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Abstract

Although stochastic volatility and GARCH models have been successful to describe the volatility dynamics of univariate asset returns, their natural extension to the multivariate models with dynamic correlations has been difficult due to several major problems. Firstly, there are too many parameters to estimate if available data are only daily returns, which results in unstable estimates. One solution to this problem is to incorporate additional observations based on intraday asset returns such as realized covariances. However, secondly, since multivariate asset returns are not traded synchronously, we have to use largest time intervals so that all asset returns are observed to compute the realized covariance matrices, where we fail to make full use of available intraday informations when there are less frequently traded assets. Thirdly, it is not straightforward to guarantee that the estimated (and the realized) covariance matrices are positive definite.

Our contributions are : (1) we obtain the stable parameter estimates for dynamic correlation models using the realized measures, (2) we make full use of intraday informations by using pairwise realized correlations, (3) the covariance matrices are guaranteed to be positive definite, (4) we avoid the arbitrariness of the ordering of asset returns, (5) propose the flexible correlation structure model (e.g. such as setting some correlations to be identically zeros if necessary), and (6) the parsimonious specification for the leverage effect is proposed. Our proposed models are applied to daily returns of nine U.S. stocks with their realized volatilities and pairwise realized correlations, and are shown to outperform the existing models with regard to portfolio performances.

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1 Introduction

Modelling time-varying volatility and correlations of multivariate time series is one of most important problems in the financial risk management, and there has been vast literature that tackles modelling time-varying volatility of univariate time series using the GARCH or stochastic volatility (SV) models. However, the extension of their models to multivariate model with dynamic correlations has not been straightforward due to the following several major problems.

Firstly, there are too many parameters to estimate if available data are only daily returns, which results in unstable estimates. An intuitive solution to reduce the number of parameters is to introduce the factor structure assuming that a small number of common factors describe the dynamics of time-varying covariance matrices as discussed in the factor stochastic volatility models (e.g. Pitt and Shephard (1999), Chib et al. (2006) and Lopes and Carvalho (2007)). However, factor modelling requires to decide the number of factors a priori and we need to restrict the structure of factors for identification of parameters (e.g. Lopes and West (2004)). Further, the estimation results and the prediction performance of the model are usually subject to the ordering of asset returns.

An alternative effective approach is to incorporate additional observations based on intraday asset returns such as realized covariances, which recently become available in financial market. In univariate SV models, realized SV (RSV) models which estimate time-varying volatilities using daily returns and realized volatility simultaneously have been proposed to show that parameter estimates are more accurate than those of SV models using only daily returns and that the RSV models outperform SV models in forecasting volatilities (e.g. Takahashi et al. (2009), Dobrev and Szerszen (2010), Koopman and Scharth (2013), Zheng and Song (2014), Omori and Watanabe (2015), Takahashi et al. (2016)). Although the realized volatilities are subject to microstructure noises and non-trading hours, and hence are biased estimates of the integrated volatilities, such biases are adjusted automatically within the proposed model. Similarly, the univariate GARCH model is extended to the realized GARCH models which incorporates the realized volatilities into the variance equations and it is shown to lead to substantial improvements in the empirical fit and quantile forecasts over the standard GARCH model that only uses daily returns (Hansen et al. (2012), Watanabe (2012)).

The extension to the multivariate RSV model is also considered in Cholesky RSV model

(Shirota et al. (2016)) where Cholesky decompositions of the realized covariance matrices are used as additional sources for measurement equations, and it modelled the dynamics for the logarithm of diagonal elements and off-diagonal elements of Cholesky decomposed covariance matrices respectively. It is shown that the portfolio performances of the proposed model outperformed other SV models without realized measures in the empirical studies, but it should also be noted that the performance of Cholesky RSV models may depend on the ordering of asset returns in the vector of the response.

Secondly, however, high-frequency data are not observed synchronously, which causes difficulties in the extension of the univariate RSV model to the multivariate RSV model. For example, in the Cholesky RSV model, it is implicitly assumed that the multivariate assets are traded synchronously to compute the realized covariance matrices. Since multivariate assets are not traded synchronously, we have to use largest time intervals so that all asset returns are observed to compute the realized covariance matrices. The non-synchronous trading leads to ignore some of frequently traded asset return data, and hence that we fail to make full use of available intraday informations when there are less frequently traded assets.

Thirdly, it is not straightforward to guarantee that the estimated (and the realized) covariance matrices are positive definite. The model parameters may be difficult to estimate in practice under the constraints which satisfy the positive definiteness. Using Cholesky decomposition of the time-varying covariance matrices is one way to guarantee the positive definiteness (Shirota et al. (2016)), but it also requires that the multivariate assets are traded synchronously to compute the realized covariance matrices as mentioned above. Also, the interpretation of each latent variable of the decomposition is not straightforward since it does not correspond to each pair of asset returns and it is subject to the ordering of the asset returns. If we use each element of realized covariances for each pair of asset returns, it may result in the non-positive definite covariance matrices.

To overcome these difficulties, we propose a multivariate realized SV (MRSV) model with pairwise realized correlations, where we incorporate dynamic latent correlation variables in addition to latent volatility variables with realized measures for each pairwise correlation and volatilities in the framework of multivariate SV models with realized volatilities. Model parameters are estimated using Markov chain Monte Carlo simulations, and we sample latent correlation variables one at a time given others so that we keep the covariance matrices positive definite. Realized Beta GARCH model proposed by Hansen et al. (2014) is such a promising multivariate GARCH model with realized measures for volatilities and covolatilities where they used measurement equations for pairwise realized correlations with market returns and modelled dynamics of Fisher transformed conditional correlation coefficients. However, they focused on pairwise correlations between the market return and an individual asset return, assuming that individual asset returns are conditionally independent given the market return. Another useful approach for the joint modelling of returns and realized covariances are based on Wishart processes (e.g. Jin and Maheu (2013), Windle et al. (2014), Jin and Maheu (2016), So et al. (2016)). The covariance matrix is assumed to follow Wishart distribution whose scale matrix depends on past realized covariance matrices which are computed using larger time intervals than necessary to be positive definite.

Our approach, on the other hand, is based on modelling individual volatilities and pairwise covariances rather than the covariance matrix simultaneously, and we are able to make full use of available intraday informations even when there are less frequently traded assets. This implies our model still can be constructed even though some of realized measures are missing. Also, our model is far more flexible in the sense that it is possible to restrict any correlation coefficients to be zero for very high dimensional asset returns data, which reduces the number of parameters and may lead to improve the forecasting performances. Among the multivariate SV models in the literature, our model is a natural extension of univariate RSV model to the multivariate model, and it gives us straightforward interpretation of estimated parameters.

Furthermore, we extend our model to incorporate the leverage effect which is well-known to exist in stock markets. The leverage effect refers to the negative correlation between an asset return and its volatility. That is, the decrease in the stock return is followed by the increase in its volatility. In the forecasting mean and covariance of asset returns for e.g. the portfolio optimization, it is expected that incorporating leverage effect in econometric models improves the accuracy of prediction. However, it may increase the number of parameters to estimate and the realized measures are not available for such an effect. Thus we also consider the parsimonious parameterization for the leverage effect.

Our contributions are : (1) we obtain the stable parameter estimates for dynamic correlation models using the realized measures, (2) we make full use of intraday informations by using pairwise realized correlations, (3) the estimated covariance matrices are guaranteed to be positive definite, (4) we avoid the arbitrariness of the ordering of asset returns, (5) propose the flexible correlation structure model such as setting some correlations to be identically zeros if necessary, and, further, (6) introduce the parsimonious specification for the leverage effect.

The structure of this paper is as follows. Section 2 introduces the multivariate realized SV model with daily returns, realized volatilities, and pairwise realized correlations. Section 3 describes estimation algorithms using Markov Chain Monte Carlo simulation. Section 4 extends it to incorporate the leverage effect. In Section 5, we give an illustrative numerical example using simulated data. Finally, in Section 6, the proposed model is applied to nine U.S. stock return data and the model with leverage effect is shown to outperform other competing models with regard to the portfolio performances.

2 Multivariate realized stochastic volatility model

This section introduces multivariate realized stochastic volatility (MRSV) model, which uses realized measures for volatility and pairwise correlations of asset returns. By using additional information of realized measure for asset returns, we can overcome the curse of dimension in estimating dynamic covariance matrices. Cholesky RSV model proposed by Shirota et al. (2016) also uses realized measure of variances and covariances (which we call a realized covariance matrix) to estimate the latent covariance matrix of asset returns. However, the realized covariance matrix is less informative when there exist less frequent series of asset returns. This is because we require synchronous observations of all the asset return series to compute the realized covariance matrix. To utilize full information of realized measure for the correlations, we propose using realized measures for the latent pairwise correlations. It should be noted that the pairwise correlation can be computed if that pair of series is synchronously observed. We call the realized measure for the correlation coefficient pairwise realized correlation. Using the realized correlations, to guarantee the positive definiteness of the latent covariance matrices, we propose MCMC algorithm where we sample latent correlation coefficients from the conditional posterior distribution so that the matrices are positive definite.

2.1 Multivariate stochastic volatility model with dynamic correlations

First, we define the multivariate SV (MSV) model without realized measures. Let $\boldsymbol{y}_t = (y_{1t}, \ldots, y_{pt})'$ and $\boldsymbol{h}_t = (h_{1t}, \ldots, h_{pt})'$ denote a $p \times 1$ stock return vector and its corresponding log volatility latent vector at time t. The basic MSV model is given by

$$\boldsymbol{y}_t = \boldsymbol{m}_t + \mathbf{V}_t^{1/2}\boldsymbol{\epsilon}_t, \quad \boldsymbol{\epsilon}_t \sim \mathrm{N}(\mathbf{0}, \mathbf{R}_t), \qquad t = 1, \ldots, T, \tag{1}$$

$$\boldsymbol{h}_{t+1} = \boldsymbol{\mu} + \boldsymbol{\Phi}(\boldsymbol{h}_t - \boldsymbol{\mu}) + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Omega}), \quad t = 1, \ldots, T-1, \tag{2}$$

$$\boldsymbol{m}_{t+1} = \boldsymbol{m}_t + \boldsymbol{\nu}_t, \quad \boldsymbol{\nu}_t \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Sigma}_m), \quad t = 1, \ldots, T-1, \tag{3}$$

$$\boldsymbol{h}_1 \sim \mathrm{N}(\boldsymbol{\mu}, \boldsymbol{\Omega}_0), \quad \boldsymbol{m}_1 \sim \mathrm{N}(\mathbf{0}, \kappa\,\boldsymbol{\Sigma}_m), \tag{4}$$

where $\mathbf{R}_t = \{\rho_{ij,t}\}$ is a correlation matrix, we assume that h_{it} follows stationary autoregressive process (with its coefficient $|\phi_j| < 1$) and that the mean process $\boldsymbol{m}_t = (m_{1t}, \ldots, m_{pt})'$ follows random walk process, and

$$\begin{aligned}
&\boldsymbol{\epsilon}_t = (\epsilon_{1t}, \ldots, \epsilon_{pt})', \quad \boldsymbol{\eta}_t = (\eta_{1t}, \ldots, \eta_{pt})', \\
&\boldsymbol{\nu}_t = (\nu_{1t}, \ldots, \nu_{pt})', \quad \mathbf{V}_t = \mathrm{diag}(\exp(h_{1t}), \ldots, \exp(h_{pt})), \\
&\boldsymbol{\Sigma}_m = \mathrm{diag}(\boldsymbol{\sigma}_m^2), \quad \boldsymbol{\sigma}_m^2 = (\sigma_{m,1}^2, \ldots, \sigma_{m,p}^2)', \\
&\boldsymbol{\Phi} = \mathrm{diag}(\boldsymbol{\phi}), \quad \boldsymbol{\phi} = (\phi_1, \ldots, \phi_p)'.
\end{aligned}$$

We denote a diagonal matrix $\mathbf{A}$ with diagonal elements $\boldsymbol{a} = (a_{11}, \ldots, a_{mm})'$ by $\mathbf{A} = \mathrm{diag}(\boldsymbol{a})$. For the initial distributions of $\boldsymbol{m}_1$ and $\boldsymbol{h}_1$, we set κ to some large constant for $\boldsymbol{m}_1$ for simplicity and set $\boldsymbol{\Omega}_0$ to satisfy the stationary condition $\boldsymbol{\Omega}_0 = \boldsymbol{\Phi}\,\boldsymbol{\Omega}_0\,\boldsymbol{\Phi} + \boldsymbol{\Omega}$ for $\boldsymbol{h}_1$ such that

$$\mathrm{vec}(\boldsymbol{\Omega}_0) = (\mathbf{I}_{p^2} - \boldsymbol{\Phi} \otimes \boldsymbol{\Phi})^{-1}\,\mathrm{vec}(\boldsymbol{\Omega}), \tag{5}$$

where $\mathbf{I}_{p^2}$ denotes a $p^2 \times p^2$ unit matrix. To model dynamics of correlation matrix, we consider the following Fisher transformation $g_{ij,t+1}$ of the correlation coefficient $\rho_{ij,t}$, and assume that it follows random walk process for simplicity:

$$g_{ij,t+1} = g_{ij,t} + \zeta_{ij,t}, \qquad \zeta_{ij,t} \sim \text{i.i.d. } \mathrm{N}(0, \sigma_{\zeta,ij}^2), \quad t = 1, \ldots, T-1, \tag{6}$$

$$g_{ij,1} \sim \mathrm{N}(0, \kappa\sigma_{\zeta,ij}^2), \qquad g_{ij,t} = \log(1 + \rho_{ij,t}) - \log(1 - \rho_{ij,t}), \tag{7}$$

for $i, j = 1, \ldots, p$ $(j < i)$ and we denote

$$\begin{aligned}
\boldsymbol{\rho}_t &= (\rho_{21,t}, \ldots, \rho_{pp-1,t})', \quad \boldsymbol{g}_t = (g_{21,t}, \ldots, g_{pp-1,t})', \\
\boldsymbol{\zeta}_t &= (\zeta_{21,t}, \ldots, \zeta_{pp-1,t})', \quad \boldsymbol{\sigma}_\zeta^2 = (\sigma_{\zeta,21}^2, \ldots, \sigma_{\zeta,pp-1}^2)'.
\end{aligned}$$

Non-arbitrary ordering of asset returns and the flexible correlation structure. We note that above specification (1) – (7) is independent of the ordering of asset returns in $\boldsymbol{y}_t$, while the conventional factor SV models or the Cholesky SV models (Shirota et al. (2016)) may be affected by the selection of the ordering. Further, it allows us to model the structure of correlations in a flexible way. For example, we can easily restrict some correlation coefficients to be exactly equal to zero when the dimensional of $\boldsymbol{y}_t$ is very high.

Remark 1. It is easy to assume that $\boldsymbol{m}_t$ and $g_{ij,t}$'s follow stationary autoregressive processes alternatively. However, since it imposes the mean reversion properties on these processes, we rather consider random walk processes without such properties for simplicity. For the long term prediction, we may need such a stationarity condition.

2.2 Realized stochastic volatilities and pairwise realized correlations

Realized measures as an additional source of information. In the MSV models above, there are too many parameters to estimate using only daily asset returns, and it is often that parameter estimates are unstable. Recently, high frequency data in the financial market has become available, and it plays more important role in the empirical studies in finance, since the realized measures of variances and covariances, are more informative estimators of true variances and covariances (see e.g. Andersen et al. (2001), Andersen et al. (2001), Barndorff-Nielsen and Shephard (2002), Barndorff-Nielsen and Shephard (2004)).

Let $x_{it} = \log RV_{it}$ and $w_{ij,t} = \log\{(1 + RCOR_{ij,t})/(1 - RCOR_{ij,t})\}$ where RV_{it} and $RCOR_{ij,t}$ are realized measures of the volatility of the i-th asset return and the correlation between i-th and j-th asset returns at time t. Thus we introduce the additional measurement equations based on realized measures:

$$x_{it} = \xi_i + h_{it} + u_{it} \qquad u_{it} \sim \mathrm{N}(0, \sigma^2_{u,i}), \quad t = 1, \ldots, T, \tag{8}$$

$$w_{ij,t} = \delta_{ij} + g_{ij,t} + v_{ij,t} \qquad v_{ij,t} \sim \mathrm{N}(0, \sigma^2_{v,ij}), \quad t = 1, \ldots, T, \tag{9}$$

for $i, j = 1, \ldots, p$ $(i > j)$. The terms ξ_j's and δ_{ij}'s are included to adjust the biases due to the microstructure noise, non-trading hours, non-synchronous trading and so forth. The multivariate realized stochastic volatility model with pairwise realized correlations are defined by (1) – (9). We denote

$$\begin{aligned}
\boldsymbol{x}_t &= (x_{1t},\ldots,x_{pt})', \quad \boldsymbol{w}_t = (w_{21,t},\ldots,w_{pp-1,t})', \\
\boldsymbol{u}_t &= (u_{1t},\ldots,u_{pt})', \quad \boldsymbol{v}_t = (v_{21,t},\ldots,v_{pp-1,t})', \\
\boldsymbol{\xi} &= (\xi_1,\ldots,\xi_p)', \qquad \boldsymbol{\delta} = (\delta_{21},\ldots,\delta_{pp-1})', \\
\boldsymbol{\sigma}_u^2 &= (\sigma_{u,1}^2,\ldots,\sigma_{u,p}^2)', \quad \boldsymbol{\sigma}_v^2 = (\sigma_{v,21}^2,\ldots,\sigma_{v,pp-1}^2)',
\end{aligned}$$

Use of pairwise realized correlations. As the realized correlation $RCOR_{ij,t}$, we will use the pairwise realized correlations. If there exist less frequent series of asset returns, the realized covariance matrix may lose a large part of information since it is calculated only when all the series are observed synchronously. On the other hand, the pairwise realized correlation coefficients can be calculated for each pair of series of returns separately, so we can use the full information of the realized measures for correlations. Moreover, we can estimate parameters even if we cannot obtain realized measures for some pairs.

Bias corrections of realized measures. Realized volatilities and pairwise realized correlations have more information about true volatilities and correlations, but there may be biases due to market microstructure noise, non-trading hours, nonsynchronous trading and so forth. To correct these biases of realized measures, we model the observation equations of realized volatilities and pairwise realized correlations with bias adjustment terms, ξ_j, δ_{ij}. Although daily returns have relatively less information about true volatilities and correlations, they are less subject to the biases caused by the high frequency data. Therefore, we can estimate biases in realized measures using the information of daily returns and, at the same time, get the additional information with regard to true volatilities and correlations using the realized measures.

3 Posterior inference

3.1 Prior distributions for parameters

Since there are many latent variables in our proposed model and hence it is difficult to evaluate the likelihood, we take Bayesian approach and estimate model parameters using Markov chain Monte Carlo simulation. First we assume the prior distribution of $\boldsymbol{\theta} \equiv (\boldsymbol{\phi}, \boldsymbol{\mu}, \boldsymbol{\xi}, \boldsymbol{\delta}, \boldsymbol{\sigma}_u^2, \boldsymbol{\sigma}_v^2, \boldsymbol{\sigma}_\zeta^2, \boldsymbol{\Sigma}_m, \boldsymbol{\Omega})$ as follows. For the prior distributions of μ_i, ξ_i and δ_{ij}, we assume multivariate independent normal distributions. The prior distributions of $\sigma_{u,i}^2, \sigma_{v,ij}^2, \sigma_{\zeta,ij}^2$

and $\sigma_{m,i}^2$ are assumed to be independent inverse gamma distributions. For ϕ_i and $\boldsymbol{\Omega}$, we assume $(1+\phi_i)/2 \sim \text{Beta}(a,b)$ and inverse Wishart distribution respectively. In summary, we assume the following prior distributions:

$$\mu_i \sim \text{N}(m_\mu, s_\mu^2), \quad \xi_i \sim \text{N}(m_\xi, s_\xi^2), \quad \delta_{ij} \sim \text{N}(m_\delta, s_\delta^2), \tag{10}$$

$$\sigma_{u,i}^2 \sim \text{IG}\left(\frac{n_u}{2}, \frac{d_u}{2}\right), \quad \sigma_{v,ij}^2 \sim \text{IG}\left(\frac{n_v}{2}, \frac{d_v}{2}\right), \quad \sigma_{\zeta,ij}^2 \sim \text{IG}\left(\frac{n_\zeta}{2}, \frac{d_\zeta}{2}\right), \tag{11}$$

$$\frac{1+\phi_i}{2} \sim \text{Beta}(a,b), \quad \sigma_{m,i}^2 \sim \text{IG}\left(\frac{n_m}{2}, \frac{d_m}{2}\right), \quad \Omega \sim \text{IW}(\nu, \boldsymbol{S}), \tag{12}$$

for $i,j = 1,\ldots,p$ $(j < i)$, and $a, b, m_\mu, s_\mu, m_\xi, s_\xi, m_\delta, s_\delta, n_u, d_u, n_v, d_v, n_\zeta, d_\zeta, n_m, d_m, \nu, \boldsymbol{S}$ are hyperparameters. In our empirical studies, these hyperparameters are chosen to reflect that we have little prior information about parameters.

3.2 Markov chain Monte Carlo simulation

Let $\boldsymbol{g} = (\boldsymbol{g}_1', \ldots, \boldsymbol{g}_T')'$, $\boldsymbol{h} = (\boldsymbol{h}_1', \ldots, \boldsymbol{h}_T')'$ and $\boldsymbol{m} = (\boldsymbol{m}_1', \ldots, \boldsymbol{m}_T')'$. Further, let $\boldsymbol{w} = (\boldsymbol{w}_1', \ldots, \boldsymbol{w}_T')'$, $\boldsymbol{x} = (\boldsymbol{x}_1', \ldots, \boldsymbol{x}_T')'$ and $\boldsymbol{y} = (\boldsymbol{y}_1', \ldots, \boldsymbol{y}_T')'$. Then, the joint posterior probability density function is given by

$$\begin{aligned}
&\pi(\boldsymbol{\theta}, \boldsymbol{g}, \boldsymbol{h}, \boldsymbol{m} \,|\, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}) \\
&\propto \prod_{t=1}^{T} |\mathbf{V}_t^{1/2}\,\mathbf{R}_t\,\mathbf{V}_t^{1/2}|^{-1/2} \exp\left\{-\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{m}_t)'(\mathbf{V}_t^{1/2}\,\mathbf{R}_t\,\mathbf{V}_t^{1/2})^{-1}(\boldsymbol{y}_t - \boldsymbol{m}_t)\right\} \\
&\quad \times |\,\boldsymbol{\Omega}_0\,|^{-1/2} \exp\left\{-\frac{1}{2}(\boldsymbol{h}_1 - \boldsymbol{\mu})'\,\boldsymbol{\Omega}_0^{-1}(\boldsymbol{h}_1 - \boldsymbol{\mu})\right\} \\
&\quad \times \prod_{t=1}^{T-1} |\boldsymbol{\Omega}|^{-1/2} \exp\left[-\frac{1}{2}\{\boldsymbol{h}_{t+1} - (\mathbf{I}_p - \boldsymbol{\Phi})\boldsymbol{\mu} - \boldsymbol{\Phi}\,\boldsymbol{h}_t\}'\boldsymbol{\Omega}^{-1}\{\boldsymbol{h}_{t+1} - (\mathbf{I}_p - \boldsymbol{\Phi})\boldsymbol{\mu} - \boldsymbol{\Phi}\,\boldsymbol{h}_t\}\right] \\
&\quad \times \prod_{t=1}^{T} |\,\boldsymbol{\Sigma}_u\,|^{-1/2} \exp\left\{-\frac{1}{2}(\boldsymbol{x}_t - \boldsymbol{\xi} - \boldsymbol{h}_t)'\,\boldsymbol{\Sigma}_u^{-1}(\boldsymbol{x}_t - \boldsymbol{\xi} - \boldsymbol{h}_t)\right\} \\
&\quad \times \prod_{i>j}^{p} \sigma_{\zeta,ij}^{-1} \exp\left(-\frac{g_{ij,1}^2}{2\kappa\sigma_{\zeta,ij}^2}\right) \prod_{t=1}^{T-1} \sigma_{\zeta,ij}^{-1} \exp\left\{-\frac{(g_{ij,t+1} - g_{ij,t})^2}{2\sigma_{\zeta,ij}^2}\right\} \\
&\quad \times \prod_{i>j}^{p} \prod_{t=1}^{T} \sigma_{v,ij}^{-1} \exp\left\{-\frac{(w_{ij,t} - \delta_{ij} - g_{ij,t})^2}{2\sigma_{v,ij}^2}\right\} \\
&\quad \times \prod_{i=1}^{p} \sigma_{m,i}^{-1} \exp\left(-\frac{m_{i,1}^2}{2\kappa\sigma_{m,i}^2}\right) \prod_{t=1}^{T-1} \sigma_{m,i}^{-1} \exp\left\{-\frac{(m_{i,t+1} - m_{i,t})^2}{2\sigma_{m,i}^2}\right\} \times \pi(\boldsymbol{\theta}),
\end{aligned} \tag{13}$$

where $\boldsymbol{\Sigma}_u = \text{diag}(\sigma_{u,1}^2, \ldots, \sigma_{u,p}^2)$ and $\pi(\boldsymbol{\theta})$ is a prior probability density function of parameters. To conduct the statistical inference on parameters, we implement Markov chain Monte Carlo simulation in nine blocks. Let $\boldsymbol{\theta}_{\backslash\boldsymbol{\beta}}$ denote the parameter $\boldsymbol{\theta}$ excluding $\boldsymbol{\beta}$. Then,

1. Initialize $\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{m}$ and $\boldsymbol{\theta}$.
2. Generate $\boldsymbol{g} \,|\, \boldsymbol{\theta}, \boldsymbol{h}, \boldsymbol{m}, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}$.
3. Generate $\boldsymbol{h} \,|\, \boldsymbol{\theta}, \boldsymbol{m}, \boldsymbol{g}, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}$.
4. Generate $\boldsymbol{m} \,|\, \boldsymbol{\theta}, \boldsymbol{h}, \boldsymbol{g}, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}$.
5. Generate $\boldsymbol{\phi} \,|\, \boldsymbol{\theta}_{\backslash\boldsymbol{\phi}}, \boldsymbol{h}, \boldsymbol{m}, \boldsymbol{g}, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}$.
6. Generate $(\boldsymbol{\mu}, \boldsymbol{\xi}, \boldsymbol{\delta}) | \, \boldsymbol{\theta}_{\backslash(\boldsymbol{\mu},\boldsymbol{\xi},\boldsymbol{\delta},)}, \boldsymbol{h}, \boldsymbol{m}, \boldsymbol{g}, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}$.
7. Generate $(\boldsymbol{\sigma}_u^2, \boldsymbol{\sigma}_v^2, \boldsymbol{\sigma}_\zeta^2, \boldsymbol{\Sigma}_m) | \, \boldsymbol{\theta}_{\backslash(\boldsymbol{\sigma}_u^2,\boldsymbol{\sigma}_v^2,\boldsymbol{\sigma}_\zeta^2,\boldsymbol{\Sigma}_m)}, \boldsymbol{h}, \boldsymbol{m}, \boldsymbol{g}, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}$.
8. Generate $\boldsymbol{\Omega} \,|\, \boldsymbol{\theta}_{\backslash\boldsymbol{\Omega}}, \boldsymbol{h}, \boldsymbol{m}, \boldsymbol{g}, \boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}$.
9. Go to Step 2.

We describe the MCMC sampling algorithm in more details below.

3.2.1 Generation of $\boldsymbol{g}_t$ for the dynamic correlation matrix $\mathbf{R}_t$

The conditional posterior probability density function of $g_{ij,t}$ given other parameters and latent variables is

$$\pi(g_{ij,t}|\cdot) \propto \exp\left\{-\frac{1}{2\sigma_{t*}^2}(g_{ij,t} - m_{t*})^2 + r(g_{ij,t})\right\}, \tag{14}$$

where

$$r(g_{ij,t}) = -\frac{1}{2}\log|\mathbf{R}_t| - \frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{m}_t)'(\mathbf{V}_t^{1/2}\,\mathbf{R}_t\,\mathbf{V}_t^{1/2})^{-1}(\boldsymbol{y}_t - \boldsymbol{m}_t), \tag{15}$$

and

$$m_{t*} = \begin{cases} \sigma_{t*}^2\left\{\sigma_{\zeta,ij}^{-2} g_{ij,2} + \sigma_{v,ij}^{-2}(w_{ij,1} - \delta_{ij})\right\}, & t = 1, \\ \sigma_{t*}^2\left\{\sigma_{\zeta,ij}^{-2}(g_{ij,t-1} + g_{ij,t+1}) + \sigma_{v,ij}^{-2}(w_{ij,t} - \delta_{ij})\right\}, & t = 2, \ldots, T-1, \\ \sigma_{t*}^2\left\{\sigma_{\zeta,ij}^{-2} g_{ij,T-1} + \sigma_{v,ij}^{-2}(w_{ij,T} - \delta_{ij})\right\}, & t = T, \end{cases} \tag{16}$$

$$\sigma_{t*}^2 = \begin{cases} \left\{(\kappa^{-1} + 1)\sigma_{\zeta,ij}^{-2} + \sigma_{v,ij}^{-2}\right\}^{-1}, & t = 1, \\ \left(2\sigma_{\zeta,ij}^{-2} + \sigma_{v,ij}^{-2}\right)^{-1}, & t = 2, \ldots, T-1, \\ \left(\sigma_{\zeta,ij}^{-2} + \sigma_{v,ij}^{-2}\right)^{-1}, & t = T. \end{cases} \tag{17}$$

Positive definiteness of $\mathbf{R}_t$. When sampling $\mathbf{R}_t$, we need sample each correlation coefficient $\rho_{ij,t}$ (or equivalently $g_{ij,t}$) so that we guarantee that the correlation matrix $\mathbf{R}_t$ to be positive definite. We first state the condition for $\rho_{ij,t}$ to guarantee that $\mathbf{R}_t$ is positive definite given other elements of $\mathbf{R}_t$ and other $\rho_{ij,s}$ $(s \neq t)$.

Proposition 1. Suppose that $\mathbf{R}_t = \{\rho_{ij,t}\}$ is a correlation matrix and let $\boldsymbol{\rho}_{it}$ denote the transpose of the i-th row vector of $\mathbf{R}_t$ excluding 1, $\boldsymbol{\rho}_{it} = (\rho_{i1,t}, \ldots, \rho_{ii-1,t}, \rho_{ii+1,t}, \ldots, \rho_{ip,t})'$, and $\mathbf{R}_{it}$ denotes the submatrix excluding the i-th row and the i-th column from $\mathbf{R}_t$. The condition for $\rho_{ij,t}$ to guarantee that $\mathbf{R}_t$ is positive definite is $\rho_{ij,t} \in (L_{ijt}, U_{ijt})$ where bounds L_{ijt} and U_{ijt} are given by

$$\frac{-\boldsymbol{b}_j' \boldsymbol{\rho}_{i,-j,t} \pm \sqrt{(\boldsymbol{b}_j' \boldsymbol{\rho}_{i,-j,t})^2 - a_j(\boldsymbol{\rho}_{i,-j,t}' \mathbf{C}_j \boldsymbol{\rho}_{i,-j,t} - 1)}}{a_j}, \tag{18}$$

and $\boldsymbol{\rho}_{i,-j,t}$ is the vector excluding the j-th element of $\boldsymbol{\rho}_{it}$, a_j is the (j,j)-th element of $\mathbf{R}_{it}^{-1}$, $\boldsymbol{b}_j$ is the vector excluding a_j from the j-th column of $\mathbf{R}_{it}^{-1}$, and $\mathbf{C}_j$ is the matrix excluding the j-th row and j-th column from $\mathbf{R}_{it}^{-1}$.

Proof: Since $\mathbf{R}_{it}$ is positive definite, its principal submatrices are all positive definite. Further, noting that

$$|\mathbf{R}_t| = |\mathbf{R}_{it}| \times |1 - \boldsymbol{\rho}_{it}' \mathbf{R}_{it}^{-1} \boldsymbol{\rho}_{it}|, \tag{19}$$

the condition for $\mathbf{R}_t$ to be positive definite is

$$1 - \boldsymbol{\rho}_{it}' \mathbf{R}_{it}^{-1} \boldsymbol{\rho}_{it} > 0, \tag{20}$$

which reduces to

$$-a_j \rho_{ij,t}^2 - 2\boldsymbol{b}_j' \boldsymbol{\rho}_{i,-j,t} \rho_{ij,t} - \boldsymbol{\rho}_{i,-j,t}' \mathbf{C}_j \boldsymbol{\rho}_{i,-j,t} + 1 > 0, \tag{21}$$

Therefore the inequality (21) implies that the lower and upper bounds for $\rho_{ij,t}$ are given by (18).

□

Thus we propose a candidate $g_{ij,t}^{\dagger}$ from normal distribution truncated on the interval (a_{ijt}, b_{ijt}), $TN_{(a_{ijt}, b_{ijt})}(m_{t*}, \sigma_{t*}^2)$, and accept it with probability $\min\{1, \exp(r(g_{ij,t}^{\dagger}) - r(g_{ij,t}))\}$, where

$$(a_{ijt}, b_{ijt}) \equiv \left(\log \frac{1 + L_{ij,t}}{1 - L_{ij,t}}, \log \frac{1 + U_{ij,t}}{1 - U_{ij,t}} \right). \tag{22}$$

3.2.2 Generation of $\boldsymbol{h}_t$ for the dynamic volatility $\mathbf{V}_t$

We use a single-move sampler for $\boldsymbol{h}_t$ where we sample $\boldsymbol{h}_t$ given other parameters and latent variables. Such a sampler is efficient when the realized measures are available as the additional information source for $\boldsymbol{h}_t$. The conditional posterior probability density function of $\boldsymbol{h}_t$ is given by

$$\pi(\boldsymbol{h}_t \,|\cdot) \propto \exp\left[-\frac{1}{2}(\boldsymbol{h}_t - \boldsymbol{m}_{t*})' \, \boldsymbol{\Omega}_{t*}^{-1}(\boldsymbol{h}_t - \boldsymbol{m}_{t*}) + l(\boldsymbol{h}_t)\right], \tag{23}$$

where

$$l(\boldsymbol{h}_t) = -\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{m}_t)'(\mathbf{V}_t^{1/2}\,\mathbf{R}_t\,\mathbf{V}_t^{1/2})^{-1}(\boldsymbol{y}_t - \boldsymbol{m}_t), \tag{24}$$

and

$$\boldsymbol{m}_{t*} = \begin{cases} \boldsymbol{\Omega}_{1*}\left[\boldsymbol{\Omega}_0^{-1}\,\boldsymbol{\mu} + \boldsymbol{\Phi}\,\boldsymbol{\Omega}^{-1}\left\{\boldsymbol{h}_2 - (\mathbf{I}_p - \boldsymbol{\Phi})\,\boldsymbol{\mu}\right\} \right. \\ \qquad\qquad\qquad \left. + \boldsymbol{\Sigma}_u^{-1}(\boldsymbol{x}_1 - \boldsymbol{\xi}) - \frac{1}{2}\,\mathbf{1}_p\right], & t = 1, \\ \boldsymbol{\Omega}_{t*}\left[\boldsymbol{\Omega}^{-1}\left\{(\mathbf{I}_p - \boldsymbol{\Phi})\,\boldsymbol{\mu} + \boldsymbol{\Phi}\,\boldsymbol{h}_{t-1}\right\}\right. \\ \qquad \left. + \boldsymbol{\Phi}\,\boldsymbol{\Omega}^{-1}\left\{\boldsymbol{h}_{t+1} - (\mathbf{I}_p - \boldsymbol{\Phi})\,\boldsymbol{\mu}\right\} + \boldsymbol{\Sigma}_u^{-1}(\boldsymbol{x}_t - \boldsymbol{\xi}) - \frac{1}{2}\,\mathbf{1}_p\right], & t = 2, \ldots, T-1, \\ \boldsymbol{\Omega}_{T*}\left[\boldsymbol{\Omega}^{-1}\left\{(\mathbf{I}_p - \boldsymbol{\Phi})\,\boldsymbol{\mu} + \boldsymbol{\Phi}\,\boldsymbol{h}_{T-1}\right\} + \boldsymbol{\Sigma}_u^{-1}(\boldsymbol{x}_T - \boldsymbol{\xi}) - \frac{1}{2}\,\mathbf{1}_p\right], & t = T, \end{cases} \tag{25}$$

$$\boldsymbol{\Omega}_{t*} = \begin{cases} \left[\boldsymbol{\Omega}_0^{-1} + \boldsymbol{\Phi}\,\boldsymbol{\Omega}^{-1}\,\boldsymbol{\Phi} + \boldsymbol{\Sigma}_u^{-1}\right]^{-1}, & t = 1, \\ \left[\boldsymbol{\Omega}^{-1} + \boldsymbol{\Phi}\,\boldsymbol{\Omega}^{-1}\,\boldsymbol{\Phi} + \boldsymbol{\Sigma}_u^{-1}\right]^{-1}, & t = 2, \ldots, T-1, \\ \left[\boldsymbol{\Omega}^{-1} + \boldsymbol{\Sigma}_u^{-1}\right]^{-1}, & t = T, \end{cases} \tag{26}$$

where $\mathbf{1}_p$ denotes a $p \times 1$ vector with all elements equal to one. Therefore, we generate a candidate $\boldsymbol{h}_t^{\dagger}$ from $\mathrm{N}(\boldsymbol{m}_{t*}, \boldsymbol{\Omega}_{t*})$, and accept it with probability $\min\{1, \exp(l(\boldsymbol{h}_t^{\dagger}) - l(\boldsymbol{h}_t))\}$.

3.2.3 Generation of $\boldsymbol{m}_t$ for the mean process of $\boldsymbol{y}_t$

Although the conditional mean of $\boldsymbol{y}_t$ given $\boldsymbol{h}_t$ is almost equal to zero in the empirical studies, it is important to consider such a non-zero mean process for the portfolio optimization as we shall see in Section 6. Noting that

$$\mathrm{E}[\boldsymbol{y}_t \,|\, \boldsymbol{h}_t] = \boldsymbol{m}_t, \quad \mathrm{Var}[\boldsymbol{y}_t \,|\, \boldsymbol{h}_t] = \mathbf{V}_t^{1/2}\,\mathbf{R}_t\,\mathbf{V}_t^{1/2} \equiv \boldsymbol{\Gamma}_t, \tag{27}$$

it can be shown that the conditional posterior distribution of $\boldsymbol{m}_t$'s is the same as that of the following linear Gaussian state space model:

$$\boldsymbol{y}_t = \boldsymbol{m}_t + \hat{\boldsymbol{\epsilon}}_t, \quad \hat{\boldsymbol{\epsilon}}_t \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Gamma}_t), \tag{28}$$

$$\boldsymbol{m}_{t+1} = \boldsymbol{m}_t + \boldsymbol{\nu}_t, \quad \boldsymbol{\nu}_t \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Sigma}_m), \tag{29}$$

where $\hat{\boldsymbol{\epsilon}}_t$ and $\boldsymbol{\nu}_t$ are independent. Thus we generate $\boldsymbol{m}$ simultaneously at one time using a simulation smoother (e.g. de Jong and Shephard (1995), Durbin and Koopman (2002)).

3.2.4 Generation of $\boldsymbol{\theta}$

See Appendix A.1.

4 Extension to incorporate the leverage effect

This section extends our model to incorporate the leverage effect. The leverage effect, which corresponds to the well-known negative correlation between asset returns and their volatilities in the stock market, is expected to improve the performance of the forecast of mean processes and volatility processes of asset returns.

4.1 Matrix variate normal distribution

We first define the matrix variate normal distribution and show its probability density function, which will be used in modelling the leverage effect.

Definition 1. The random matrix $\mathbf{X}$ $(p \times n)$ is said to have a matrix variate normal distribution with mean matrix $\mathbf{M}$ $(p \times n)$ and covariance matrix $\boldsymbol{\Psi} \otimes \boldsymbol{\Sigma}$ where $\boldsymbol{\Psi}$ $(p \times p)$ and $\boldsymbol{\Sigma}$ $(n \times n)$ are positive definite matrices if $\mathrm{vec}\,(\mathbf{X}') \sim \mathrm{N}(\mathrm{vec}\,(\mathbf{M}'), \boldsymbol{\Psi} \otimes \boldsymbol{\Sigma})$. We denote such matrix as $\mathbf{X} \sim \mathrm{N}_{p,n}(\mathbf{M}, \boldsymbol{\Psi} \otimes \boldsymbol{\Sigma})$.

If $\mathbf{X} \sim \mathrm{N}_{p,n}(\mathbf{M}, \boldsymbol{\Psi} \otimes \boldsymbol{\Sigma})$, the probability density function for $\mathbf{X}$ is given by

$$f(\mathbf{X}) = (2\pi)^{-np/2} |\,\boldsymbol{\Psi}\,|^{-n/2} |\,\boldsymbol{\Sigma}\,|^{-p/2} \times \exp\left\{ -\frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{\Psi}^{-1}(\mathbf{X}-\mathbf{M})\,\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})' \right) \right\}, \tag{30}$$

since

$$\begin{aligned} \mathrm{tr}\left(\boldsymbol{\Psi}^{-1}(\mathbf{X}-\mathbf{M})\,\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\mathbf{M})' \right) &= \mathrm{vec}\left((\mathbf{X}-\mathbf{M})' \right)' \left(\boldsymbol{\Psi}^{-1} \otimes \boldsymbol{\Sigma}^{-1} \right) \mathrm{vec}\left((\mathbf{X}-\mathbf{M})' \right) \\ &= \left\{ \mathrm{vec}(\mathbf{X}') - \mathrm{vec}(\mathbf{M}') \right\}' \left(\boldsymbol{\Psi}^{-1} \otimes \boldsymbol{\Sigma}^{-1} \right) \left\{ \mathrm{vec}(\mathbf{X}') - \mathrm{vec}(\mathbf{M}') \right\}. \end{aligned}$$

4.2 Modeling the leverage effect

We extend our proposed model to incorporate the leverage effect as follows. The joint distribution of $(\boldsymbol{y}_t, \boldsymbol{h}_{t+1})$ is given by

$$\begin{pmatrix} \boldsymbol{y}_t \\ \boldsymbol{h}_{t+1} \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \boldsymbol{m}_t \\ \boldsymbol{\mu} + \boldsymbol{\Phi}(\boldsymbol{h}_t - \boldsymbol{\mu}) \end{pmatrix}, \begin{pmatrix} \mathbf{V}_t^{1/2}\,\mathbf{R}_t\,\mathbf{V}_t^{1/2} & \mathbf{V}_t^{1/2}\,\mathbf{R}_t^{1/2}\,\boldsymbol{\Lambda}' \\ \boldsymbol{\Lambda}\,\mathbf{R}_t^{1/2'}\,\mathbf{V}_t^{1/2} & \boldsymbol{\Psi} + \boldsymbol{\Lambda}\,\boldsymbol{\Lambda}' \end{pmatrix} \right). \tag{31}$$

The marginal distributions of $\boldsymbol{y}_t$ and $\boldsymbol{h}_{t+1}$ given $\boldsymbol{h}_t$ are the same as before with $\boldsymbol{\Omega} = \boldsymbol{\Psi} + \boldsymbol{\Lambda}\,\boldsymbol{\Lambda}'$, but we note that

$$\boldsymbol{h}_{t+1} | \boldsymbol{y}_t, \boldsymbol{h}_t, \boldsymbol{\theta} \sim N\left(\boldsymbol{\mu} + \boldsymbol{\Phi}(\boldsymbol{h}_t - \boldsymbol{\mu}) + \boldsymbol{\Lambda}\,\mathbf{R}_t^{-1/2}\,\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t - \boldsymbol{m}_t), \boldsymbol{\Psi} \right).$$

If $\boldsymbol{\Lambda} = \mathbf{O}$, it reduces to the model without leverage effect. The matrix $\boldsymbol{\Lambda}$ is the coefficient of the leverage for

$$\boldsymbol{z}_t = \mathbf{R}_t^{-1/2}\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t - \boldsymbol{m}_t). \tag{32}$$

We assume that the prior distribution of $\boldsymbol{\Lambda}$ given $\boldsymbol{\Psi}$ is $\mathrm{N}_{p,p}(\mathbf{M}_0, \boldsymbol{\Psi} \otimes \boldsymbol{\Gamma}_0)$. That is,

$$\boldsymbol{\Lambda} | \boldsymbol{\Psi} \sim \mathrm{N}_{p,p}(\mathbf{M}_0, \boldsymbol{\Psi} \otimes \boldsymbol{\Gamma}_0). \tag{33}$$

Remark 2. There are several ways to choose $\mathbf{R}_t^{1/2}$. For example, we can use the spectral decomposition of the correlation matrix $\mathbf{R}_t = \mathbf{P}_t\,\mathbf{Q}_t\,\mathbf{P}_t'$ and $\mathbf{R}_t^{1/2} = \mathbf{P}_t\,\mathbf{Q}_t^{1/2}$ where the i-th diagonal element of the diagonal matrix $\mathbf{Q}_t$ is the i-th largest eigenvalue of $\mathbf{R}_t$ and the i-th column of $\mathbf{P}_t$ is the corresponding i-th eigenvector (and we set the first elements of the eigenvectors to be positive for the identification purpose). Thus the i-th element of $\boldsymbol{z}_t$ can be interpreted as the i-th market factor among p asset returns. Alternatively, Cholesky decomposition, $\mathbf{R}_t = \mathbf{R}_t^{1/2}\,\mathbf{R}_t^{1/2'}$, can be used so that $\mathbf{R}_t^{1/2}$ is a lower triangular matrix where all the diagonal elements are equal to one, but we note that it is affected by the ordering of the asset returns.

Remark 3. The alternative specification for the leverage effect is to consider

$$\begin{pmatrix} \boldsymbol{y}_t \\ \boldsymbol{h}_{t+1} \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} \boldsymbol{m}_t \\ \boldsymbol{\mu} + \boldsymbol{\Phi}(\boldsymbol{h}_t - \boldsymbol{\mu}) \end{pmatrix}, \begin{pmatrix} \mathbf{V}_t^{1/2}\,\mathbf{R}_t\,\mathbf{V}_t^{1/2} & \mathbf{V}_t^{1/2}\,\boldsymbol{\Lambda}' \\ \boldsymbol{\Lambda}\,\mathbf{V}_t^{1/2} & \boldsymbol{\Psi} + \boldsymbol{\Lambda}\,\mathbf{R}_t^{-1}\,\boldsymbol{\Lambda}' \end{pmatrix} \right),$$

so that

$$\boldsymbol{h}_{t+1} | \boldsymbol{y}_t, \boldsymbol{h}_t, \boldsymbol{\theta} \sim N\left(\boldsymbol{\mu} + \boldsymbol{\Phi}(\boldsymbol{h}_t - \boldsymbol{\mu}) + \boldsymbol{\Lambda}\,\mathbf{R}_t^{-1}\,\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t - \boldsymbol{m}_t), \boldsymbol{\Psi} \right).$$

However, we do not use this specification since it is difficult to interpret $\mathbf{\Lambda}$ and to justify the time-varying unconditional covariance matrix for $\boldsymbol{h}_{t+1}$'s where $\boldsymbol{h}_t$ may not be stationary and its initial distribution is unknown.

4.2.1 Generation of $\mathbf{\Lambda}$

The conditional posterior distribution of $\mathbf{\Lambda}$ is derived in the following Proposition.

Proposition 2. Suppose that the prior distribution of $\mathbf{\Lambda}$ given $\mathbf{\Psi}$ is $\mathrm{N}_{p,p}(\mathbf{M}_0, \mathbf{\Psi} \otimes \mathbf{\Gamma}_0)$. Then the conditional posterior distribution of $\mathbf{\Lambda}$ given other parameters and latent variables is $\mathrm{N}_{p,p}(\mathbf{M}_1, \mathbf{\Psi} \otimes \mathbf{\Gamma}_1)$ where

$$\mathbf{M}_1 = \left(\mathbf{A} + \mathbf{\Gamma}_0^{-1}\right)^{-1}\left(\mathbf{B} + \mathbf{\Gamma}_0^{-1}\mathbf{M}_0\right), \quad \mathbf{\Gamma}_1 = \left(\mathbf{A} + \mathbf{\Gamma}_0^{-1}\right)^{-1}, \tag{34}$$

$$\mathbf{A} = \sum_{t=1}^{T-1} \boldsymbol{z}_t \boldsymbol{z}_t', \quad \mathbf{B} = \sum_{t=1}^{T-1} \boldsymbol{z}_t \boldsymbol{\eta}_t', \tag{35}$$

and $\boldsymbol{z}_t = \mathbf{R}_t^{-1/2}\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t - \boldsymbol{m}_t)$ and $\boldsymbol{\eta}_t = \boldsymbol{h}_{t+1} - \boldsymbol{\mu} - \mathbf{\Phi}(\boldsymbol{h}_t - \boldsymbol{\mu})$.

Proof: Since

$$\begin{aligned} & f(\boldsymbol{h}_{t+1}|\boldsymbol{y}_t, \boldsymbol{g}_t, \boldsymbol{h}_t, \boldsymbol{m}_t, \boldsymbol{\theta}) \\ & \propto |\mathbf{\Psi}|^{-1/2} \exp\left\{-\frac{1}{2}\left(\boldsymbol{h}_{t+1} - \boldsymbol{\mu} - \mathbf{\Phi}(\boldsymbol{h}_t - \boldsymbol{\mu}) - \mathbf{\Lambda}\,\mathbf{R}_t^{-1/2}\,\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t - \boldsymbol{m}_t)\right)' \right. \\ & \qquad \left. \mathbf{\Psi}^{-1}\left(\boldsymbol{h}_{t+1} - \boldsymbol{\mu} - \mathbf{\Phi}(\boldsymbol{h}_t - \boldsymbol{\mu}) - \mathbf{\Lambda}\,\mathbf{R}_t^{-1/2}\,\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t - \boldsymbol{m}_t)\right)\right\}, \end{aligned} \tag{36}$$

and

$$\pi(\mathbf{\Lambda}|\mathbf{\Psi}) \propto |\,\mathbf{\Psi}\,|^{-p/2}|\mathbf{\Gamma}_0|^{-p/2} \exp\left\{-\frac{1}{2}\,\mathrm{tr}\left(\mathbf{\Psi}^{-1}(\mathbf{\Lambda} - \mathbf{M}_0)\mathbf{\Gamma}_0^{-1}(\mathbf{\Lambda} - \mathbf{M}_0)'\right)\right\}, \tag{37}$$

the conditional posterior probability density function of $\mathbf{\Lambda}$ is

$$\begin{aligned} \pi(\mathbf{\Lambda}\,|\cdot) & \propto \exp\left[-\frac{1}{2}\left\{\sum_{t=1}^{T-1}(\boldsymbol{y}_t - \boldsymbol{m}_t)'\,\mathbf{V}_t^{-1/2}\,\mathbf{R}_t^{-1/2'}\,\mathbf{\Lambda}'\,\mathbf{\Psi}^{-1}\,\mathbf{\Lambda}\,\mathbf{R}_t^{-1/2}\,\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t - \boldsymbol{m}_t) \right.\right. \\ & \qquad \left.\left. -2\sum_{t=1}^{T-1}(\boldsymbol{h}_{t+1} - \boldsymbol{\mu} - \mathbf{\Phi}(\boldsymbol{h}_t - \boldsymbol{\mu}))'\,\mathbf{\Psi}^{-1}\,\mathbf{\Lambda}\,\mathbf{R}_t^{-1/2}\,\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t - \boldsymbol{m}_t)\right\}\right] \times \pi(\mathbf{\Lambda}\,|\mathbf{\Psi}) \\ & \propto \exp\left[-\frac{1}{2}\left\{\mathrm{tr}(\mathbf{\Psi}^{-1}\,\mathbf{\Lambda}(\mathbf{A} + \mathbf{\Gamma}_0^{-1})\,\mathbf{\Lambda}') - 2\,\mathrm{tr}(\mathbf{\Psi}^{-1}\,\mathbf{\Lambda}(\mathbf{B} + \mathbf{\Gamma}_0^{-1}\,\mathbf{M}_0))\right\}\right] \\ & \propto \exp\left[-\frac{1}{2}\,\mathrm{tr}\left\{\mathbf{\Psi}^{-1}(\mathbf{\Lambda} - \mathbf{M}_1)(\mathbf{A} + \mathbf{\Gamma}_0^{-1})(\mathbf{\Lambda} - \mathbf{M}_1)'\right\}\right], \end{aligned} \tag{38}$$

and the result follows.

$\square$

From Proposition 2, we generate $\mathrm{vec}(\mathbf{\Lambda})|\cdot \sim \mathrm{N}(\mathrm{vec}(\mathbf{M}_1'), \mathbf{\Psi}\otimes\mathbf{\Gamma}_1)$.

4.2.2 Generation of $\mathbf{\Psi}$

As a prior distribution of $\mathbf{\Psi}$, we assume $\mathbf{\Psi} \sim \mathrm{IW}(\nu_\psi, \mathbf{S}_\psi)$. Then the conditional posterior probability density function of $\mathbf{\Psi}$ is

$$\pi(\mathbf{\Psi}|\cdot) \propto m(\mathbf{\Psi}) \times |\mathbf{\Psi}|^{-(\tilde{\nu}_\psi+p+1)/2} \exp\left\{-\frac{1}{2}\,\mathrm{tr}\left(\mathbf{\Psi}^{-1}\tilde{\mathbf{S}}_\psi\right)\right\}, \tag{39}$$

where

$$\begin{aligned} m(\mathbf{\Psi}) &= |\,\mathbf{\Omega}_0\,|^{-1/2} \exp\left(-\frac{1}{2}(\boldsymbol{h}_1-\boldsymbol{\mu})'\,\mathbf{\Omega}_0^{-1}(\boldsymbol{h}_1-\boldsymbol{\mu})\right), &(40)\\ \tilde{\nu}_\psi &= \nu_\psi + p + T - 1, &(41)\\ \tilde{\mathbf{S}}_\psi &= \mathbf{S}_\psi + (\mathbf{\Lambda}-\mathbf{M}_0)\mathbf{\Gamma}_0^{-1}(\mathbf{\Lambda}-\mathbf{M}_0)' \\ &\quad + \sum_{t=1}^{T-1}\left\{\boldsymbol{h}_{t+1}-\boldsymbol{\mu}-\mathbf{\Phi}(\boldsymbol{h}_t-\boldsymbol{\mu})-\mathbf{\Lambda}\,\mathbf{R}_t^{-1/2}\,\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t-\boldsymbol{m}_t)\right\} \\ &\qquad \left\{\boldsymbol{h}_{t+1}-\boldsymbol{\mu}-\mathbf{\Phi}(\boldsymbol{h}_t-\boldsymbol{\mu})-\mathbf{\Lambda}\,\mathbf{R}_t^{-1/2}\,\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t-\boldsymbol{m}_t)\right\}'. &(42)\end{aligned}$$

Thus we propose a candidate $\mathbf{\Psi}^\dagger$ from $\mathrm{IW}(\tilde{\nu}_\psi, \tilde{\mathbf{S}}_\psi)$, and accept it with probability $\min\{1, m(\mathbf{\Psi}^\dagger)/m(\mathbf{\Psi})\}$.

For the generations of other parameters and latent variables, see Appendix A.2 .

4.3 Parsimonious specification of the leverage effect

This subsection proposes the parsimonious specification for $\mathbf{\Lambda} = [\boldsymbol{\lambda}_1, \cdots, \boldsymbol{\lambda}_p]$, to reduce the number of leverage parameters from p^2 to pq ($q \ll p$) by setting $\mathbf{\Lambda} = [\boldsymbol{\lambda}_1, \cdots, \boldsymbol{\lambda}_q, \mathbf{0}, \ldots, \mathbf{0}]$ since we do not have additional measurement equations for the leverage effect. Using the spectral decomposition to compute $\mathbf{R}_t^{1/2}$, we can interpret the i-th column corresponds to the i-th market factor among asset returns ($i = 1, \ldots, q$). The number of factors, q, is expected to be small, e.g., $q = 1$ or $q = 2$.

4.3.1 Generation of $\mathbf{\Lambda} = [\boldsymbol{\lambda}_1, \cdots, \boldsymbol{\lambda}_q, \mathbf{0}, \ldots, \mathbf{0}]$

The following proposition and the corollary shows the conditional posterior distribution of parameters for the leverage effect under parsimonious specifications.

Proposition 3. Let $\mathbf{\Lambda} = [\boldsymbol{\lambda}_1, \cdots, \boldsymbol{\lambda}_q, \mathbf{0}, \ldots, \mathbf{0}]$ and $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1', \ldots, \boldsymbol{\lambda}_q')'$. If the prior distribution of $\boldsymbol{\lambda}$ is assumed to be normal, $\boldsymbol{\lambda} \sim \mathrm{N}(\boldsymbol{m}_0, \mathbf{\Gamma}_0)$, then the conditional posterior distribution of $\boldsymbol{\lambda}$ is $\boldsymbol{\lambda}\,|\cdot \sim \mathrm{N}(\boldsymbol{m}_1, \mathbf{\Gamma}_1)$ where

$$\boldsymbol{m}_1 = \mathbf{\Gamma}_1 \left\{ \mathbf{\Gamma}_0^{-1} \boldsymbol{m}_0 + (\mathbf{I}_q \otimes \mathbf{\Psi}^{-1} \mathbf{B}')\mathrm{vec}\left(\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_q\}\right) \right\}, \tag{43}$$

$$\mathbf{\Gamma}_1 = \left(\mathbf{\Gamma}_0^{-1} + \mathbf{A}_{1:q,1:q} \otimes \ \mathbf{\Psi}^{-1} \right)^{-1}, \tag{44}$$

$\mathbf{A}, \mathbf{B}$ are defined in (35), $\mathbf{A}_{1:q,1:q}$ denotes the first q rows and the q columns of $\mathbf{A}$, $\mathrm{vec}(\mathbf{X}) \equiv (\boldsymbol{x}_1', \ldots, \boldsymbol{x}_m')'$ denotes a vectorization of the matrix $\mathbf{X} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\}$, and $\otimes$ denotes Kronecker product.

Proof: Using $\mathbf{\Lambda} = \sum_{j=1}^q \boldsymbol{e}_j' \otimes \boldsymbol{\lambda}_j$ where $\boldsymbol{e}_j$ is the $p \times 1$ unit vector with the j-th element being one, the posterior probability density function of $\mathbf{\Lambda}$ is

$$\begin{aligned}
&\pi(\boldsymbol{\lambda}|\cdot) \\
&\propto \exp\left[-\frac{1}{2} \left\{ \sum_{t=1}^{T-1} \boldsymbol{z}_t' \left(\sum_{j=1}^q \boldsymbol{e}_j \otimes \boldsymbol{\lambda}_j' \right) \mathbf{\Psi}^{-1} \left(\sum_{j=1}^q \boldsymbol{e}_j' \otimes \boldsymbol{\lambda}_j \right) \boldsymbol{z}_t - 2 \sum_{t=1}^{T-1} \boldsymbol{\eta}_t' \mathbf{\Psi}^{-1} \left(\sum_{j=1}^q \boldsymbol{e}_j' \otimes \boldsymbol{\lambda}_j \right) \boldsymbol{z}_t \right\} \right] \\
&\qquad \times \pi(\boldsymbol{\lambda}) \\
&\propto \exp\left[-\frac{1}{2} \left\{ \sum_{t=1}^{T-1} \left(\sum_{j=1}^q \boldsymbol{z}_t' \boldsymbol{e}_j \otimes \boldsymbol{\lambda}_j' \right) \mathbf{\Psi}^{-1} \left(\sum_{j=1}^q \boldsymbol{e}_j' \boldsymbol{z}_t \otimes \boldsymbol{\lambda}_j \right) - 2 \sum_{t=1}^{T-1} \boldsymbol{\eta}_t' \mathbf{\Psi}^{-1} \left(\sum_{j=1}^q \boldsymbol{e}_j' \boldsymbol{z}_t \otimes \boldsymbol{\lambda}_j \right) \right\} \right] \\
&\qquad \times \pi(\boldsymbol{\lambda}) \\
&\propto \exp\left[-\frac{1}{2} \left\{ \sum_{t=1}^{T-1} \sum_{i=1}^q \sum_{j=1}^q (\boldsymbol{e}_i' \boldsymbol{z}_t \boldsymbol{z}_t' \boldsymbol{e}_j) \boldsymbol{\lambda}_i' \mathbf{\Psi}^{-1} \boldsymbol{\lambda}_j - 2 \sum_{t=1}^{T-1} \sum_{j=1}^q \boldsymbol{e}_j' \boldsymbol{z}_t \boldsymbol{\eta}_t' \mathbf{\Psi}^{-1} \boldsymbol{\lambda}_j \right\} \right] \times \pi(\boldsymbol{\lambda}) \\
&\propto \exp\left[-\frac{1}{2} \left\{ \sum_{i=1}^q \sum_{j=1}^q (\boldsymbol{e}_i' \mathbf{A} \boldsymbol{e}_j) \boldsymbol{\lambda}_i' \mathbf{\Psi}^{-1} \boldsymbol{\lambda}_j - 2 \sum_{j=1}^q \boldsymbol{\lambda}_j' \mathbf{\Psi}^{-1} \mathbf{B}' \boldsymbol{e}_j \right\} \right] \times \pi(\boldsymbol{\lambda}) \\
&\propto \exp\left[-\frac{1}{2} \left\{ \boldsymbol{\lambda}' \left(\mathbf{\Gamma}_0^{-1} + \ \mathbf{A}_{1:q,1:q} \otimes \mathbf{\Psi}^{-1} \right) \boldsymbol{\lambda} - 2 \boldsymbol{\lambda}' \left(\mathbf{\Gamma}_0^{-1} \boldsymbol{m}_0 + (\mathbf{I}_q \otimes \mathbf{\Psi}^{-1} \mathbf{B}')\mathrm{vec}\left(\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_q\}\right) \right) \right\} \right],
\end{aligned} \tag{45}$$

and the result follows.

□

We derive the conditional posterior distribution for the case $q = 1$ in the following corollary.

Corollary 1. Let $\mathbf{\Lambda} = [\boldsymbol{\lambda}, \mathbf{0}, \ldots, \mathbf{0}]$. If the prior distribution of $\boldsymbol{\lambda}$ is assumed to be normal, $\boldsymbol{\lambda} \sim \mathrm{N}(\boldsymbol{m}_0, \mathbf{\Gamma}_0)$, then the conditional posterior distribution of $\boldsymbol{\lambda}$ is $\boldsymbol{\lambda}|\cdot \sim \mathrm{N}(\boldsymbol{m}_1, \mathbf{\Gamma}_1)$ where

$$\begin{aligned} \boldsymbol{m}_1 &= \mathbf{\Gamma}_1 \left\{ \mathbf{\Gamma}_0^{-1} \boldsymbol{m}_0 + \mathbf{\Psi}^{-1} \boldsymbol{b} \right\}, \quad \mathbf{\Gamma}_1 = \left(\mathbf{\Gamma}_0^{-1} + a \times \mathbf{\Psi}^{-1} \right)^{-1}, \qquad (46) \\ a &= \sum_{t=1}^{T-1} z_{1t}^2, \quad \boldsymbol{b} = \sum_{t=1}^{T-1} z_{1t} \left\{ \boldsymbol{h}_{t+1} - \boldsymbol{\mu} - \mathbf{\Phi}(\boldsymbol{h}_t - \boldsymbol{\mu}) \right\}, \end{aligned}$$

and z_{1t} is the first element of $\boldsymbol{z}_t = \mathbf{R}_t^{-1/2} \mathbf{V}_t^{-1/2} (\boldsymbol{y}_t - \boldsymbol{m}_t)$.

4.3.2 Generation of $\mathbf{\Psi}$

As the prior distribution of $\mathbf{\Lambda}$ is changed, the conditional posterior probability density function of $\mathbf{\Psi}$ is now replaced by

$$\pi(\mathbf{\Psi}|\cdot) \propto m(\mathbf{\Psi}) \times |\mathbf{\Psi}|^{-(\tilde{\nu}_\psi + p + 1)/2} \exp\left\{ -\frac{1}{2} \mathrm{tr}\left(\mathbf{\Psi}^{-1} \tilde{\mathbf{S}}_\psi \right) \right\}, \qquad (47)$$

where

$$\begin{aligned} m(\mathbf{\Psi}) &= |\,\mathbf{\Omega}_0\,|^{-1/2} \exp\left(-\frac{1}{2} (\boldsymbol{h}_1 - \boldsymbol{\mu})' \mathbf{\Omega}_0^{-1} (\boldsymbol{h}_1 - \boldsymbol{\mu}) \right), \qquad (48) \\ \tilde{\nu}_\psi &= \nu_\psi + T - 1, \qquad (49) \\ \tilde{\mathbf{S}}_\psi &= \mathbf{S}_\psi + \sum_{t=1}^{T-1} \left\{ \boldsymbol{h}_{t+1} - \boldsymbol{\mu} - \mathbf{\Phi}(\boldsymbol{h}_t - \boldsymbol{\mu}) - \mathbf{\Lambda} \mathbf{R}_t^{-1/2} \mathbf{V}_t^{-1/2} (\boldsymbol{y}_t - \boldsymbol{m}_t) \right\} \\ &\qquad \left\{ \boldsymbol{h}_{t+1} - \boldsymbol{\mu} - \mathbf{\Phi}(\boldsymbol{h}_t - \boldsymbol{\mu}) - \mathbf{\Lambda} \mathbf{R}_t^{-1/2} \mathbf{V}_t^{-1/2} (\boldsymbol{y}_t - \boldsymbol{m}_t) \right\}'. \qquad (50) \end{aligned}$$

Thus we propose a candidate $\mathbf{\Psi}^\dagger$ from $\mathrm{IW}(\tilde{\nu}_\psi, \tilde{\mathbf{S}}_\psi)$, and accept it with probability $\min\{1, m(\mathbf{\Psi}^\dagger)/m(\mathbf{\Psi})\}$.

5 Illustrative example using simulated data

In this section, we give illustrative examples for our proposed multivariate stochastic volatility models with leverage effect using simulated data where we use the spectral decomposition to compute $\mathbf{R}_t^{-1/2}$. The number of observations is set $T = 2000$ and the dimension of the daily returns is $p = 9$, which is the same as in empirical studies. The parsimonious specification is used such that $\mathbf{\Lambda} = [\boldsymbol{\lambda}, \mathbf{0}, \ldots, \mathbf{0}]$ with $q = 1$. We set true values of parameters as

$$\begin{aligned} \boldsymbol{\mu} &= 0.5 \times \mathbf{1}_p, \quad \boldsymbol{\xi} = 0.1 \times \mathbf{1}_p, \quad \boldsymbol{\delta} = 0.1 \times \mathbf{1}_{p(p-1)/2}, \quad \boldsymbol{\lambda} = -0.1 \times \mathbf{1}_p, \\ \boldsymbol{\phi} &= 0.9 \times \mathbf{1}_p, \quad \boldsymbol{\sigma}_u = 0.1 \times \mathbf{1}_p, \quad \boldsymbol{\sigma}_v = 0.3 \times \mathbf{1}_{p(p-1)/2}, \\ \boldsymbol{\sigma}_\zeta &= 0.005 \times \mathbf{1}_{p(p-1)/2}, \quad \boldsymbol{\sigma}_m = 0.05 \times \mathbf{1}_p, \quad \mathbf{\Psi} = 0.1 \times \mathbf{I}_p, \end{aligned}$$

where $\boldsymbol{\sigma}_u = (\sigma_{u,1}, \ldots, \sigma_{u,p})'$, $\boldsymbol{\sigma}_v = (\sigma_{v,21}, \ldots, \sigma_{v,p\,p-1})'$, $\boldsymbol{\sigma}_m = (\sigma_{m,1}, \ldots, \sigma_{m,p})'$ and $\boldsymbol{\sigma}_\zeta = (\sigma_{\zeta,21}, \ldots, \sigma_{\zeta,p\,p-1})'$. The prior distributions are assumed to be vague and flat to reflect the fact that we have little information with regard to the parameters, and are given by

$$\begin{aligned}
&\mu_i \sim \mathrm{N}(0, 10^4), \quad \xi_i \sim \mathrm{N}(0, 10^4), \quad \delta_{ij} \sim \mathrm{N}(0, 10^4), \quad \lambda_i \sim \mathrm{N}(0, 10^4) \\
&\frac{1+\phi_i}{2} \sim \mathrm{Beta}(1,1), \quad \sigma^2_{u,i} \sim \mathrm{IG}(10^{-5}/2, 10^{-5}/2), \quad \sigma^2_{v,ij} \sim \mathrm{IG}(10^{-5}/2, 10^{-5}/2), \\
&\sigma^2_{\zeta,ij} \sim \mathrm{IG}(10^{-6}/2, 10^{-6}/2), \quad \sigma^2_{m,i} \sim \mathrm{IG}(10^{-6}/2, 10^{-6}/2), \qquad \boldsymbol{\Psi} \sim \mathrm{IW}(10, \mathbf{I}_9),
\end{aligned}$$

for $i = 1, \ldots, p$, $j = 1, \ldots, i-1$.

The number of iteration for MCMC is 12000, and the first 2000 samples are discarded as burn-in samples. Table 1 shows posterior means, 95% credible intervals and inefficiency factors (IF)[1] for $\boldsymbol{\mu}$, $\boldsymbol{\xi}$, $\boldsymbol{\phi}$, $\boldsymbol{\sigma}_u$, $\boldsymbol{\sigma}_m$ and $\boldsymbol{\lambda}$. The posterior means are close to their corresponding true values in the sense that almost all of them are included in 95% credible intervals. Also the inefficiency factors are not very high and our sampling algorithms worked well. Similar results are obtained for $\boldsymbol{\delta}$, $\boldsymbol{\sigma}_v$, $\boldsymbol{\sigma}_\zeta$ and $\boldsymbol{\Psi}$ as shown in Tables 2 and 3. The inefficiency factors for $\boldsymbol{\mu}$, $\boldsymbol{\xi}$, $\boldsymbol{\phi}$, $\boldsymbol{\sigma}_u$, $\boldsymbol{\sigma}_m$ and $\boldsymbol{\lambda}$ are at most 135, which are much smaller than those for other multivariate stochastic volatility models.

Figure 1 shows the time series plot of 95% credible intervals and true values for $h_{1t}, \ldots, h_{9t}$. The credible intervals indicate that we succeed to capture the dynamics of h_{jt}'s. Similarly, we also show the time series plot of 95% credible intervals and true values for the selected $\rho_{ij,t}$'s in Figure 2. The transitions of $\rho_{ij,t}$'s are covered by the credible intervals successfully throughout the sample period.

[1]The inefficiency factor is defined as $1 + 2\sum_{g=1}^{\infty} \rho(g)$, where $\rho(g)$ is the sample autocorrelation at lag g. This is interpreted as the ratio of the numerical variance of the posterior mean from the chain to the variance of the posterior mean from hypothetical uncorrelated draws. The smaller the inefficiency factor becomes, the closer the MCMC sampling is to the uncorrelated sampling.

Table 1: Simulated data. Posterior means, 95% credible intervals and inefficiency factors for $\boldsymbol{\mu}$, $\boldsymbol{\xi}$, $\boldsymbol{\phi}$, $\boldsymbol{\sigma}_u$, $\boldsymbol{\sigma}_m$ and $\boldsymbol{\lambda}$. Spectral decomposition is used to compute $\mathbf{R}_t^{-1/2}$.

Par.	True	Mean	95% interval	IF	Par.	True	Mean	95% interval	IF
μ_1	0.5	0.584	[0.428,0.737]	16	$\sigma_{u,1}$	0.1	0.102	[0.072,0.134]	98
μ_2	0.5	0.626	[0.482,0.771]	13	$\sigma_{u,2}$	0.1	0.082	[0.035,0.115]	120
μ_3	0.5	0.549	[0.387,0.711]	21	$\sigma_{u,3}$	0.1	0.071	[0.031,0.105]	130
μ_4	0.5	0.605	[0.460,0.751]	22	$\sigma_{u,4}$	0.1	0.107	[0.076,0.132]	95
μ_5	0.5	0.332	[0.192,0.478]	24	$\sigma_{u,5}$	0.1	0.105	[0.081,0.127]	81
μ_6	0.5	0.386	[0.250,0.519]	36	$\sigma_{u,6}$	0.1	0.058	[0.030,0.084]	123
μ_7	0.5	0.516	[0.339,0.692]	15	$\sigma_{u,7}$	0.1	0.092	[0.066,0.122]	100
μ_8	0.5	0.385	[0.237,0.540]	23	$\sigma_{u,8}$	0.1	0.074	[0.024,0.110]	127
μ_9	0.5	0.432	[0.276,0.587]	19	$\sigma_{u,9}$	0.1	0.092	[0.055,0.122]	117
ξ_1	0.1	0.069	[0.017,0.127]	112	$\sigma_{m,1} \times 10$	0.5	0.487	[0.380,0.636]	76
ξ_2	0.1	0.053	[-0.002,0.094]	115	$\sigma_{m,2} \times 10$	0.5	0.440	[0.307,0.584]	89
ξ_3	0.1	0.069	[0.007,0.115]	128	$\sigma_{m,3} \times 10$	0.5	0.461	[0.355,0.605]	81
ξ_4	0.1	0.046	[-0.016,0.102]	120	$\sigma_{m,4} \times 10$	0.5	0.503	[0.370,0.652]	82
ξ_5	0.1	0.162	[0.097,0.227]	120	$\sigma_{m,5} \times 10$	0.5	0.516	[0.401,0.677]	84
ξ_6	0.1	0.125	[0.074,0.192]	135	$\sigma_{m,6} \times 10$	0.5	0.417	[0.299,0.565]	96
ξ_7	0.1	0.072	[0.011,0.126]	119	$\sigma_{m,7} \times 10$	0.5	0.472	[0.339,0.619]	87
ξ_8	0.1	0.039	[-0.032,0.094]	126	$\sigma_{m,8} \times 10$	0.5	0.497	[0.357,0.669]	96
ξ_9	0.1	0.109	[0.056,0.174]	122	$\sigma_{m,9} \times 10$	0.5	0.457	[0.354,0.594]	78
ϕ_1	0.9	0.897	[0.876,0.918]	17	λ_1	-0.1	-0.095	[-0.111,-0.080]	2
ϕ_2	0.9	0.891	[0.870,0.912]	21	λ_2	-0.1	-0.095	[-0.110,-0.080]	2
ϕ_3	0.9	0.903	[0.883,0.922]	21	λ_3	-0.1	-0.109	[-0.124,-0.095]	2
ϕ_4	0.9	0.895	[0.876,0.914]	14	λ_4	-0.1	-0.104	[-0.119,-0.089]	2
ϕ_5	0.9	0.896	[0.876,0.915]	12	λ_5	-0.1	-0.107	[-0.122,-0.093]	4
ϕ_6	0.9	0.873	[0.852,0.895]	8	λ_6	-0.1	-0.099	[-0.113,-0.084]	2
ϕ_7	0.9	0.911	[0.893,0.929]	12	λ_7	-0.1	-0.010	[-0.115,-0.085]	2
ϕ_8	0.9	0.891	[0.870,0.911]	19	λ_8	-0.1	-0.099	[-0.114,-0.084]	2
ϕ_9	0.9	0.897	[0.876,0.918]	18	λ_9	-0.1	-0.095	[-0.110,-0.080]	3

Table 2: Simulated data. Posterior means for $\boldsymbol{\delta}$ and $\boldsymbol{\sigma}_v$.

True values: $\boldsymbol{\delta} = 0.1 \times \mathbf{1}_{36}$, $\boldsymbol{\sigma}_v = 0.3 \times \mathbf{1}_{36}$.

δ_{ij}	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$
$i=2$	0.084							
$i=3$	0.086	0.097						
$i=4$	0.082	0.097	0.087					
$i=5$	0.101	0.095	0.111	0.103				
$i=6$	0.104	0.093	0.090	0.110	0.097			
$i=7$	0.089	0.090	0.112	0.109	0.111	0.106		
$i=8$	0.087	0.091	0.101	0.097	0.099	0.010	0.105	
$i=9$	0.111	0.098	0.112	0.077	0.090	0.113	0.125	0.102

$\sigma_{v,ij}$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$
$i=2$	0.294							
$i=3$	0.301	0.300						
$i=4$	0.295	0.294	0.293					
$i=5$	0.297	0.296	0.294	0.302				
$i=6$	0.300	0.304	0.298	0.298	0.299			
$i=7$	0.304	0.304	0.304	0.301	0.295	0.300		
$i=8$	0.299	0.288	0.307	0.309	0.299	0.303	0.303	
$i=9$	0.300	0.301	0.308	0.301	0.307	0.301	0.307	0.297

*Red figures indicate that 95% credible interval includes true value.

Table 3: Simulated data. Posterior means for $\boldsymbol{\sigma}_\zeta$ and $\boldsymbol{\Psi}$.

True value: $\boldsymbol{\sigma}_\zeta \times 10^2 = 0.5 \times \mathbf{1}_{36}$, $\boldsymbol{\Psi} = 0.1 \times \mathbf{I}_9$.

$\sigma_{\zeta,ij} \times 10^2$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$
$i=2$	0.611							
$i=3$	0.593	0.422						
$i=4$	0.616	0.589	0.643					
$i=5$	0.323	0.488	0.536	0.422				
$i=6$	0.511	0.520	0.539	0.391	0.512			
$i=7$	0.459	0.490	0.679	0.310	0.526	0.497		
$i=8$	0.641	0.667	0.387	0.687	0.404	0.485	0.589	
$i=9$	0.285	0.410	0.497	0.629	0.417	0.465	0.598	0.417

ψ_{ij}	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$
$i=1$	0.107								
$i=2$	0.005	0.106							
$i=3$	0.006	0.002	0.102						
$i=4$	0.000	0.004	0.001	0.098					
$i=5$	0.001	-0.002	-0.003	0.002	0.090				
$i=6$	-0.006	0.005	0.001	-0.001	-0.002	0.106			
$i=7$	0.000	-0.003	-0.002	0.002	-0.002	0.000	0.109		
$i=8$	0.002	0.001	0.005	0.002	0.001	-0.002	0.002	0.111	
$i=9$	0.000	0.003	-0.003	0.004	0.001	0.000	0.003	0.003	0.106

*Red figures indicate that 95% credible interval includes true value.

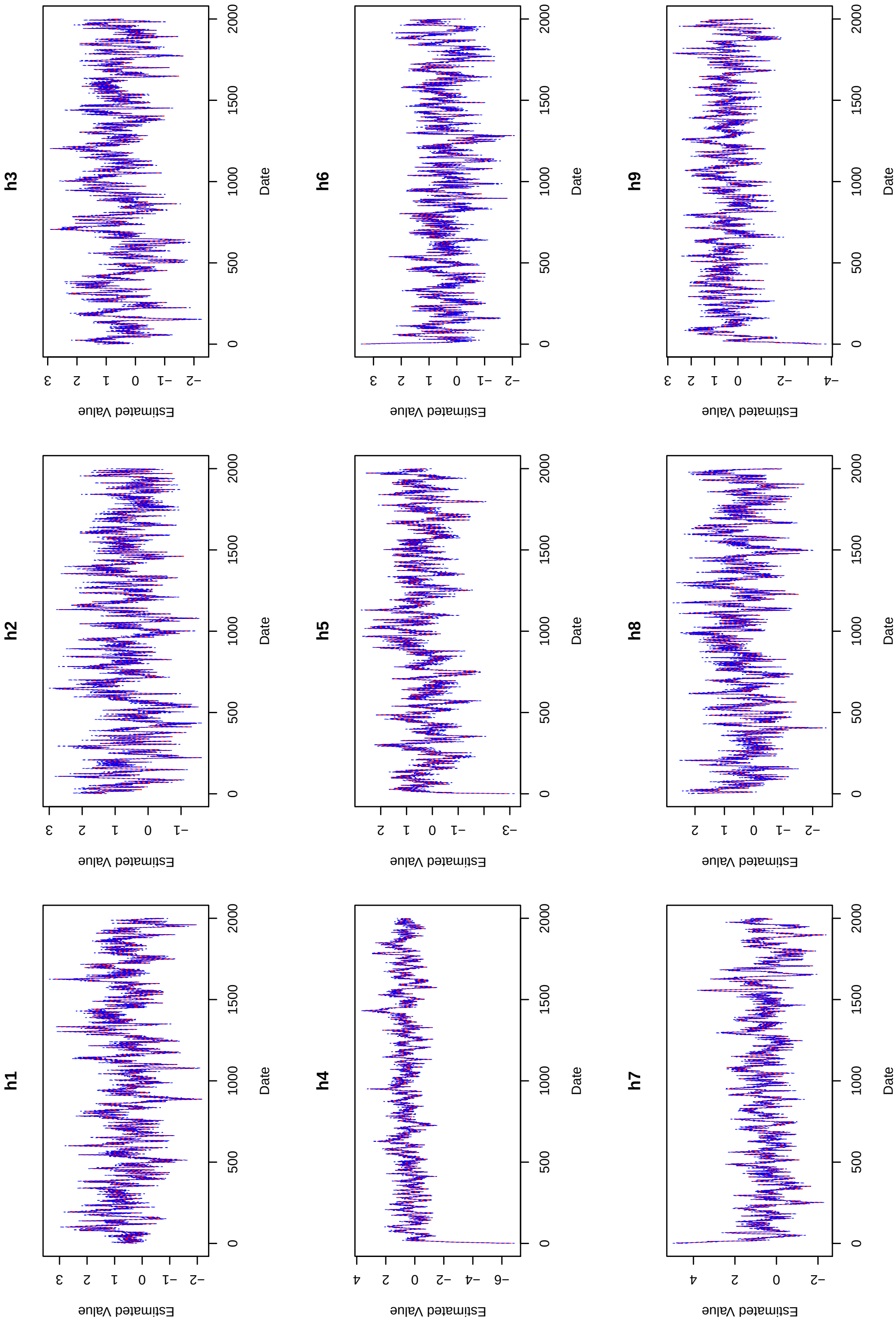


Figure 1: Time series plot of 95% credible intervals (blue) and true values (red) for $h_{1t}, \ldots, h_{9t}$.

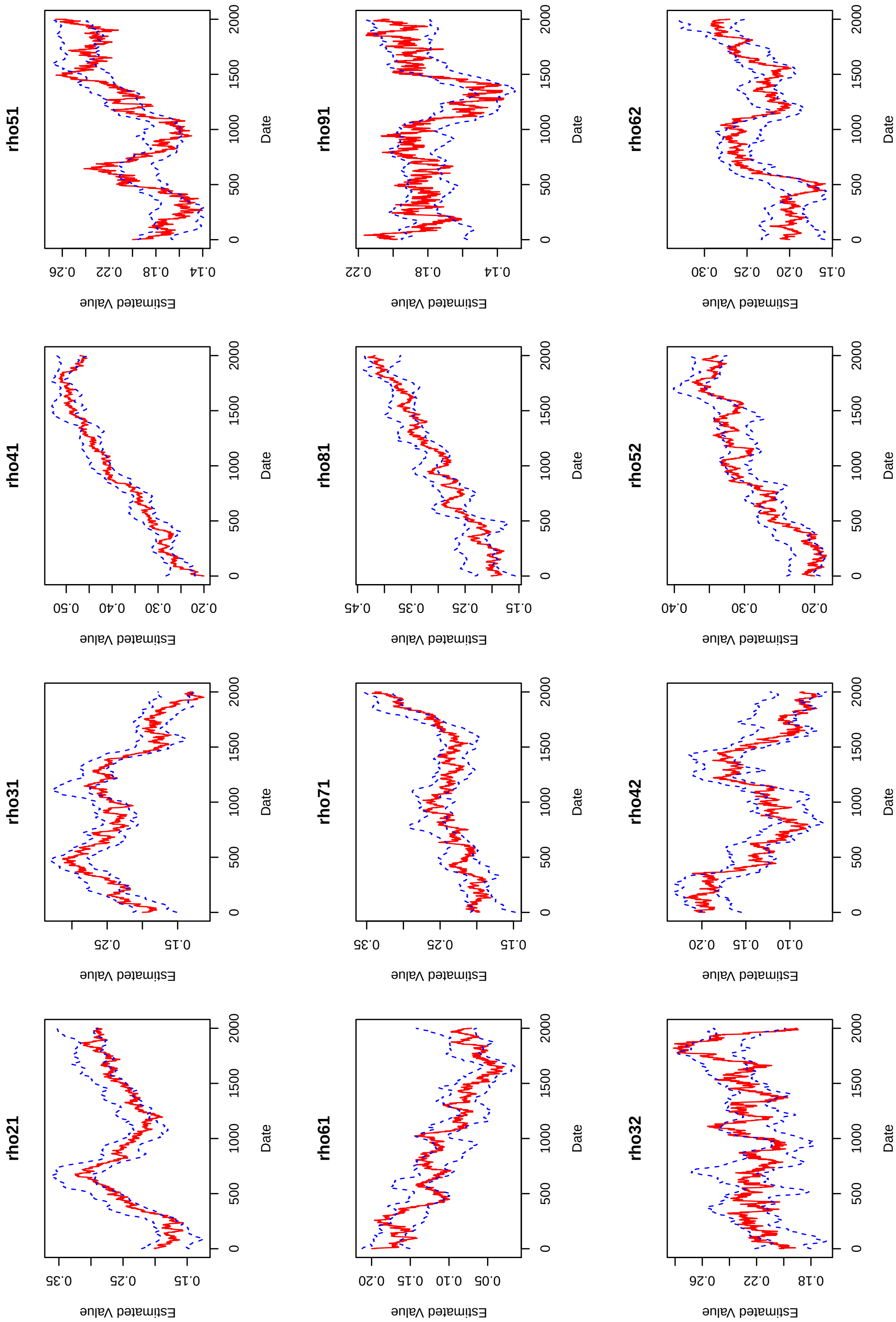


Figure 2: Time series plot of 95% credible intervals (blue) and true values (red) for $\rho_{21,t}, \ldots, \rho_{62,t}$

6 Application to U.S. stock returns

This section applies our proposed model to daily returns of nine U.S. stocks ($p = 9$) with realized volatilities and pairwise realized correlations. The nine series of stock returns are JP Morgan (JPM), International Business Machine (IBM), Microsoft (MSFT), Exxon Mobil (XOM), Alcoa (AA), American Express (AXP), Du Pont (DD), General Electric (GE), and Coca Cola (KO). The sample period is from February 1, 2001 to December 31, 2009, and the number of observation is $T = 2242$. The daily returns for the i-th stocks are defined as $y_{it} = 100 \times (\log p_{it} - \log p_{i,t-1})$, where p_{it} is the closing price of the i-th asset at time t.

Time series plots of y_{it}'s are shown in Figure 3, where there is a very high volatility period in 2008 (the financial crisis when Lehman Brothers filed for Chapter 11 bankruptcy protection). Also, there are other relatively high volatility periods in 2001 (the dot-com bubble and the September 11 attacks) and in 2002 (the market turmoil during which Worldcom filed for Chapter 11 bankruptcy protection).

The realized volatilities and pairwise realized correlations are computed from the realized covariance matrices for these assets which can be downloaded from Oxford Man Institute website (see, Noureldin et al. (2012)). We assume the same prior distributions as in Section 5 except assuming

$$\sigma^2_{\zeta,ij} \sim \mathrm{IG}(10^{-5}/2, 10^{-5}/2),$$

for $i = 1, \ldots, p$, $j = 1, \ldots, i-1$. These prior distributions are still found to be vague enough to reflect the fact we have little prior information regarding these parameters as we shall see in the estimation results. The proposed model is estimated where we use the parsimonious specification of the leverage effect with the number of factors $q = 1$.

6.1 Estimation results

We run 12,000 MCMC iterations and the first 2,000 iterations are discarded as the burn-in period. Table 4 shows the posterior means, 95% credible intervals and inefficiency factors for $\boldsymbol{\mu}, \boldsymbol{\xi}, \boldsymbol{\phi}, \boldsymbol{\sigma}_u, \boldsymbol{\sigma}_m$ and $\boldsymbol{\lambda}$. The inefficiency factors are relatively small (less than 130) in the multivariate stochastic volatility models and our algorithm works well. The posterior means and posterior standard deviations for $\boldsymbol{\delta}$, $\boldsymbol{\sigma}_v$, $\boldsymbol{\sigma}_\zeta$ and $\boldsymbol{\Psi}$ are shown in Tables 5, 6, 7 and 8.

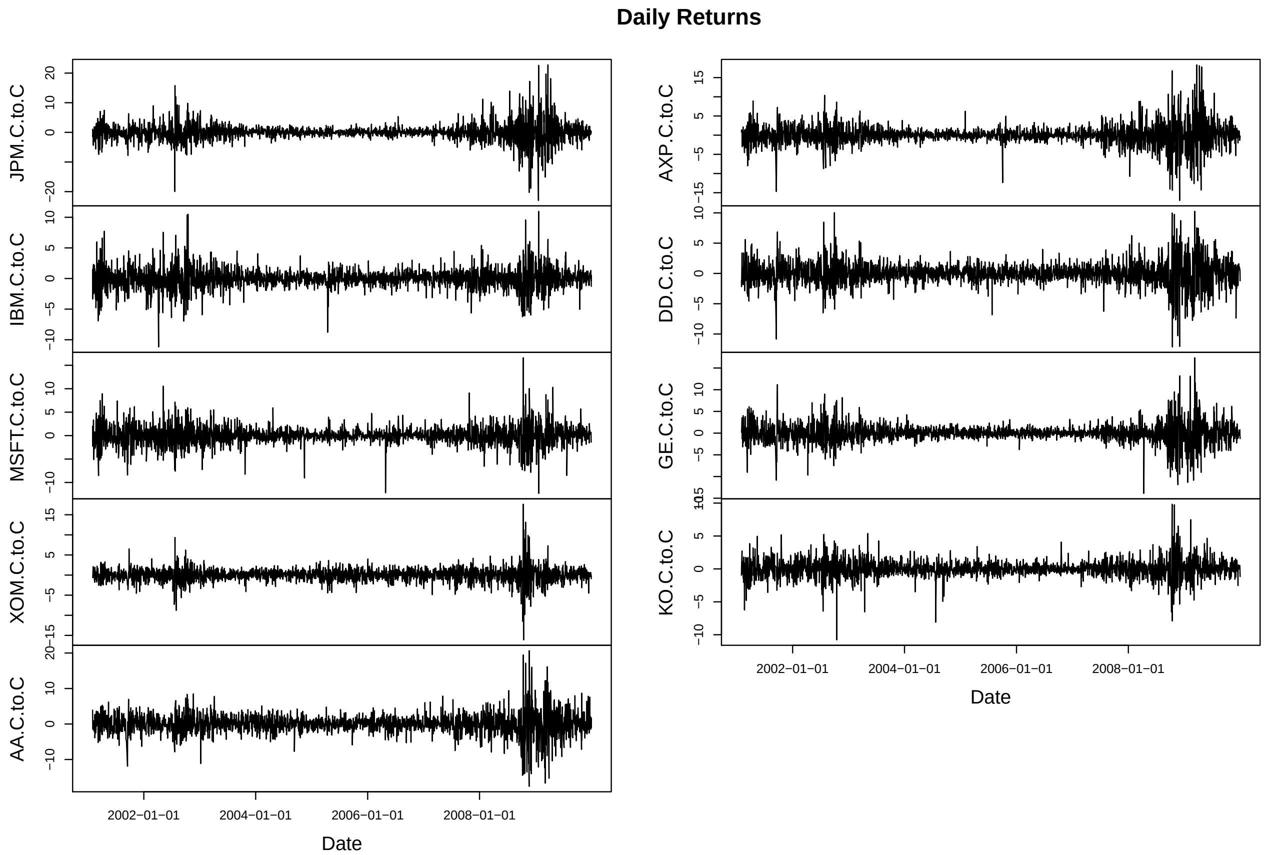


Figure 3: Time series plots of nine US stock (close-to-close) returns.

Mean processes and volatilities. The posterior means of $\sigma_{m,i}$'s are around $0.067 \sim 0.098$, which indicates the magnitude of the mean process, $\boldsymbol{m}_t$, is much smaller than that of the stochastic volatility component, $\mathbf{V}_t^{1/2}\boldsymbol{\epsilon}_t$, as we expected. The unconditional means of log volatilities, μ_i's, are estimated to be from 0.203 to 1.582 where the posterior mean of μ_5 (corresponding to Alcoa) is much larger than those of others. The stock returns of Alcoa are found to be the most volatile among others, while those of Coca Cola are the least volatile. Since all posterior means of autoregressive coefficients, ϕ_i's, are around 0.9, log volatilities are found to have high persistence. The elements of $\boldsymbol{\Psi}$ (the conditional covariance matrix of $\boldsymbol{h}_{t+1}$ given $\boldsymbol{y}_t$) are all around 0.1 and the probability that ψ_{ij} is positive are greater than 0.975 for all i, j. The log volatilities, $h_{i,t+1}$'s, are positively correlated with each other given $\boldsymbol{y}_t$. Figure 4 shows 95% credible intervals for h_{it} with $x_{it} - \xi_i$ where ξ_i is the estimated posterior mean of the i-th bias correction term. The estimated 95 % credible intervals have smaller fluctuation than that of bias-adjusted realized measures. These estimates succeeded to extract the mean trends of volatilities and adjust the measurement errors automatically. Overall, 95% credible intervals captures the traceplot of the (bias-corrected) realized volatilities, suggesting that our proposed model is successful to describe the dynamics of latent log volatilities.

Dynamic correlations. The posterior means of the standard deviations of the disturbance terms in the state equations corresponding to the dynamic correlations, $\sigma_{\zeta,ij}$'s, are shown in Table 7. They are $0.039 \sim 0.071$ and much smaller than those posterior means of the standard deviations for the measurement errors of the realized measures, $\sigma_{u,i}$'s and $\sigma_{v,ij}$'s (shown in Tables 4 and 6) which are found to be similar for all i, j around 0.30. Figure 5 shows time series plots of 95% credible intervals of the selected dynamic correlations, $\rho_{21,t}, \ldots, \rho_{62,t}$ with $\{\exp(x_{21,t} - \delta_{21}) - 1\}/\{\exp(x_{21,t} - \delta_{21}) + 1\}, \ldots, \{\exp(x_{62,t} - \delta_{62}) - 1\}/\{\exp(x_{62,t} - \delta_{62}) + 1\}$ where δ_{ij} is the estimated posterior mean. Again, the estimated 95% credible intervals of $\rho_{ij,t}$ have much smaller fluctuation than that of bias-adjusted realized measures, $\{\exp(x_{ij,t} - \delta_{ij}) - 1\}/\{\exp(x_{ij,t} - \delta_{ij}) + 1\}$. These intervals seem to extract the mean trends of the bias-adjusted realized measures which has relatively large noises in the measurement equation. The correlations between asset returns are found to be time-varying in the sample period, and they seem to increase after the financial crisis in 2008. This result corresponds to our intuition that each asset return has a larger positive correlation with others when the market faces the stress than in an usual period.

Biases in realized volatilities and correlations. The bias correction terms, ξ_i's, of the realized volatilities are estimated to be negative, indicating that the realized volatilities have downward biases and underestimate the volatilities by ignoring the overnight non-trading hours. Since the realized volatilities tend to overestimate the volatilities due to the microstructure noises, the effect of non-trading hours seems to dominate in the direction of the biases. We also note that the magnitude of the bias depends on the series of stock returns. Table 5 shows the estimation result of the bias term $\boldsymbol{\delta}$ of correlation coefficients. All δ_{ij}'s are estimated to be negative, and the posterior probability that δ_{ij} is negative is greater than 0.975. This implies that the realized correlations underestimate the latent correlations, suggesting the existence of Epps effect.

Table 4: Stock returns data. Posterior means, 95% credible intervals and inefficiency factors for $\boldsymbol{\mu}$, $\boldsymbol{\xi}$, $\boldsymbol{\phi}$, $\boldsymbol{\sigma}_u$, $\boldsymbol{\sigma}_m$ and $\boldsymbol{\lambda}$. Spectral decomposition is used to compute $\mathbf{R}_t^{-1/2}$.

Par.	Mean	95% interval	IF	Par.	Mean	95% interval	IF
μ_1	1.221	[1.020,1.430]	8	$\sigma_{u,1}$	0.285	[0.270,0.300]	40
μ_2	0.644	[0.495,0.794]	16	$\sigma_{u,2}$	0.286	[0.271,0.303]	57
μ_3	0.914	[0.759,1.070]	9	$\sigma_{u,3}$	0.291	[0.278,0.304]	14
μ_4	0.684	[0.541,0.827]	11	$\sigma_{u,4}$	0.274	[0.261,0.287]	20
μ_5	1.582	[1.430,1.730]	8	$\sigma_{u,5}$	0.318	[0.304,0.332]	17
μ_6	1.136	[0.918,1.360]	9	$\sigma_{u,6}$	0.307	[0.293,0.321]	20
μ_7	0.873	[0.726,1.020]	12	$\sigma_{u,7}$	0.291	[0.278,0.304]	20
μ_8	0.861	[0.670,1.050]	10	$\sigma_{u,8}$	0.303	[0.289,0.317]	18
μ_9	0.203	[0.055,0.352]	15	$\sigma_{u,9}$	0.294	[0.281,0.308]	18
ξ_1	-0.520	[-0.582,-0.470]	102	$\sigma_{m,1}$	0.087	[0.067,0.108]	118
ξ_2	-0.554	[-0.610,-0.495]	101	$\sigma_{m,2}$	0.070	[0.053,0.089]	126
ξ_3	-0.549	[-0.594,-0.501]	90	$\sigma_{m,3}$	0.076	[0.053,0.106]	129
ξ_4	-0.442	[-0.487,-0.394]	88	$\sigma_{m,4}$	0.087	[0.067,0.106]	117
ξ_5	-0.537	[-0.582,-0.494]	78	$\sigma_{m,5}$	0.098	[0.073,0.135]	129
ξ_6	-0.586	[-0.651,-0.533]	109	$\sigma_{m,6}$	0.088	[0.071,0.118]	119
ξ_7	-0.428	[-0.480,-0.376]	102	$\sigma_{m,7}$	0.087	[0.063,0.111]	123
ξ_8	-0.535	[-0.589,-0.474]	100	$\sigma_{m,8}$	0.078	[0.060,0.108]	125
ξ_9	-0.322	[-0.376,-0.263]	93	$\sigma_{m,9}$	0.067	[0.048,0.088]	124
ϕ_1	0.914	[0.904,0.924]	19	λ_1	-0.0626	[-0.0852,-0.0403]	7
ϕ_2	0.888	[0.874,0.901]	22	λ_2	-0.0541	[-0.0757,-0.0330]	7
ϕ_3	0.900	[0.887,0.913]	19	λ_3	-0.0430	[-0.0638,-0.0216]	7
ϕ_4	0.890	[0.876,0.904]	21	λ_4	-0.0518	[-0.0722,-0.0311]	6
ϕ_5	0.907	[0.895,0.920]	20	λ_5	-0.0424	[-0.0625,-0.0219]	7
ϕ_6	0.926	[0.916,0.935]	25	λ_6	-0.0518	[-0.0735,-0.0303]	6
ϕ_7	0.899	[0.886,0.911]	22	λ_7	-0.0536	[-0.0736,-0.0331]	9
ϕ_8	0.908	[0.897,0.920]	21	λ_8	-0.0538	[-0.0767,-0.0308]	8
ϕ_9	0.903	[0.889,0.916]	21	λ_9	-0.0436	[-0.0637,-0.0235]	10

Table 5: Stock returns data. Posterior means (posterior standard deviations) of $\boldsymbol{\delta}$.

δ_{ij}	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$
$i=2$	**-0.313**							
	(0.029)							
$i=3$	**-0.265**	**-0.333**						
	(0.031)	(0.053)						
$i=4$	**-0.301**	**-0.180**	**-0.149**					
	(0.048)	(0.035)	(0.065)					
$i=5$	**-0.410**	**-0.227**	**-0.246**	**-0.287**				
	(0.065)	(0.027)	(0.038)	(0.043)				
$i=6$	**-0.629**	**-0.265**	**-0.295**	**-0.353**	**-0.391**			
	(0.040)	(0.032)	(0.048)	(0.045)	(0.036)			
$i=7$	**-0.472**	**-0.276**	**-0.219**	**-0.312**	**-0.532**	**-0.477**		
	(0.047)	(0.029)	(0.030)	(0.025)	(0.035)	(0.032)		
$i=8$	**-0.526**	**-0.370**	**-0.319**	**-0.340**	**-0.440**	**-0.563**	**-0.478**	
	(0.053)	(0.041)	(0.065)	(0.037)	(0.029)	(0.048)	(0.033)	
$i=9$	**-0.194**	**-0.081**	**-0.098**	**-0.249**	**-0.150**	**-0.246**	**-0.158**	**-0.172**
	(0.043)	(0.033)	(0.026)	(0.050)	(0.029)	(0.030)	(0.026)	(0.057)

*Bold figures indicate that the 95% credible interval does not include zero.

Table 6: Stock returns data. Posterior means (posterior standard deviations) of $\boldsymbol{\sigma}_v$.

$\sigma_{v,ij}$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$
$i=2$	0.329							
	(0.006)							
$i=3$	0.325	0.325						
	(0.005)	(0.005)						
$i=4$	0.329	0.329	0.310					
	(0.005)	(0.006)	(0.005)					
$i=5$	0.329	0.322	0.314	0.328				
	(0.005)	(0.005)	(0.005)	(0.005)				
$i=6$	0.350	0.331	0.315	0.319	0.332			
	(0.006)	(0.006)	(0.005)	(0.006)	(0.006)			
$i=7$	0.338	0.338	0.319	0.334	0.344	0.338		
	(0.006)	(0.006)	(0.005)	(0.006)	(0.006)	(0.006)		
$i=8$	0.334	0.320	0.304	0.322	0.317	0.332	0.326	
	(0.006)	(0.006)	(0.005)	(0.006)	(0.005)	(0.006)	(0.005)	
$i=9$	0.314	0.329	0.307	0.317	0.320	0.321	0.337	0.328
	(0.005)	(0.006)	(0.005)	(0.006)	(0.005)	(0.005)	(0.006)	(0.006)

Table 7: Stock returns data. Posterior means (posterior standard deviations) of $\boldsymbol{\sigma}_\zeta$.

$\sigma_{\zeta,ij}$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$
$i=2$	0.047							
	(0.005)							
$i=3$	0.039	0.043						
	(0.004)	(0.004)						
$i=4$	0.061	0.062	0.061					
	(0.004)	(0.006)	(0.005)					
$i=5$	0.047	0.044	0.042	0.047				
	(0.004)	(0.004)	(0.004)	(0.004)				
$i=6$	0.058	0.046	0.044	0.069	0.047			
	(0.005)	(0.005)	(0.005)	(0.006)	(0.004)			
$i=7$	0.041	0.044	0.041	0.061	0.053	0.049		
	(0.004)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)		
$i=8$	0.048	0.052	0.055	0.071	0.049	0.054	0.049	
	(0.005)	(0.005)	(0.005)	(0.005)	(0.004)	(0.006)	(0.005)	
$i=9$	0.049	0.042	0.040	0.064	0.043	0.045	0.044	0.048
	(0.005)	(0.005)	(0.004)	(0.006)	(0.004)	(0.005)	(0.005)	(0.004)

Table 8: Stock returns data. Posterior means (posterior standard deviations) of $\boldsymbol{\Psi}$.

ψ_{ij}	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$
$i=1$	**0.165**								
	(0.011)								
$i=2$	**0.129**	**0.142**							
	(0.009)	(0.009)							
$i=3$	**0.119**	**0.114**	**0.127**						
	(0.008)	(0.007)	(0.009)						
$i=4$	**0.111**	**0.105**	**0.097**	**0.124**					
	(0.008)	(0.007)	(0.007)	(0.008)					
$i=5$	**0.104**	**0.088**	**0.082**	**0.089**	**0.103**				
	(0.007)	(0.006)	(0.006)	(0.006)	(0.008)				
$i=6$	**0.137**	**0.112**	**0.105**	**0.102**	**0.094**	**0.142**			
	(0.009)	(0.008)	(0.008)	(0.007)	(0.007)	(0.010)			
$i=7$	**0.116**	**0.102**	**0.097**	**0.101**	**0.089**	**0.108**	**0.115**		
	(0.008)	(0.007)	(0.007)	(0.007)	(0.006)	(0.007)	(0.008)		
$i=8$	**0.141**	**0.124**	**0.116**	**0.109**	**0.100**	**0.128**	**0.113**	**0.157**	
	(0.009)	(0.008)	(0.008)	(0.008)	(0.007)	(0.009)	(0.008)	(0.011)	
$i=9$	**0.101**	**0.095**	**0.087**	**0.086**	**0.069**	**0.091**	**0.087**	**0.099**	**0.103**
	(0.007)	(0.007)	(0.006)	(0.006)	(0.005)	(0.006)	(0.006)	(0.007)	(0.007)

*Bold figures indicate that the 95% credible interval does not include zero.

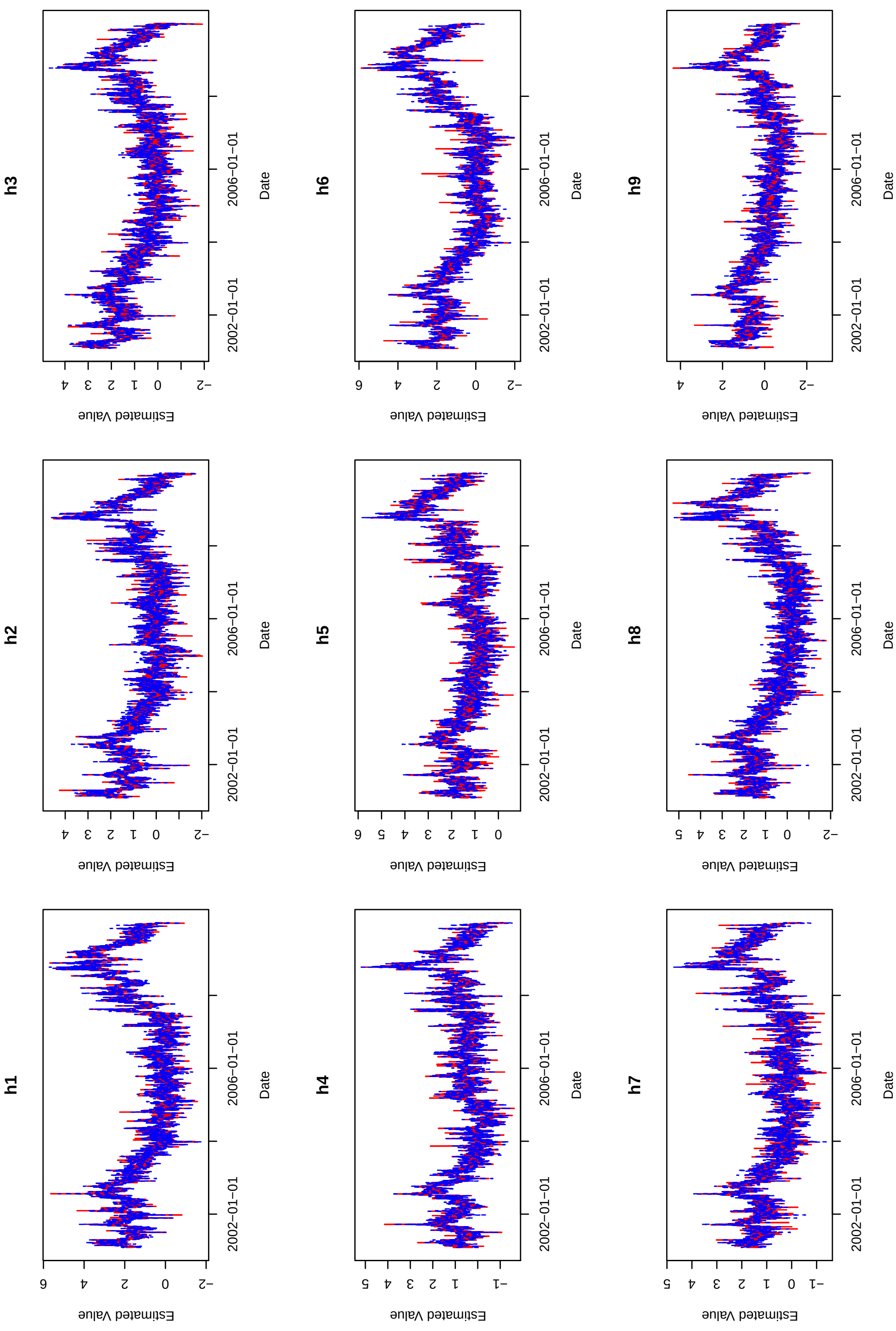


Figure 4: Stock returns data. Time series plots of 95% credible intervals (blue) of h_{it}, and $x_{it} - \xi_i$ (red) where ξ_i is the estimated posterior mean for $i = 1, \ldots, 9$.

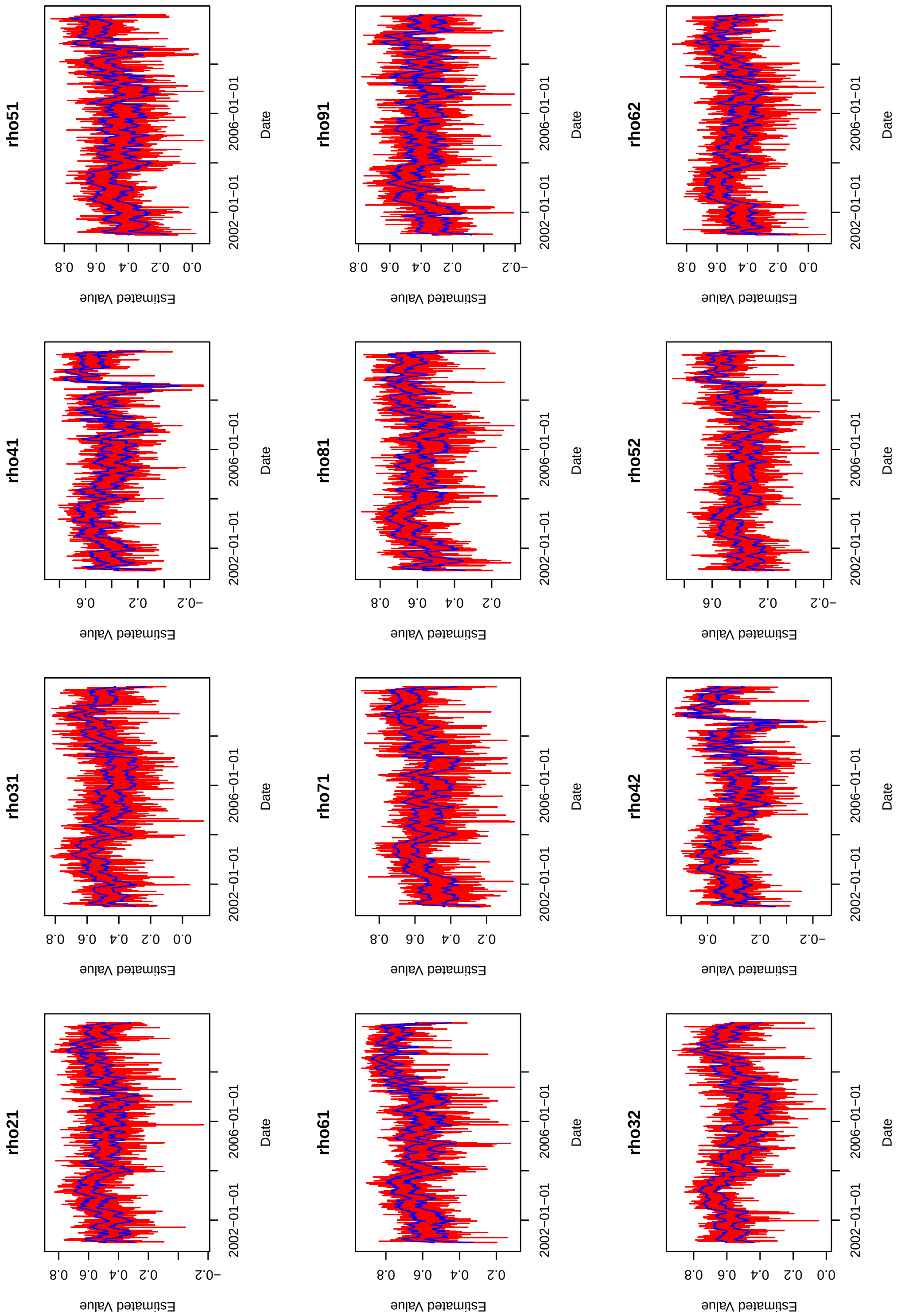


Figure 5: Time series plots of 95% credible intervals (blue) of $\rho_{21,t},\ldots,\rho_{62,t}$ and $\{\exp(x_{21,t}-\delta_{21})-1\}/\{\exp(x_{21,t}-\delta_{21})+1\},\ldots,\{\exp(x_{62,t}-\delta_{62})-1\}/\{\exp(x_{62,t}-\delta_{62})+1\}$ where δ_{ij} is the estimated posterior mean.

Leverage effect and the selection of the number of factors q. The parameters for leverage effect, λ_i's, are estimated to be negative in Table 4 and the posterior probability that λ_i is negative is greater than 0.975 for all i. This implies the existence of the leverage effect. Table 9 also shows the estimation results for the correlation between the first element of $\boldsymbol{z}_t = \mathbf{V}_t^{-1/2}\mathbf{R}_t^{-1/2}\boldsymbol{y}_t$ and $h_{i,t+1}$, *i.e.*, $\rho_i^* = Corr(z_{1t}, h_{i,t+1}) = \lambda_{ii}/\sqrt{\lambda_{ii}^2 + \psi_{ii}}$ for $i = 1, \ldots, 9$. The posterior means of ρ_i^*'s are estimated to be negative, from -0.22 to -0.15. If we regard z_{1t} as the market factor, the increase in the log volatility ($h_{i,t+1}$) is followed by the decrease in the market return (z_{1t}), which implies the existence of the leverage effect.

To investigate whether the number of factors is $q = 1$, we also fit the proposed model with $q = 2$ where we set $\boldsymbol{\Lambda} = [\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \mathbf{0}, \ldots, \mathbf{0}]$ for the leverage effect. Table 10 shows the posterior means, 95% credible intervals and inefficiency factors for $\rho_{1i}^* = Corr(z_{1t}, h_{i,t+1})$ and $\rho_{2i}^* = Corr(z_{2t}, h_{i,t+1})$ for $i = 1, \ldots, 9$ where $\boldsymbol{z}_t = \mathbf{R}_t^{-1/2}\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t - \boldsymbol{m}_t)$. The estimation results for ρ_{1i}^*'s are almost the same as those for ρ_i^*'s in Table 9. On the other hand, the posterior means of ρ_{2i}^*'s are close to zeros, and 95% credible intervals include zero. This suggests that one factor ($q = 1$) is enough to describe the leverage effect for our dataset.

Cholesky and spectral decompositions to compute $\mathbf{R}_t^{-1/2}$. We also estimated our proposed models with $q = 1$ and 2 using Cholesky decomposition instead of the spectral decomposition. The estimation results using Cholesky decomposition are very similar to those using the spectral decomposition (and hence omitted) except for the parameters of the leverage effect. Table 11 shows the estimation results for the correlation, ρ_i^*, with $q = 1$. All posterior means are estimated to be negative and the posterior probability that ρ_i^* is negative is greater than 0.975 for all i. However, we note that the absolute values of ρ_i^* are smaller than those in the model using the spectral decomposition. Table 12 shows the estimation results for the correlations, ρ_{1i}^* and ρ_{2i}^*, with $q = 2$. The estimation results for ρ_{1i}^*'s are similar to those for ρ_i^*'s in Table 11, but all posterior means of ρ_{2i}^* are estimated to be negative, and the posterior probability that ρ_{2i}^* is negative is greater than 0.975 for $i = 2, 3, 7$. This implies that we need to include more factors when we use Cholesky decomposition. Since the number of factors q depends on the order of the asset return, we have to find the order which minimizes q for the parsimonious specification. On the other hand, the spectral decomposition does not depend on the order of the asset returns and it is much faster to find a parsimonious specification. We will compare these models using different decompositions in portfolio performances.

Table 9: Stock returns data. Posterior means, 95% credible intervals and inefficiency factors for $\rho_i^* = Corr(z_{1t}, h_{i,t+1})$ where $\boldsymbol{z}_t = \mathbf{R}_t^{-1/2}\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t - \boldsymbol{m}_t)$ and $q = 1$. Spectral decomposition is used.

Par.	Mean	95% interval	IF
ρ_1^*	-0.199	[-0.306,-0.113]	8
ρ_2^*	-0.187	[-0.302,-0.100]	8
ρ_3^*	-0.153	[-0.261,-0.067]	7
ρ_4^*	-0.199	[-0.327,-0.101]	7
ρ_5^*	-0.179	[-0.322,-0.076]	8
ρ_6^*	-0.177	[-0.288,-0.089]	8
ρ_7^*	-0.224	[-0.374,-0.114]	10
ρ_8^*	-0.171	[-0.278,-0.087]	8
ρ_9^*	-0.187	[-0.334,-0.083]	11

Table 10: Stock returns data. Posterior means, 95% credible intervals and inefficiency factors for $\rho_{1i}^* = Corr(z_{1t}, h_{i,t+1})$ and $\rho_{2i}^* = Corr(z_{2t}, h_{i,t+1})$ where $q = 2$. Spectral decomposition is used.

Par.	Mean	95% interval	IF	Par.	Mean	95% interval	IF
ρ_{11}^*	-0.195	[-0.305,-0.107]	6	ρ_{21}^*	-0.009	[-0.103,0.071]	35
ρ_{12}^*	-0.183	[-0.305,-0.091]	10	ρ_{22}^*	0.016	[-0.084,0.096]	41
ρ_{13}^*	-0.147	[-0.254,-0.063]	8	ρ_{23}^*	-0.001	[-0.141,0.106]	64
ρ_{14}^*	-0.199	[-0.334,-0.097]	6	ρ_{24}^*	0.037	[-0.054,0.116]	41
ρ_{15}^*	-0.187	[-0.343,-0.080]	11	ρ_{25}^*	0.025	[-0.096,0.114]	51
ρ_{16}^*	-0.166	[-0.276,-0.079]	7	ρ_{26}^*	-0.013	[-0.109,0.070]	31
ρ_{17}^*	-0.225	[-0.400,-0.111]	8	ρ_{27}^*	-0.012	[-0.125,0.081]	45
ρ_{18}^*	-0.173	[-0.284,-0.084]	8	ρ_{28}^*	0.021	[-0.079,0.099]	39
ρ_{19}^*	-0.183	[-0.337,-0.075]	13	ρ_{29}^*	0.054	[-0.066,0.143]	55

Table 11: Stock returns data. Posterior means, 95% credible intervals and inefficiency factors for $\rho_i^* = Corr(z_{1t}, h_{i,t+1})$ where $\boldsymbol{z}_t = \mathbf{R}_t^{-1/2}\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t - \boldsymbol{m}_t)$ and $q = 1$. Cholesky decomposition is used.

Par.	Mean	95% interval	IF
ρ_1^*	-0.170	[-0.266,-0.092]	7
ρ_2^*	-0.118	[-0.199,-0.049]	4
ρ_3^*	-0.091	[-0.174,-0.021]	8
ρ_4^*	-0.118	[-0.206,-0.045]	7
ρ_5^*	-0.123	[-0.239,-0.044]	7
ρ_6^*	-0.119	[-0.205,-0.048]	7
ρ_7^*	-0.141	[-0.244,-0.061]	9
ρ_8^*	-0.144	[-0.237,-0.067]	10
ρ_9^*	-0.148	[-0.271,-0.058]	8

Table 12: Stock returns data. Posterior means, 95% credible intervals and inefficiency factors for $\rho_{1i}^* = Corr(z_{1t}, h_{i,t+1})$ and $\rho_{2i}^* = Corr(z_{2t}, h_{i,t+1})$ where $q = 2$. Cholesky decomposition is used.

Par.	Mean	95% interval	IF	Par.	Mean	95% interval	IF
ρ_{11}^*	-0.168	[-0.271,-0.087]	7	ρ_{21}^*	-0.029	[-0.089, 0.024]	5
ρ_{12}^*	-0.132	[-0.230,-0.056]	6	ρ_{22}^*	-0.113	[-0.200,-0.043]	7
ρ_{13}^*	-0.100	[-0.189,-0.029]	7	ρ_{23}^*	-0.073	[-0.147,-0.009]	6
ρ_{14}^*	-0.120	[-0.212,-0.046]	4	ρ_{24}^*	-0.019	[-0.080, 0.036]	7
ρ_{15}^*	-0.128	[-0.240,-0.045]	9	ρ_{25}^*	-0.044	[-0.117, 0.018]	8
ρ_{16}^*	-0.125	[-0.215,-0.051]	8	ρ_{26}^*	-0.028	[-0.093, 0.029]	5
ρ_{17}^*	-0.149	[-0.262,-0.064]	7	ρ_{27}^*	-0.081	[-0.160,-0.016]	8
ρ_{18}^*	-0.145	[-0.239,-0.066]	6	ρ_{28}^*	-0.011	[-0.071, 0.042]	5
ρ_{19}^*	-0.151	[-0.269,-0.058]	10	ρ_{29}^*	-0.003	[-0.069, 0.052]	7

6.2 Comparison of portfolio performances

To compare the performance of forecasts among our proposed models and other existing models, we consider three portfolio strategies: (1) minimum-variance strategy (2) mean-variance strategy (3) maximum-expected return strategy (see Han (2006)). Denote the conditional mean and the conditional covariance matrix of the stock return $\boldsymbol{y}_{t+1}$ given the information set $\mathcal{F}_t$ at time t by

$$\begin{aligned}\boldsymbol{m}_{t+1|t} &\equiv \mathrm{E}[\boldsymbol{y}_{t+1}\,|\,\mathcal{F}_t] = \boldsymbol{m}_t,\\ \boldsymbol{\Sigma}_{t+1|t} &\equiv \mathrm{Var}[\boldsymbol{y}_{t+1}\,|\,\mathcal{F}_t] = \mathbf{V}_{t+1}^{1/2}\,\mathbf{R}_{t+1}\,\mathbf{V}_{t+1}^{1/2} + \boldsymbol{\Sigma}_m\,.\end{aligned}$$

Let $r_{p,t+1}$ denote the portfolio return at time $t+1$. Further, denote the conditional mean and conditional variance of $r_{p,t+1}$ given the information set $\mathcal{F}_t$ at time t by

$$\begin{aligned}\mu_{p,t+1} &\equiv \mathrm{E}[\boldsymbol{w}_t'\,\boldsymbol{y}_{t+1} + (1-\boldsymbol{w}_t'\,\mathbf{1}_p)r_f\,|\,\mathcal{F}_t] = \boldsymbol{w}_t'\,\boldsymbol{m}_{t+1|t} + (1-\boldsymbol{w}_t'\,\mathbf{1}_p)r_f,\\ \sigma_{p,t+1}^2 &\equiv \mathrm{Var}[\boldsymbol{w}_t'\,\boldsymbol{y}_{t+1} + (1-\boldsymbol{w}_t'\,\mathbf{1}_p)r_f\,|\,\mathcal{F}_t] = \boldsymbol{w}_t'\,\boldsymbol{\Sigma}_{t+1|t}\,\boldsymbol{w}_t,\end{aligned}$$

where r_f is the risk free asset return, and $\boldsymbol{w}_t$ is a portfolio weight vector for the stock return $\boldsymbol{y}_{t+1}$. The weight $\boldsymbol{w}_t$ is optimized depending on the portfolio strategies as follows.

Minimum-variance strategy

In this strategy, we minimize the conditional variance $\sigma_{p,t+1}^2$ for the target level μ_p^* of conditional expected return $\mu_{p,t+1}$. The solution is given by

$$\hat{\boldsymbol{w}}_t = \boldsymbol{\Sigma}_{t+1|t}^{-1}(\boldsymbol{m}_{t+1|t} - r_f\,\mathbf{1}_p)\frac{\mu_p^* - r_f}{\kappa_t},$$

where

$$\kappa_t = (\boldsymbol{m}_{t+1|t} - r_f\,\mathbf{1}_p)'\,\boldsymbol{\Sigma}_{t+1|t}^{-1}(\boldsymbol{m}_{t+1|t} - r_f\,\mathbf{1}_p).$$

Mean-variance strategy

This strategy solves the following utility maximization problem:

$$\max_{\boldsymbol{w}_t}\left\{\mu_{p,t+1} - \frac{\gamma}{2}\sigma_{p,t+1}^2\right\},$$

where γ is the coefficient of absolute risk aversion. The solution is given by

$$\hat{\boldsymbol{w}}_t = \frac{1}{\gamma}\,\boldsymbol{\Sigma}_{t+1|t}^{-1}(\boldsymbol{m}_{t+1|t} - r_f\,\mathbf{1}_p).$$

Maximum expected return strategy

We maximize the conditional expected return $\mu_{p,t+1}$ given the target conditional variance $\sigma^2_{p,t+1} = \sigma^{2*}_p$. The solution is given by

$$\hat{\boldsymbol{w}}_t = \boldsymbol{\Sigma}^{-1}_{t+1|t}(\boldsymbol{m}_{t+1|t} - r_f\,\mathbf{1}_p)\sqrt{\frac{\sigma^{2*}_p}{\kappa_t}}.$$

We compare the portfolio performances based on the rolling forecast:

Step 1. First, we estimate parameters using the first 1742 observations from February 1, 2001 to January 8, 2008 and forecast the mean, the volatility and the correlation of the multiple stock returns for January 9, 2008. Use them to obtain the optimal weights of the assets for the above portfolio strategies where the federal funds rate is used for the risk free asset return r_f.

Step 2. Next, we drop the first observation (February 1, 2001) from the sample period and add the new observation (January 9, 2008). The new sample period is from February 2, 2001 to January 9, 2008. We estimate parameters using these observations and forecast the mean, the volatility and the correlation for January 10, 2008. Use them to obtain the optimal weights in a similar manner.

Step 3. We iterate these rolling forecasts until December 31, 2009 to obtain the 500 one-day ahead forecasts and corresponding weights.

To compute the optimal weight $\hat{\boldsymbol{w}}_t$, we also need the estimates of $\boldsymbol{m}_{t+1|t}$ and $\boldsymbol{\Sigma}_{t+1|t}$. Let N denote the number of MCMC iterations, and $(\theta^{(i)}, \{\boldsymbol{h}^{(i)}_t\}^T_{t=1}, \{\mathbf{R}^{(i)}_t\}^T_{t=1}, \{\boldsymbol{m}^{(i)}_t\}^T_{t=1})$ denote the i-th MCMC sample ($i = 1, \ldots, N$). Using $\boldsymbol{m}^{(i)}_{t+1|t}, \mathbf{V}^{(i)}_{t+1|t}, \mathbf{R}^{(i)}_{t+1|t}, \boldsymbol{\Sigma}^{(i)}_m$, we estimate $\boldsymbol{m}_{t+1|t}$ and $\boldsymbol{\Sigma}_{t+1|t}$ by

$$\hat{\boldsymbol{m}}_{t+1|t} = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{m}^{(i)}_{t+1|t},$$

$$\hat{\boldsymbol{\Sigma}}_{t+1|t} = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Sigma}^{(i)}_{t+1|t} = \frac{1}{N}\sum_{i=1}^{N}\left(\mathbf{V}^{(i)1/2}_{t+1|t}\,\mathbf{R}^{(i)}_{t+1|t}\,\mathbf{V}^{(i)1/2}_{t+1|t} + \boldsymbol{\Sigma}^{(i)}_m\right).$$

In our empirical study, we set $N = 1500$ where we discard 500 samples as burn-in period for each MCMC rolling estimation (Steps 2 and 3)[2]. We compare the five multivariate stochastic

[2]The number of samples being discarded as burn-in period is sufficient after we obtain MCMC posterior samples from the previous sample period since we use the posterior means of parameters and latent variables for the initial values of the next MCMC runs.

volatility models as follows.

1. MSV model: Basic multivariate stochastic volatility model without leverage, realized variances and correlations.

2. MRSV model: Multivariate stochastic volatility model without leverage and with realized variances and pairwise realized correlations.

3. MRSV-L1-C model : Multivariate stochastic volatility model with leverage, realized variances and pairwise realized correlations. The parsimonious specification is assumed to model the leverage effect, $\mathbf{\Lambda} = [\boldsymbol{\lambda}_1, \mathbf{0}, \ldots, \mathbf{0}]$ with $q = 1$. Cholesky decomposition is used to compute $\mathbf{R}_t^{-1/2}$.

4. MRSV-L2-C model : Multivariate stochastic volatility model with leverage, realized variances and pairwise realized correlations. The parsimonious specification is assumed to model the leverage effect, $\mathbf{\Lambda} = [\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \mathbf{0}, \ldots, \mathbf{0}]$ with $q = 2$. Cholesky decomposition is used to compute $\mathbf{R}_t^{-1/2}$.

5. MRSV-L1-S model : Multivariate stochastic volatility model with leverage, realized variances and pairwise realized correlations. The parsimonious specification is assumed to model the leverage effect, $\mathbf{\Lambda} = [\boldsymbol{\lambda}_1, \mathbf{0}, \ldots, \mathbf{0}]$ with $q = 1$. The spectral decomposition is used to compute $\mathbf{R}_t^{-1/2}$.

Cumulative realized objective functions. Table 13 shows the cumulative values of the realized objective functions for five multivariate SV models under three portfolio strategies. Under the minimum-variance strategy and the maximum-return strategy, the MRSV-L1-S model outperforms other models. Among MRSV models, the models with leverage outperform the models without leverage, indicating the existence of the leverage effect. MRSV-L2-C model outperforms MRSV-L1-C model, but its performance is not as good as that of MRSV-L1-S model. We could improve the performance of the MRSV models using Cholesky decomposition by changing the order of the assets or increasing the number of non-zero columns of $\mathbf{\Lambda}$, but it is more efficient to use the spectral decomposition to compute $\mathbf{R}_t^{-1/2}$. Under the mean-variance strategy, MSV model outperforms other competing models, but this is because γ is relatively large (*i.e.*, $\hat{\boldsymbol{w}}_t$ becomes very small) and most of the weights are allocated to the risk free asset whose weight is $1 - \hat{\boldsymbol{w}}_t'\mathbf{1}_p$. The time series plots of the cumulative realized objective functions of five models are also shown in Figure 6.

Table 13: The cumulative values of realized objective functions.

Minimum-Variance Strategy			
	$\mu_p^* = 0.004$	$\mu_p^* = 0.01$	$\mu_p^* = 0.1$
MSV	1.454	8.598	1275
MRSV	0.733	4.556	637
MRSV-L1-C	0.346	2.110	295
MRSV-L2-C	0.333	2.057	288
MRSV-L1-S	**0.316**	**1.914**	**268**
Mean-Variance Strategy			
	$\gamma = 6$	$\gamma = 10$	$\gamma = 15$
MSV	**0.730**	**1.001**	**1.137**
MRSV	0.024	0.577	0.854
MRSV-L1-C	−1.399	−0.277	0.285
MRSV-L2-C	−1.134	−0.118	0.391
MRSV-L1-S	−1.333	−0.237	0.311
Maximum-Return Strategy			
	$\sigma_p^{2*} = 0.001$	$\sigma_p^{2*} = 0.01$	$\sigma_p^{2*} = 0.1$
MSV	0.663	−0.947	−6.039
MRSV	0.916	−0.146	−3.507
MRSV-L1-C	1.093	0.414	−1.735
MRSV-L2-C	1.192	0.726	−0.746
MRSV-L1-S	**1.324**	**1.143**	**0.571**

*Bold figures indicates the optimal values. For the maximum return strategy, the cumulative realized returns are computed as $\sum_{t=1742}^{2241}\{\hat{\boldsymbol{\omega}}_t'\boldsymbol{y}_{t+1} + (1-\hat{\boldsymbol{\omega}}_t'\mathbf{1}_p)r_f\}$ where $\hat{\boldsymbol{\omega}}_t$ is the vector of portfolio weights estimated at time t.

Figure 6: Time series plot of the cumulative realized objective functions of MSV (yellow), MRSV (green), MRSV-L1-C (blue) , MRSV-L2-C (purple) and MRSV-L1-S (red).

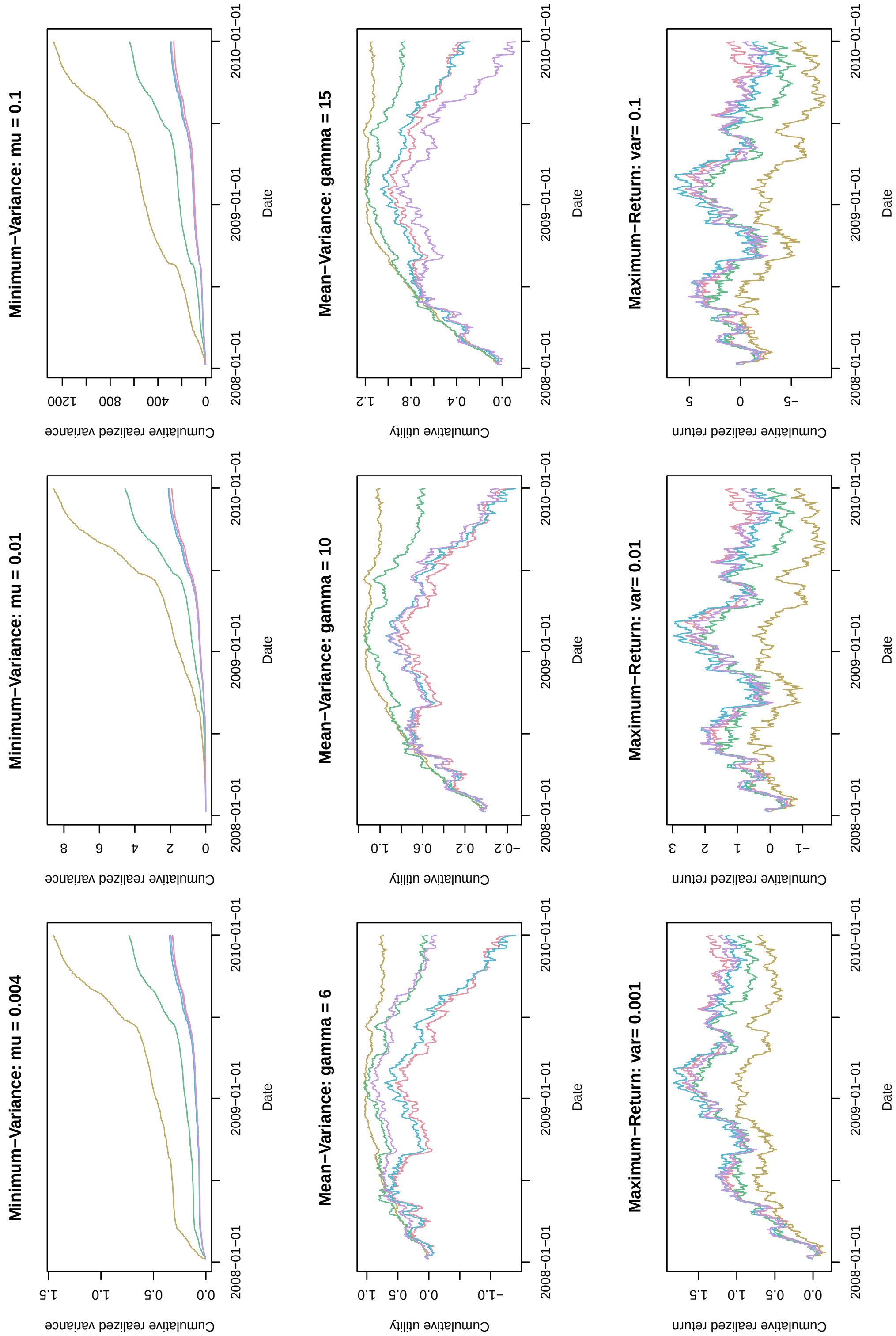


Figure 7: Time series plot of portfolio weights for minimum-variance strategy in MRSV-L1-S: $\mu_p^* = 0.004$ (red), $\mu_p^* = 0.01$ (green) and $\mu_p^* = 0.1$ (blue).

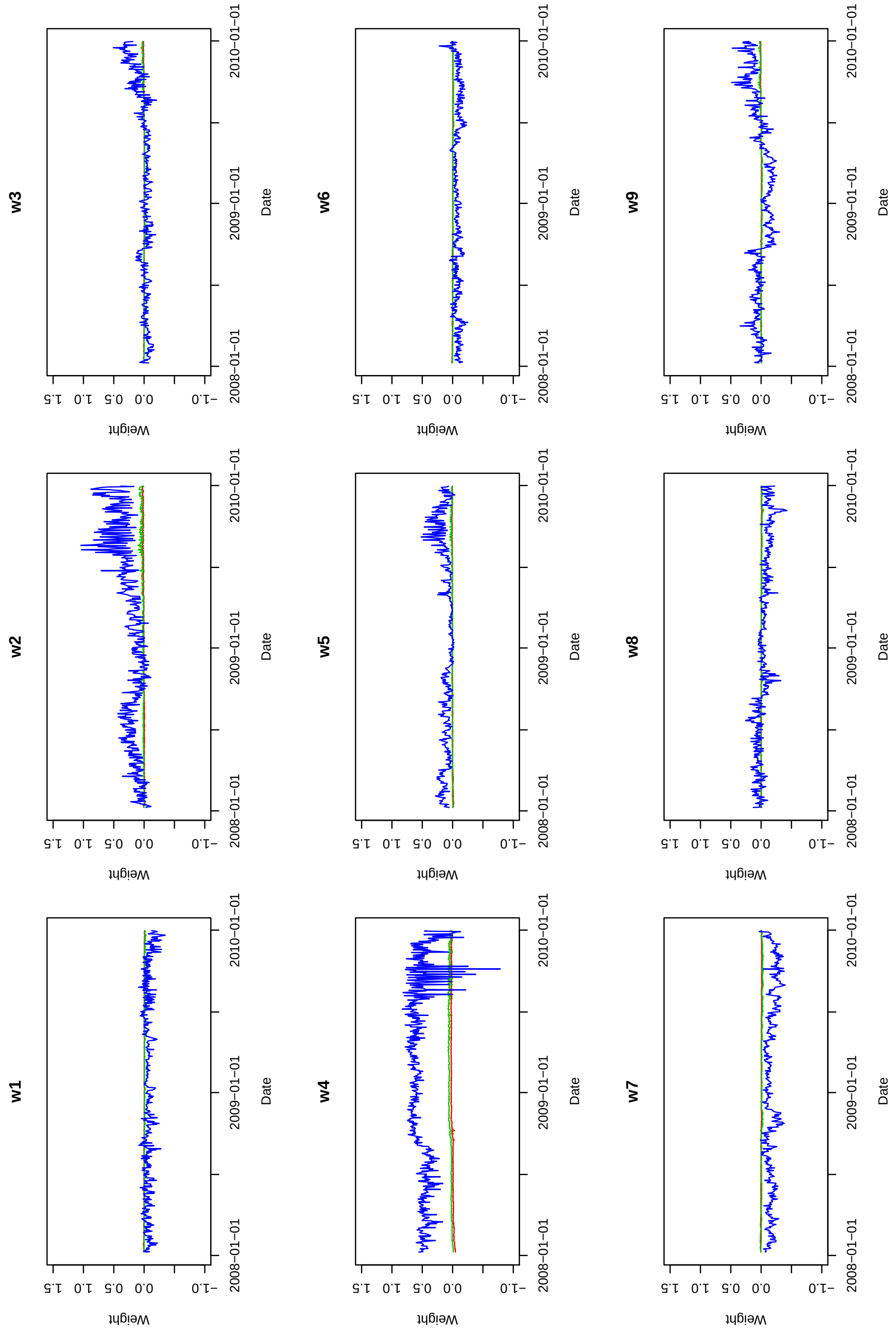


Figure 8: Time series plot of portfolio weights for mean-variance strategy in MRSV-L1-S: $\gamma = 6$ (red), $\gamma = 10$ (green) and $\gamma = 15$ (blue).

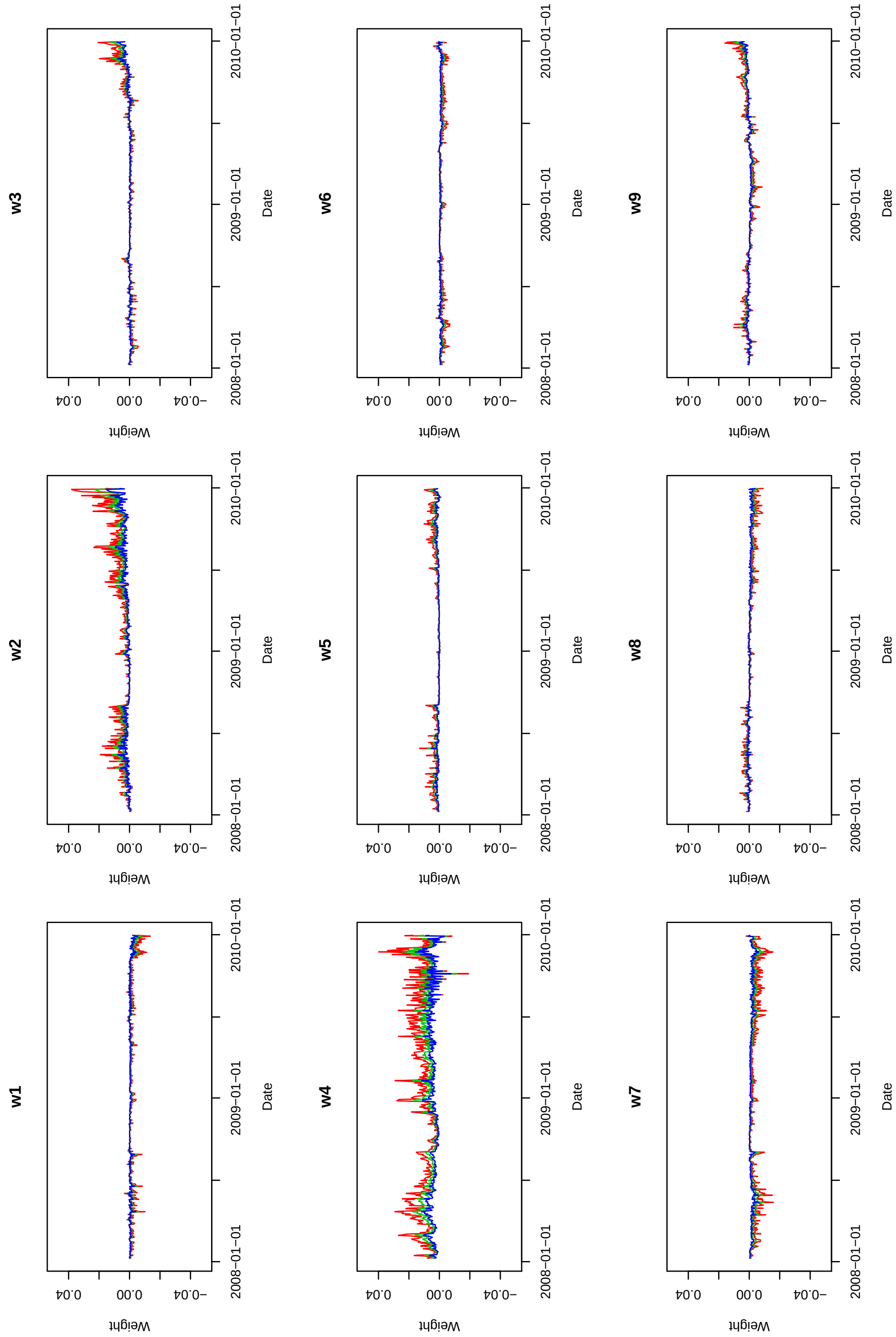
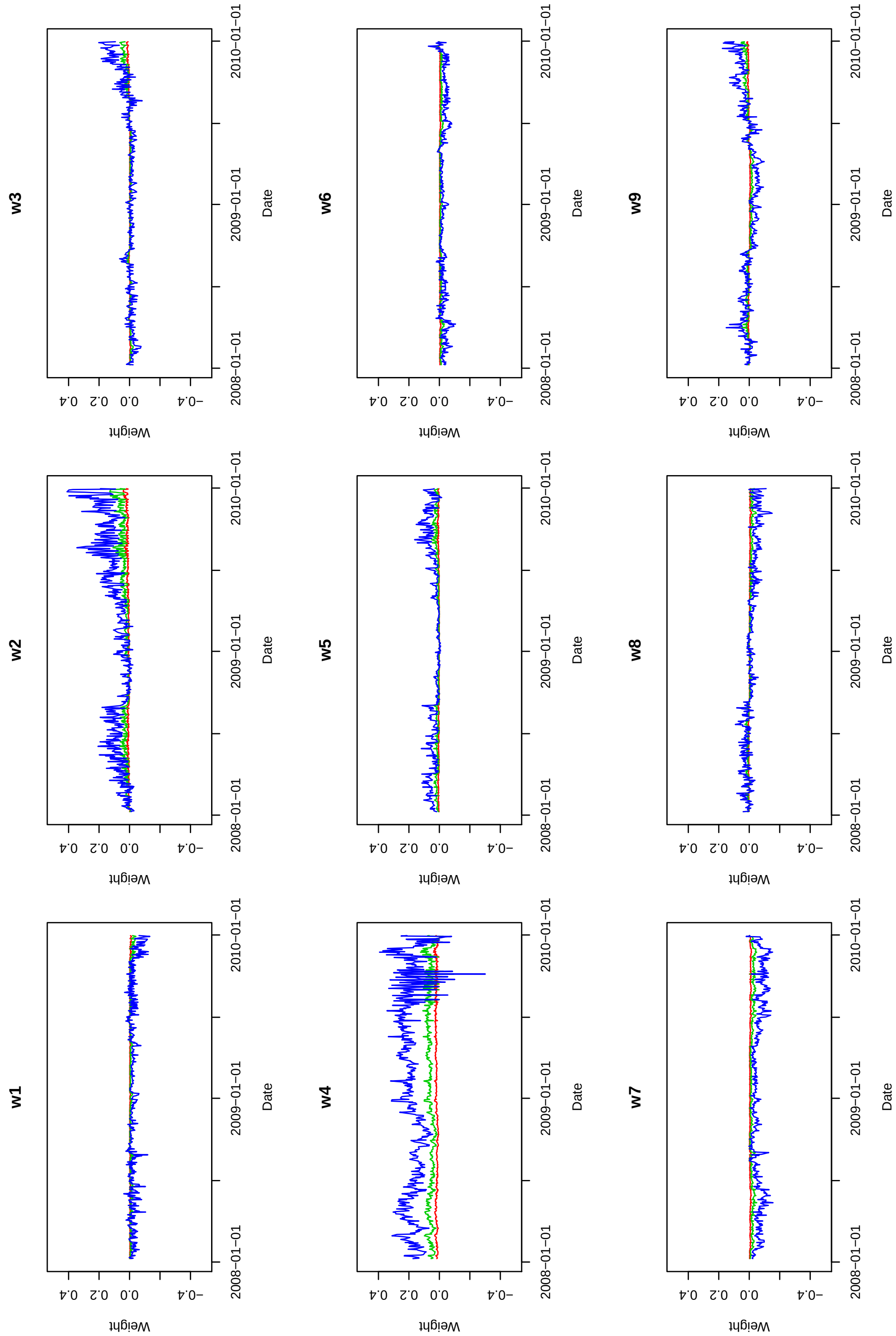


Figure 9: Time series plot of portfolio weights for maximum-return strategy in MRSV-L1-S: $\sigma_p^{*2} = 0.001$ (red), $\sigma_p^{*2} = 0.01$ (green) and $\sigma_p^{*2} = 0.1$ (blue).



Time series plots of portfolio weights. Tables 7, 8 and 9 show the time series plots of portfolio weights for three strategies in MRSV-L1-S. For the minimum-variance strategy, the weights for Exxon Mobil (w_{4t}) are overall largest throughout the forecasting period, but toward the end of the period, the weights for IBM, Microsoft and Alcoa (w_{2t}, w_{3t}, w_{5t}) tend to become large. For the mean-variance strategy, we note that all weights ($w_{1t} - w_{9t}$) for the stock returns are very small and most of the weights are allocated to the risk free asset. Finally, for the maximum-return strategy, we found similar weight patterns to those for the minimum-variance strategy.

7 Conclusion

The multivariate SV model with flexible dynamic correlation structures is proposed with Markov chain Monte Carlo estimation method. Making full use of the realized variances and realized pairwise correlations, we obtain the stable parameter estimates where the covariance matrices are guaranteed to be positive definite. The spectral decomposition is used for the correlation matrices to avoid the arbitrariness of the ordering of asset returns. The parsimonious specification for the leverage effect is also proposed. Our models are applied to daily returns of nine U.S. stocks with their realized volatilities and pairwise realized correlations, and are shown to outperform the existing models with regard to portfolio optimizations under minimum-variance and maximum-return strategies.

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Appendix

A MCMC algorithm

A.1 MRSV model without the leverage effect

A.1.1 Generation of $\boldsymbol{\phi}$

It can be shown that the conditional posterior probability density function of $\boldsymbol{\phi}$ is

$$\pi(\boldsymbol{\phi}|\cdot) \propto k(\boldsymbol{\phi}) \times \exp\left(-\frac{1}{2}(\boldsymbol{\phi}-\boldsymbol{\mu}_{\boldsymbol{\phi}})'\boldsymbol{\Sigma}_{\boldsymbol{\phi}}^{-1}(\boldsymbol{\phi}-\boldsymbol{\mu}_{\boldsymbol{\phi}})\right) \times I\left\{|\phi_i|<1, i=1,\ldots,p\right\}, \tag{51}$$

where $I(B)$ is an indicator function such that $I(B)=1$ if B is true and 0 otherwise,

$$k(\boldsymbol{\phi}) = |\boldsymbol{\Omega}_0|^{-1/2}\prod_{i=1}^{p}\left(\frac{1+\phi_i}{2}\right)^{a-1}\left(\frac{1-\phi_i}{2}\right)^{b-1}\exp\left(-\frac{1}{2}(\boldsymbol{h}_1-\boldsymbol{\mu})'\boldsymbol{\Omega}_0^{-1}(\boldsymbol{h}_1-\boldsymbol{\mu})\right), \tag{52}$$

$$\boldsymbol{\mu}_{\boldsymbol{\phi}} = \boldsymbol{\Sigma}_{\boldsymbol{\phi}}\boldsymbol{b}, \quad \boldsymbol{\Sigma}_{\boldsymbol{\phi}}^{-1} = \boldsymbol{\Omega}^{-1}\odot\mathbf{A}, \tag{53}$$

$$\mathbf{A} = \sum_{t=1}^{T-1}(\boldsymbol{h}_t-\boldsymbol{\mu})(\boldsymbol{h}_t-\boldsymbol{\mu})', \quad \boldsymbol{b} = \text{diagonal}\left\{\sum_{t=1}^{T-1}(\boldsymbol{h}_t-\boldsymbol{\mu})(\boldsymbol{h}_{t+1}-\boldsymbol{\mu})'\boldsymbol{\Omega}^{-1}\right\}, \tag{54}$$

$\odot$ is Hadamard product, and diagonal($\mathbf{B}$) denotes a column vector with diagonal elements of $\mathbf{B}$. We propose a candidate $\boldsymbol{\phi}^{\dagger} \sim \text{TN}_R(\boldsymbol{\mu}_{\boldsymbol{\phi}}, \boldsymbol{\Sigma}_{\boldsymbol{\phi}})$, where $R=\{\boldsymbol{\phi}: |\phi_i|<1, i=1,\ldots,p\}$, and accept it with probability $\min\{1, k(\boldsymbol{\phi}^{\dagger})/k(\boldsymbol{\phi})\}$.

A.1.2 Generation of $\boldsymbol{\mu}, \boldsymbol{\xi}, \boldsymbol{\delta}$

The $\boldsymbol{\mu}, \boldsymbol{\xi}$ and $\boldsymbol{\delta}$ are conditionally independent and we generate them from the following normal distributions:

$$\boldsymbol{\mu}|\cdot \sim N(\tilde{\boldsymbol{m}}_{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Omega}}_{\boldsymbol{\mu}}), \quad \boldsymbol{\xi}|\cdot \sim N(\tilde{\boldsymbol{m}}_{\boldsymbol{\xi}}, \tilde{\boldsymbol{\Sigma}}_{\boldsymbol{\xi}}), \quad \boldsymbol{\delta}|\cdot \sim N(\tilde{\boldsymbol{m}}_{\boldsymbol{\delta}}, \tilde{\boldsymbol{\Sigma}}_{\boldsymbol{\delta}}), \tag{55}$$

where

$$\tilde{\boldsymbol{m}}_{\boldsymbol{\mu}} = \tilde{\boldsymbol{\Omega}}_{\boldsymbol{\mu}}\left[s_{\mu}^{-2}\boldsymbol{m}_{\boldsymbol{\mu}} + \boldsymbol{\Omega}_0^{-1}\boldsymbol{h}_1 + (\mathbf{I}_p-\boldsymbol{\Phi})\boldsymbol{\Omega}^{-1}\sum_{t=1}^{T-1}(\boldsymbol{h}_{t+1}-\boldsymbol{\Phi}\,\boldsymbol{h}_t)\right], \tag{56}$$

$$\tilde{\boldsymbol{\Omega}}_{\boldsymbol{\mu}} = \left[s_{\mu}^{-2}\,\mathbf{I}_p + \boldsymbol{\Omega}_0^{-1} + (T-1)(\mathbf{I}_p-\boldsymbol{\Phi})\boldsymbol{\Omega}^{-1}(\mathbf{I}_p-\boldsymbol{\Phi})\right]^{-1}, \tag{57}$$

$$\tilde{\boldsymbol{m}}_{\xi} = \tilde{\boldsymbol{\Sigma}}_{\xi}\left[s_{\xi}^{-2}\,\boldsymbol{m}_{\xi} + \boldsymbol{\Sigma}_u^{-1}\sum_{t=1}^{T}(\boldsymbol{x}_t-\boldsymbol{h}_t)\right], \quad \tilde{\boldsymbol{\Sigma}}_{\xi} = (s_{\xi}^{-2}\,\mathbf{I}_p + T\,\boldsymbol{\Sigma}_u^{-1})^{-1}, \tag{58}$$

$$\tilde{\boldsymbol{m}}_{\delta} = \tilde{\boldsymbol{\Sigma}}_{\delta}\left[s_{\delta}^{-2}\,\boldsymbol{m}_{\delta} + \boldsymbol{\Sigma}_v^{-1}\sum_{t=1}^{T}(\boldsymbol{w}_t-\boldsymbol{g}_t)\right], \quad \tilde{\boldsymbol{\Sigma}}_{\delta} = (s_{\delta}^{-2}\,\mathbf{I}_p + T\,\boldsymbol{\Sigma}_v^{-1})^{-1}. \tag{59}$$

A.1.3 Generation of $(\boldsymbol{\sigma}_u^2, \boldsymbol{\sigma}_v^2, \boldsymbol{\sigma}_\zeta^2, \boldsymbol{\Sigma}_m)$

The $\sigma_{u,i}^2, \sigma_{v,ij}^2, \sigma_{\zeta,ij}^2, \sigma_{m,i}^2$ are conditionally independent and we generate them from inverse gamma distributions:

$$\sigma_{u,i}^2 \sim \mathrm{IG}\left(\frac{\tilde{n}_{ui}}{2}, \frac{\tilde{d}_{ui}}{2}\right), \quad \sigma_{v,ij}^2 \sim \mathrm{IG}\left(\frac{\tilde{n}_{v,ij}}{2}, \frac{\tilde{d}_{v,ij}}{2}\right), \tag{60}$$

$$\sigma_{\zeta,ij}^2 \sim \mathrm{IG}\left(\frac{\tilde{n}_{\zeta,ij}}{2}, \frac{\tilde{d}_{\zeta,ij}}{2}\right), \quad \sigma_{m,i}^2 \sim \mathrm{IG}\left(\frac{\tilde{n}_{mi}}{2}, \frac{\tilde{d}_{mi}}{2}\right), \tag{61}$$

where

$$\tilde{n}_{ui} = n_u + T, \quad \tilde{d}_{ui} = d_u + \sum_{t=1}^{T}(x_{it} - \xi_i - h_{it})^2, \tag{62}$$

$$\tilde{n}_{v,ij} = n_v + T, \quad \tilde{d}_{v,ij} = d_v + \sum_{t=1}^{T}(w_{ij,t} - \delta_{ij} - g_{ij,t})^2, \tag{63}$$

$$\tilde{n}_{\zeta,ij} = n_\zeta + T, \quad \tilde{d}_{\zeta,ij} = d_\zeta + \kappa^{-1} g_{ij,1}^2 + \sum_{t=1}^{T-1}(g_{ij,t+1} - g_{ij,t})^2, \tag{64}$$

$$\tilde{n}_{mi} = n_m + T, \quad \tilde{d}_{mi} = d_m + \kappa^{-1} m_{i1}^2 + \sum_{t=1}^{T-1}(m_{i,t+1} - m_{it})^2, \tag{65}$$

for $i, j = 1, \ldots, p$ $(i > j)$.

A.1.4 Generation of $\boldsymbol{\Omega}$

The conditional posterior probability density function of $\boldsymbol{\Omega}$ is

$$\pi(\boldsymbol{\Omega}|\boldsymbol{h}, \boldsymbol{\mu}, \boldsymbol{\phi}) \propto m(\boldsymbol{\Omega}) \times |\boldsymbol{\Omega}|^{-(\tilde{\nu}+p+1)/2} \exp\left\{-\frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{\Omega}^{-1}\tilde{\mathbf{S}}\right)\right\}, \tag{66}$$

where

$$m(\boldsymbol{\Omega}) = |\,\boldsymbol{\Omega}_0\,|^{-1/2} \exp\left(-\frac{1}{2}(\boldsymbol{h}_1 - \boldsymbol{\mu})'\,\boldsymbol{\Omega}_0^{-1}(\boldsymbol{h}_1 - \boldsymbol{\mu})\right), \quad \tilde{\nu} = \nu + T - 1, \tag{67}$$

$$\tilde{\boldsymbol{S}} = \boldsymbol{S} + \sum_{t=1}^{T-1}\{\boldsymbol{h}_{t+1} - (\mathbf{I}_p - \boldsymbol{\Phi})\boldsymbol{\mu} - \boldsymbol{\Phi}\,\boldsymbol{h}_t\}\{\boldsymbol{h}_{t+1} - (\mathbf{I}_p - \boldsymbol{\Phi})\boldsymbol{\mu} - \boldsymbol{\Phi}\,\boldsymbol{h}_t\}'. \tag{68}$$

We propose a candidate $\boldsymbol{\Omega}^\dagger \sim \mathrm{IW}(\tilde{\nu}, \tilde{\boldsymbol{S}})$, and accept it with probability $\min\{1, m(\boldsymbol{\Omega}^\dagger)/m(\boldsymbol{\Omega})\}$.

A.2 MRSV model with the leverage effect

We need to modify the sampling procedures of $\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{m}, \boldsymbol{\phi}$ and $\boldsymbol{\mu}$ for the model with the leverage effect. Generations of other parameters are the same as in the previous section.

A.2.1 Generation of $\boldsymbol{g}_t$

We only need to modify (15) in Section 3.2.1 as follows.

$$
\begin{aligned}
r(g_{ij,t}) = &-\frac{1}{2}\log|\mathbf{R}_t| - \frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{m}_t)'(\mathbf{V}_t^{1/2}\,\mathbf{R}_t\,\mathbf{V}_t^{1/2})^{-1}(\boldsymbol{y}_t - \boldsymbol{m}_t) \\
&- \frac{1}{2}\,\boldsymbol{y}_t'\,\mathbf{V}^{-1/2}\,\mathbf{R}_t^{-1/2\prime}\,\boldsymbol{\Lambda}'\,\boldsymbol{\Psi}^{-1}\,\boldsymbol{\Lambda}\,\mathbf{R}_t^{-1/2}\,\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t - \boldsymbol{m}_t) + \boldsymbol{y}_t'\,\mathbf{V}_t^{-1/2}\,\mathbf{R}_t^{-1/2\prime}\,\boldsymbol{\Lambda}'\,\boldsymbol{\Psi}^{-1}\,\boldsymbol{\eta}_t,
\end{aligned}
\tag{69}
$$

for $t = 1, \ldots, T-1$ where $\boldsymbol{\eta}_t = \boldsymbol{h}_{t+1} - \boldsymbol{\mu} - \boldsymbol{\Phi}(\boldsymbol{h}_t - \boldsymbol{\mu})$.

A.2.2 Generation of $\boldsymbol{h}_t$

The conditional posterior probability density function of $\boldsymbol{h}_t$ is given by

$$
\pi(\boldsymbol{h}_t\,|\cdot) \propto \exp\left[-\frac{1}{2}(\boldsymbol{h}_t - \boldsymbol{m}_{t*})'\,\boldsymbol{\Omega}_{t*}^{-1}(\boldsymbol{h}_t - \boldsymbol{m}_{t*}) + l(\boldsymbol{h}_t)\right], \tag{70}
$$

where

$$
\begin{aligned}
&l(\boldsymbol{h}_t) \\
&= \begin{cases}
-\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{m}_t)'\,\mathbf{V}^{-1/2}\,\mathbf{R}_t^{-1}\,\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t - \boldsymbol{m}_t) - \frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{m}_t)'\,\mathbf{V}_t^{-1/2}\,\mathbf{R}_t^{-1/2\prime}\,\boldsymbol{\Lambda}'\,\boldsymbol{\Psi}^{-1} \\
\quad \times\left\{\boldsymbol{\Lambda}\,\mathbf{R}_t^{-1/2}\,\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t - \boldsymbol{m}_t) - 2(\boldsymbol{h}_{t+1} - (\mathbf{I} - \boldsymbol{\Phi})\,\boldsymbol{\mu} - \boldsymbol{\Phi}\,\boldsymbol{h}_t)\right\}, \\
\hfill t = 1, \ldots, T-1, \\
-\frac{1}{2}(\boldsymbol{y}_T - \boldsymbol{m}_T)'\,\mathbf{V}_T^{-1/2}\,\mathbf{R}_T^{-1}\,\mathbf{V}_T^{-1/2}(\boldsymbol{y}_T - \boldsymbol{m}_T), \hfill t = T,
\end{cases}
\end{aligned}
\tag{71}
$$

and

$$
\boldsymbol{m}_{t*} = \begin{cases}
\boldsymbol{\Psi}_{1*}\left[\boldsymbol{\Omega}_0^{-1}\,\boldsymbol{\mu} + \boldsymbol{\Phi}\,\boldsymbol{\Psi}^{-1}\left\{\boldsymbol{h}_2 - (\mathbf{I}_p - \boldsymbol{\Phi})\,\boldsymbol{\mu}\right\} + \boldsymbol{\Sigma}_u^{-1}(\boldsymbol{x}_1 - \boldsymbol{\xi}) - \frac{1}{2}\,\mathbf{1}_p\right], & t = 1, \\
\boldsymbol{\Psi}_{t*}\Big[\boldsymbol{\Psi}^{-1}\left\{\boldsymbol{\Lambda}\,\mathbf{R}_{t-1}^{-1/2}\,\mathbf{V}_{t-1}^{-1/2}(\boldsymbol{y}_{t-1} - \boldsymbol{m}_{t-1}) + (\mathbf{I}_p - \boldsymbol{\Phi})\,\boldsymbol{\mu} + \boldsymbol{\Phi}\,\boldsymbol{h}_{t-1}\right\} \\
\qquad + \boldsymbol{\Phi}\,\boldsymbol{\Psi}^{-1}\left\{\boldsymbol{h}_{t+1} - (\mathbf{I}_p - \boldsymbol{\Phi})\,\boldsymbol{\mu}\right\} + \boldsymbol{\Sigma}_u^{-1}(\boldsymbol{x}_t - \boldsymbol{\xi}) - \frac{1}{2}\,\mathbf{1}_p\Big], \\
& t = 2, \ldots, T-1, \\
\boldsymbol{\Psi}_{T*}\Big[\boldsymbol{\Psi}^{-1}\left\{\boldsymbol{\Lambda}\,\mathbf{R}_{T-1}^{-1/2}\,\mathbf{V}_{T-1}^{-1/2}(\boldsymbol{y}_{T-1} - \boldsymbol{m}_{T-1}) + (\mathbf{I}_p - \boldsymbol{\Phi})\,\boldsymbol{\mu} + \boldsymbol{\Phi}\,\boldsymbol{h}_{T-1}\right\} \\
\qquad + \boldsymbol{\Sigma}_u^{-1}(\boldsymbol{x}_T - \boldsymbol{\xi}) - \frac{1}{2}\,\mathbf{1}_p\Big], & t = T,
\end{cases}
\tag{72}
$$

$$
\boldsymbol{\Psi}_{t*} = \begin{cases}
\left[\boldsymbol{\Omega}_0^{-1} + \boldsymbol{\Phi}\,\boldsymbol{\Psi}^{-1}\,\boldsymbol{\Phi} + \boldsymbol{\Sigma}_u^{-1}\right]^{-1}, & t = 1, \\
\left[\boldsymbol{\Psi}^{-1} + \boldsymbol{\Phi}\,\boldsymbol{\Psi}^{-1}\,\boldsymbol{\Phi} + \boldsymbol{\Sigma}_u^{-1}\right]^{-1}, & t = 2, \ldots, T-1, \\
\left[\boldsymbol{\Psi}^{-1} + \boldsymbol{\Sigma}_u^{-1}\right]^{-1}, & t = T.
\end{cases}
\tag{73}
$$

We generate a candidate $\boldsymbol{h}_t^{\dagger}$ from $\mathrm{N}(\boldsymbol{m}_{t*}, \boldsymbol{\Omega}_{t*})$, and accept it with probability $\min\{1, \exp(l(\boldsymbol{h}_t^{\dagger}) - l(\boldsymbol{h}_t))\}$.

A.2.3 Generation of $\boldsymbol{m}$

Noting that

$$\begin{aligned} E[\boldsymbol{y}_t \,|\, \boldsymbol{h}_t, \boldsymbol{h}_{t+1}, \boldsymbol{\theta}] &= \boldsymbol{m}_t + \mathbf{V}_t^{1/2}\mathbf{R}_t^{-1/2}\boldsymbol{\Lambda}'(\boldsymbol{\Psi} + \boldsymbol{\Lambda}\boldsymbol{\Lambda}')^{-1}\{\boldsymbol{h}_{t+1} - \boldsymbol{\mu} - \boldsymbol{\Phi}(\boldsymbol{h}_t - \boldsymbol{\mu})\}, & (74)\\ \mathrm{Var}[\boldsymbol{y}_t \,|\, \boldsymbol{h}_t, \boldsymbol{h}_{t+1}, \boldsymbol{\theta}] &= \mathbf{V}_t^{1/2}\,\mathbf{R}_t\,\mathbf{V}_t^{-1/2} - \mathbf{V}_t^{1/2}\mathbf{R}_t^{-1/2}\boldsymbol{\Lambda}'(\boldsymbol{\Psi} + \boldsymbol{\Lambda}\boldsymbol{\Lambda}')^{-1}\boldsymbol{\Lambda}\mathbf{R}_t^{-1/2\prime}\mathbf{V}_t^{1/2} \equiv \boldsymbol{\Gamma}_t, & (75) \end{aligned}$$

we define

$$\hat{\boldsymbol{y}}_t = \boldsymbol{y}_t - \boldsymbol{\Lambda}\mathbf{R}_t^{-1/2\prime}\mathbf{V}_t^{1/2}(\boldsymbol{\Psi} + \boldsymbol{\Lambda}\boldsymbol{\Lambda}')^{-1}\{\boldsymbol{h}_{t+1} - \boldsymbol{\mu} - \boldsymbol{\Phi}(\boldsymbol{h}_t - \boldsymbol{\mu})\}, \tag{76}$$

and consider the linear Gaussian state space model (28) and (29) with $\boldsymbol{\Gamma}_t$ in (75). We generate $\boldsymbol{m}$ simultaneously using a simulation smoother.

A.2.4 Generation of ϕ

In Section A.1.1, we replace $\boldsymbol{\mu}_{\boldsymbol{\phi}}$ and $\boldsymbol{\Sigma}_{\boldsymbol{\phi}}$ as follows.

$$\boldsymbol{\mu}_{\boldsymbol{\phi}} = \boldsymbol{\Sigma}_{\boldsymbol{\phi}}\boldsymbol{b}, \quad \boldsymbol{\Sigma}_{\boldsymbol{\phi}}^{-1} = \boldsymbol{\Psi}^{-1} \odot \mathbf{A},$$

where

$$\begin{aligned} \mathbf{A} &= \sum_{t=1}^{T-1}(\boldsymbol{h}_t - \boldsymbol{\mu})(\boldsymbol{h}_t - \boldsymbol{\mu})', \\ \boldsymbol{b} &= \text{diagonal}\left[\sum_{t=1}^{T-1}(\boldsymbol{h}_t - \boldsymbol{\mu})\left\{\boldsymbol{h}_{t+1} - \boldsymbol{\mu} - \boldsymbol{\Lambda}\,\mathbf{R}_t^{-1/2}\,\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t - \boldsymbol{m}_t)\right\}'\boldsymbol{\Psi}^{-1}\right]. \end{aligned}$$

A.2.5 Generation of $\boldsymbol{\mu}$

We generate $\boldsymbol{\mu}|\cdot \sim N(\tilde{\boldsymbol{m}}_{\boldsymbol{\mu}}, \tilde{\boldsymbol{\Psi}}_{\boldsymbol{\mu}})$, where

$$\begin{aligned} \tilde{\boldsymbol{m}}_\mu &= \tilde{\boldsymbol{\Psi}}_\mu\left[s_\mu^{-2}\boldsymbol{m}_{\boldsymbol{\mu}} + \boldsymbol{\Omega}_0^{-1}\,\boldsymbol{h}_1 + (\mathbf{I} - \boldsymbol{\Phi})\boldsymbol{\Psi}^{-1}\sum_{t=1}^{T-1}\left\{\boldsymbol{h}_{t+1} - \boldsymbol{\Phi}\,\boldsymbol{h}_t - \boldsymbol{\Lambda}\,\mathbf{R}_t^{-1/2}\,\mathbf{V}_t^{-1/2}(\boldsymbol{y}_t - \boldsymbol{m}_t)\right\}\right] \\ \tilde{\boldsymbol{\Psi}}_\mu &= \left[s_\mu^{-2}\,\mathbf{I} + \boldsymbol{\Omega}_0^{-1} + (T-1)(\mathbf{I} - \boldsymbol{\Phi})\boldsymbol{\Psi}^{-1}(\mathbf{I} - \boldsymbol{\Phi})\right]^{-1}. \end{aligned}$$

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