates the case of partial detection. Section 8 investigates a class of incomplete information at the second stage. Section 9 discusses the generalization of detection. Section 10 concludes.

2. Literature Review

The basic framework for the implementation problem was explored by Hurwicz (1972) and Maskin (1999). Maskin showed that monotonicity is a necessary condition for a social choice correspondence to be fully implementable in Nash equilibrium. This result should be regarded as being negative, because monotonicity is a quite demanding condition for a deterministic SCF. In fact, with some additional restrictions, any

5 Bergemann and Morris (2009) investigated not exact, but virtual, implementation. Our arguments do not utilize any device of virtualness at all. The requirement of exactness is very crucial for our arguments.

deterministic SCF that is fully implementable in Nash equilibrium must be dictatorial. The purpose of this paper is therefore to show permissive results for full, or unique, implementation.

Matsushima (1988) and Abreu and Sen (1991) showed a permissive result that any full-support stochastic SCF is monotonic, and therefore, fully implementable in Nash equilibrium. Hence, if the tiny probability to select unwanted allocations even on the equilibrium path is permitted, even any deterministic SCF becomes fully implementable in Nash equilibrium, not in the exact sense, but in the virtual sense. In contrast, this paper sticks to exact implementation.

Moore and Repullo (1988) and Palfrey and Srivastava (1991) replaced Nash equilibrium with their respective refinements such as subgame perfect equilibrium and undominated Nash equilibrium. Abreu and Matsushima (1994) replaced Nash equilibrium with weak iterative dominance, or according to the terminology of Moulin (1979), dominance solvability, and then showed a permissive result for unique implementation by permitting just tiny monetary transfers off the equilibrium path. Chen, Kunimoto, and Chung (2015) extended this result to the Bayesian environments. Because of the use of refinement, these works commonly permitted the existence of Nash equilibria that fail to achieve the value of the SCF. In contrast, this paper will replace Nash equilibrium with an even weaker equilibrium concept, namely iterative dominance in the ex-post term, eliminating all unwanted Bayesian Nash equilibria irrespective of the specification of the full-support prior distribution.

Many previous works in the implementation literature have constructed mechanisms that have "implausible" features where the mechanisms are incorporated with constructions that have no equilibrium, such as the integer games. To exclude such constructions that are implausible, or according to the terminology of Jackson (1992), are unbounded, Abreu and Matsushima (1992a) innovated a new method, namely the AM mechanism design, that makes the mechanism bounded by permitting only a finite strategy space for each player. Abreu and Matsushima then showed a very permissive result for unique virtual implementation in iterative dominance. Abreu and Matsushima (1992b) extend this result to the Bayesian environments. Abreu and Matsushima (1994) and Chen, Kunimoto, and Chung (2015) also utilized the AM mechanism design for exact implementation, by replacing iterative dominance with weak iterative dominance. The present paper will apply the AM bounded mechanism design for exact implementation without replacing iterative dominance with any stronger equilibrium concept such as weak iterative dominance.

In the Bayesian framework, the designed mechanisms generally depend on the fine details of a fixed prior distribution. Bergemann and Morris (2009) emphasized the importance of detail-free mechanism design and the usage of ex-post equilibrium concepts. This paper defines the iterative dominance notion on the ex-post term and usage of design mechanisms that are detail-free in terms of the prior distribution.

Based on these backgrounds, this paper is the first work to show the permissive result for exact implementation of SCFs by using only detail-free bounded mechanisms with tiny transfers.

The construction in this paper is divided into two parts, i.e., the application of the AM mechanism design, and the establishment of the truthful reference. The technical contribution of this paper is mainly devoted to the latter part.

To establish the reference truthfully, the pioneering works such as Abreu and Matsushima (1992a, 1992b, 1994) have utilized the incentive devices of "virtualness". Bergemann and Morris (2009) investigated virtual implementation from the viewpoint of robustness. Alternatively, Matsushima (2008a, 2008b) assumed the presence of a tiny psychological cost of dishonesty for a player, and then incentivized him to make truthful announcements for the reference. In contrast to these works, this paper will not utilize either the incentive device of virtualness or the psychological cost of dishonesty.

To establish the truthful reference, this paper demonstrates an alternative approach by assuming that some partial information about the state is verifiable. There exist many previous works, such as Hansen (1985), Mezzetti (2004), DeMarzo, Kremer, and Skrzypaz (2005), Mylovanov and Zapechelnyuk (2014), Deb and Mishra (2014), and Carroll (2015), that incorporated such verification into the problems of mechanism design. These works commonly showed that the presence of verification makes incentive compatibility more easily satisfied. In contrast, this paper's concern is the impact of verification, not on incentive compatibility, but rather on uniqueness of equilibrium. In this respect, this paper is the first attempt to incorporate verification into the unique or full implementation theory.

This paper makes an important contribution to dynamic mechanism design, which is currently a growing concern; for example, see the survey by Krähmer and Strausz (2005). In particular, Penta (2015) is related to this paper, because Penta investigated full implementation in the incomplete information environments where players receive information over time. Penta selected perfect Bayesian equilibrium as the solution concept, which is much stronger than the concept of iterative dominance in this paper, while Penta carefully considered the robustness issue. Compared with Penta, we confine our attention to a more special class of incomplete information environments, where players share information. This confinement serves to highlight the contrast between static mechanism design and dynamic mechanism design in terms of uniqueness. It might be important for future research to extend the analysis of this paper to more general classes of incomplete information environments.

The literature of persuasion games is also related to this paper, where players voluntarily reveal verifiable information, i.e., hard evidence, which can partially prove their announcements to be correct, encouraging correct public decision making; for instance, see Grossman (1981) and Kamenica and Gentzkov (2011). Kartik and Tercieux (2012) and Ben-Porath and Lipmann (2012) investigated full implementation with such hard evidence, stating that the great degree to which hard evidence directly proves players' announcements to be correct is crucial in implementing a wide variety of SCFs. In contrast, this paper emphasizes that a wide variety of SCFs are implementable even if the verifiable information is quite limited.

To show such permissive results even with limited verification, this paper assumes that the state space is common knowledge and the central planner can make the mechanism dependent on the state space. We then demonstrate a condition concerning the shape of state space, namely full detection, which guarantees any SCF to be implementable.

Full detection assumes that there exists a rare event, to which each player assigns a probability of occurrence of zero, and therefore the event can be ignored. As pointed out by the authors in behavioral economics, such as Camerer and Kunreuther (1989), real people tend to assign a rare event with probability zero because of their psychological biases such as the optimistic bias. This justifies the relevancy of full detection.

The literature on models of knowledge has discussed "the puzzle of the hats," the idea that a tiny information release has a big influence on players' reasoning. In contrast, this paper focuses on the influence of tiny information releases, i.e., verified information, on player's incentives. This information should be hidden from players when they make announcements for the purpose of establishing the reference.

The method of the AM mechanism design has long been criticized without any formal analysis, because of the conjecture that this method crucially depends on the expected utility assumption. This paper will prove that this criticism is groundless. The functioning of the AM mechanism relies just on the local linearity of preferences, implying the irrelevance of global linearity such as expected utility and quasi-linearity. Hence, this paper will be expected to promote the popularity of this essentially powerful method[6].

3. The Model

We consider a situation in which the central planner determines an allocation and makes monetary transfers. Let $N \equiv \{1,...,n\}$ denote the finite set of all players, where we assume $n \geq 3$ except for Section 8. Let A denote the finite set of all allocations. Let Δ denote the set of all lotteries over allocations[7]. Let Ω denote the finite set of all states, i.e., the finite state space. A social choice function, an SCF, is defined as $f : \Omega \to \Delta$.

We define the state-contingent utility function for each player $i \in N$ as

$$u_i : \Delta \times R \times \Omega \to R.$$

where $u_i(\alpha, t_i, \omega)$ implies the utility for player i when he (or she) expects the state ω to occur, and the central planner to determine an allocation according to the lottery $\alpha \in \Delta$, and make a monetary transfer $t_i \in R$ to player i. Let $u \equiv (u_i)_{i \in N}$.

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[6] There are controversies on the practical usage of Abreu-Matsushima mechanisms that are related to level-K reasoning and focal points. See Glazer and Rosenthal (1992) and Abreu and Matsushima (1992c).

[7] We denote $\alpha \in \Delta$. We write $\alpha = a$ if $\alpha(a) = 1$.

We assume that $u_i(\alpha, t_i, \omega)$ is *continuous* with respect to $\alpha \in \Delta$ and $t_i \in R$, and that $u_i(\alpha, t_i, \omega)$ is *increasing* in t_i. Importantly, this paper does not assume expected utility and quasi-linearity. Let U_i denote the set of all utility functions for player i.

4. Full Verification

As a benchmark for this paper's analysis, this section will assume *full verification* as follows. After the central planner determines an allocation, but before he makes monetary transfers, the state becomes public and verifiable to the court. The central planner can make the monetary transfers contingent on the state as well as the players' announcements, but he cannot make the allocation choice contingent on the state.

We define a static mechanism, or shortly, a *mechanism*, as $\mathrm{G} \equiv (M, g, x)$, where $M \equiv \underset{i \in N}{\times} M_i$, M_i denotes the set of all *messages* of player i, $g : M \to \Delta$ denotes the *allocation rule*, $x \equiv (x_i)_{i \in N}$ denotes the *transfer rule*, and $x_i : M \times \Omega \to R$ denotes the transfer rule for player i. We confine our attention to mechanisms such that M_i is *finite* for all $i \in N$; i.e., we focus on a class of mechanisms that are so-called bounded.

This section assumes complete information in that each player observes the state $\omega \in \Omega$, while the central planner cannot observe it before his allocation choice. Each player $i \in N$ announces a message $m_i \in M_i$ that is contingent on the state ω. The central planner then determines an allocation according to the lottery $g(m) \in \Delta$, where $m \equiv (m_i)_{i \in N} \in M$ denotes the message profile. After the state ω becomes verifiable, the central planner receives the monetary transfer $x_i(m, \omega) \in R$ from each player i.

A *strategy for each player* i *in a mechanism* G is defined as $s_i : \Omega \to M_i$. Player i announces the message $s_i(\omega) \in M_i$ when he observes ω. Let S_i denote the set of all strategies for player i. Let $S \equiv \underset{i \in N}{\times} S_i$ and $s \equiv (s_i)_{i \in N} \in S$.

4.1. Iterative Dominance

We introduce the equilibrium concept, namely *iterative dominance*, which is defined as the survival of iterative removal of messages that are dominated with strict inequality in the ex-post term, in the following manner. For every $i \in N$ and $\omega \in \Omega$, let

$$M_i(0,\omega) \equiv M_i .$$

Recursively, for each $h \geq 1$, we define a subset of player $i's$ messages $M_i(h,\omega) \subset M_i$ in the manner that $m_i \in M_i(h,\omega)$ if and only if there exists no $m_i' \in M_i(h-1,\omega)$ such that for every $m_{-i} \in M_{-i}(h-1,\omega)$,

$$u_i(g(m), -x_i(m,\omega), \omega) < u_i(g(m_i', m_{-i}), -x_i(m_i', m_{-i}, \omega), \omega),$$

where $M_{-i}(h-1,\omega) \equiv \underset{j \in N \setminus \{i\}}{\times} M_j(h-1,\omega)$. Here, we require each player i to prefer m_i' to m_i irrespective of $m_{-i} \in M_{-i}(h-1,\omega)$. We require strict inequalities for the iterative steps of eliminating dominated messages.

Note that $M_i(h,\omega) \subset M_i(h-1,\omega)$. Define

$$M_i(\infty,\omega) \equiv \bigcap_{h=0}^{\infty} M_i(h,\omega).$$

Definition 1: A strategy $s_i \in S_i$ for player i is said to be *iteratively undominated in* G *with full verification* if

$$s_i(\omega) \in M_i(\infty,\omega) \text{ for all } \omega \in \Omega .$$

Because of the requirement of strict inequalities, the order of elimination does not matter in the definition of iterative dominance. By requiring the uniqueness of an iteratively undominated strategy profile, we define unique implementation in iterative dominance with full verification as follows.

Definition 2: A mechanism G is said to *uniquely implement an SCF* f *in iterative dominance with full verification* if there exists the unique iteratively undominated strategy profile $s \in S$ in G, i.e.,

$$\bigcap_{h=0}^{\infty} M_i(h,\omega) = \{s_i(\omega)\} \text{ for all } \omega \in \Omega \text{ and } i \in N,$$

and it induces the value of the SCF, i.e.,

$$g(s(\omega)) = f(\omega) \text{ for all } \omega \in \Omega .$$

The definition of iterative dominance is independent of the specification of the prior distribution on Ω. Hence, the mechanism that uniquely implements an SCF in iterative dominance with full verification is independent of the specification of the prior distribution on Ω.

4.2. Construction of Mechanisms

Fix arbitrary real numbers $\eta_1 > 0$ and $\eta_2 > 0$. Fix an arbitrary integer $K > 0$. We construct a mechanism $G^* = G^*(f, \eta_1, \eta_2, K) = (M, g, x)$ in the following manner. For every $i \in N$, let

$$M_i = \underset{k=1}{\overset{K}{\times}} M_i^k ,$$

and

$$M_i^k = \Omega \text{ for all } k \in \{1,...,K\} .$$

Player $i \in N$ announces K *sub-messages* $m_i^k \in M_i^k$ at once.

For each $k \in \{1,...,K\}$, we define $g^k : M^k \to \Delta$ in the manner that for each $\omega \in \Omega$,

$$g^k(m^k) = f(\omega) \qquad \text{if } m_i^k = \omega \text{ for at least } n-1 \text{ players,}$$

and

$$g^k(m^k) = a^* \qquad \text{if there exists no such } \omega,$$

where $a^* \in A$ is an arbitrary allocation, which is regarded as the status quo allocation. This specification is well-defined because we assumed $n \geq 3$. Let

$$g(m) = \frac{\sum_{k=1}^{K} g^k(m^k)}{K} .$$

The central planner randomly selects an integer k from $\{1,...,K\}$, and determines an allocation according to $g^k(m^k) \in \Delta$. The central planner selects an allocation according to the value of the SCF, i.e., $f(\omega)$, if at least $n-1$ players i announce $m_i^k = \omega$, where we assumed $n \geq 3$. Otherwise, he selects the status quo allocation a^*.

Let

$$x_i(m,\omega) = \eta_1 + \frac{r_i}{K}\eta_2 \qquad \text{if there exists } k \in \{1,...,K\} \text{ such that}$$
$$m_i^k \neq \omega \text{, and}$$
$$m_j^{k'} = \omega \quad \text{for all } k' < k \text{ and } i \in N,$$

and

$$x_i(m,\omega) = \frac{r_i}{K}\eta_2 \qquad \text{if there exists no such } k \in \{1,...,K\},$$

where $r_i \in \{0,...,K\}$ denotes the number of integers $k \in \{1,...,K\}$ such that $m_i^k \neq \omega$.

If a player is one of the first deviants from ω, i.e., one of the players who tell lies as the earliest sub-message among all liars, he is fined the monetary amount η_1. Any player $i \in N$ is fined the monetary amount $\frac{r_i}{K}\eta_2$. That is, by announcing any single sub-message dishonestly, the player is fined the monetary amount $\frac{\eta_2}{K}$.

Since

$$0 \leq x_i(m,\omega) \leq \eta_1 + \eta_2,$$

by selecting $\eta_1 + \eta_2$ close to zero, we can make the monetary transfer $x_i(m,\omega)$ as close to zero as possible.

We denote a strategy $s_i = (s_i^k)_{k=1}^K$, where $s_i^k : \Omega \to M_i^k$. We define the *honest* strategy for player i, $s_i^* = (s_i^{*k})_{k=1}^K$, as

$$s_i^{*k}(\omega) = \omega \quad \text{for all } k \in \{1,...,K\} \text{ and } \omega \in \Omega.$$

The honest strategy profile $s^* \equiv (s_i^*)_{i \in N}$ induces the value of the SCF f in G^*, i.e.,

$$g(s^*(\omega)) = f(\omega) \quad \text{for all } \omega \in \Omega,$$

and no monetary transfers, i.e.,

$$x_i(s^*(\omega),\omega) = 0 \quad \text{for all } i \in N \text{ and } \omega \in \Omega.$$

The construction of the mechanism G^* is based on the bounded mechanism design that originates in Abreu and Matsushima (1992a, 1992b, 1994). Abreu and Matsushima demonstrated the basic concepts relevant to G^*, such that each player announces multiple sub-messages at once, and the central planner randomly selects one sub-message profile and he fines the first deviants.

There is a substantial difference between this paper and Abreu and Matsushima in that we do not utilize any incentive device of "virtualness" that originates in Matsushima (1988) and Abreu and Sen (1991). Virtualness permits the selection of an undesirable allocation even on the equilibrium path. In contrast, this paper does not permit such selections at all, i.e., it requires a mechanism to achieve, not virtually, but exactly, the value of the SCF.[8]

4.3. Possibility Theorem

Since $u_i(\alpha, t_i, \omega)$ is continuous in (α, t_i) and increasing in t_i, we can select a sufficient K such that whenever

$$\max_{a \in A} |\alpha(a) - \alpha'(a)| \leq \frac{1}{K},$$

then

(1) $$u_i(\alpha, -t_i, \omega) > u_i(\alpha', -t_i - \eta_1, \omega) \text{ for all } t_i \in [0, \eta_2] \text{ and } \omega \in \Omega.$$

The inequalities in (1) imply that a first-deviant's loss from the monetary fine η_1 is always greater than his gain from the change of allocation caused by his lying.

The following theorem shows that G^* uniquely implements f in iterative dominance with full verification. Since G^* is well-defined, we can conclude that *with full verification, any SCF is uniquely implementable in iterative dominance, where we need almost no monetary transfers off the equilibrium path, and need no monetary transfers on the equilibrium path.*

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[8] Abreu and Matsushima (1994) showed a possibility theorem in exact implementation, where the iterative removal of strictly dominated strategies was replaced with the iterative removal of weakly dominated strategies.

Theorem 1: *The honest strategy profile* s^* *is the unique iteratively undominated strategy profile in* G^* *with full verification.*

Proof: We can show that each player $i \in N$ prefers $m_i^1 = \omega$. Suppose that there exists a player $j \in N \setminus \{i\}$ who announces $m_j^1 \neq \omega$. Then, by announcing $m_i^1 \neq \omega$ instead of ω, player i is fined η_1 or even more. From (1), the impact of the fine η_1 on his welfare is greater than the impact of the resultant change of allocation.

Next, suppose that there exists no player $j \in N \setminus \{i\}$ who announces $m_j^1 \neq \omega$. Then, by announcing $m_i^1 \neq \omega$ instead of ω, player i is fined $\frac{\eta_2}{K}$ or even more. (Even if he announces $m_i^1 = \omega$, he may still be one of the first deviants, and therefore, he may not avoid the fine η_1 in this case.) From the specification of g and the assumption of $n \geq 3$, there is no resultant change of allocation. These observations imply that he prefers $m_i^1 = \omega$ regardless of the other players' announcements.

Fix an arbitrary integer $h \in \{2,...,K\}$. We can repeat the same logic as above in the following manner. Suppose that each player $i \in N$ announces $m_i^{h'} = \omega$ for all $h' \in \{1,...,h-1\}$. According to the same manner as above, we can show that he prefers $m_i^h = \omega$. Suppose that there exists a player $j \in N \setminus \{i\}$ who announces $m_j^h \neq \omega$. Then, by announcing $m_i^h \neq \omega$ instead of ω, player i is fined η_1 or even more. From (1), the impact of the fine η_1 on his welfare is greater than the impact of the resultant change of allocation. Next, suppose that there exists no player $j \in N \setminus \{i\}$ who announces $m_j^h \neq \omega$. Then, by announcing $m_i^h \neq \omega$ instead of ω, player i is fined $\frac{\eta_2}{K}$ or even more. The specification of g implies that there is no resultant change of allocation in this case. Hence, he prefers $m_i^h = \omega$.

Q.E.D.

5. Partial Verification

From this section, let us describe a state as

$$\omega = (\omega_0, \omega_1, ..., \omega_n).$$

For each $i \in N \bigcup \{0\}$, Ω_i denotes the set of possible ω_i. Let $\Omega_{-i} \equiv \underset{j \in N \bigcup \{0\} \setminus \{i\}}{\times} \Omega_j$, $\omega_{-i} \equiv (\omega_j)_{j \in N \bigcup \{0\} \setminus \{i\}} \in \Omega_{-i}$, $\Omega_{-i-j} \equiv \underset{l \in N \bigcup \{0\} \setminus \{i,j\}}{\times} \Omega_l$, and $\omega_{-i-j} \equiv (\omega_l)_{l \in N \bigcup \{0\} \setminus \{i,j\}} \in \Omega_{-i-j}$.

We assume that Ω is a proper subset of $\underset{i \in N \bigcup \{0\}}{\times} \Omega_i$. Each player $i \in N$ regards the set difference $\underset{i \in N \bigcup \{0\}}{\times} \Omega_i \setminus \Omega \neq \phi$ as the rare event the occurrence of which can be ignored.

Let $\Omega_{-i}(\omega_i) \subset \Omega_{-i}$ denote the set of possible ω_{-i} such that $(\omega_i, \omega_{-i}) \in \Omega$. We assume that for every $\omega_i \in \Omega_i$, $\Omega_{-i}(\omega_i)$ is nonempty.

We assume *partial verification* as follows. After the central planner determines an allocation, but before he determines monetary transfers, only ω_0 becomes public and verifiable to the court. The remaining part of the state, ω_{-0}, is unverifiable throughout. In this case, players have incentive to announce dishonestly about ω_{-0}. Hence, as far as we stick to static mechanism design, it is generally impossible to derive a permissive result in exact implementation with no verification about ω_{-0}. This motivates us to investigate dynamic mechanisms instead of static mechanisms, which will be addressed in the next subsection.

5.1. Dynamic Mechanisms

We consider the following two-stage procedure. At the first stage, each player $i \in N$ observes ω_i, i.e., observes ω_i earlier than $\omega_{-i} = (\omega_j)_{j \in N \bigcup \{0\} \setminus \{i\}}$, as his private information. Each player i does not observe ω_{-i} at this stage. Hence, player i is the first person who observes ω_i among all players. The central planner then requires each player i to announce about ω_i as his first announcement.

At the second stage, each player $i \in N$ observes the remaining part of the state ω_{-i}. Hence, at the second stage, the state ω becomes common knowledge among all players.

The central planner then requires each player i to announce about the state ω as his second announcement. In other words, this section assumes incomplete information at the first stage, but assumes complete information at the second stage.

Based on the above-mentioned requirements of two announcements, we define a *dynamic mechanism* as $\Gamma \equiv (M^0, M, g, x)$, where $M^0 \equiv \underset{i \in N}{\times} M_i^0$, M_i^0 denotes the set of possible first announcements by player i, $M \equiv \underset{i \in N}{\times} M_i$, M_i denotes the set of possible second announcements by player i, $g: M^0 \times M \to \Delta$ denotes the allocation rule, $x \equiv (x_i)_{i \in N}$ denotes the transfer rule, and $x_i : M^0 \times M \times \Omega_0 \to R$ denotes the transfer rule for player i. We assume that both M_i^0 and M_i are finite sets for each $i \in N$; i.e., we focus on a set of bounded dynamic mechanisms.

After observing ω_i, but before observing ω_{-i}, i.e., at the first stage, each player i makes his first announcement $m_i^0 \in M_i^0$. After observing ω_{-i}, i.e., at the second stage, each player i makes his second announcement $m_i \in M_i$. The central planner then selects an allocation according to $g(m_0, m) \in \Delta$. After ω_0 becomes verifiable, the central planner receives $x_i(m^0, m, \omega_0) \in R$ from each player i.[9]

We will assume imperfect information in that each player cannot observe the other players' first and second announcements.

A *strategy for player* i *in a dynamic mechanism* Γ is defined as $\psi_i \equiv (s_i^0, s_i)$, where $s_i^0 : \Omega_i \to M_i^0$ and $s_i : \Omega \to M_i$. He announces $s_i^0(\omega_i) \in M_i^0$ as his first announcement. He announces $s_i(\omega) \in M_i$ as his second announcement, provided he announced $s_i^0(\omega_i)$ at the first stage. Because of the imperfect information assumption, his second announcement does not depend on the other players' announcements. Let S_i^0 denote the set of possible s_i^0. Let $\Psi_i \equiv S_i^0 \times S_i$ denote the set of all strategies for player i in Γ. Denote $S^0 \equiv \underset{i \in N}{\times} S_i^0$, $\Psi \equiv \underset{i \in N}{\times} \Psi_i$, and $\psi \equiv (\psi_i)_{i \in N} \in \Psi$.

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[9] This paper assumes that ω_0 becomes verifiable after the allocation is selected. This assumption is crucial in the case of full verification. Without this assumption, implementation is trivial in the case of full verification, while it is still problematic in the case of partial verification. For the case of partial verification, we can eliminate this assumption without any substantial change of the arguments.

5.2. Iterative Dominance

With partial verification, we define *iterative dominance* as follows. For every $i \in N$, let

$$\hat{M}_i^0(0,\omega_i) \equiv M_i^0 \text{ for all } \omega_i \in \Omega_i ,$$

and

$$\hat{M}_i(0,\omega) \equiv M_i \text{ for all } \omega \in \Omega .$$

Let

$$\hat{M}^0(0,\omega_{-0}) \equiv \underset{i\in N}{\times} \hat{M}_i^0(0,\omega_i), \quad \hat{M}_{-i}^0(0,\omega_{-i-0}) \equiv \underset{j\in N\setminus\{i\}}{\times} \hat{M}_j^0(0,\omega_j),$$

$$\hat{M}(0,\omega_{-0}) \equiv \underset{i\in N}{\times} \hat{M}_i^0(0,\omega_i), \text{ and } \hat{M}_{-i}^0(0,\omega_{-i-0}) \equiv \underset{j\in N\setminus\{i\}}{\times} \hat{M}_j^0(0,\omega_j).$$

Recursively, for each $h \geq 1$, we define a subset of player $i's$ first messages $\hat{M}_i^0(h,\omega_i) \subset \hat{M}_i^0$ in the manner that $m_i^0 \in \hat{M}_i^0(h,\omega_i)$ if and only if there exists no $\tilde{m}_i^0 \in \hat{M}_i^0(h-1,\omega_i)$ such that for every $\omega_{-i} \in \Omega_{-i}(\omega_i)$, $m_{-i}^0 \in \hat{M}_{-i}^0(h-1,\omega_{-i-0})$, and $m \in \hat{M}(h-1,\omega)$,

$$u_i(g(m^0,m),-x_i(m^0,m,\omega_0),\omega)$$
$$\leq u_i(g(m_{-i}^0,\tilde{m}_i^0,m),-x_i(m_{-i}^0,\tilde{m}_i^0,m,\omega_0),\omega),$$

and there exists $\omega_{-i} \in \Omega_{-i}(\omega_i)$ such that for every $m \in \hat{M}(h-1,\omega)$ and $m_{-i}^0 \in \hat{M}_{-i}^0(h-1,\omega_{-i-0})$,

$$u_i(g(m^0,m),-x_i(m^0,m,\omega_0),\omega)$$
$$< u_i(g(m_{-i}^0,\tilde{m}_i^0,m),-x_i(m_{-i}^0,\tilde{m}_i^0,m,\omega_0),\omega).$$

We define a subset of player $i's$ second messages $\hat{M}_i(h,\omega) \subset M_i$ in the manner that $m_i \in \hat{M}_i(h,\omega)$ if and only if there exists no $m_i' \in \hat{M}_i(h-1,\omega)$ such that for every $m_{-i} \in \hat{M}_{-i}(h-1,\omega)$ and $m^0 \in \hat{M}^0(h-1,\omega_{-0})$,

$$u_i(g(m^0,m),-x_i(m^0,m,\omega_0),\omega)$$
$$< u_i(g(m^0,m_i',m_{-i}),-x_i(m^0,m_i',m_{-i},\omega_0),\omega).$$

Let

$$\hat{M}^0(h,\omega_{-0}) \equiv \underset{i\in N}{\times} \hat{M}_i^0(h,\omega_i)\,,\ \ \hat{M}_{-i}^0(h,\omega_{-i-0}) \equiv \underset{j\in N\setminus\{i\}}{\times} \hat{M}_j^0(h,\omega_j)\,,$$

$$\hat{M}(h,\omega_{-0}) \equiv \underset{i\in N}{\times} \hat{M}_i^0(h,\omega_i)\,, \text{ and } \ \hat{M}_{-i}^0(h,\omega_{-i-0}) \equiv \underset{j\in N\setminus\{i\}}{\times} \hat{M}_j^0(h,\omega_j)\,.$$

Note that $\hat{M}_i^0(h,\omega_i) \subseteq \hat{M}_i^0(h-1,\omega_i)$, $\hat{M}_i(h,\omega) \subset \hat{M}_i(h-1,\omega)$. Define

$$\hat{M}_i^0(\infty,\omega_i) \equiv \bigcap_{h=0}^{\infty} \hat{M}_i^0(h,\omega_i) \ \text{ and } \ \hat{M}_i(\infty,\omega) \equiv \bigcap_{h=0}^{\infty} \hat{M}_i(h,\omega)\,,$$

both of which are non-empty because of the finiteness. Note that we can eliminate a second message only if it is dominated irrespective of the first announcement profile m^0. Hence, we can regard the set of all iteratively undominated strategies for player i as the Cartesian product given by $(\underset{\omega_i\in\Omega_i}{\times} \hat{M}_i^0(\infty,\omega_i)) \times (\underset{\omega\in\Omega}{\times} \hat{M}_i(\infty,\omega))$.

Definition 3: A strategy $\psi_i = (s_i^0, s_i) \in \Psi_i$ for player i is said to be *iteratively undominated in* Γ *with partial verification* if

$$s_i^0(\omega_i) \in \hat{M}_i^0(\infty,\omega_i) \ \text{ for all } \ \omega_i \in \Omega_i\,,$$

and

$$s_i(\omega) \in \hat{M}_i(\infty,\omega) \ \text{ for all } \ \omega \in \Omega\,.$$

Suppose that $\Omega_i = \Omega_0$ for all $i \in N$, and

$$\omega_i = \omega_0 \ \text{ for all } \ i \in N \ \text{ if and only if } \ \omega \in \Omega\,.$$

This supposition corresponds to the full verification case, where $\Omega_{-i}(\omega_i)$ is a singleton for all $i \in N$ and $\omega_i \in \Omega_i$. The definition of iterative dominance in this section is equivalent to that of iterative dominance with full verification.

Note that the order of elimination does not matter in the definition of iterative dominance with partial verification. Even if we change the order of eliminating strategies, the set of eventually survived strategies is unchanged. The reason for this irrelevance is that for every $\omega_i \in \Omega_i$, there exists $\omega_{-i} \in \Omega_{-i}(\omega_i)$ for which the strict inequalities hold for players' incentives irrespective of the other players' announcements.

Definition 4: A dynamic mechanism Γ is said to *uniquely implement an SCF f in iterative dominance with partial verification* if there exists the unique iteratively undominated strategy profile $\psi \in \Psi$ in Γ with partial verification, and this profile induces the value of the SCF, i.e.,

$$g(\psi(\omega)) = f(\omega) \text{ for all } \omega \in \Omega .$$

The definition of iterative dominance is independent of the specification of the prior distribution on Ω. That is, the dynamic mechanism that uniquely implements an SCF in iterative dominance with partial verification is "detail-free" with respect to the prior distribution on Ω.

At the first stage, each player $i \in N$ prefers a (ω_i)-contingent choice $(f^{\omega_i}, t_i^{\omega_i})$ to another (ω_i)-contingent choice $(f'^{\omega_i}, t_i'^{\omega_i})$ if $(f^{\omega_i}, t_i^{\omega_i})$ makes a more preferable choice of allocation and transfer for player i than $(f'^{\omega_i}, t_i'^{\omega_i})$ irrespective of ω_{-i}, i.e., if for every $\omega_{-i} \in \Omega_{-i}(\omega_i)$,

$$u_i(f_i^{\omega_i}(\omega_{-i}), t_i^{\omega_i}(\omega_{-i}), \omega) > u_i(f_i'^{\omega_i}(\omega_{-i}), t_i'^{\omega_i}(\omega_{-i}), \omega),$$

where $f_i^{\omega_i} : \Omega_{-i}(\omega_i) \to \Delta$ and $t_i^{\omega_i} : \Omega_{-i}(\omega_i) \to R$. Hence, we can say that any message that is eliminated through the iterative procedure is regarded as a message that is dominated with strict inequality irrespective of the specification of the "full-support" prior distribution on Ω. In the definition of iterative dominance with partial verification, we required strict inequalities for not all, but some states. This implies that any eliminated message is weakly dominated for all non-full-support distributions, while it is strictly dominated for all full-support distributions.

6. Full Detection

This section demonstrates a sufficient condition under which any SCF is uniquely implementable in iterative dominance with partial verification. This section is the main part of this paper.

6.1. Definitions

For each $i \in N$, let us denote $\chi_{-i} : \Omega_{-i} \to 2^{\Omega_{-i}}$, where

$$\omega_{-i} \in \chi_{-i}(\omega_{-i}),$$

and

$$\tilde{\omega}_0 = \omega_0 \text{ for all } \tilde{\omega}_{-i} \in \chi_{-i}(\omega_{-i}).$$

We regard the function χ_{-i} as describing the pattern of the announcements made by all players other than player i. That is, they announce a profile that belongs to $\chi_{-i}(\omega_{-i}) \subset \Omega_{-i}$ when ω_{-i} occurs. Here, we regard player 0 as the dummy player who always announces ω_0 truthfully.

We introduce a notion on χ_{-i}, namely *detection*, as follows.

Definition 5: A function χ_{-i} is said to *detect player* i *for* ω_i *against* ω_i' if there exists $\omega_{-i} \in \Omega_{-i}(\omega_i)$ such that

(2) $$\chi_{-i}(\tilde{\omega}_{-i}) \cap \chi_{-i}(\omega_{-i}) = \phi \text{ for all } \tilde{\omega}_{-i} \in \Omega_{-i}(\omega_i').$$

Suppose that ω_i is correct, but player i announces $\omega_i' \neq \omega_i$ incorrectly. Suppose that for every ω_{-i}, the other players announce according to $\chi_{-i}(\omega_{-i}) \subset \Omega_{-i}$; i.e., they announce a profile ω_{-i-0}' that satisfies $(\omega_0, \omega_{-i-0}') \in \chi_{-i}(\omega_{-i})$. Note that if player $i's$ announcement ω_i' is correct, the other players announce according to $\chi_{-i}(\tilde{\omega}_{-i})$ for some $\tilde{\omega}_{-i} \in \Omega_{-i}(\omega_i')$.

Suppose that player i expects $\omega_{-i} = (\omega_0, \omega_{-i-0}) \in \Omega_{-i}(\omega_i)$ to occur. Then, player i expects the other players to announce according to $\chi_{-i}(\omega_{-i})$. However, the condition of (2) implies that if player $i's$ announcement ω_i' is correct, the other players never announce according to $\chi_{-i}(\omega_{-i})$. This is a contradiction. In this case, we can recognize that player $i's$ announcement ω_i' is incorrect. Hence, χ_{-i} detects player i for ω_i ag*ainst* ω_i'.

Based on this detection notion, we define *full detection* as follows. For every $h \in \{0,1,...\}$ and $i \in N \cup \{0\}$, we specify $\chi_i(h): \Omega_i \to 2^{\Omega_i}$ and $\chi_{-i}(h): \Omega_{-i} \to 2^{\Omega_{-i}}$ in the following manner. Let

$$\chi_0(h)(\omega_0) = \{\omega_0\} \text{ for all } \omega_0 \in \Omega_0 \text{ and } h \in \{0,1,...\},$$

and

$$\chi_i(0)(\omega_i) = \Omega_i \text{ and } \chi_{-i}(0)(\omega_{-i}) = \{\omega_0\} \times \Omega_{-i} \text{ for all } i \in N \text{ and } \omega \in \Omega.$$

Recursively, for each $h \in \{1,2,...\}$, we define $\chi_i(h)(\omega_i) \subset \chi_i(h-1)(\omega_i)$ and $\chi_{-i}(h)(\omega_{-i}) \subset \chi_{-i}(h-1)(\omega_{-i})$ in the manner that for every $\omega_i' \in \chi_i(h-1)(\omega_i)$,

$\omega_i' \in \chi_i(h)(\omega_i)$ if and only if $\chi_{-i}(h-1)$ fails to detect player i for ω_i against ω_i', i.e.,

$$\{\bigcup_{\tilde{\omega}_{-i} \in \Omega_{-i}(\omega_{-i}')} \chi_{-i}(h-1)(\tilde{\omega}_{-i})\} \cap \chi_{-i}(h-1)(\omega_{-i}) \neq \phi \text{ for all } \omega_{-i} \in \Omega_{-i}(\omega_i),$$

and for every $\omega_{-i} \in \chi_{-i}(h-1)(\omega_{-i})$,

$\omega_{-i} \in \chi_{-i}(h)(\omega_{-i})$ if and only if $\omega_j \in \chi_j(h)(\tilde{\omega}_j)$ for all $j \in N \cup \{0\} \setminus \{i\}$.

Here, $\chi_{-i}(h)$ is the set of all announcements that can survive through the h-round iterative removal of detected lies.

Full Detection: For every $i \in N$ and $\omega_i \in \Omega_i$,

$$\bigcap_{h \to \infty} \chi_i(h)(\omega_i) = \{\omega_i\}.$$

The sequence $((\chi_i(h), \chi_{-i}(h-1))_{i \in N})_{h=0}^{\infty}$ describes the iterative removal of detected lies. Full detection implies that the iterative removal of detected lies eventually eliminates all lies. Truth-telling is therefore the only announcement that survives through such a removal procedure. Since Ω is finite, there exists a positive integer h^* such that

$$\chi_i(h)(\omega_i) = \{\omega_i\} \text{ for all } h \geq h^*.$$

We demonstrate the following tractable sufficient condition for full detection. Let us describe a state as

$$\omega = (\theta_0, ..., \theta_L),$$

where L is a positive integer. We assume that there exists a function $\tau : \{0,...,L\} \to 2^{N \cup \{0\}}$ such that for every $i \in N \cup \{0\}$

$$\omega_i = \theta_{L(i)},$$

where we denote

$$L(i) \equiv \{l \in \{0,...,L\} \mid i \in \tau(l)\},$$

and

$$\theta_C \equiv (\theta_l)_{l \in C} \text{ for each } C \subset \{0,...,L\}.$$

Note that $\tau(l) \subset N$ is the set of all players who observe θ_l at the first stage, and $L(i) \subset \{1,...,L\}$ is the set of all components of the state that player i observes at the first stage. That is, at the first stage, each player i observes any component θ_l of the state such that $l \in L(i)$. For each component θ_l, there may exist multiple players who can observe it at the first stage; i.e., it might be the case that $|\tau(l)| \geq 2$. (For convenience, we assume that $L(0) = \{0\}$, i.e., $\omega_0 = \theta_0$.)

Let Ξ_l denote the set of possible θ_l. Hence, $\Omega \subset \underset{l \in \{0,...,L\}}{\times} \Xi_l$.

For every $i \in N \cup \{0\}$ and $l \in L(i)$, let

$$L(l,i) \equiv \{\tilde{l} \in \{0,...,l-1\} \mid i \notin \tau(\tilde{l})\},$$

which is the set of all components of the state that are earlier than l and player i cannot observe at the first stage. For every $i \in N \cup \{0\}$, $l \in L(i)$, and $\omega_i \in \Omega_i$, let

$$\theta_{L(l,i)} \in \Xi_{L(l,i)}(\omega_i) \text{ if and only if there exists } \tilde{\omega} \in \Omega \text{ such that}$$

$$\tilde{\omega}_i = \omega_i \text{ and } \tilde{\theta}_{L(l,i)} = \theta_{L(l,i)}.$$

Note that $\Xi_{L(l,i)}(\omega_i)$ is the set of all $\theta_{L(l,i)}$ that are consistent with player $i's$ observation ω_i at the first stage.

The following proposition shows a sufficient condition for full detection, which implies that for every $l \in \{1,...,L\}$, there exists a player who can observe the $l-th$

component θ_l at the first stage, and whenever he observes different $l-th$ components θ_l, then different profiles of earlier components $\theta_{L(l,i)}$ might occur. With this condition, his lie about the $l-th$ component is detected through the observation of $\theta_{L(l,i)}$.

Proposition 2: *Suppose that for every $l \in \{1,...,L\}$, there exists $i \in \tau(l)$ such that for every $\omega_i \in \Omega_i$ and $\tilde{\omega}_i \in \Omega_i$,*

$$\Xi_{L(l,i)}(\omega_i) \not\subset \Xi_{L(l,i)}(\tilde{\omega}_i) \text{ if } \theta_l \neq \tilde{\theta}_l \text{ and } \theta_{l'} = \tilde{\theta}_{l'} \text{ for all } l' \in L(i) \setminus \{l\}.$$

Then, Ω satisfies full detection.

Proof: Consider $l = 1$. In this case, $\Xi_{L(l,i)}(\omega_i) \not\subset \Xi_{L(l,i)}(\tilde{\omega}_i)$ implies that θ_0 detects player $i = \tau(1)$ for ω_i against $\tilde{\omega}_i$ whenever $\theta_1 \neq \tilde{\theta}_1$, where we must note that the dummy player 0 always tells the truth about θ_0 because of its verification.

Fix an arbitrary $h \in \{2,...,L\}$. Suppose that for every $h' \in \{0,...,h-1\}$, player $\tau(l')$ tells the truth about $\theta_{l'}$. In this case, $\Xi_{L(l,i)}(\omega_i) \not\subset \Xi_{L(l,i)}(\tilde{\omega}_i)$ implies that $\theta_{L(l,i)}$ detects player $i = \tau(l)$ for ω_i against $\tilde{\omega}_i$ whenever $\theta_l \neq \tilde{\theta}_l$.

Q.E.D.

A special case of the sufficient condition in Proposition 2 is introduced as follows. Suppose that

$$\Xi_l = \Xi_0 \text{ for all } l \in \{1,...,L\},$$

and each component of the state is always different from its neighbors, i.e., for every $\omega \in \underset{l \in \{0,...,L\}}{\times} \Xi_l$,

$$\omega \in \Omega \text{ if and only if } \theta_{l-1} \neq \theta_l \text{ for all } l \in \{1,...,L\}.$$

Moreover, suppose that for every $l \in \{1,...,L\}$, there exists $i = \iota(l) \in N$ such that

$$l \in L(i),$$

$$\{l-2, l-1\} \not\in L(i) \text{ if } l \geq 2,$$

and

$0 \notin L(i)$ if $l = 1$.

Hence, player $\iota(l)$ cannot observe θ_{l-1} and θ_{l-2} at the first stage.

Clearly, Ω satisfies the sufficient condition in Proposition 2, i.e., full detection, in this special case. A player's lie about θ_l is detected through the observation of θ_{l-1}. Since θ_0 is verifiable, we can eliminate all lies about θ_1 through the observation of θ_0. Recursively, for every $l \in \{2,...,L\}$, we can eliminate all lies about θ_l through the truthful announcement about θ_{l-1}[10].

Let us further consider an example of this special case, where we assume

$\tau(i) = \{i\}$, $\Xi_i = \{1,2,3\}$, and

$\omega \in \Omega$ if and only if $\theta_i \neq \theta_{i-1}$ for all $i \in \{1,...,n\}$,

where each player i observes only θ_i at the first stage, i.e., $\omega_i = \theta_i$. This example corresponds to the example addressed in the introduction of this paper. The verified information ω_0 can directly detect any lie by player 1 about θ_1, because player 1 cannot exclude the possibility that his lie $\theta_1' \neq \theta_1$ is equivalent to θ_0, i.e., $\theta_1' = \theta_0$. This motivates player 1 to tell the truth about θ_1. In the same manner, the truthful announcement by player 1 about θ_1 can detect any lie by player 2 about θ_2. This motivates player 2 to tell the truth about θ_2. Recursively, any player $i \in \{1,...,n\}$ is motivated to tell the truth about θ_i, implying full detection.

In the process of such iterative removal of detected lies, it is crucial to assume that each player i is not informed of θ_{i-1} at the first stage. Otherwise, he can find a way to escape from detection by announcing $\tilde{\theta}_i \notin \{\theta_i, \theta_{i-1}\}$. Hence, it is crucial to assume that any player i does not observe either θ_{i-1} or θ_{i-2}. This is exactly what the special case implies.[11]

6.2. The Theorem

[10] We will show some generalization of this special case in Subsection 9.1.
[11] With only tiny transfers permitted, full detection does not imply incentive compatibility.

Fix arbitrary real numbers, $\eta_1(h) > 0$ for each $h \in \{1,...,h^*\}$, $\eta_2 > 0$, and $\eta_3 > 0$. Let $\eta_1 \equiv (\eta_1(h))_{h=1}^{h^*}$. Fix an arbitrary integer $K > 1$. To uniquely implement an SCF f in iterative dominance with partial verification, we construct a dynamic mechanism $\Gamma^* = \Gamma^*(f,\eta_1,\eta_2,\eta_3,K) = (M^0, M, g, x)$ as follows. Let

$$M_i^0 = \Omega_i,$$

and

$$M_i = \prod_{k=1}^{K} M_i^k \quad \text{and} \quad M_i^k = \Omega \quad \text{for all } k \in \{1,...,K\}.$$

For each $k \in \{2,...,K\}$, we define $g^k : M^k \to \Delta$ in the manner that for each $\omega \in \Omega$,

$$g^k(m^k) = f(\omega) \qquad \text{if } m_i^k = \omega \text{ for at least } n-1 \text{ players,}$$

and

$$g^k(m^k) = a^* \qquad \text{if there exists no such } \omega.$$

Let

$$g(m^0, m) = \frac{\sum_{k=1}^{K} g^k(m^k)}{K}.$$

The allocation choice does not depend on the first announcements m^0.

Let

$$x_i(m^0, m, \omega_0) = \sum_{h=1}^{h^*} x_i^h(m^0, \omega_0) + z_i(m^0, m, \omega_0),$$

where

$$x_i^h(m^0,\omega_0) = \eta_1(h) \qquad \text{if } (\omega^0, m_{-i}^0) \notin \bigcup_{\tilde{\omega}_{-i} \in \Omega_{-i}(m_i^0)} \chi_{-i}(h-1)(\tilde{\omega}_{-i}),$$

$$x_i^h(m^0,\omega_0) = 0 \qquad \text{if } (\omega^0, m_{-i}^0) \notin \bigcup_{\tilde{\omega}_{-i} \in \Omega_{-i}(m_i^0)} \chi_{-i}(h-1)(\tilde{\omega}_{-i}),$$

$$z_i(m^0, m, \omega_0) = \eta_2 + \frac{r_i}{K}\eta_3 \qquad \text{if there exists } k \in \{1,...,K\} \text{ such that}$$

$m_i^k \neq (m^0, \omega_0)$, and

$m_j^{k'} = (m^0, \omega_0)$ for all $k' < k$ and

$$j \in N \setminus \{i\},$$

and

$$z_i(m^0, m, \omega_0) = \frac{r_i}{K}\eta_3 \qquad \text{if there exists no such } k \in \{1,...,K\},$$

where $r_i \in \{0,...,K\}$ implies the number of $k \in \{2,...,K\}$ such that $m_i^k \neq (m^0, \omega_0)$.

We select (η_1, η_2, η_3) such that

(3) $$\eta_1(\tilde{h}) > \sum_{h=1}^{\tilde{h}-1} \eta_1(h) + \eta_2 + \eta_3 \quad \text{for all } \tilde{h} \in \{1,...,h^*\}.$$

Since

$$0 \leq x_i(m^0, m, \omega_0) \leq \sum_{h=1}^{h^*} \eta_1(h) + \eta_2 + \eta_3 \quad \text{for all } i \in N \text{ and } (m^0, m, \omega_0),$$

by choosing $\sum_{h=1}^{h^*} \eta_1(h) + \eta_2 + \eta_3$ close to zero, we can make $x_i(m^0, m, \omega_0)$ as close to zero as possible.

We define the *honest* strategy for player i in Γ^*, $\psi_i^* = (s_i^{0*}, s_i^*)$, as

$$s_i^{0*}(\omega_i) = \omega_i \quad \text{for all } \omega_i \in \Omega_i,$$

and

$$s_i^{*k}(\omega) = \omega \quad \text{for all } k \in \{1,...,K\} \text{ and } \omega \in \Omega.$$

The honest strategy profile $\psi^* \equiv (\psi_i^*)_{i \in N}$ induces the value of the SCF f, i.e.,

$$g(\psi^*(\omega)) = f(\omega) \quad \text{for all } \omega \in \Omega,$$

and no monetary transfers, i.e.,

$$x_i(\psi^*(\omega), \omega) = 0 \quad \text{for all } i \in N \text{ and } \omega \in \Omega.$$

Because of the continuity assumption, we can select a sufficient K such that whenever $\max_{a \in A} |\alpha(a) - \alpha'(a)| \leq \frac{1}{K}$, then

(4) $$u_i(\alpha, -t_i, \omega_i) > u_i(\alpha', -t_i - \eta_2, \omega_i) \quad \text{for all } t_i \in [0, \sum_{h=1}^{h^*} \eta_1(h) + \eta_3] \text{ and } \omega_i \in \Omega_i.$$

The inequalities in (4) imply that η_2 is close to zero but is sufficient compared with the change of allocation within the $\frac{1}{K}$ – limit.

We can see this paper's main technical contribution in the arguments about the players' incentives at the first stage. It is crucial in incentives to assume that each player i makes the first announcement before he observes ω_{-i}. According to x_i^h, any player $i \in N$ is fined the monetary amount $\eta_1(h)$ if he makes a first announcement that is detected by $\chi_{-i}(h-1)$ that describes the profiles of the other players' announcements which survived through the $(h-1)-$round iterative removals of detected messages. This holds true irrespective of h. Hence, from (3) and full detection, it follows that each player is willing to announce an undetected message, i.e., the honest message, as his first announcement. This is reason we can utilize the first announcements and the verified information as the reference to judge whether the second announcements are honest.

According to z_i, any first deviant from the combination of the profile of first announcements and the verified information (ω^0, m_0) in the second announcement stage is fined the monetary amount η_2. Any player is additionally fined $\frac{\eta_3}{K}$ whenever he deviates from (ω^0, m_0). We apply the bounded mechanism design that originates in Abreu and Matsushima (1992a, 1992b, 1994), showing that by setting the first announcement and the verified information as the reference, any player is willing to make a truthful second announcement.

Based on these observations, we can demonstrate the following theorem, which states that under full detection, the dynamic mechanism Γ^* uniquely implements f in iterative dominance with partial verification. Since Γ^* is well-defined, we can conclude that *with full detection, any SCF is uniquely implementable in iterative dominance with partial verification, where we need almost no monetary transfers off the equilibrium path, and no monetary transfers on the equilibrium path.*

Theorem 3: *Under full detection, the honest strategy profile ψ^* is the unique iteratively undominated strategy profile in Γ^*. That is, Γ^* uniquely implements f in iterative dominance with partial verification.*

Proof: Suppose that player i observes ω_i and announces $m_i^0 \notin \chi_i(1)(\omega_i)$ as his first announcement. In this case, $\chi_{-i}(0)$ detects him for ω_i against m_i^0; there exists $\omega_{-i} \in \Omega_{-i}(\omega_i)$ such that

$$\chi_{-i}(0)(\tilde{\omega}_{-i}) \cap \chi_{-i}(0)(\omega_{-i}) = \phi \text{ for all } \tilde{\omega}_{-i} \in \Omega_{-i}(m_i^0).$$

Since $\chi_{-i}(0)(\omega_{-i}) = \{\omega_0\} \times M_{-0-i}$, the announcement by any other player $j \in N \setminus \{i\}$ belongs to $\chi_j(0)(\omega_j)$. This implies that, by announcing m_i^0, he is fined $\eta_1(1)$. In contrast, he can avoid this fine by announcing ω_i truthfully. Since the announcement of m_i^0 is irrelevant to the allocation choice and $\eta_1(1)$ is large enough to satisfy (3), it follows that player i never announces any element that does not belong to $\chi_i(1)(\omega_i)$.

Consider an arbitrary $h \in \{2,...,h^*\}$. Suppose that any player $i \in N$ announces a message for the first announcement that belongs to $\chi_i(h-1)(\omega_i)$. Suppose that player i observes ω_i and announces $m_i^0 \notin \chi_i(h)(\omega_i)$. In this case, $\chi_{-i}(h-1)$ detects him for ω_i against m_i^0. That is, there exists $\omega_{-i} \in \Omega_{-i}(\omega_i)$ such that

$$\chi_{-i}(h-1)(\tilde{\omega}_{-i}) \cap \chi_{-i}(h-1)(\omega_{-i}) = \phi \text{ for all } \tilde{\omega}_{-i} \in \Omega_{-i}(m_i^0).$$

Since the announcement by any other player $j \in N \setminus \{i\}$ belongs to $\chi_j(h-1)(\omega_j)$, he is fined $\eta_1(h)$. In contrast, he can avoid this fine by announcing ω_i truthfully. Since the announcement of m_i^0 is irrelevant to the allocation choice and $\eta_1(h)$ is large enough to satisfy (3), it follows that player i never announces any element that does not belong to $\chi_i(h)(\omega_i)$.

From the above arguments, we have proved that if ψ_i is strictly iteratively undominated, then,

$$s_i^0(\omega_i) \in \bigcap_{h \to \infty} \chi_i(h)(\omega_i) \text{ for all } \omega_i \in \Omega_i,$$

which, along with full detection, implies that

$$s_i^0(\omega_i) = \omega_i \text{ for all } \omega_i \in \Omega_i.$$

Since all players tell the truth for their first announcements, i.e., any player $i \in N$ announces $m_i^0 = \omega_i$, we can prove in the same manner as in Theorem 1 that for each $i \in N$, if ψ_i is strictly iteratively undominated, then,

$$s_i^k(\omega) = \omega \text{ for all } \omega \in \Omega \text{ and } k \in \{1,...,K\},$$

where we utilized the inequality (4) for deriving this statement.

From these observations, we have proved that ψ^* is the unique strict iteratively undominated strategy profile in Γ^*.

Q.E.D.

In the proof, we utilize the basic concept of bounded mechanism design that originates in Abreu and Matsushima (1992a, 1992b, 1994) such that the central planner requires each player to make multiple announcements at one time, selects one profile from their announcements, and fines the first deviants from the reference. Once we can establish the truthful reference, the mechanism á la Abreu-Matsushima can successfully implement the SCF in iterative dominance.

The remaining problem is to show how the central planner can establish such truthful reference. This problem becomes substantial once we require the mechanism to be "detail-free" in terms of the prior distribution. In fact, this is an easy problem to solve if we permit a particular prior distribution $p: \Omega \to [0,1]$ to be common knowledge. For instance, let us denote by $p_i(\cdot | \omega_i): \Omega_0 \to [0,1]$ the $(\omega_i)-$conditional distribution on Ω_0 induced by p. Assume that for each $i \in N$,

$$p_i(\cdot | \omega_i) \neq p_i(\cdot | \omega_i') \text{ whenever } \omega_i \neq \omega_i'.$$

In this case, by introducing a device of proper scoring rule, the central planner can incentivize each player i to reveal ω_i truthfully, establishing the truthful reference.

Since this paper assumes no such common knowledge about the prior distribution, we need to utilize a proper-subset nature of the state space, such as full detection, in more complicated manners than the above-mentioned differences in conditional distributions.

7. Partial Detection

This section considers the case in which Ω does not satisfy full detection. We weaken implementation by replacing the uniqueness of iteratively undominated strategy profile with the uniqueness of outcome induced by iteratively undominated strategy profiles. In this manner, we define full implementation in iterative dominance with partial verification as follows.

Definition 6: A dynamic mechanism Γ is said to *fully implement an SCF* f *in iterative dominance with partial verification* if every iteratively undominated strategy profile $\psi \in \Psi$ in Γ induces the value of the SCF, i.e.,

$$g(\psi(\omega)) = f(\omega) \text{ for all } \omega \in \Omega .$$

Full implementation permits the multiplicity of iteratively undominated strategies. However, it requires any profile of iteratively undominated strategies to correctly achieve the value of the SCF.

A *partition on* Ω_i is defined as $\Phi_i : \Omega_i \to 2^{\Omega_i} \setminus \{\phi\}$, where for every $\omega_i \in \Omega_i$ and $\omega_i' \in \Omega_i$,

$$\text{either } \Phi_i(\omega_i) = \Phi_i(\omega_i') \text{ or } \Phi_i(\omega_i) \cap \Phi_i(\omega_i') = \phi .$$

We can regard a partition Φ_i as the set of subsets $\phi_i \subset \Omega_i$ such that

$$\phi_i \in \Phi_i \text{ if and only if } \phi_i = \Phi_i(\omega_i) \text{ for some } \omega_i \in \Omega_i .$$

Let $\Phi \equiv (\Phi_i)_{i \in N \cup \{0\}}$. denote $\Phi(\omega) = (\Phi_i(\omega_i))_{i \in N \cup \{0\}}$.

We specify Φ_i^* as the finest partition on Ω_i satisfying that

$$\bigcap_{h \to \infty} \chi_i(h)(\omega_i) \subset \Phi_i^*(\omega_i) \text{ for all } \omega_i \in \Omega_i ,$$

where $\chi_i(h)$ was introduced in the definition of full detection. Note that full detection holds if and only if for every $i \in N \cup \{0\}$, Φ_i^* is the full partition, i.e., $\Phi_i^*(\omega_i) = \{\omega_i\}$ for all $\omega_i \in \Omega_i$.

Definition 7: An SCF f is said to be *measurable* if for every $\omega \in \Omega$ and $\omega' \in \Omega$,

$$f(\omega) = f(\omega') \text{ whenever } \Phi_i^*(\omega_i) = \Phi_i^*(\omega_i') \text{ for all } i \in N \bigcup \{0\}.$$

Measurability implies that the value of f is the same between ω_i and ω_i' whenever both belong to the same cell of Φ^*. The following theorem shows that the measurability is a sufficient condition for an SCF to be fully implementable in iterative dominance with partial verification.

Theorem 4: *Suppose that an SCF* f *is measurable. Then, it is fully implementable in iterative dominance with partial verification.*

Proof: See Appendix A.

8. Incomplete Information at the Second Stage

Throughout the previous sections, we assumed complete information at the second stage. This section eliminates this assumption, and instead assumes *incomplete information at the second stage*. That is, even at the second stage, each player does not know all about the state.

To be precise, for each $i \in N$, let us fix an arbitrary set $C(i) \subset N \bigcup \{0\}$, where we assume $i \in C(i)$. At the second stage, each player i observes $\omega_{C(i)} = (\omega_j)_{j \in C(i)}$, while he (or she) cannot observe $\omega_{N \bigcup \{0\} \setminus C(i)}$. He makes his second announcement contingent only on $\omega_{C(i)}$.

We define $\Omega_C(\omega_{N \bigcup \{0\} \setminus C}) \subset \underset{i \in C}{\times} \Omega_i$ in the manner that

$$\omega_C \in \Omega_C(\omega_{N \bigcup \{0\} \setminus C}) \text{ if and only if } (\omega_C, \omega_{N \bigcup \{0\} \setminus C}) \in \Omega.$$

We redefine a strategy $\psi_i = (s_i^0, s_i)$ for each player i by replacing $s_i : \Omega \to M_i$ with $s_i : \Omega_{C(i)} \to M_i$.

We further redefine iterative dominance as follows. For every $i \in N$, let

$$\hat{M}_i^0(0, \omega_i) \equiv M_i^0 \text{ for all } \omega_i \in \Omega_i,$$

and

$$\hat{M}_i(0,\omega_{C(i)}) \equiv M_i \quad \text{for all} \quad \omega \in \Omega .$$

Let $\hat{M}(0,\omega) \equiv \underset{j\in N}{\times} \hat{M}_j(0,\omega_{C(j)})$ and $\hat{M}_{-i}(0,\omega) \equiv \underset{j\in N\setminus\{i\}}{\times} \hat{M}_j(0,\omega_{C(j)})$. Recursively, for each $h \geq 1$, we define $\hat{M}_i^0(h,\omega_i) \subset \hat{M}_i^0$ in the manner that $m_i^0 \in \hat{M}_i^0(h,\omega_i)$ if and only if there exists no $\tilde{m}_i^0 \in \hat{M}_i^0(h-1,\omega_i)$ such that for every $\omega_{-i} \in \Omega_{-i}(\omega_i)$, $m \in \hat{M}(h-1,\omega)$, and $m_{-i}^0 \in \hat{M}_{-i}^0(h-1,\omega_{-i-0})$,

$$u_i(g(m^0,m),-x_i(m^0,m,\omega_0),\omega)$$
$$\leq u_i(g(m_{-i}^0,\tilde{m}_i^0,m),-x_i(m_{-i}^0,\tilde{m}_i^0,m,\omega_0),\omega),$$

and there exists $\omega_{-i} \in \Omega_{-i}(\omega_i)$ such that for every $m \in \hat{M}(h-1,\omega)$ and $m_{-i}^0 \in \hat{M}_{-i}^0(h-1,\omega_{-i-0})$,

$$u_i(g(m^0,m),-x_i(m^0,m,\omega_0),\omega)$$
$$< u_i(g(m_{-i}^0,\tilde{m}_i^0,m),-x_i(m_{-i}^0,\tilde{m}_i^0,m,\omega_0),\omega).$$

We define $\hat{M}_i(h,\omega_{C(i)}) \subset M_i$ in the manner that $m_i \in \hat{M}_i(h,\omega)$ if and only if there exists no $m_i' \in \hat{M}_i(h-1,\omega_{C(i)})$ such that for every $\omega_{N\cup\{0\}\setminus C(i)} \in \Omega_{N\cup\{0\}\setminus C(i)}(\omega_{C(i)})$, $m_{-i} \in \hat{M}_{-i}(h-1,\omega)$, and $m^0 \in \hat{M}^0(h-1,\omega_{-0})$,

$$u_i(g(m^0,m),-x_i(m^0,m,\omega_0),\omega)$$
$$\leq u_i(g(m^0,m_i',m_{-i}),-x_i(m^0,m_i',m_{-i},\omega_0),\omega),$$

and there exists $\omega_{N\cup\{0\}\setminus C(i)} \in \Omega_{N\cup\{0\}\setminus C(i)}(\omega_{C(i)})$ such that for every $m_{-i} \in \hat{M}_{-i}(h-1,\omega)$ and $m^0 \in \hat{M}^0(h-1,\omega_{-0})$,

$$u_i(g(m^0,m),-x_i(m^0,m,\omega_0),\omega)$$
$$< u_i(g(m^0,m_i',m_{-i}),-x_i(m^0,m_i',m_{-i},\omega_0),\omega).$$

A strategy $\psi_i = (s_i^0,s_i) \in \Psi_i$ for player i is said to be *iteratively undominated in* Γ *with partial verification under incomplete information at the second stage* if

$$s_i^0(\omega_i) \in \bigcap_{h=0}^{\infty} \hat{M}_i^0(h,\omega_i) \quad \text{for all} \quad \omega_i \in \Omega_i ,$$

and

$$s_i(\omega) \in \bigcap_{h=0}^{\infty} \hat{M}_i(h,\omega) \text{ for all } \omega \in \Omega .$$

Definition 8: An SCF f is said to be *strictly incentive compatible* if there exists a positive real number $\xi > 0$ and a function $\tilde{f} : \times_{i \in N} \Omega_{C(i)} \to \Delta$ such that for every $\omega \in \Omega$,

$$\tilde{f}((\omega_{C(i)})_{i\in N}) = f(\tilde{\omega}) \text{ whenever } \omega_{C(i)} = \tilde{\omega}_{C(i)} \text{ for all } i \in N \text{, and}$$

$$\bigcup_{i\in N} C(i) = N \cup \{0\}$$

and for every $i \in N$, $\omega'_{C(i)} \in \Omega_{C(i)}$, and $t_i \in [0,\xi]$,

$$u_i(\tilde{f}((\omega_{C(j)})_{j\in N}), -t_i, \omega) \geq u_i(\tilde{f}((\omega_{C(j)})_{j\in N\setminus\{i\}}, \omega'_{C(i)}), -t_i, \omega) .$$

Note that the "strictness" in Definition 8 is different from the standard usage of this word. Strict incentive compatibility in Definition 8 implies that irrespective of the constant transfer $-t_i$ within the $\xi-$limit, truth-telling is a weakly (not necessarily strictly) dominant strategy for each player i in the direct mechanism given by $\tilde{f}$. We do not require strict inequality in this definition. Clearly, this strict incentive compatibility holds under complete information at the second stage, provided that $n \geq 3$ and $\tilde{f}$ is replaced with the rule that selects the allocation implied by a state whenever $n-1$ players have the same private information as this state.

Note also from the continuity assumption that an SCF f is strictly incentive compatible if there exists $\tilde{f}$ such that for every $\omega \in \Omega$,

$$\tilde{f}((\omega_{C(i)})_{i\in N}) = f(\tilde{\omega}) \text{ whenever } \omega_{C(i)} = \tilde{\omega}_{C(i)} \text{ for all } i \in N \text{, and}$$

$$\bigcup_{i\in N} C(i) = N \cup \{0\}$$

and for every $i \in N$ and $\omega'_{C(j)} \in \Omega_{C(i)}$,

$$u_i(\tilde{f}((\omega_{C(j)})_{j\in N}), 0, \omega) > u_i(\tilde{f}((\omega_{C(j)})_{j\in N\setminus\{i\}}, \omega'_{C(i)}), 0, \omega) \text{ if}$$

$$\tilde{f}((\omega_{C(j)})_{j\in N}) \neq \tilde{f}((\omega_{C(j)})_{j\in N\setminus\{i\}}, \omega'_{C(i)}) .$$

The following theorem states that with full detection and strict incentive compatibility, we can construct a dynamic mechanism that uniquely implements the SCF

f in iterative dominance with partial verification under incomplete information at the second stage. We need almost no monetary transfers off the equilibrium path, and no monetary transfers on the equilibrium path. The following theorem is a natural extension of Theorem 3.

Theorem 5: *Assume incomplete information at the second stage, full detection, and strict incentive compatibility. Then, the SCF f is uniquely implementable in iterative dominance with partial verification under incomplete information at the second stage. That is, there exists a dynamic mechanism Γ that has the unique iteratively undominated strategy profile ψ, and*

$$g(\psi(\omega)) = f(\omega) \text{ for all } \omega \in \Omega .$$

We need almost no monetary transfers off the equilibrium path, and no monetary transfers on the equilibrium path.

Proof: See Appendix B.

We further introduce a condition on an SCF as a combination of measurability and ex-post incentive compatibility as follows.

Definition 9: An SCF f is said to be *strict measurable incentive compatible* if there exists a positive real number $\xi > 0$ and a function $\tilde{f} : \underset{i \in N}{\times} \Omega_i \to \Delta$ such that for every $\omega \in \Omega$,

$$\tilde{f}(\omega_{-0}) = f(\omega_{-0}),$$

and for every $i \in N$, $\tilde{\omega}_i \in \Omega_i$, and $t_i \in [0, \xi]$,

$$u_i(\tilde{f}(\omega_{-0}), -t_i, \omega) \geq u_i(\tilde{f}(\tilde{\omega}_i, \omega_{-i-0}), -t_i, \omega),$$

where $\hat{f}$ is measurable in that for every $\omega \in \underset{i \in N}{\times} \Omega_i$ and $\omega' \in \underset{i \in N}{\times} \Omega_i$,

$$f(\omega) = f(\omega') \text{ whenever } \Phi_i^*(\omega_i) = \Phi_i^*(\omega_i') \text{ for all } i \in N \cup \{0\}.$$

From the continuity assumption, it follows that an SCF f is strict measurable incentive compatible if there exists $\tilde{f}:\underset{i\in N}{\times}\Omega_i\rightarrow\Delta$ such that for every $\omega\in\Omega$,

$$\tilde{f}(\omega_{-0})=f(\omega_{-0}),$$

and for every $i\in N$ and $\tilde{\omega}_i\in\Omega_i$,

$$u_i(\tilde{f}(\omega_{-0}),0,\omega)>u_i(\tilde{f}(\tilde{\omega}_i,\omega_{-i-0}),0,\omega) \text{ whenever } \tilde{f}(\omega_{-0})\neq\tilde{f}(\tilde{\omega}_i,\omega_{-i-0}),$$

where $\hat{f}$ is measurable.

The following theorem shows that the strict measurable incentive compatibility is a sufficient condition for an SCF to be fully implementable in iterative dominance with partial verification under incomplete information at the second stage. The following theorem is a natural extension of Theorem 4.

Theorem 6: *Assume incomplete information at the second stage and strict measurable incentive compatibility. Then, the SCF f is fully implementable in iterative dominance with partial verification under incomplete information at the second stage.*

Proof: See Appendix C.

The case of $C\{i\}=\{i\}$ for all $i\in N$ corresponds to Bergemann and Morris (2009), which investigates virtual implementation instead of exact implementation. Note that the difference of exact implementation and virtual implementation is very crucial. Note also that in this case we do not need any dynamics of mechanism design. By permitting the overlap in information across players, and by utilizing the dynamic aspect of information acquisition, we can weaken the restriction of incentive compatibility with no harm in uniqueness.

Multi-Round Announcements

In the previous sections, we assumed as a dynamic aspect of mechanism design that the central planner requires each player $i\in N$ to make the first announcement about ω_i

at the first stage. This section will assume that the central planner can require each player i to make announcements even before the first stage, i.e., even when he observes only partial information about ω_i. Even for the establishment of the reference, we will utilize dynamic, not static mechanism design. This section shows a weaker sufficient condition than full detection for guaranteeing unique implementation in iterative dominance.

Let us consider the following modification of the special case in Subsection 6.1. Before the first stage, there exist T multiple rounds, i.e., round 1, round 2, …, and round T, where round T corresponds to the first stage. Each player i can observe all components of $\omega_i = (\theta_l)_{l \in L(i)}$ by round T.

For each $i \in N$ and $l \in L(i)$, player i observes θ_l at round $t = t(i,l) \in \{1,...,T\}$. (For convenience, denote $t(i,l) = \infty$ for each $l \notin L(i)$.) We assume that there exists a mapping $\iota : \{1,...,L\} \to N$ such that

$$t(\iota(1),1) < t(\iota(1),0),$$

and for every $l \in \{2,...,L\}$,

$$\iota(l) \neq \iota(l-1),$$

$$t(\iota(l),l) < t(\iota(l),l-1), \text{ and } t(\iota(l),l) < t(\iota(l),l-2).$$

Hence, player $\iota(l)$ can observe θ_l earlier than θ_{l-1} and θ_{l-2}. Note that we do not exclude the case in which player $\iota(l)$ can observe, not only θ_l, but also θ_{l-1} and θ_{l-2} by the first stage (round T). This is the main point of difference from the case of single-round, i.e., two-stage announcements, where full detection excludes the case in which player $\iota(l)$ observes θ_{l-1} and θ_{l-2} at the first stage.

Let us consider the T-round procedure, in which, for every $l \in \{1,...,L\}$, the central planner requires player $i = \iota(l)$ to make an announcement about θ_l at round $t(i,l)$. Since $\theta_l \neq \theta_{l-1}$, it is clear that player $\iota(l-1)'s$ announcement about θ_{l-1} detects player $\iota(l)'s$ lies about θ_l. This can make unique implementation in iterative dominance possible to achieve by constructing a dynamic mechanism with multiple rounds.

To understand that the assumption of this section is weaker than the sufficient condition in Subsection 6.1, let us consider the example in which

$$L = 2n,$$

for every $i \in N$,

$$L(i) = \{2i-1, 2i+1\}\text{, i.e., } \omega_i = (\theta_{2i-1}, \theta_{2i+1}) \quad \text{if } i \text{ is odd,}$$

and

$$L(i) = \{2(i-1), 2i\}\text{, i.e., } \omega_i = (\theta_{2(i-1)}, \theta_{2i}) \quad \text{if } i \text{ is even.}$$

Note that for every $l \in \{1,...,L\}$ and $i \in N$,

$$l-2 \in L(i) \text{ whenever } l \in L(i),$$

which violates the sufficient condition in Subsection 6.1. Let us further suppose that $T = 2$, and each player i observes

$$\theta_{2i+1} \text{ at round 1 and } \theta_{2i-1} \text{ at round 2} \qquad \text{if } i \text{ is odd,}$$

and

$$\theta_{2i} \text{ at round 1 and } \theta_{2(i-1)} \text{ at round 2} \qquad \text{if } i \text{ is even.}$$

Then, this case satisfies the assumption of this subsection, and therefore, we can make unique implementation in iterative dominance possible to achieve.

10. Conclusion

We investigated unique exact implementation of an SCF, where we required a mechanism to be bounded and detail-free, utilize only tiny transfers, and satisfy uniqueness of iteratively undominated strategy profile. We defined the iterative dominance notion on the ex-post terms, and required strict inequalities for all full-support distributions as the incentive constraints.

We assumed that there exists partial information about the state that is verifiable. We considered the dynamic aspect of players' information acquisition. We permitted the central planner to design a dynamic mechanism, in which a player is required to announce what he knows at multiple stages with sufficient intervals, where each player was less informed at the early stage than at the later stage. Using this approach, the central planner could establish the reference truthfully in a wide class of environments. With the establishment of truthful reference, we could successfully apply the bounded mechanism design proposed by Abreu and Matsushima (1992).

To be precise, we demonstrated a sufficient condition on the state space and the dynamic aspect of players' information acquisition, namely full detection, under which any SCF was uniquely implementable. In contrast to static mechanism design, a wide variety of SCFs were uniquely implementable by dynamic mechanism design. This permissive result held even if the range of players' lies that the verified information can directly detect was quite narrow.

This is the first paper to demonstrate permissive results in exact implementation with uniqueness of Nash equilibrium. This is also the first paper to investigate bounded mechanism design with the uniqueness of mixed strategy Nash equilibrium without the expected utility hypothesis.

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Appendix A: Proof of Theorem 4

Fix arbitrary real numbers $\eta_1(h) > 0$ for each $h \in \{1,...,h^*\}$, $\eta_2 > 0$, and $\eta_3 > 0$. Let $\eta_1 \equiv (\eta_1(h))_{h=1}^{h^*}$. Fix an arbitrary integer $K > 1$. We construct a dynamic mechanism $\Gamma^{**} = \Gamma^{**}(f, \eta_1, \eta_2, \eta_3, K) = (M^0, M, g, x)$ as follows. Let

$$M_i^0 = \Omega_i ,$$

and

$$M_i = \prod_{k=1}^{K} M_i^k \text{ and } M_i^k = \Phi^* \text{ for all } k \in \{1,...,K\}.$$

In contrast with the dynamic mechanism Γ^* specified in Subsection 6.2, each player i announces an element of the partition Φ_i^* as a sub-message in the second announcement stage instead of announcing an element of Ω_i.

For each $k \in \{2,...,K\}$, we define $g^k : M^k \to \Delta$ in the manner that for each $\omega \in \Omega$,

$$g^k(m^k) = f(\omega) \quad \text{if } m_i^k = \Phi^*(\omega) \text{ for at least } n-1 \text{ players,}$$

and

$$g^k(m^k) = a^* \quad \text{if there exists no such } \omega.$$

Let

$$g(m^0, m) = \frac{\sum_{k=1}^{K} g^k(m^k)}{K}.$$

Let

$$x_i(m^0, m, \omega_0) = \sum_{h=1}^{h^*} x_i^h(m^0, \omega_0) + z_i(m^0, m, \omega_0),$$

where

$$x_i^h(m^0, \omega_0) = \eta_1(h) \quad \text{if } (\omega^0, m_{-i}^0) \notin \bigcup_{\tilde{\omega}_{-i} \in \Omega_{-i}(m_i^0)} \chi_{-i}(h-1)(\tilde{\omega}_{-i}),$$

$$x_i^h(m^0, \omega_0) = 0 \quad \text{if } (\omega^0, m_{-i}^0) \in \bigcup_{\tilde{\omega}_{-i} \in \Omega_{-i}(m_i^0)} \chi_{-i}(h-1)(\tilde{\omega}_{-i}),$$

$$z_i(m^0, m, \omega_0) = \eta_2 + \frac{r_i}{K}\eta_3 \text{ if there exists } k \in \{1,...,K\} \text{ such that}$$

$(m^0, \omega_0) \notin m_i^k$, and

$(m^0, \omega_0) \in m_j^{k'}$ for all $k' < k$ and

$j \in N \setminus \{i\}$,

and

$$z_i(m^0, m, \omega_0) = \frac{r_i}{K}\eta_3 \qquad \text{if there exists no such } k \in \{1,...,K\},$$

where $r_i \in \{0,...,K\}$ implies the number of $k \in \{2,...,K\}$ such that $(m^0, \omega_0) \notin m_i^k$. According to z_i, any first player i who reports an element of Φ_i^* as his sub-message in the second announcement stage that does not include the combination of the profile of first announcements and the verified information (ω^0, m_0) is fined the tiny amount given by η_2.

We select (η_1, η_2, η_3) such that

$$\eta_1(\tilde{h}) > \sum_{h=1}^{\tilde{h}-1} \eta_1(h) + \eta_2 + \eta_3 \quad \text{for all } \tilde{h} \in \{1,...,h^*\}.$$

With this, in the same manner as the proof of Theorem 3, we can prove that for every $i \in N$, if ψ_i is strictly iteratively undominated, then

$$s_i^0(\omega_i) \in \Phi_i^*(\omega_i) \quad \text{for all } \omega_i \in \Omega_i,$$

and

$$s_i^k(\omega) = \Phi^*(\omega) \quad \text{for all } k \in \{1,...,K\} \text{ and } \omega \in \Omega.$$

This along with the measurability implies that if ψ is a strictly iteratively undominated strategy profile, then

$$g(\psi(\omega)) = f(\omega) \quad \text{for all } \omega \in \Omega,$$

and

$$x_i(\psi(\omega), \omega) = 0 \quad \text{for all } i \in N \text{ and } \omega \in \Omega.$$

Appendix B: Proof of Theorem 5

Fix arbitrary real numbers, $\eta_1(h) > 0$ for each $h \in \{1,...,h^*\}$, $\eta_2 > 0$, and $\eta_3 > 0$. Let $\eta_1 = (\eta_1(h))_{h=1}^{h^*}$. Fix an arbitrary integer $K > 1$. We construct a dynamic mechanism, denoted by $\hat{\Gamma}^* = \hat{\Gamma}^*(f, \eta_1, \eta_2, \eta_3, K) = (M^0, M, g, x)$, as follows. Let

$$M_i^0 = \Omega_i,\ M_i = \prod_{k=1}^{K} M_i^k, \text{ and } M_i^k = \Omega_{C(i)} \text{ for all } k \in \{1,...,K\}.$$

Let

$$g(m) = \frac{\sum_{k=2}^{K} \tilde{f}(m^k)}{K-1} \text{ for all } m \in M,$$

which $\tilde{f}$ is the function introduced in Definition 6. Let

$$x_i(m, \omega_0) = \sum_{h=1}^{h^*} x_i^h(m^0, \omega_0) + z_i(m^0, m, \omega_0),$$

where

$$x_i^h(m^0, \omega_0) = \eta_1(h) \quad \text{if } (m_{-i}^0, \omega_0) \notin \bigcup_{\tilde{\omega}_{-i} \in \Omega_{-i}(m_i^1)} \chi_{-i}(h)(\tilde{\omega}_{-i}),$$

$$x_i^h(m^0, \omega_0) = 0 \quad \text{if } (m_{-i}^0, \omega_0) \notin \bigcup_{\tilde{\omega}_{-i} \in \Omega_{-i}(m_i^1)} \chi_{-i}(h)(\tilde{\omega}_{-i}),$$

$$z_i(m^0, m, \omega_0) = \eta_2 + \frac{r_i}{K}\eta_3 \quad \text{if there exists } k \in \{1,...,K\} \text{ such that}$$

$$m_i^k \neq m_{C(i)}^0, \text{ and}$$

$$m_j^{k'} = m_{C(j)}^0 \text{ for all } k' < k \text{ and } j \in N \setminus \{i\},$$

and

$$z_i(m^0, m, \omega_0) = \frac{r_i}{K}\eta_3 \quad \text{if there exists no such } k \in \{1,...,K\},$$

where we denote $m_0^0 = \omega_0$, and $r_i \in \{0,...,K\}$ implies the number of $k \in \{1,...,K\}$ satisfying $m_i^k \neq m_{C(i)}^0$. We select (η_1, η_2, η_3) such that

$$\eta_1(\tilde{h}) > \sum_{h=1}^{\tilde{h}-1} \eta_1(h) + \eta_2 + \eta_3 \text{ for all } \tilde{h} \in \{1,...,h^*\}.$$

Note that

$$0 \le x_i(m,\omega_0) \le \sum_{h=1}^{h^*} \eta_1(h) + \eta_2 + \eta_3 \quad \text{for all} \quad i \in N \quad \text{and} \quad (m,\omega_0).$$

Hence, by choosing $\sum_{h=1}^{h^*} \eta_1(h) + \eta_2 + \eta_3$ close to zero, we can make the monetary transfer $x_i(m,\omega_0)$ close to zero, i.e., lesser than ξ, where ξ was the real number in Definition 6, which was selected close to zero.

We define the *honest* strategy for player i, denoted by $\hat{\psi}_i^* = (\hat{s}_i^{0*}, (\hat{s}_i^{*k})_{k=1}^K)$, as

$$\hat{s}_i^{*0}(\omega_i) = \omega_i \quad \text{for all} \quad \omega_i \in \Omega_i,$$

and

$$\hat{s}_i^{*k}(\omega_{C(i)}) = \omega_{C(i)} \quad \text{for all} \quad k \in \{1,\dots,K\} \quad \text{and} \quad \omega_i \in \Omega_i.$$

The honest strategy profile $\hat{\psi}^* = (\hat{\psi}_i^*)_{i\in N}$ always induces the value of the SCF f and no monetary transfers.

Because of the continuity assumption, we can select a sufficiently large K such that whenever $\max_{a\in A} |\alpha(a) - \alpha'(a)| \le \frac{1}{K}$, then

(B-1) $$u_i(\alpha, -t_i, \omega_i) > u_i(\alpha', -t_i - \eta_2, \omega_i) \quad \text{for all} \quad t_i \in [0, \sum_{h=1}^{h^*} \eta_1(h) + \eta_3] \quad \text{and} \quad \omega_i \in \Omega_i.$$

In the same manner as in Theorem 3, we can prove that if s_i is strictly iteratively undominated in $\hat{G}^*$, then, $s_i(\omega_i) \in \bigcap_{h\to\infty} \chi_i(h)(\omega_i)$, that is,

$$s_i^1(\omega_i) = \omega_i \quad \text{for all} \quad \omega_i \in \Omega_i.$$

Suppose $m_j^1 = \omega_j$ for all $j \in N$. We can show that if s_i is strictly iteratively undominated, then,

$$s_i^2(\omega_i) = \omega_i \quad \text{for all} \quad \omega_i \in \Omega_i.$$

Suppose that there exists a player $j \in N \setminus \{i\}$ who announces $m_j^2 \ne m_j^1$, i.e., $m_j^2 \ne \omega_j$. Then, by announcing $m_i^2 \ne \omega_i$ instead of ω_i, player i is fined the monetary amount given by η_2 or more. From (B-1), the impact of the monetary fine η_2 on his welfare is greater than the impact of the resultant change of allocation. Next, suppose that there exists no player $j \in N \setminus \{i\}$ who announces $m_j^2 \ne \omega_j$. Then, by announcing $m_i^2 \ne \omega_i$

instead of ω_i, player i is fined the monetary amount given by η_3. Because of strict incentive compatibility, the resultant change of allocation never improves his welfare. Hence, player i prefers $m_i^2 = \omega_i$.

Fix an arbitrary $h \geq 3$. Suppose that $m_j^{h'} = \omega_j$ for all $j \in N$ and $h' < h$. In the same manner as above, we can show that each player i prefers $m_i^h = \omega_i$. These observations imply that if s_i is strictly iteratively undominated, then,

$$s_i^k(\omega_i) = \omega_i \text{ for all } \omega_i \in \Omega_i \text{ and all } k \in \{1,...,K\},$$

that is, $s_i = \hat{s}_i^*$.

Appendix C: Proof of Theorem 6

Fix arbitrary real numbers, $\eta_1(h) > 0$ for each $h \in \{1,...,h^*\}$, $\eta_2 > 0$, and $\eta_3 > 0$. Let $\eta_1 = (\eta_1(h))_{h=1}^{h^*}$. Fix an arbitrary integer $K > 1$. We construct a mechanism denoted by $\hat{G}^{**} = \hat{G}^{**}(f, \eta_1, \eta_2, \eta_3, K) = (M, g, x)$ as follows. Let

$$M_i = \prod_{k=1}^{K} M_i^k,$$

$$M_i^k = \Omega_i,$$

and

$$M_i^k = \Phi_i^* \text{ for all } k \in \{2,...,K\}.$$

In contrast with $\hat{G}^*$, each player i announces not ω_i but $\phi_i \in \Phi_i^*$ for all sub-messages except the first sub-message m_i^1. Let

$$g(m) = \frac{\sum_{k=2}^{K} \tilde{f}(m^k)}{K-1} \text{ for all } m \in M,$$

which $\tilde{f}$ is the function introduced in Definition 9. We will write $\tilde{f}(\phi) = \tilde{f}(\omega)$ if $\omega_i \in \phi_i$ for all $i \in N$, where we denote $\phi = (\phi_i)_{i \in N}$. Let

$$x_i(m,\omega_0)=\sum_{h=1}^{h^*}x_i^h(m^1,\omega_0)+z_i(m),$$

where

$x_i^h(m^1,\omega_0)=\eta_1(h)$ if $(m_{-i}^1,\omega_0)\notin \bigcup_{\tilde{\omega}_{-i}\in\Omega_{-i}(m_i^1)}\chi_{-i}(h)(\tilde{\omega}_{-i})$,

$x_i^h(m^1,\omega_0)=0$ if $(m_{-i}^1,\omega_0)\notin \bigcup_{\tilde{\omega}_{-i}\in\Omega_{-i}(m_i^1)}\chi_{-i}(h)(\tilde{\omega}_{-i})$,

$z_i(m)=\eta_2+\frac{r_i}{K-1}\eta_3$ if there exists $k\in\{2,...,K\}$ such that

$(m_i^1,\omega_0)\notin m_i^k$, and

$(m_j^1,\omega_0)\in m_j^{k'}$ for all $k'<k$ and $j\in N\setminus\{i\}$,

and

$z_i(m)=\frac{r_i}{K-1}\eta_3$ if there exists no such $k\in\{2,...,K\}$,

where $r_i\in\{0,...,K-1\}$ implies the number of $k\in\{2,...,K\}$ satisfying $m_i^k\neq m_i^1$. We select (η_1,η_2,η_3) such that

$$\eta_1(\tilde{h})>\sum_{h=1}^{\tilde{h}-1}\eta_1(h)+\eta_2+\eta_3 \text{ for all } \tilde{h}\in\{1,...,h^*\}.$$

In the same manner as the proof of Theorem 5, we can prove that for every $i\in N$, if ψ_i is strictly iteratively undominated, then

$$s_i^1(\omega_i)\in\Phi_i^*(\omega_i) \text{ for all } \omega_i\in\Omega_i,$$

and

$$s_i^k(\omega)=\Phi^*(\omega) \text{ for all } k\in\{2,...,K\} \text{ and } \omega\in\Omega.$$

This combined withmeasurability implies that if ψ is a strictly iteratively undominated strategy profile, then

$$g(\psi(\omega))=f(\omega) \text{ for all } \omega\in\Omega,$$

and

$$x_i(\psi(\omega),\omega)=0 \text{ for all } i\in N \text{ and } \omega\in\Omega.$$