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The Anglo-Dutch Auction

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Abstract

I analyze discrete-valuation model of the Anglo-Dutch auction with exogenous entry and derive a complete ranking between this auction and its components - the ascending and first-price auctions. I find that the Anglo-Dutch auction can revenue-dominate for a small set of parameters, and similarly ranks revenue-last in a small number of cases. For most parameter values the Anglo-Dutch auction ranks as intermediate. I also show that the auction performs particularly well when bidders face small entry costs and almost-common values.

JEL Classification: D44

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The Anglo-Dutch auction introduced by Klemperer (1995, 1998) is both appealing in theory, and has been successfully applied in practice. This practical relevance has not, unfortunately, been matched by prevalence in economic literature, and it is the aim of this paper redress this imbalance by providing the first theoretical model that allows for a complete revenue and efficiency comparison of the Anglo-Dutch auction with both of its component auctions, the ascending and the first-price auction. In a three-bidder discrete-value setting, I show that the Anglo-Dutch auction is revenue-dominant for small range of parameters, but equally, it is rarely revenue-worst. The auction also performs particularly well when bidders face small entry costs and almost-common values. In terms of policy implications, my results suggest that such a hybrid auction may be a good compromise when the auctioneer is uncertain about the relative valuations amongst the bidders.

I describe the rules and properties of the Anglo-Dutch auction in Section 1, and introduce the discrete-valuation setup that underlies my analysis in Section 2. Equilibrium bidding strategies for the first-price, ascending and Anglo-Dutch auctions are derived in Sections 3, 4 and 5 respectively. I present analytical revenue rankings, for parameter values that admit such solutions, in Section 6, and proceed to present numerical results in Section 7. I discuss the efficiency characteristics of the three auctions in Section 8, while Section 9 concludes.

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1. Rules and Properties of the Anglo-Dutch Auction

An Anglo-Dutch auction for $K \geq 1$ objects, when each bidder demands only one unit, proceeds in two phases. First a simultaneous ascending (English or Japanese) auction is run, until all but K+1 bidders drop out. The price at which the last bidder drops out is noted, and it is set as the reserve price in a first price auction among the remaining K+1 bidders.[2] After these 'best and final' offers have been submitted, the bidders with the K highest bids are selected as winners, and each pays their bid.

There are three main areas where the Anglo-Dutch auction aims to improve over its alternatives: firstly in encouraging entry and thus boosting seller revenue, secondly in furthering efficiency and in preventing collusion. Thirdly, the Anglo-Dutch auction uses the first-price phase to take advantage of risk-aversion.[3]

The first-price stage at the end of the Anglo-Dutch auction encourages entry by the following logic. In a pure ascending auction advantaged bidders always get a chance to out-bid weaker rivals, hence weaker entrants can never win. This is not true in the Anglo-Dutch auction, where the first-price phase gives the entrants a chance to out-bid a stronger rival if he bids cautiously. Hence entrants have more of an incentive to participate in the auction.

The first-price phase of the Anglo-Dutch auction also helps to prevent collusion by making the final bids unobservable, and hence non-punishable by others. Even if bidders can observe each others' bids in the ascending phase, and even if they bid collusively in that phase, the final sealed-bid stage gives each bidder an incentive to renege on a collusive agreement, without the threat of punishment. Brunner et al. (2006) find supporting evidence for this phenomenon in an experimental setting.[4]

Information revealed in the ascending phase of the Anglo-Dutch auction is also useful to bidders when common values are present. Here the points at which each (non-winning) bidder drops out in the ascending phase provides the remaining bidders with useful information relating to the value of the object they are bidding for. This is particularly useful if bidders are risk averse and consequently concerned about winner's curse. The additional information reduces the cost of risk, alleviates winner's curse and stimulates more aggressive bidding.

Previous theoretical work on Anglo-Dutch auctions, by Azacis and Burguet (2005) and Bustos and Costinot (2003) has considered models with endogenous entry, but such models have been unable to produce a complete ranking between the ascending, first-price and Anglo-Dutch auctions. Both of these papers show that Anglo-Dutch auctions can revenue-dominate ascending auctions, and induce more entry. However, both also use models of asymmetry which favor the first-price auction over the ascending auction; it is thus unsurprising that the Anglo-Dutch auction out-performs its ascending counterpart, due to the presence of the first-price stage.[5]The interesting - and unanswered - question

[2]With respect to terminology, I will use the labels of "Dutch" and "first-price" auctions interchangeably, since the two auctions are strategically equivalent. I also use the phrases "English" or an "Anglo-Auction" to refer to the ascending auction.

[3]This paper offers a short description of how the auction achieves these effects; a more thorough discussion is presented in Marszalec (2011).

[4]The experimental paper by Abbink et al (2001), conversely, cannot find a significant difference between the performance of the Anglo-Dutch and multi-item ascending auctions. However, the paper does not specifically look for collusion effects.

[5]These results are also not robust to a small perturbation in the valuation model: when the incumbent has an arbitrarily small, but certain, valuation advantage (analogous to Klemperer's (1998) "almost-common values") no entry will occur in any purely ascending auction types, and the revenue from

in this context is how the Anglo-Dutch auction fares compared to the first-price.

My aim is to provide a full ranking of the Anglo-Dutch auction relative to both the ascending and first-price auctions using a valuation model that is robust to small perturbations. I construct a discrete valuation model where a single object is sold to three risk-neutral bidders, two of whom are 'weak' and one of whom is 'strong'. This setting could be considered to model an auction with a single incumbent and two entrants, or a market that has one clearly advantaged bidder. The model is biased against the Anglo-Dutch auction, as it excludes aspects such as entry and risk-aversion. I furthermore show that in cases in which the strong bidder has a significant value-advantage, the model will be favorable to the first-price auction, and when the strong bidder is relatively weak, the model favors the ascending auction. My setup is thus close to a worst-case assessment of the Anglo-Dutch auction's performance.

An analytical ranking of revenues is possible for some parameter models in my model, and for the remaining cases a comparison can be obtained numerically. I find that there exist parameter values for which the Anglo-Dutch auction revenue-dominates both its rivals - and for these parameter values it is also more efficient than the first-price auction. While I show that there is also a range of parameter values for which the Anglo-Dutch auction performs revenue-worst, this range is usually small, and for the majority of parameter values the auction ranks as intermediate, and hence performs well on average across a wide set of parameters. I also show that in a variant of my model where small entry costs are present, the Anglo-Dutch auction will perform particularly well, beating both the first-price and ascending auctions.

2. The Value Model

My model considers a situation where the weak bidders' valuations are random, but the strong bidder's value is known. I investigate how the level of the strong bidder's valuation impacts on the revenue from three auctions: the ascending (ASC), the first-price sealed-bid (FPS) and the Anglo-Dutch (AD) auctions. My analysis hence examines one main aspect Klemperer's (1998) two chief motivations for the Anglo-Dutch auction: how revenues are affected by the presence of a (possibly advantaged) incumbent.

I assume there are two weak (W), and one strong (S) bidder. The realized value (or type) of a weak bidder, v^w is either high (H) or low (L), with probability μ and $1-\mu$ respectively. The strong bidder's value, v, is common knowledge. Subsequent to the weak bidders' receiving their value signals, the three bidders participate in a one-off auction. I assume that bidders are risk-neutral, and do not face budget constraints. In what follows, I use b_i to denote bidder type i's bidding function.

While I assume that $v > L$, I do not insist that $v > H$, though for a non-degenerate first-price equilibrium to exist it will be necessary that $v > \bar{w} = \mu H + (1-\mu) L$: For a non-trivial analysis, the strong bidder's value must exceed the expected value of the weak bidders' value.[6] This model can be naturally interpreted as a model with a single advantaged incumbent and two entrants who have already decided to bid. There are no entry costs in the base model, though an extension that considers entry costs is discussed in Section 6.2.

Due to the discreteness of the valuation setup in my model, there exists the possibility that more than one bidder will submit exactly the same bid, and in this case a tie-breaking rule is necessary to determine a winner. The rule that I use is as follows:

ascending auctions will be minimal.

[6] See Appendix C, Section 12.1.

- if the tied bidder have the same value, the object is assigned randomly between them with equal probability
- if an L-type weak bidder ties with either an H-type weak bidder, or a strong bidder, the L-type weak bidder loses
- if an H-type weak bidder ties with a strong bidder, the weak bidder wins.

If the weak bidders are viewed as entrants, and the strong bidder as an incumbent, this tie-break rule embodies a slight pro-entrant bias if an entrant and incumbent are tied. In the other other two cases it is either neutral, or supports the higher value bidder.[7]

3. The First-Price Sealed Bid Auction

Given the nature of asymmetry used in my model, both a pure strategy and a hybrid equilibrium is possible. Depending on where the precise value of v is relative to an interval $[v_\alpha, v_\beta]$, defined below, one of three cases will apply. When $v > v_\beta$, then v will be a lot higher than $\bar{w}$ and a pure strategy equilibrium will exist. Here the strong bidder's value is so high that he doesn't want to risk competing with the weak bidders at all, and prefers to win for sure by bidding $b_S = H$; the weak bidders then bid their value, and never win.[8]

When $v \in [v_\alpha, v_\beta]$, the strong bidder's value is moderate, and a hybrid equilibrium will exist. Here the L-type weak bidder will always bid his value (L), while the H-type weak bidders and the strong bidder mix over an interval.This interval must be common to both of these bidder types by the following reasoning. Suppose one of the bidders bids over a closed interval. Then the rival has no incentive to bid above the supremum of that interval (since that only decreases his expected surplus when he wins). Submitting a bid below the infimum of such an interval, conversely, would never win. Hereby the mixing interval must be the same for both types. By standard arguments, I can rule out the presence of atoms at the supremum of the mixing interval, as well as in the interior of the interval itself. Similarly, I can rule out the case where both types' distributions have an atom at the infimum of the interval, but I cannot exclude the case in which at most one type has such an atom.[9]

Finally, when $v < v_\alpha$ the strong bidder's value is very low, and he will not compete with the H-type bidders at all. Instead the strong bidder bids $b_S = L$, hoping to win in

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[7] There are two other intuitions for justifying this tie-breaking rule.

Firstly, it can be considered analogous to a price-setting mechanism in a Bertrand game with players with different costs.

In a two-player version of this game, the price is set equal to the marginal cost of the higher-cost player, and the low-cost player attracts all the buyers. What is usually meant in this situation is that the low-cost player actually undercuts the other player by some tiny ε, without specifying the magnitude of ε.

Secondly, the situation can be viewed in terms of pricing on a discrete grid. Preempting later parts of the paper, I note that the tie-breaking rule is particularly important since the bidder whose valuation is common knowledge will have an atom in his bidding distribution at L. The tie-breaking rule is constructed in such a way that if prices were set on a discrete grid, the strong bidder would have an atom at L, and mix thereafter, while the H-type weak bidder would mix over a range which starts from L + (1 increment).

[8] In both of these pure strategy equilibria the strong bidder wins the auction due to the tie-breaking rule. However, in both cases the dependence on tie-break rule can be removed by specifying that the strong bidder bids $H + \varepsilon$ instead.

[9] If both players did have an atom at the infimum, one of them could deviate such that his mixing distribution starts just an ε above the opponent's atom. This reduces the expected surplus only by ε, but increases winning probability by much more (since now the deviating bidder now beats his rival's atom).

the eventuality that both weak bidders have a value of L. The H-type bidders then bid according to a mixing distribution over an interval. Proposition 1 summarizes the three possible types of equilibria.

Proposition 1. *Let the boundaries v_α and v_β be defined by:* $v_\alpha = H - \frac{(1-\mu)^2}{1+(1-\mu)^2}(H-L)$ *and* $v_\beta = H + \frac{(1-\mu)^2}{1-(1-\mu)^2}(H-L)$. *Then equilibrium of the FPS auction, for a given μ is characterized as follows:*

When $v \in [v_\alpha, v_\beta]$:

- *Type L weak bidders bid L. Type H weak bidders and the strong bidder mix over an interval* $\left[L, \bar{b}\right]$, following the distributions G_H and G_S, respectively:

$$G_H(b) = \frac{1-\mu}{\mu}\left(\frac{\sqrt{v-L}-\sqrt{v-b}}{\sqrt{v-b}}\right) \qquad G_S(b) = \frac{1}{1-\mu}\frac{\left(H-\bar{b}\right)}{\sqrt{v-L}}\frac{\sqrt{v-b}}{(H-b)} \tag{1}$$

$$\text{where } \bar{b} = v-(1-\mu)^2(v-L)$$

- *The expected revenue is:*

$$R_F = v-(1-\mu)^2(v-L)-(1-\mu)\left(H-\bar{b}\right)(v-L)^{\frac{1}{2}}\int_L^{\bar{b}}\frac{1}{(H-t)(v-t)^{\frac{1}{2}}}dt \tag{2}$$

When $v \in [L, v_\alpha)$

- *Type L weak bidders bid L*, and the strong bidder also bids L.
- *Type H weak bidders bid according to the distribution* $G_{H\alpha}(b)$, over the interval $[L, L+\mu(H-L)]$:

$$G_{H\alpha}(b) = \frac{1-\mu}{\mu}\frac{b-L}{H-b}$$

- *The expected revenue is :*

$$R_{F\alpha} = (1-\mu)^2\left[L+(H-L)\int_L^{L+\mu(H-L)}\frac{(H-L)+(t-L)}{(H-t)^3}tdt\right]$$

When $v > v_\beta$:

- *Type L weak bidders bid L, Type H weak bidders bid H*, and *the strong bidder bids H*,
- The *expected revenue is H.*

Proof. See Appendix C, Section 12. □

In the case where $v \in [v_\alpha, v_\beta]$ the strong bidder has an atom at L. Its derivative with respect to μ is given by:

$$\frac{d\left(G_S(L)\right)}{d\mu} = \frac{1}{(1-\mu)^2(H-L)}\left(H+L-2v-(2-\mu)\mu(L-v)\right)$$

This will always be negative when $v > H$, so the weight of the atom is decreasing in μ when the strong bidder's valuation is high. An increase in μ makes it more likely that the strong bidder is bidding against H-type weak bidders, and hence gives the strong bidder an incentive to bid more aggressively. However, when $v \in (v_\alpha, H)$, the mass of the atom varies non-monotonically with μ : the derivative is negative for small μ, and positive for large μ. This is because when μ is high, the competition from the H-type weak bidders is so intense that the strong bidder prefers to not compete with them: even if the strong bidder would win, the expected surplus would be small. Thus the strong bidder will instead play L with a higher probability and hope for a larger surplus in the case in which both weak bidders turn out to be type L. In the limit, when μ is sufficiently high (and $v < v_\alpha$), it becomes equilibrium behavior for the strong bidder to bid L for sure.

4. The Ascending Auction

In an ascending auction, the optimal strategy for each bidder is to bid up to their true value. The auction stops once all but one bidders have dropped out, whereby the object is sold to the highest value bidder, for a price equal to the bid of the bidder with the second highest value. I assume that if the two top bidders share the exact same value (i.e. they would quit simultaneously at some price, leaving no bidders in the auction), the object is assigned randomly among them at the prevailing price.

Depending on the position of v relative to H, there are two possible outcomes. Firstly, when $v \in (L, H]$, if both weak bidders are of type L, the auction stops at price L, with the strong bidder winning the object. If precisely one weak bidder is of L type, then the auction proceeds up to the strong bidder's valuation, v, and terminates there, with the H-type weak bidder obtaining the object. Finally, if both weak bidders are of type H, the auction will terminate at H.

Second, when $v \in (H, \infty)$, the strong bidder is dominant and will always out-bid the weak bidders in the ascending auction. Thus the winning price will be determined by the highest realized valuation held by a weak bidder. With probability $(1-\mu)^2$, both weak bidders have valuation L, and with probability $(1-(1-\mu)^2)$ at least one of them has value H. Summarizing the above two arguments, Proposition 2 gives the expected revenue from the ascending auction.

Proposition 2. *The revenue from an ASC auction is:*

$$R_{ASC} = (1-\mu)^2 L + 2\mu(1-\mu)\min(v, H) + \mu^2 H \tag{3}$$

When $v > H$, in this auction, nobody except the strong bidder expects positive surplus, and then the revenue does not depend on the particular value of v. This is because the strong bidder will never actually drop out in an ascending auction when $v > H$.

5. The Anglo-Dutch Auction

Modeling the Anglo-Dutch auction has two main differences from the first-price auction: firstly, only two of the three bidders will be present in the final bidding stage, and secondly, the remaining bidders will have more information (conditional on being in the Dutch stage) since they will have seen which of the bidders has dropped out. This observed drop-out point also serves as a reserve price in the Dutch phase of the auction.

I assume that if two bidders drop out simultaneously in the ascending phase of the Anglo-Dutch auction, one of these two will be selected at random to play in the Dutch phase. Furthermore, I assume that the remaining bidder does not know whether or not his opponent in the Dutch phase previously tried to drop out.

The first observation in deriving the equilibrium is that the L-type bidder will always bid up to L in the ascending phase of the Anglo-Dutch auction - and provided he is allowed into the second stage, he will also submit a bid of L. Any bid larger than that gives negative expected surplus. However, since the strong bidder always has a value greater than L, the L-type will never actually win.

Secondly, I observe that the Anglo phase of the auction will never terminate above the level of $\min(v, H)$. If an H-type weak bidder is still present in the Anglo-Dutch auction when the Anglo-phase terminates at $\min(v, H)$, he knows for sure that the rival he is facing has a valuation of at least H.[10] In such a situation the H-type bidder must bid H in the first-price phase of the auction. Similar reasoning applies to the strong bidder. If the auction terminates at $\min(v, H)$, then provided that the strong bidder is admitted to the first-price stage, he will submit a sealed bid of $\min(v, H)$.[11]

In the only remaining case, the ascending phase terminates at L. This can occur either because one of the weak bidders is an L-type, or both of them are. As in the first-price auction, there are two possible types of equilibria, depending on how high the strong bidder's value is. If v is very high, a pure strategy equilibrium will prevail in the first-price stage of the Anglo-Dutch auction: the strong bidder will bid H, and win the item for sure, while the H-type weak bidder also bids H. If the value of v is not extremely high, both the strong bidder and the H-type weak bidder will mix. Proposition 3 summarizes the two types of equilibria fully.

Proposition 3. *Let the boundary value v_γ be defined by: $v_\gamma = H + \frac{1-\mu}{2\mu}(H - L)$. Then the equilibrium strategies in an AD-auction are:*

When $v \in (L, v_\gamma)$ *(i.e. $b^* \leq H$)*

- *The L-type weak bidder bids up to L in the ascending phase, and bids L in the Dutch phase*
- *The strong bidder bids up to v in the ascending phase. If the ascending phase terminates at $\min(v, H)$, the strong bidder submits a bid of $\min(v, H)$ in the sealed-bid stage. If the ascending phase terminates at L, the strong bidder will submit sealed bids according to the distribution:*

$$G_S^*(b) = \frac{H - b^*}{H - b} \tag{4}$$

- *The H-type weak bidder bids up to H in the ascending phase. If the ascending phase terminates at $\min(v, H)$, the H-type bidder submits a sealed bid of H in the Dutch*

[10]This is because if the Anglo phase terminates at v, it must be the case that $v < H$, and both remaining bidders have a true valuation of H. If, on the other hand the Anglo-phase terminates at H, then $v \geq H$ for sure, and so the H-type bidder will be facing a strong bidder with valuation $v \geq H$ in the second round.

[11]If the ascending phase terminates at v, it must be true that $v < H$, and the strong bidder is not admitted to the second round. If the ascending phase terminates at H, then the bidder knows that his opponent in the second round has valuation H. In this case, due to the tie-breaking rule, the strong bidder can guarantee winning the object by bidding $H = \min(H, v)$.

phase. If the ascending phase terminates at L, the H-type bidder will submit sealed bids according to the distribution:

$$G_H^*(b) = \frac{1-\mu}{2\mu}\left(\frac{b-L}{v-b}\right) \tag{5}$$

- *If the ascending phase stopped at v, in the Dutch phase the H-type weak bidder bids H*
- *The expected revenue is:*[12]

$$\begin{aligned} R_{AD} &= \mu^2 H + \left(1-\mu^2\right) v - (1-\mu)^2 (v-L) \\ &\quad - (1-\mu)^2 (H-b^*)(v-L)\int_L^{b^*} \frac{1}{(H-t)(v-t)} dt \end{aligned} \tag{6}$$

- Where

$$b^* = v - \frac{(1-\mu)^2}{1-\mu^2}(v-L)$$

When $v > v_\gamma$ *(i.e.* $b^* > H$) :

- *Type L of weak bidder bids L in both the ascending and Dutch phases of the auction*
- *Type H of weak bidder bids up to H in the ascending phase, and bids H in the Dutch phase*
- *Strong bidder bids up to v in the ascending phase, and bids H in the Dutch phase*
- *The expected revenue is H.*

The expression allows me to formulate a proposition that I will use in intuitive explanations of my revenue rankings.

Proof. See Appendix D, Sections 13. □

Proposition 4. *If for given parameter values a mixing equilibrium exists in the first-price auction, then a mixing equilibrium also exists in the Anglo-Dutch auction, for the same parameters.*

Proof. Immediate from the fact that $b^* \leq \bar{b}$. □

Corollary 5. *There will exist sets of parameter values such that a mixing equilibrium exists in the Anglo-Dutch auction, but not in the first-price auction. When $v > v_\alpha$, for the parameters where a mixing equilibrium does not exist in the first-price auction, the strong bidder always bids H in that auction.*

The corollary captures the intuition that in expectation the strong bidder faces stricter competition under an outright first-price auction, than in Anglo-Dutch. In the first-price, the strong bidder know that he can be facing up to two H-types, whereas in Anglo-Dutch, conditional on getting to the sealed-bid phase, he will face at most one H-type. Thus

[12] *Derivation in Appendix 13.4.*

in the first-price auction the strong bidder will switch to the "always bid H" equilibrium under a broader range of parameters.

Looking at the behavior of the strong bidder's atom in the Anglo-Dutch auction, I find that:

$$\frac{d\left(G_S\left(L\right)\right)}{d\mu} = -\frac{2}{\left(1+\mu\right)^2\left(H-L\right)}\left(v-L\right) < 0$$

This shows that the mass of the strong type's atom is always decreasing in μ, so the Anglo-Dutch auction does not exhibit the kind of non-monotonicity in the behavior of the atom's mass as did the first-price auction.

It is straightforward to show that G_H first-order stochastically dominates G_H^*, which suggests that H-type bidders bid more aggressively in the first-price auction, provided that a mixing equilibrium exists in both the Anglo-Dutch and first-price auctions. If a mixing equilibrium occurs in the Anglo-Dutch auction, then the H-type weak bidder knows he is only bidding against one strong bidder, for sure. However, in a mixing equilibrium in the first-price auction an H-type knows he is facing a strong bidder for sure, but with probability μ he will also be facing another H-type weak bidder - thus an H-type expects more competition in the first-price auction, and so bids more aggressively. No similar stochastic dominance ranking is available for the distributions of the strong bidder's bids, G_S and G_S^*; the relative shapes of these two distributions depend on μ and v, and for most $(\mu, v)-$ pairs the two distribution functions intersect at some $b \in (L, b^*)$.

6. Analytical Revenue Comparisons

The revenue functions for the first-price and the Anglo-Dutch are not analytically integrable in general, but for specific parameter values some analytical comparisons can be made, as indicated by Propositions 6 to 6.

Proposition 6. *When $v = H$, ASC generates most revenue, followed by AD. The FPS auction gives least revenue.*

When $v \in [v_\beta, v_\gamma)$, such that $\bar{b} \geq H$, FPS gives higher revenue than both ASC and AD.

When $v > v_\gamma$, such that $b^* \geq H$, AD and FPS do equally well, and both give higher revenue than ASC.

Proof. See Appendix E, Section 14.4. □

In Proposition 6 the ascending auction gives revenue of H in all cases where at least one weak bidder is H. The Anglo-Dutch auction only gives revenue of H if both weak bidders are H, and the first-price auction never gives such high revenue. This effect dominates over the higher revenues given by ASC and FPS when more of the weak bidders are of L-type. Propositions 6 and 6 relate to the switch-over points in first-price and Anglo-Dutch auctions respectively. When $v \in [v_\beta, v_\gamma)$, the strong bidder switches to always bidding H in the first-price auction, while this has not yet occurred in the Anglo-Dutch auction. Hence the first-price revenue is always H, which is more than in the other auctions.[13] At the point where $v > v_\gamma$, the strong bidder always bids H in the Dutch phase of the AD auction also, giving the same expected as in the first-price. While I have presented Propositions 6 to 6 in terms of cut-off values for v, a dual set of propositions could be presented in terms of μ, as these parameters play a dual role in determining $\bar{b}$ and b^*.

[13]The relative position of ascending and Anglo-Dutch auctions in this case is ex-ante ambiguous, and depends on the model's parameters, as discussed in Section 7.

6.1. Relating the Results to Maskin and Riley (2000)

The intuition behind the revenue results in my model can be explained using the framework of Maskin and Riley (2000) on asymmetric auctions. They consider thee types of asymmetries, two of which are relevant to my model. When the strong bidder's value distribution is a stretch of the weak bidder's distribution, the authors find that the first-price auction out-performs the ascending auction. However, when the weak bidder's distribution has been obtained from the strong bidder's distribution by shifting some mass from the upper end to the lower end, the ascending auction performs better.

In my model, the weak bidder has a binary distribution on (L, H), and the strong bidder has a (degenerate) distribution at the point v. To obtain the strong bidder's distribution from the weak bidder's distribution, I can first "stretch" the upper end of the weak bidder's distribution such that it terminates at v, rather than H. Second, I move $(1-\mu)$ of mass from the lower end (L), to the upper end of the distribution, v. This two-step transformation gives a degenerate distribution at v. The magnitude of the stretch-effect will depend on the difference between H and v, while the magnitude of the mass-reallocation effect grows as μ becomes smaller, as then more mass is shifted to v.

The Maskin and Riley framework can thus explain why the first-price auction performs better when v is larger. In the case when $v > H$, the stretch effect, and the mass-shift effect, both work in a way that favors the first-price auction; the effects become more favorable as v increases. On the other hand, when $v < H$, the two effects work in opposite directions: the stretch effect now works to "decrease" the upper end of the distribution, which favors an ascending auction, but the mass-reallocation effect moves $(1-\mu)$ of probability from L to $v > L$, and so still favors the first-price auction. Thus the first-price auction is likely to do particularly badly when v is low. So in my model, when v is large, the first-price auction is favored by the value structure; when v is low, the value-structure favors the ascending auction. The results from my model are consistent with those derived in Maskin and Riley (2000).

6.2. Small Entry Costs and Almost-Common Values

It is easy to slightly modify my model in a way that makes the AD auction rank first, by introducing small entry costs for the weak bidders. Recall that for the case when $v = H$, the revenue ranking is $R_F < R_{AD} < R_{ASC}$. Suppose that the two weak bidders in my model are instead "potential entrants", and they have to pay a small entrance cost c to participate in the auction and observe their value, while the strong bidder pays no such cost. There will be no entry in the ascending auction, since none of the entrants have a positive surplus conditional on entering; the revenue from a no-reserve ascending auction will be minimal. However, for c small enough, both entrants will enter in both the Anglo-Dutch and first-price auctions. Conditional on having both bidders participating, I have shown in Proposition 6 that in this case the Anglo-Dutch auction outperforms the first-price auction with respect to revenue, and so would rank as first in this restricted context. Since the expected revenue functions from the first-price and Anglo-Dutch auctions are continuous in v, the above argument also extends to the case when $v = H + \varepsilon$, in which case I have an "almost-common value" model with an advantaged incumbent. My results thus show that the Anglo-Dutch auction performs particularly well when entry costs and almost-common values are an issue.

7. Numerical Revenue Comparisons

The analytical results above give only a partial picture of the performance of the Anglo-Dutch auction against the other two competitors. Indeed, for some cases only a partial ranking was possible. In this section I present two kinds of graphs which more broadly exhibit the behavior of the three auctions at hand.[14] The first kind of graph shows how revenues vary with v changing, for a given fixed values of μ, L and H. The second kind of graph shows how revenue behaves when μ is varied, for a particular fixed set of L,H, and v. While my graphs are drawn with L and H fixed at 0 and 1, respectively, this can be assumed without loss of generality: choosing different (L,H) pairs would only stretch and shift the graphs. Qualitatively, the graphs would have exactly the same shape, and the same relative relationships would hold.

7.1. Revenue Varying with v, with other Parameters Fixed

I will plot the revenues for varying values of v, $c = L$. In my first plot I select μ to be relatively high ($\mu = 0.8$). Based on the analytical results from Section 6, I would expect that the Anglo-Dutch auction performs quite poorly for this parameter value. Note also how the behaviour of the first-price auction changes depending on whether $v < v_\alpha$: this jump in behavior is due to the strong bidder essentially exiting the market, and not actively trying to win against an H-type.

Indeed we see that the for these parameter values the Anglo-Dutch auction never performs best, and ranks as the worst of the three auctions for v approximately in the range $[1.02, 1.07]$. However, when v is large enough (e.g. greater than 1.126), the optimal strategy for the strong bidder in both the first-price and Anglo-Dutch auctions is to bid simply H, and so for very large v the first-price and Anglo-Dutch auctions both outperform the ascending auction.

When μ is decreased, the range over which the Anglo-Dutch auction performs worst becomes relatively smaller. This is shown in Figures1 and 2.

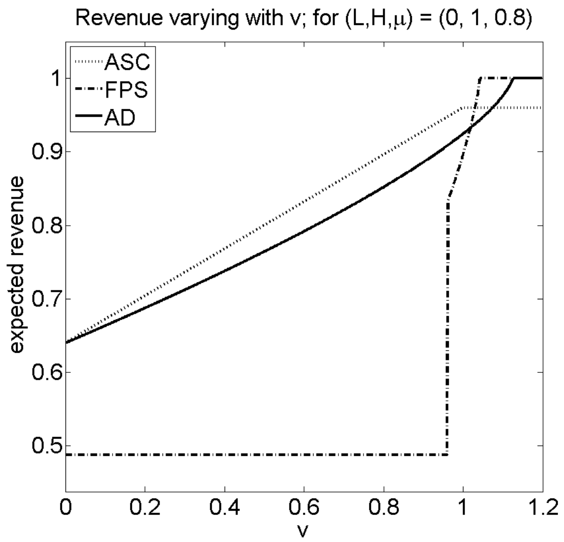


Figure 1: AD performs poorly when μ is high.

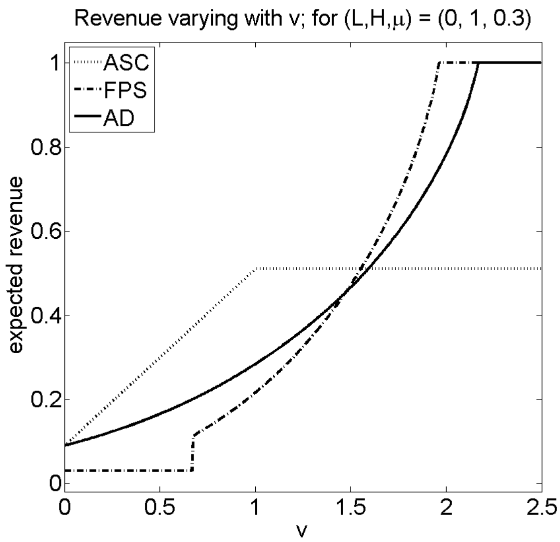


Figure 2: Performance of AD improves when μ decreases

When μ becomes even smaller, the Anglo-Dutch auction will rank first for a range of v. The largest value of μ for which this occurs is $\mu = 0.16$ (not shown). I display below, in Figure 3, a case in which the superiority of the Anglo-Dutch auction is visible more clearly.A close-up of the range over which the Anglo-Dutch auction revenue-dominates its rivals is given in Figure 4.

[14]All numerical computations have been performed using Matlab.

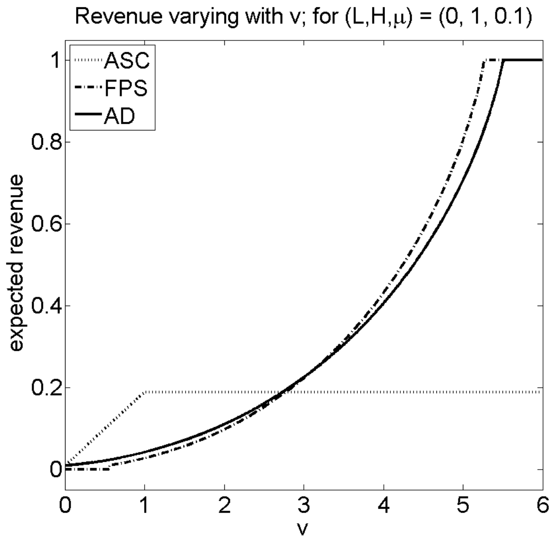


Figure 3: AD can be revenue-dominant for small μ

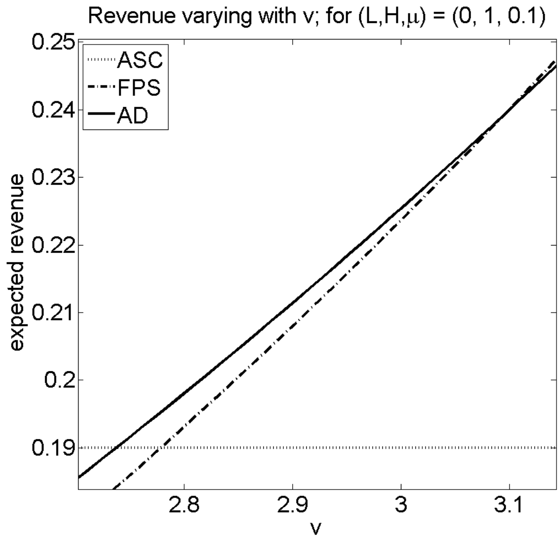


Figure 4: Close up of parameter space where AD is revenue-dominant

As suggested by my analytical comparisons, it is thus possible to obtain a set of parameters for which the Anglo-Dutch auction revenue-dominates both the first-price and the ascending auction. These parameter values require a relatively high value of v, and a low value of μ, for a given pair of H and L. Under such parameters, the ascending auction performs poorly, since the revenue in that auction never depends on the value of v. In the auctions which have a first-price element, however, a higher value of v leads to more aggressive bidding by the strong bidder - and this is so even if, in fact, his opponent's signal realization is low.

7.2. Revenue Varying with μ, with Other Parameters Fixed.

This section will show how the three auctions perform relative to each other when the probability of a weak bidder's being an H-type is changed. I start with the case when the strong bidder's valuation is between that of the two weak bidders. For this purpose, I choose $v = 0.95$, shown in Figure 5.

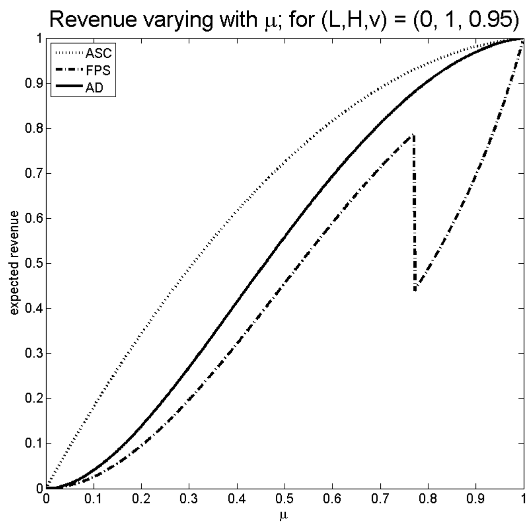


Figure 5: AD ranks intermediate when $v < H$.

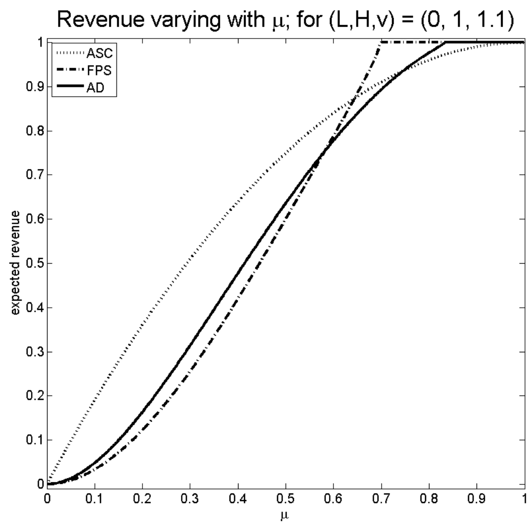


Figure 6: When $v < H$, auctions with a first-price stage can dominate.

This shape of the graphs is not specific to my choice of $v = 0.95$, but rather generic to all values of $v \in (L, H)$. We see that the Anglo-Dutch auction always ranks above the first-price auction, but below the ascending auction. The downward jump in revenue of the first-price auction is due to the strong bidder stopping to compete with the H-types: as μ increases with $v < H$, it is more likely that we end up in the $v < v_\alpha$ case. With high μ and low v, the strong bidder prefers to bid very low, and hope to make a profit if both of his opponents turn out to be L-types. The consequent fall in revenue is substantial.

When I select a value of $v > H$, there will be a range of values of μ for which the auctions with a first-price component dominate the ascending auction. Figure 6 depicts such a case, where $v = 1.1$. When μ is very large, both the Anglo-Dutch and first-price auctions dominate the ascending auction, however, there is also a range of $\mu-$ values for which the Anglo-Dutch auction performs worst.

Finally, by picking v appropriately, I can also generate a range of $\mu-$values for which the Anglo-Dutch auction outperforms its rivals. Figure 7 illustrates one such case, with$v = 2.8$. A close-up of the section where the Anglo-Dutch auction outperforms the other two auctions is presented in Figure 8.

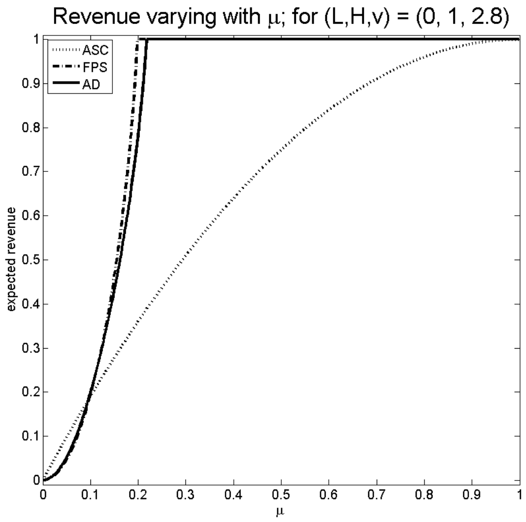


Figure 7: AD can be revenue-dominant for a range of μ.

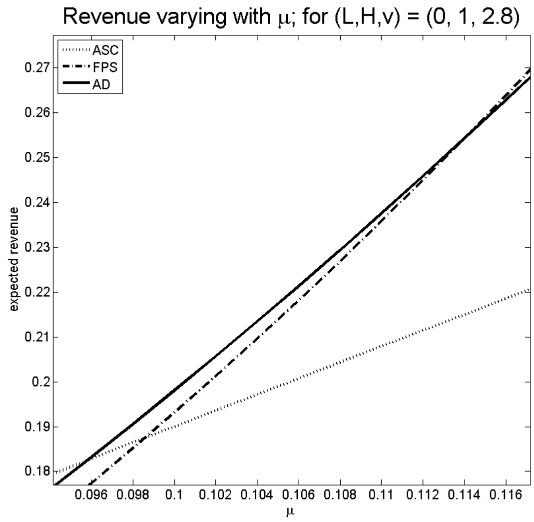
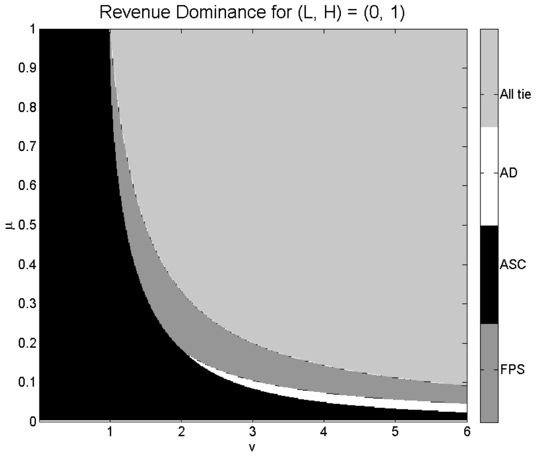


Figure 8: Close-up of AD's dominance range.

7.3. The Overall Picture

To summarize the findings of the two previous sub-sections, in Figures 9 and 10 I present a pair of contour plots showing which auction is revenue-best or revenue-worst, for each pairing of μ and v. The set of parameters over which the Anglo-Dutch auction is revenue-dominant is comparatively small, but equally, it is also ranks worst in a similarly small area.



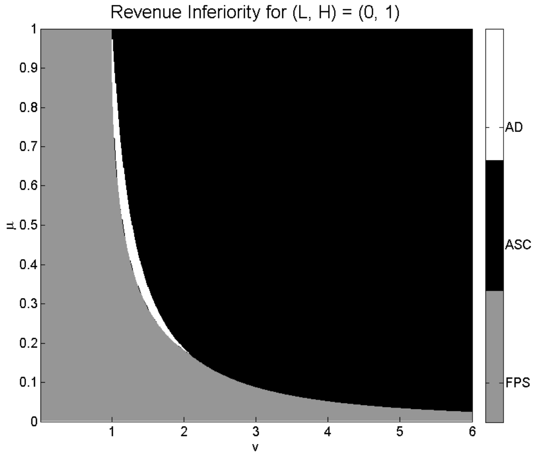


Figure 9: Areas of revenue-dominance, by auction. AD is dominant for a small subset of parameters only.

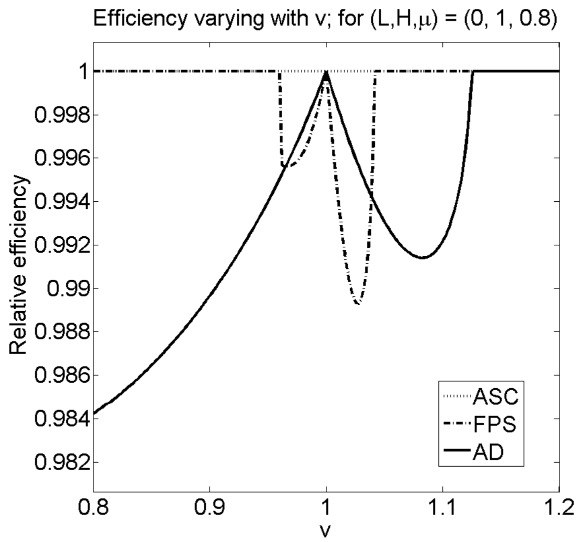
Figure 10: Areas of revenue-inferiority, by auction. AD performs worst for a similarly small subset of parameters.

Note that as v increases,the areas where the Anglo-Dutch auction dominates grows, while it never performs worst when v exceeds 2. While the other two auctions are revenue-dominant for much greater sets of parameters, as suggested by Figure 9, they also rank last in a larger number of cases (as shown in Figure 10). Furthermore, if small entry costs were present, the Anglo-Dutch auction would also be revenue-dominant in the whole area where the ascending auction is now dominant (and the ascending auction would, correspondingly, rank revenue-last, instead of the first-price). Overall, the Anglo-Dutch auction is rarely best, but even more rarely worst. In the context of my model, if there is some uncertainty on the auctioneers part as to what the relevant parameters are, picking such an auction could be a sensible policy decision.

8. Efficiency Comparisons - Numerical

In this section I compare the relative efficiency of the three auctions, using numerical analysis as in Section 7. I interpret the notion of efficiency in the same way as do Bustos and Costinot (2003), in that I measure efficiency by the expected value of the winning bidder's valuation. To obtain a "relative" measure, I take these expected-valuation numbers from the Anglo-Dutch and first-price auctions, and divide them by the valuation of the winning bidder in the ascending auction. I know ex-ante that the ascending auction will always be efficient, and in my relative comparison it will always obtain relative efficiency of 1; the other two auctions will rank below that. I should also note that at the point $v = H$, all of the auctions will be equally efficient, since irrespective the identity of the winner, he will have a valuation of H - this is an artifact of the discrete valuation model. All auctions will also be efficient when $v > v_\gamma$, since in that case the strong bidder has the highest valuation by far, and always wins in all three auctions.

Figure 11 shows that when μ is high, the Anglo-Dutch auction may be more efficient than the first-price auction for some values of $v < H$, and is certainly more efficient for moderately high $v > H$.



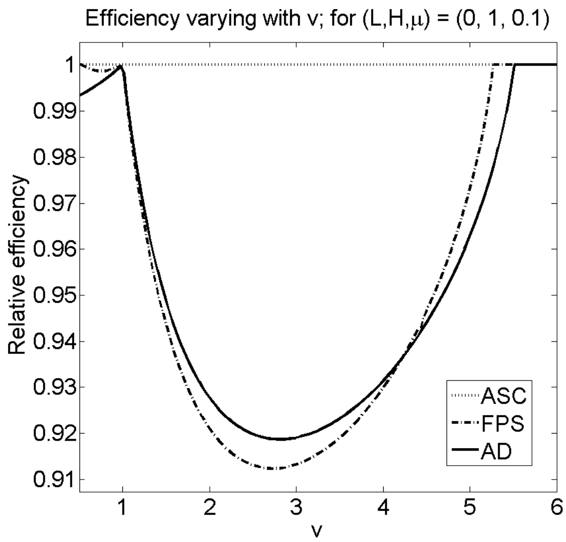
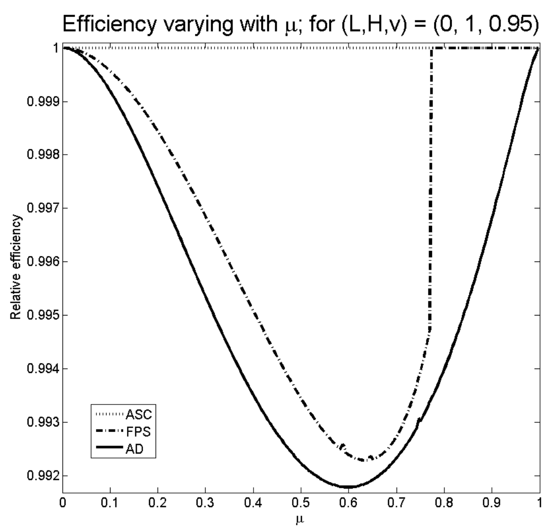


Figure 11: Efficiency comparison with a high μ. No clear dominance pattern emerges.

Figure 12: Efficiency comparison with μ, where AD outperforms FPS for most $v > H$.

When μ is decreased, we see from Figure 12 that the Anglo-Dutch auction is less efficient than the first-price for $v < H$, but is still more efficient for a range of $v > H$. In the range where the Anglo-Dutch revenue-dominates its rivals, it is also more efficient than the first-price.

From figure 13 we see that for $v < H$, the first-price auction will be more efficient than the Anglo-Dutch auction for most μ.



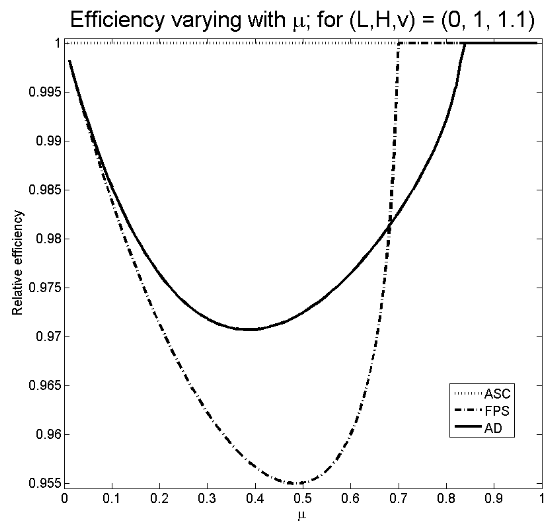


Figure 13: AD is relatively inefficient in the case when $v < H$.

Figure 14: When $v > H$, AD is relatively efficient for most μ.

Figure 14 shows that the conclusions are reversed when $v > H$, and for most μ-values the Anglo-Dutch auction is more efficient than the first-price.

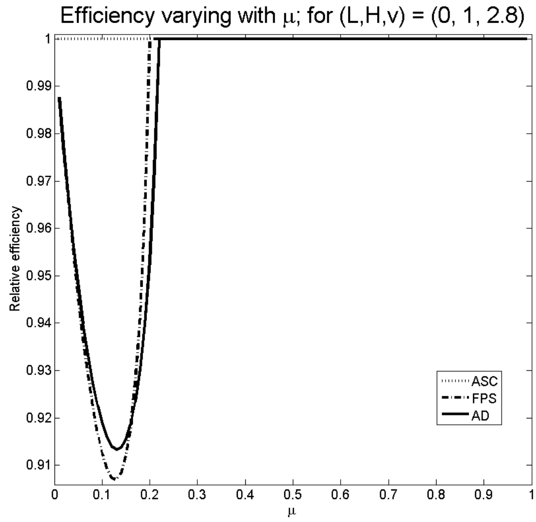


Figure 15: AD is more efficient than FPS at those parameters which make AD revenue-dominant.

In Figure 15 we see that with v fixed at $v = 2.8$, in the range in which the Anglo-Dutch auction revenue-dominated the first-price auction (for $\mu < 0.16$), the Anglo-Dutch auction is also more efficient.

The relative efficiency of the Anglo-Dutch and first-price auction appear sensitive to my assumption of discrete values. I would expect that the Anglo-Dutch is more efficient than the first-price, because in the Anglo-Dutch auction it is always one of the two highest-valuing bidders who win. However, when v is very large, the equilibrium in the first-price auction more readily switches to $\min(v, H)$ being played by the strong bidder, and H being played by the H-type - so the first-price becomes more efficient. This effect also dominates for values of v (and μ) for which v is slightly less than v_β. The argument for the case when $v < H$ is analogous to the argument for revenue: the Anglo-Dutch auction performs better than the first-price for the cases when both weak bidders obtain a value of H, but performs worse in the case when one weak bidder is H and the other is L, since the H-types will bid more aggressively in the first-price auction. The relative magnitude of these two effects depends on the parameters of the model, and either one can dominate, as can be seen by comparing Figures 1-10 and 1-11.

8.1. A Short Comment on welfare

Since I am using the expected valuation of the winning bidder as a measure of efficiency, I can easily consider a particular kind of welfare function, which considers both efficiency and revenue. As used by Bustos and Costinot (2003), the form of the welfare function can be:

$$\lambda(\text{revenue}) + (1 - \lambda)\,(\text{efficiency})$$

Using this welfare function, I can consider the question for what kind of welfare-preferences is the Anglo-Dutch auction a good idea. Since my model does not allow endogenous entry, if I wish to find a case where the Anglo-Dutch auction welfare-dominates its rivals, I must look at a case where the Anglo-Dutch auction is revenue-superior. We have a situation like this depicted in Figure 4, and from there we can see a set of parameters which is favorable to the Anglo-Dutch auction. In general, if the Anglo-Dutch auction revenue-dominates its rivals, then for λ sufficiently close to 1, it will also welfare-dominate. The range of λ for which this occurs is usually small. For example, in the case when $L = 0$, $H = 1$, $\mu = 0.1$ and $v = 3.05$ (as depicted in Figure 1-4), the Anglo-Dutch auction is welfare-superior for values of λ in the interval $\lambda \in [0.86, 1]$.

My results thus indicate that when endogenous entry is not modeled, the Anglo-Dutch auction is most likely to be welfare-best in cases where revenue is considered more important than efficiency. However, I should also note that while Anglo-Dutch auction is quite rarely welfare-best, it is also very unlikely to rank as welfare-worst. Thus if I performed across-model-averaging, in that I would postulate some distributions on the μ and v parameters, the Anglo-Dutch auction may perform best on average, though it is not necessarily always best in a case-by-case comparison. In this sense it is more robust than its competitors, and could be preferred by a policymaker who does not know the model's parameters exactly.

9. Conclusions

In this paper I constructed a theoretical model which did not entail most of the elements which would inherently favor the Anglo-Dutch auction. Most significantly, I did not model endogenous entry or risk aversion - so my model to produce results which are likely to be conservative with respect to the performance of the Anglo-Dutch auction. The assumptions of discrete valuations and exogenous entry allowed me to rank all three auctions of interest, which is an important contribution to the existing literature, since previous work has only produced rankings of the Anglo-Dutch auction relative to the ascending auction.

I showed analytically that the Anglo-Dutch auction ranks above one of its rivals under particular conditions, and numerical analysis showed that the auction can dominate both of its component-auctions for a range of parameters, and is relatively efficient in this case. While there are parameter values for which the Anglo-Dutch auction ranked as worst, the range of these parameters is narrow, and the Anglo-Dutch auction usually ranks as intermediate; this conclusion was robust to slight changes of the structure of my base model. For this reason I proposed that the Anglo-Dutch auction may be a good alternative for a policymaker who is uncertain as to the exact values of the model's parameters.

In the base model the range of parameters in which the Anglo-Dutch auction performed very well was narrow. However, if I extend the model to consider small entry costs, it is easy to obtain examples in which the Anglo-Dutch auction would dominate its rivals - the assumption of almost-common values falls in this category. While extending my base model to consider budget constraints, and an alternative process of information revelation, did not improve the relative performance of the Anglo-Dutch auction, the auction appears quite robust. Across all variants of my model, I found the range of parameters for which the Anglo-Dutch auction performs worst of all three auctions is usually small, or non-existent. Thus if a policymaker has a relatively inaccurate prior information as to the likely true parameters of the model, it is possible that Anglo-Dutch auction could perform better on average than both of its rivals.My extension of the base model to consider small entry costs and almost-common values provided us with another setting in which the Anglo-Dutch auction performed particularly well.

References

[1] **Abbink, K., B. Irlenbusch, P. Pezanis-Christou, B. Rockenbach, A. Sadrieh, and R. Selten**, "An experimental test of design alternatives for the British 3G/UMTS auction," *European Economic Review*, 2005, *49* (2), 505–530.

[2] **Āzacis, H. and R. Burguet**, "Incumbency and Entry in License Auctions: The Anglo-Dutch Auction Meets Other Simple Alternatives," *Cardiff Economics Working Papers*, Dec 2005, (E2005/11).

[3] ___ **and** ___ , "Incumbency and entry in license auctions: The Anglo–Dutch auction meets another simple alternative," *International Journal of Industrial Organization*, 2008, *26* (3), 730–745.

[4] **Brunner, C., J.K Goeree, C. Holt, and J.O. Ledyard**, "Combinatorial Auctioneering," *Working Paper, taken from J.K. Goeree's website, at http://www.hss.caltech.edu/ jkg/*, 2006.

[5] **Bustos, A. and A. Costinot**, "The Anglo-Dutch Auction," *Coursework paper, Princeton University*, 2003.

[6] **Cramton, P. and J.A. Schwartz**, "Collusive Bidding: Lessons from the FCC Spectrum Auctions," *Journal of Regulatory Economics*, 2000, *17* (3), 229–252.

[7] **Kagel, J.H.**, "Auctions: A Survey of Experimental Research," *The Handbook of Experimental Economics*, 1995, pp. 501–585.

[8] **Klemperer, P.**, "Comments on 'Putting Auction Theory to Work'," *Discussant's Comments on Paul Milgrom's 1995 Churchill Lectures, working paper, Oxford University, March 1995*, 1995.

[9] ___ , "Auctions with Almost Common Values: The "Wallet Game" and its Applications," *European Economic Review*, 1998, *42* (3), 757–69.

[10] **Lee, D.**, "Lessons from the Nigerian GSM auction," *Telecommunications Policy*, 2003, *27* (5), 407–416.

[11] **Levin, D. and L. Ye**, "Hybrid auctions revisited," *Economics Letters*, 2008, *99* (3), 591–594.

[12] **Maskin, Eric and John Riley**, "Asymmetric Auctions," *Review of Economic Studies*, July 2000, *67* (3), 413–38.

[13] **McMillan, J.**, "Selling Spectrum Rights," *Journal of Economic Perspectives*, 1994, *8* (3), 145–162.

[14] **Milgrom, P.R. and R.J. Weber**, "A Theory of Auctions and Competitive Bidding," *Econometrica*, 1982, *50* (5), 1089–1122.

[15] **Riley, J.G.**, "Expected Revenue from Open and Sealed Bid Auctions," *The Journal of Economic Perspectives*, 1989, *3* (3), 41–50.

[16] **Woo, C.K., D. Lloyd, M. Borden, R. Warrington, and C. Baskette**, "A robust internet-based auction to procure electricity forwards," *Energy*, 2004, *29* (1), 1–11.

10. Appendix A – Modifications of the Base Model

This section discusses three possible extensions of my base model. Due to space constraints, the derivations of the theoretical results for the modified model are omitted, but quantitative and qualitative implications of those modifications will be fully presented.

10.1. Different Pattern of Information Revelation - the L^+/L^- Model

My basic model of the Anglo-Dutch auction assumes that both of the weak bidders are identical. This assumption is analogous to saying that the strong bidder "doesn't see" the identity of the bidder that drops out in the first round, if the ascending phase ends at L. In practice this is unlikely to be true, and indeed the identity of the drop-out bidders can be identified. To model a situation where the identity of the drop-out can be seen by the strong bidder, I can modify the valuation structure in the model. Instead of assuming that both weak type bidders are ex-ante identical, I now assume that there are two ex-ante types of weak bidders. One of the weak bidders is of type W^+, and has value of H with probability of μ, and L^+ with probability of $(1-\mu)$. The other bidder is of type W^-, and has valuation of H with probability μ, and L^- with probability $(1-\mu)$. I assume that L^+ is just above L, and L^- is just below L.

Given the new value structure, the strong bidder now gains more information after the ascending stage is over. In the case in which the ascending auction terminates at L^-, he still doesn't know whether his rival has a value of L^+ or H; the conditional probability of the rival's being an H-type, however, changes. Is is straightforward to show that the conditional probability of the rival's being H is lower in the L$^+$/L$^-$ variant, so I would expect less aggressive bidding when a mixing equilibrium exists. Yet if the ascending auction ends at L^+, the bidder knows that the only situation in which the W^- type would stay in at a price of L^+ is when his valuation is in fact H. Thus in this case the strong bidder will behave "as if" he had observed a price of $\min(v, H)$ at the end of the ascending phase. He will thus bid $\min(v, H)$ for sure, rather than engaging in mixing (as was the case in the standard model). The equilibrium mixing distributions are given by:

$$\begin{aligned}
\hat{G}_H(b) &= \frac{(1-\mu)}{\mu}\left(\frac{b-L}{(v-b)}\right) \\
\hat{G}_S(b) &= \frac{H-\hat{b}}{H-b} \\
\hat{b} &= v-(1-\mu)(v-L) < b^*
\end{aligned}$$

Thus a mixing equilibrium will exist in the L$^+$/L$^-$ model, whenever it existed in the basic model. Furthermore, mixing will now occur for a broader range of v. Comparing $\hat{G}_H$ and $\hat{G}_S$ with G_H^* and G_S^* it is clear that the G^* distributions first-order stochastically dominate their $\hat{G}$ equivalents. This suggests that bidding is more aggressive in the base model. The reason for this finding is that in the L$^+$/L$^-$ model, it is less likely that de-facto mixing will occur in the second stage: in the base model mixing occurs with probability $1-\mu^2$, whereas in the new model it occurs only with probability $\mu(1-\mu)$. Conditional on a mixing equilibrium being played, it is more likely in the basic model that the strong bidder is facing an H-type in the second round (this probability is $\frac{2\mu(1-\mu)}{1-\mu^2}$), compared with the L$^+$/L$^-$ variant (where the probability is $\mu < \frac{2\mu(1-\mu)}{1-\mu^2}$, $\forall\mu \in (0,1)$). There will thus be two new effects on revenue, when I compare the L$^+$/L$^-$ model with the base case. Firstly, since the probability of a mixing equilibrium occurring is now lower, the probability of the revenue being directly $\min(v, H)$ will be larger - this effect enhances expected revenue. However, in the remaining case when mixing does occur, bidding will be less aggressive, and so expected revenue from the mixing scenario will fall.

It is possible to show analytically that the expected revenue from the L$^+$/L$^-$ model is less than it would be in the base model, so the modified version of the Anglo-Dutch auction performs revenue-wise worse than the base model. I find that in the modified model the range of parameters for which the Anglo-Dutch auction performs worse than both rivals is now increased, but for majority of parameter values the Anglo-Dutch auction still ranks in the middle. Despite being revenue-inferior to the base model, the L^+/L^- model of the Anglo-Dutch auction may nonetheless be more efficient than the base model, for some parameter values.

10.2. Budget Constraints Model with Common Values

The model I have discussed in this paper can be straightforwardly extended to consider different budget constraints, instead of different valuations. I thus assume in this section that the values of L, H and v are in fact budget constraints of the different types of bidders, and the object has a true common value of x to each bidder. All bidders know the value of x, and I assume it is larger than all the budget constraints. I use the same tie-breaking rule as before, though now I make all the decisions contingent on budgets rather than valuations. Deriving equilibrium bidding distributions in this model is exactly analogous to the standard model, and in fact when a mixing equilibrium exists in the first-price auction, bidding distributions for the first-price and Anglo-Dutch auctions can be obtained simply by substituting x for v and H in equations (1), (4) and (5). The interesting new comparison in this model is to keep L, H, v and μ fixed, and vary x.

Results from the budget constraints model are not very favorable for the Anglo-Dutch auction. The auction never ranks strictly first in terms of expected revenue, but will rank as last for a relatively small range of parameters. However, as before, for the majority of parameter values the Anglo-Dutch auction ranks between its two competitors. Hence while the Anglo-Dutch auction would never be strictly preferred by a policymaker who knows x exactly, it might nonetheless be a desirable option when the policymaker is uncertain of the position of x. The Anglo-Dutch auction could still perform best "on average", depending on the range and distribution of x considered by the policymaker.

10.3. Correlation in Weak Bidders' Values

Another possible extension of my model is to consider situations where the valuations of weak bidders are correlated. While it is possible to generalize the derivations of equilibrium bidding distributions to the correlated case, this generates substantial difficulties for the first-price equilibrium. In particular, the problem of negative marginal densities, as considered in Section 12.1, becomes more severe and occurs at much broader range of parameter values. Other new difficulties with respect to the shape of the equilibrium bidding distribution also present themselves for the first-price auction. For example, under a broad range of parameter values the mass of the atom in the strong bidder's distribution will not be less than one, which suggests that the equilibrium distribution is not well defined. While it was possible to derive conditions on v for which these kinds of problems did not occur in my base model, deriving similar conditions of the correlated model is difficult, and I have not found an analytical expression for these constraints. It is possible, by trial and error, to find sets of parameters for which bidding distributions are well-defined in the first-price auction, but this is not sufficient to allow a systematic investigation of the model.

Thus the range of parameters for which a first-price equilibrium would exist is severely restricted, and I have not found an analytical solution which would constrain the model to be well-behaved. While the equilibrium in Anglo-Dutch and ascending auctions occur under a much broader range of parameters, the problems with the first-price auction deter me from obtaining a proper ranking. Modifying the basic model in way that gets the first-price equilibrium to behave better for correlated valuations is a thus a topic for future research.

11. Appendix B: The Degenerate Case, when v$\in$ [0,L]

11.1. First-price auction

Here the strong bidder's value is so low that his maximum bid is of no viable threat to the weak bidders. Hence, the lowest "active" valuation against which weak bidders have to compete is L, which is the lowest possible valuation for a weak bidder. Thus I can construct an equilibrium similar to that of Azacis and Burguet (2005). The properties of the equilibrium are that:

- Strong bidder bids v, and never wins
- Weak bidder of type L bids L, and can only win of both weak bidders are of L-type.
- H-type weak bidders mix on $[L, \bar{b}_0]$, with cumulative bidding density $G_{H0}(b)$.

By similar reasoning as in the cases when $v > v_\alpha$ for the first-price auction, at most one of the bidders can have an atom at L, so it follows that G_{H0} cannot have such an atom (else, in the case when there are two H types, both players would have an atom, which is a contradiction). Given the above description, the profit function for an H-type is:

$$\Pi_{H0}(b) = ((1-\mu) + \mu G_{H0}(b))(H-b) \tag{7}$$

Using the fact that, $G_{H0}(L) = 0$ and $G_{H0}\left(\bar{b}\right) = 1$, and the fact that all values in the support of the mixing distribution give the same payoff, I have:

$$\Pi_{H0}(L) = \Pi_{H0}\left(\bar{b}\right) = \Pi_{H0}(b)$$

Hence:

$$\begin{aligned} (1-\mu)(H-L) &= \left(H-\bar{b}_0\right) \\ &\implies \bar{b}_0 = H-(1-\mu)(H-L) \end{aligned}$$

And:

$$\begin{aligned} (1-\mu)(H-L) &= ((1-\mu)+\mu G_{H0}(b))(H-b) \\ &\implies G_{H0}(b) = \frac{1-\mu}{\mu}\left(\frac{b-L}{H-b}\right) \end{aligned}$$

Proposition 7. *The equilibrium in the FPS for $v < L$ is thus fully characterized as follows:*
Strong bidder always plays v
L-type weak bidder always plays L
H-type weak bidder mixes according to:

$$G_{H0}(b) = \frac{1-\mu}{\mu}\left(\frac{b-L}{H-b}\right)$$

The expected revenue in Case A is:

$$R^0_{FPS} = \left(1-\mu^2\right)L + \mu^2 H \tag{8}$$

11.2. Ascending Auction

In this case, the strong bidder's value is below the lowest conceivable value for the weak bidders, so the strong bidder never wins in the auction - indeed, he never even determines the final price.

If both weak bidders are of type H, then they will both bid up to H, and this is where the ascending auction terminates; the probability of this event is μ^2. In all the remaining cases, when at least one bidder is of type L, the auction stops at L; the probability of this happening is $(1-\mu^2)$. Hence:

Proposition 8. *The revenue from an ASC auction, when $v < L$ is:*

$$R^0_{ASC} = (1-\mu^2)L + \mu^2 H = R^0_{FPS} \tag{9}$$

11.3. Anglo-Dutch auction

In this case the Anglo-Dutch auction will be equivalent to the first-price auction with $v \in (0, L]$: the dropping bidder is always the incumbent, so in the Dutch phase it is the two entrants playing against each other, which is essentially the same as playing in a first-price auction, since the "reserve price" has no bite. That is, in the first-price auction the incumbent also always bids v, which has the same effect as the reserve price in the Anglo-Dutch. In this context, the additional information revealed by the Anglo-Dutch auction has no value.

Proposition 9. *The equilibrium of the Anglo-Dutch auction when $v < L$ is analogous to the equilibrium of the FPS auction for the case when $v < L$. More fully, the complete strategies require the strong bidder to bid up to v in the Anglo-stage, and bid v in the Dutch stage also. The L-type weak bidder bids up to L in the Anglo phase, and submits a bid equal to L in the Dutch phase. Finally, the H-type weak bidder bid up to H in the Anglo-phase, and bid according to G_{H0} in the Dutch phase.*

11.4. Comparison

Comparing the revenues from the first-price and ascending auctions, and noting that the first-price and the Anglo-Dutch auction are revenue-equivalent, we see that indeed all three auctions are revenue equivalent for when $v \in [0, L]$. This result is not surprising, and can be related to Riley's (1989) derivation of revenue equivalence in discrete valuation models. In my model, as it stands when $v < L$, what we essentially have is a setting where two ex-ante symmetric weak bidders bid against each other, and they both have valuations drawn from the same discrete distribution; we also have risk-neutrality, and other IPV assumptions. So my model satisfies the assumptions of Riley's model, and hence the conclusion of revenue-equivalence should follow. My explicit derivations of revenues confirms that this is indeed the case.

12. Appendix C: Details on the First Price Auction in the Non-Degenerate Case (when $v > L$).

12.1. Deriving the FPS Equilibrium (Proposition 1)

Proposition 1 covers all values of v, except the case when $v < L$, which generates revenue-equivalence among the three auctions, is treated in Appendix B, Section 11. It should be also noted that the behavior of the expected revenue at the two boundary points, v_α and v$_\beta$ will be very different. At v_β, all mixing types switch to bidding H

in a continuous manner - the mixing distributions put more and more mass at close to H, until it becomes optimal for the strong bidder to bid this value for sure. However, when v falls below v_α, the change in revenue is discrete: the strong bidder decides 'not to participate', so while the H-type weak bidders still mix in equilibrium, they only expect to be bidding against one viable opponent at most. The lower revenue is thus caused by loss of competition if the value of the strong bidder is too low.

The main characteristics of the equilibrium will be thus:

- L-type weak bidder bids L, and expects no surplus.
- The H-type weak bidders and the strong bidder mix on an interval $[L, \bar{b}]$, according to the cumulative distributions G_H and G_S respectively.
- The strong bidder's bidding distribution has an atom at L.

I use the following arguments to generate the profit functions. If a high type is to win, he must beat the strong bidder, and either beats an L-type opponent for sure (happens with probability $(1-\mu)$), or he must bid higher than another H-type, which happens with probability $\mu G_H(b)$. Thus:

$$\Pi_H(b) = G_S(b)\left((1-\mu) + \mu G_H(b)\right)(H-b) \tag{10}$$

On the other hand, if the strong bidder is to win, he must either beat two L-types (occurs with probability $(1-\mu)^2$), or he must beat one H type, and one L type (occurs with probability $2\mu(1-\mu)G_H(b)$), or he must beat two H-types (with probability $\mu^2 G_H^2(b)$). Hereby:

$$\Pi_S(b) = \left((1-\mu)^2 + 2\mu(1-\mu)G_H(b) + \mu^2 G_H^2(b)\right)(v-b)$$

Given the structure of the surplus functions above, it is easy to see that it is the strong bidder who must have an atom at L. Indeed, if the valuation realizations were such that we would have two H-types bidding in the auction, we would end up in a situation where two bidders have an atom at L - but I argued above that such a case is a violation of my equilibrium.

Using the fact that, $G_H(L) = 0$ and $G_H(\bar{b}) = G_S(\bar{b}) = 1$, and the fact that all values in the support of the mixing distribution give the same payoff, I have, $\Pi_S(L) = \Pi_S(\bar{b})$ and so:

$$\begin{aligned} (1-\mu)^2(v-L) &= v-\bar{b} \\ &\implies \bar{b} = v-(1-\mu)^2(v-L) \end{aligned}$$

To obtain the cumulative bidding distributions, I solve the two equations for G_S and G_H:

$$\begin{aligned} \Pi_S(b) &= \Pi_S(L) \\ \Pi_H(b) &= \Pi_H(L) \end{aligned}$$

This yields:[15]

$$G_H(b) = \frac{(1-\mu)}{\mu}\frac{\sqrt{v-L}-\sqrt{v-b}}{\sqrt{v-b}} \tag{11}$$

[15] Derivations in Appendix 13.

$$G_S(b) = \frac{1}{1-\mu}\frac{\left(H-\bar{b}\right)}{\sqrt{v-L}}\frac{\sqrt{(v-b)}}{(H-b)} \tag{12}$$

To complete the definition of equilibrium for $v > L$, I must consider two further issues. Firstly, when v is very small, the above bidding distribution for the strong bidder is not well-defined. Secondly, when v is very large, $\bar{b} > H$, and a mixing equilibrium cannot exist, since it would require bidding distributions where the H-type bidder bids above his valuation with positive probability.

With respect to the first problem I notice that for v close to L, G_S is first increasing, and then decreasing. Clearly, in this case it cannot be a well-defined (cumulative) equilibrium bidding distribution. To check for the conditions when I have an admissible cumulative density, I look at the marginal density:

$$\frac{\partial G_S}{\partial b} = \frac{1}{1-\mu}\frac{\left(H-\bar{b}\right)}{\sqrt{v-L}}\frac{1}{(H-b)(v-b)^{\frac{1}{2}}}\left(\frac{v-b}{H-b}-\frac{1}{2}\right)$$

The expression in brackets is decreasing in b, whence it takes the minimum value when $b = \bar{b}$, so the condition for a well-defined cumulative density can be re-written as:

$$\begin{aligned} v - L &\geq \frac{1}{(1-\mu)^2}(H - v_\alpha) \\ v &\geq \left(H - \frac{(1-\mu)^2}{1+(1-\mu)^2}(H-L)\right) \end{aligned}$$

I can thus define the lower bound for v, such that a mixing equilibrium exists, as:

$$v_\alpha = H - \frac{(1-\mu)^2}{1+(1-\mu)^2}(H-L)$$

Subtracting from this $\mu H + (1-\mu)L$ gives us an indication of the magnitude of v_α relative to the (ex-ante) expected valuation of a weak bidder. I thus find:

$$v_\alpha - (\mu H + (1-\mu)L) = \left(1 - \frac{(1-\mu)}{1+(1-\mu)^2}\right)(1-\mu)(H-L) > 0$$

We thus see that v_α is always larger than the expectation of the weak bidder's valuation. One could interpret this requirement as suggesting that the strong bidder must be "strong enough" for my equilibrium to exist. Observe that v_α depends on three parameters of my model - so it will change whenever L, H or μ is changed. In the rest of the paper I will only consider values of v such that $v \geq v_\alpha$.

With respect to the second problem, I observe that for some parameter values $\bar{b} > H$ (for example, when v is very large). In this event, the "top" of the mixing distribution exceeds the valuation of the H-type weak bidder - so no H-type bidder would play according to such a distribution in equilibrium. There is a natural way for the players to behave in this case: the H-type bidder bids H, and the strong bidder also bids H (and obtains the good via the tie-breaking rule). The rationale for this switching of behavior is that when v is high enough, the strong bidder doesn't want to "risk" losing to the H-type weak bidder, and bids H (and wins) for sure. I show, in Appendix D, Section 12.3, that when $\bar{b} > H$, then playing H for sure gives the strong bidder a higher surplus than a mixing strategy (of the above sort) would. The values of v at which the strong bidder prefers to always bid H satisfy the following inequality.

$$\begin{aligned}
\bar{b} &\geq H \\
v - (1-\mu)^2 (v_\beta - L) &\geq H \\
v &\geq \left(H + \frac{(1-\mu)^2}{1-(1-\mu)^2} (H-L) \right) > H
\end{aligned}$$

The boundary value of v at which the switch-over in behavior occurs can is thus:

$$v_\beta = H + \frac{(1-\mu)^2}{1-(1-\mu)^2} (H-L)$$

Observe again that this critical value depends on L, H and μ - and will change if we change one of those parameters. Combining the above arguments, I obtain a full specification of equilibrium behavior in the first-price auction.

The surplus/profit and density functions for the H-type weak bidder will be indexed by H, and those for the strong bidder will be indexed by S.

12.2. First Price Auction

For the strong bidder, the profit function is:

$$\Pi_S(b) = \left((\mu)^2 G_H(b)^2 + 2\mu(1-\mu) G_H(b) + (1-\mu)^2 \right) (v-b)$$

For the H-type weak bidder:

$$\Pi_H(b) = (\mu G_H(b) + (1-\mu)) G_S(b) (H-b)$$

Using $G_H(L) = 0$:

$$\Pi_I(L) = (1-\mu)^2 (v-L) = \Pi_I(b)$$

The last equality follows due to the fact that all bids which the bidder submits with positive probability, must give the same expected payoff. Writing this out:

$$\left((\mu)^2 G_H(b)^2 + 2\mu(1-\mu) G_H(b) + (1-\mu)^2 \right) (v-b) = (1-\mu)^2 (v-L)$$

This is a quadratic in G_H.

$$G_H(b)^2 \underbrace{(\mu)^2 (v-b_I)}_{A} + G_H(b) \underbrace{2\mu(1-\mu)(v-b_I)}_{B} + \underbrace{(1-\mu)^2 [L-b_I]}_{C} = 0$$

Applying the quadratic formula to the above equation, taking the positive root, gives us:

$$\begin{aligned}
G_H(b) &= \frac{-B+\sqrt{B^2-4AC}}{2A} \\
&= \frac{-2\mu(1-\mu)(v-b)}{2(\mu)^2(v-b)} + \\
&\quad + \frac{\sqrt{(2\mu(1-\mu)(v-b))^2 - 4(\mu)^2(v-b)\left[(1-\mu)^2[L-b]\right]}}{2(\mu)^2(v-b)} \\
&= \frac{(1-\mu)}{\mu} \left[\frac{\sqrt{v-L} - \sqrt{(v-b)}}{\sqrt{(v-b)}} \right]
\end{aligned}$$

This gives one of the bidding distributions. For the other, I observe that since $G_H\left(\bar{b}\right) = 1$:

$$\begin{aligned} \Pi_S\left(\bar{b}\right) &= \left(v - \bar{b}\right) = (1-\mu)^2 (v - L) \\ &\implies \bar{b} = v - (1-\mu)^2 (v - L) \end{aligned}$$

Again, since all bids that are played with positive probability must give the same expected payoff, we have:

$$\Pi_H(b) = \Pi_H\left(\bar{b}\right) = \left(H - \bar{b}\right)$$

Writing this out, and solving for G_S :

$$\begin{aligned} \left(\mu G_H(b) + (1-\mu)\right) G_S(b)(H-b) &= \left(H - \bar{b}\right) \\ G_S(b) &= \frac{1}{(1-\mu)} \frac{\left(H-\bar{b}\right)}{\sqrt{v-L}} \frac{\sqrt{(v-b)}}{(H-b)} \end{aligned}$$

For a mixing equilibrium to exist I need:

$$\begin{aligned} \bar{b} &\leq H \\ &\implies v - (1-\mu)^2 (v-L) \leq H \\ &\implies v \leq H + \frac{(1-\mu)^2}{1-(1-\mu)^2}(H-L) \end{aligned}$$

This means that the strong bidder's valuation cannot be "too large" (since then he will just bid H, and win always).

12.3. Justifying "switch-over" of strong bidder's strategy when $\bar{b} > H$

I now look for an equilibrium in the case when $\bar{b} > H$. Then:

$$v > H + \frac{(1-\mu)^2}{1-(1-\mu)^2}(H-L)$$

The profit the strong bidder would then get by "mixing" would be:

$$(1-\mu)^2 (v-L)$$

Whereas by bidding H he gets the item for sure (via the tie-breaking rule):

$$v - H$$

The difference between these two cases is

$$\begin{aligned} \Delta &= (v-H) - (1-\mu)^2 (v-L) \\ &= \left(2\mu - \mu^2\right) v + (1-\mu)^2 L - H \\ &> \left(2\mu - \mu^2\right)\left(H + \frac{(1-\mu)^2}{1-(1-\mu)^2}(H-L)\right) + (1-\mu)^2 L - H = 0 \end{aligned}$$

Hence my desired conclusion of $\Delta > 0$. That is:

$$\begin{aligned} v - H &> (1-\mu)^2 (v-L) \\ when &: \bar{b} > H \end{aligned}$$

12.4. Revenue in the First Price Auction

12.4.1. When $v \in [0, L]$

I consider different combinations of weak bidders separately, and then aggregate to get total revenue.

Both weak bidders are L. This happens with probability: $(1-\mu)^2$. Revenue is then L.

One weak bidder is L, other is H. This happens with probability $2\mu(1-\mu)$. The cumulative density is of the winning bid is then $G_H(b)$. The density is thus:

$$G'_H = \frac{1-\mu}{\mu}\left(\frac{H-L}{(H-b)^2}\right)$$

The expected revenue can be obtained as:

$$\frac{1-\mu}{\mu}\int_L^{\bar{b}} t\frac{H-L}{(H-t)^2}dt$$

Both weak bidders are H. This happens with probability μ^2. The winning bid has cumulative density $G_H(b)^2$. Hence density of the winning bid is:

$$\begin{aligned} & 2G_H(b)\,G'_H(b) \\ = & 2\left(\frac{1-\mu}{\mu}\right)^2(H-L)\frac{b-L}{(H-b)^3} \end{aligned}$$

So the expected revenue is:

$$2\left(\frac{1-\mu}{\mu}\right)^2(H-L)\int_L^{\bar{b}} t\frac{t-L}{(H-t)^3}dt$$

Total expected revenue. Aggregating the above expressions, after proper pre-multiplication by event probabilities, I have:

$$\begin{aligned} R^0_{FPS} &= (1-\mu)^2 L + \\ &+2\mu(1-\mu)\frac{1-\mu}{\mu}\int_L^{\bar{b}} t\frac{H-L}{(H-t)^2}dt \\ &+\mu^2 2\left(\frac{1-\mu}{\mu}\right)^2(H-L)\int_L^{\bar{b}} t\frac{t-L}{(H-t)^3}dt \end{aligned}$$

$$\begin{aligned} R^0_{FPS} &= (1-\mu)^2 L + \\ &+2(1-\mu)^2(H-L)\int_L^{\bar{b}} t\left(\frac{1}{(H-t)^2}+\frac{t-L}{(H-t)^3}\right)dt \end{aligned}$$

Using partial fractions:

$$\begin{aligned} R^0_{FPS} &= (1-\mu)^2 L + \\ &+2(1-\mu)^2(H-L)\int_L^{\bar{b}}\left(H\frac{H-L}{(H-t)^3}-\frac{H-L}{(H-t)^2}\right)dt \end{aligned}$$

$$\begin{aligned} R^0_{FPS} &= (1-\mu)^2 L + \\ &\quad +2(1-\mu)^2 (H-L) \frac{1}{2} \left[\frac{(L-H)}{(t-H)^2} (H-2t) \right]_L^{\bar{b}} \end{aligned}$$

$$\begin{aligned} R^0_{FPS} &= (1-\mu)^2 L + \\ &\quad +2(1-\mu)^2 (H-L) \frac{1}{2} \left[\begin{array}{r} \frac{(L-H)}{((1-\mu)(H-L))^2} (H - 2(H-(1-\mu)(H-L))) \\ -\frac{(L-H)}{(L-H)^2} (H-2L) \end{array} \right] \end{aligned}$$

Simplifying the above I get:

$$R^0_{FPS} = (1-\mu)^2 L + \mu^2 H + 2\mu(1-\mu) L = R^A_{ASC}$$

12.4.2. When $v > v_\alpha$

The following derivation only applies under parameter values under which the strong bidder's bidding distribution is well behaved. The equilibrium bidding behavior is defined in Proposition 1. I proceed by calculating revenue separately from all the possible combinations of realized strong and weak bidder valuations, and then I aggregate

Expected revenue calculations - the sub-cases. The following list of cases is jointly exhaustive (and mutually exclusive):

- Two weak bidders are L, strong bidder plays Atom
- Two weak bidders are L, strong bidder mixes
- Exactly one weak bidder is L, one is H, strong bidder plays Atom
- Exactly one weak bidder is L, one is H, strong bidder mixes
- Both weak bidders are H, strong bidder plays Atom
- Both weak bidders are H, strong bidder mixes

Two weak bidders are L, strong bidder plays Atom. The probability of this event is:

$$(1-\mu)^2 \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}$$

Revenue in this case is

$$R_{FPS}(2L, atom) = L$$

Two weak bidders are L, strong bidder mixes. The probability of this event is:

$$(1-\mu)^2 \left(1 - \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right)$$

The cumulative density of the winning bid is then:

$$\frac{G_S(b) - G_S(L)}{1 - G_S(L)}$$

The corresponding density of winning bid is:

$$\frac{1}{1-G_S(L)}G_S'(b)$$

The relevant derivative is given by:

$$\begin{aligned}
G_S'(b) &= \left(\frac{1}{1-\mu}\right)\frac{\left(H-\bar{b}\right)}{(v-L)^{\frac{1}{2}}}\left(\frac{-\frac{1}{2}(v-b)^{-\frac{1}{2}}(H-b)+(v-b)^{\frac{1}{2}}}{(H-b)^2}\right) \\
&= \left(\frac{1}{1-\mu}\right)\frac{\left(H-\bar{b}\right)}{(v-L)^{\frac{1}{2}}}\left(\frac{-\frac{1}{2}(v-b)^{-\frac{1}{2}}}{(H-b)}+\frac{(v-b)^{\frac{1}{2}}}{(H-b)^2}\right)
\end{aligned}$$

Revenue under the appropriate density is then:

$$R_{FPS}(2L,mix) = \frac{1}{\left(1-\frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right)}\left(\frac{1}{1-\mu}\right)\frac{\left(H-\bar{b}\right)}{(v-L)^{\frac{1}{2}}}\cdot$$
$$\cdot\int_L^{\bar{b}} t\left(\frac{-\frac{1}{2}(v-t)^{-\frac{1}{2}}}{(H-t)}+\frac{(v-t)^{\frac{1}{2}}}{(H-t)^2}\right)dt$$

Exactly one weak bidder is L, one is H, strong bidder plays Atom. The probability of this event is:

$$2\mu(1-\mu)\frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)} \tag{13}$$

The cumulative density of a winning bid is then simply $G_H(b)$. Hence the density of the winning bid is:

$$\begin{aligned}
G_H'(b) &= \left(\frac{1-\mu}{\mu}\right)\left(\frac{\frac{1}{2}(v-b)^{-\frac{1}{2}}(v-b)^{\frac{1}{2}}+\frac{1}{2}(v-b)^{-\frac{1}{2}}\left((v-L)^{\frac{1}{2}}-(v-b)^{\frac{1}{2}}\right)}{(v-b)}\right) \\
&= \left(\frac{1-\mu}{2\mu}\right)\left(\frac{1}{(v-b)}+\frac{(v-L)^{\frac{1}{2}}-(v-b)^{\frac{1}{2}}}{(v-b)^{\frac{3}{2}}}\right)
\end{aligned}$$

The expected revenue is then:

$$R_{FPS}(1L,Atom) = \left(\frac{1-\mu}{2\mu}\right)\int_L^{\bar{b}} t\left(\frac{1}{(v-t)}+\frac{(v-L)^{\frac{1}{2}}-(v-t)^{\frac{1}{2}}}{(v-t)^{\frac{3}{2}}}\right)dt$$

Exactly one weak bidder is L, one is H, strong bidder mixes. This occurs with probability

$$2\mu(1-\mu)\left(1-\frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right)$$

The cumulative density of the winning bid is:

$$\begin{aligned}
F &= \frac{G_S(b) - G_S(L)}{1 - G_S(L)} G_H(b) \\
&= \left[\frac{1}{1 - G_S(L)} \frac{1}{\mu} \frac{\left(H - \bar{b}\right)}{(v - L)^{\frac{1}{2}}}\right] \left[\frac{(v-b)^{\frac{1}{2}}}{(H-b)} - \frac{(v-L)^{\frac{1}{2}}}{(H-L)}\right] \frac{(v-L)^{\frac{1}{2}} - (v-b)^{\frac{1}{2}}}{(v-b)^{\frac{1}{2}}}
\end{aligned}$$

Differentiating:

$$\begin{aligned}
F' &= \left[\frac{1}{1 - G_S(L)} \frac{1}{\mu} \frac{\left(H - \bar{b}\right)}{(v - L)^{\frac{1}{2}}}\right] \cdot \\
&\cdot \left[\frac{(v-L)^{\frac{1}{2}} - (v-b)^{\frac{1}{2}}}{(H-b)^2} - \frac{\frac{1}{2}(v-L)}{(H-L)(v-b)^{\frac{3}{2}}} + \frac{\frac{1}{2}(v-b)^{\frac{1}{2}}}{(H-b)(v-b)}\right] \\
&= \left[\frac{1}{1 - \frac{(H-\bar{b})}{(1-\mu)(H-L)}} \frac{1}{\mu} \frac{\left(H - \bar{b}\right)}{(v - L)^{\frac{1}{2}}}\right] \cdot \\
&\cdot \left[\frac{(v-L)^{\frac{1}{2}} - (v-b)^{\frac{1}{2}}}{(H-b)^2} - \frac{\frac{1}{2}(v-L)}{(H-L)(v-b)^{\frac{3}{2}}} + \frac{\frac{1}{2}(v-b)^{\frac{1}{2}}}{(H-b)(v-b)}\right]
\end{aligned}$$

The expected revenue is then:

$$\begin{aligned}
R_{FPS}(1L, mix) &= \left[\frac{1}{1 - \frac{(H-\bar{b})}{(1-\mu)(H-L)}} \frac{1}{\mu} \frac{\left(H - \bar{b}\right)}{(v - L)^{\frac{1}{2}}}\right] * \\
&* \int_L^{\bar{b}} t \left[\begin{array}{c} \frac{(v-L)^{\frac{1}{2}} - (v-t)^{\frac{1}{2}}}{(H-t)^2} - \\ -\frac{\frac{1}{2}(v-L)}{(H-L)(v-t)^{\frac{3}{2}}} + \frac{\frac{1}{2}(v-t)^{\frac{1}{2}}}{(H-t)(v-t)} \end{array}\right] dt
\end{aligned}$$

Both weak bidders are H, strong bidder plays atom. The event has probability:

$$\mu^2 \left(\frac{\left(H - \bar{b}\right)}{(1-\mu)(H-L)}\right)$$

In this case the winning bid has the cumulative density $G_H(b)^2$. The relevant density is then:

$$2 G_H(b) G'_H(b)$$

Substituting in for G_H and G'_H:

$$\begin{aligned}
& 2\left(\frac{1-\mu}{\mu}\right) \frac{(v-L)^{\frac{1}{2}} - (v-b)^{\frac{1}{2}}}{(v-b)^{\frac{1}{2}}} \left(\frac{1-\mu}{2\mu}\right) \left(\frac{1}{(v-b)} + \frac{(v-L)^{\frac{1}{2}} - (v-b)^{\frac{1}{2}}}{(v-b)^{\frac{3}{2}}}\right) \\
= & \left(\frac{1-\mu}{\mu}\right)^2 \left(\frac{(v-L)^{\frac{1}{2}} - (v-b)^{\frac{1}{2}}}{(v-b)^{\frac{3}{2}}} + \frac{\left((v-L) + (v-b) - 2(v-L)^{\frac{1}{2}}(v-b)^{\frac{1}{2}}\right)}{(v-b)^2}\right)
\end{aligned}$$

The revenue is then:

$$R_{FPS}(2H, Atom) = \left(\frac{1-\mu}{\mu}\right)^2 \int_L^{\bar{b}} t \left(\begin{array}{c} \frac{(v-L)^{\frac{1}{2}}-(v-t)^{\frac{1}{2}}}{(v-t)^{\frac{3}{2}}} \\ +\frac{\left((v-L)+(v-t)-2(v-L)^{\frac{1}{2}}(v-t)^{\frac{1}{2}}\right)}{(v-t)^2} \end{array} \right) dt$$

Both weak bidders are H, strong bidder mixes. The probability of this event is:

$$\mu^2 \left(1 - \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right)$$

The cumulative density of the winning bid is:

$$\begin{aligned} & \frac{G_S(b) - G_S(L)}{1-G_S(L)} \left[G_H(b)\right]^2 \\ = \; & \frac{1}{1-G_S(L)} \left[\left(\frac{1}{1-\mu}\right) \frac{\left(H-\bar{b}\right)}{(v-L)^{\frac{1}{2}}} \frac{(v-b)^{\frac{1}{2}}}{(H-b)} - \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right] \cdot \\ & \cdot \left[\left(\frac{1-\mu}{\mu}\right) \frac{(v-L)^{\frac{1}{2}} - (v-b)^{\frac{1}{2}}}{(v-b)^{\frac{1}{2}}}\right]^2 \\ = \; & \frac{\left(H-\bar{b}\right)}{1-G_S(L)} \left(\frac{1-\mu}{\mu^2}\right) \left[\frac{1}{(v-L)^{\frac{1}{2}}} \frac{(v-b)^{\frac{1}{2}}}{(H-b)} - \frac{1}{(H-L)}\right] \cdot \\ & \cdot \left[\frac{(v-L)+(v-b)-2(v-b)^{\frac{1}{2}}(v-L)^{\frac{1}{2}}}{(v-b)}\right] \end{aligned}$$

The corresponding density is:

$$\begin{aligned} & \frac{\left(H-\bar{b}\right)}{1-G_S(L)} \left(\frac{1-\mu}{\mu^2}\right) * \\ & \frac{1}{(v-b)} \left(\begin{array}{c} \left[\frac{\left[(v-L)(v-b)^{\frac{1}{2}}+(v-b)^{\frac{3}{2}}-2(v-b)(v-L)^{\frac{1}{2}}\right]}{(v-L)^{\frac{1}{2}}(H-b)^2}\right] \\ +\left[\frac{\frac{1}{2}(v-L)-\frac{1}{2}(v-b)}{(v-L)^{\frac{1}{2}}(H-b)(v-b)^{\frac{1}{2}}} - \frac{(v-L)-(v-b)^{\frac{1}{2}}(v-L)^{\frac{1}{2}}}{(v-b)(H-L)}\right] \end{array} \right) \end{aligned}$$

Whereby the expected revenue is:

$$\begin{aligned} & R_{FPS}(2H, mix) = \frac{\left(H-\bar{b}\right)}{1-\frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}} \left(\frac{1-\mu}{\mu^2}\right) * \\ & \int_L^{\bar{b}} t \left(\begin{array}{c} \left[\frac{\left[(v-L)(v-t)^{\frac{1}{2}}+(v-t)^{\frac{3}{2}}-2(v-t)(v-L)^{\frac{1}{2}}\right]}{(v-L)^{\frac{1}{2}}(H-t)^2(v-t)}\right] \\ +\left[\frac{\frac{1}{2}(v-L)-\frac{1}{2}(v-t)}{(v-L)^{\frac{1}{2}}(H-t)(v-t)^{\frac{3}{2}}} - \frac{(v-L)-(v-t)^{\frac{1}{2}}(v-L)^{\frac{1}{2}}}{(v-t)(H-L)(v-t)}\right] \end{array} \right) dt \end{aligned}$$

Overall Expected Revenue.

$$
\begin{aligned}
& R_{FPS}(Total) \\
= \; & (1-\mu)^2 \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)} L \\
& +(1-\mu)^2 \left(1-\frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right) \frac{1}{\left(1-\frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right)} \left(\frac{1}{1-\mu}\right) \frac{\left(H-\bar{b}\right)}{(v-L)^{\frac{1}{2}}} * \\
& \quad * \int_L^{\bar{b}} t \left(\frac{-\frac{1}{2}(v-t)^{-\frac{1}{2}}}{(H-t)} + \frac{(v-t)^{\frac{1}{2}}}{(H-t)^2}\right) dt \\
& +2\mu(1-\mu) \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)} \left(\frac{1-\mu}{2\mu}\right) \int_L^{\bar{b}} t \left(\frac{1}{(v-t)} + \frac{(v-L)^{\frac{1}{2}}-(v-t)^{\frac{1}{2}}}{(v-t)^{\frac{3}{2}}}\right) dt \\
& +2\mu(1-\mu) \left(1-\frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right) \left[\frac{1}{1-\frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}} \frac{1}{\mu} \frac{\left(H-\bar{b}\right)}{(v-L)^{\frac{1}{2}}}\right] * \\
& \quad * \int_L^{\bar{b}} t \left[\frac{(v-L)^{\frac{1}{2}}-(v-t)^{\frac{1}{2}}}{(H-t)^2} - \frac{\frac{1}{2}(v-L)}{(H-L)(v-t)^{\frac{3}{2}}} + \frac{\frac{1}{2}(v-t)^{\frac{1}{2}}}{(H-t)(v-t)}\right] dt \\
& +\mu^2 \left(\frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right) \left(\frac{1-\mu}{\mu}\right)^2 * \\
& \quad * \int_L^{\bar{b}} t \left(\frac{(v-L)^{\frac{1}{2}}-(v-t)^{\frac{1}{2}}}{(v-t)^{\frac{3}{2}}} + \frac{\left((v-L)+(v-t)-2(v-L)^{\frac{1}{2}}(v-t)^{\frac{1}{2}}\right)}{(v-t)^2}\right) dt \\
& +\mu^2 \left(1-\frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}\right) \frac{\left(H-\bar{b}\right)}{1-\frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)}} \left(\frac{1-\mu}{\mu^2}\right) * \\
& \quad * \int_L^{\bar{b}} t \left(\begin{array}{c} \left[\frac{\left[(v-L)(v-t)^{\frac{1}{2}}+(v-t)^{\frac{3}{2}}-2(v-t)(v-L)^{\frac{1}{2}}\right]}{(v-L)^{\frac{1}{2}}(H-t)^2(v-t)}\right] \\ +\left[\frac{\frac{1}{2}(v-L)-\frac{1}{2}(v-t)}{(v-L)^{\frac{1}{2}}(H-t)(v-t)^{\frac{3}{2}}} - \frac{(v-L)-(v-t)^{\frac{1}{2}}(v-L)^{\frac{1}{2}}}{(v-t)(H-L)(v-t)}\right] \end{array} \right) dt
\end{aligned}
$$

After some algebra, the above simplifies to:

$$
\begin{aligned}
R_{FPS} \; = \; & (1-\mu)^2 \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)} L + \\
& +(1-\mu)\left(H-\bar{b}\right)(v-L)^{\frac{1}{2}} \int_L^{\bar{b}} t \left(\frac{(v-t)^{\frac{1}{2}}}{(H-t)^2(v-t)} + \frac{\frac{1}{2}(v-t)^{\frac{1}{2}}}{(H-t)(v-t)^2}\right) dt
\end{aligned}
$$

Integrating by parts:

$$
\begin{aligned}
R_{FPS} \; = \; & (1-\mu)^2 \frac{\left(H-\bar{b}\right)}{(1-\mu)(H-L)} L + \\
& +(1-\mu)\left(H-\bar{b}\right)(v-L)^{\frac{1}{2}} \left(\left[\frac{t}{(H-t)(v-t)^{\frac{1}{2}}}\right]_L^{\bar{b}} - \int_L^{\bar{b}} \frac{1}{(H-t)(v-t)^{\frac{1}{2}}} dt\right)
\end{aligned}
$$

Substituting for boundaries:

$$\begin{aligned} R_{FPS} &= (1-\mu)\frac{\left(H-\bar{b}\right)}{(H-L)}L+ \\ &+(1-\mu)\left(H-\bar{b}\right)(v-L)^{\frac{1}{2}}\left(\left(\frac{\bar{b}}{\left(H-\bar{b}\right)(v-\bar{t})^{\frac{1}{2}}}\right)-\left(\frac{L}{(H-L)(v-L)^{\frac{1}{2}}}\right)\right) \\ &-(1-\mu)\left(H-\bar{b}\right)(v-L)^{\frac{1}{2}}\int_L^{\bar{b}}\frac{1}{(H-t)(v-t)^{\frac{1}{2}}}dt \end{aligned}$$

Simplifying and substituting for $\bar{\text{b}}$:

$$\begin{aligned} R_{FPS} &= (1-\mu)(v-L)^{\frac{1}{2}}\left(\frac{v-(1-\mu)^2(v-L)}{(1-\mu)(v-L)^{\frac{1}{2}}}\right) \\ &\quad -(1-\mu)\left(H-\bar{b}\right)(v-L)^{\frac{1}{2}}\int_L^{\bar{b}}\frac{1}{(H-t)(v-t)^{\frac{1}{2}}}dt \end{aligned}$$

Canceling the appropriate terms, I finally obtain:

$$R_{FPS}=v-(1-\mu)^2(v-L)-(1-\mu)\left(H-\bar{b}\right)(v-L)^{\frac{1}{2}}\int_L^{\bar{b}}\frac{1}{(H-t)(v-t)^{\frac{1}{2}}}dt$$

13. Appendix D: Details of the Anglo-Dutch Auction in the Non-Degenerate Case (when $v > L$).

13.1. Deriving the Anglo-Dutch Equilibrium (Proposition 3)

The profit functions in this case are as constructed as follows.

If the ascending phase terminated at L, and an H-type bidder is present in the first-price stage, he knows that he is the only H-type left in the auction, and he is bidding against the single strong bidder. Given a bid of b, the H-type will beat the strong bidder with probability $G_S^*(b)$. The surplus from bidding b is thus:

$$\Pi_H^*(b)=G_S^*(b)(H-b)$$

If the strong bidder is present in the first-price stage, he doesn't know his opponent's identity for sure - the only thing he knows at this point is that not both of his opponents were H-types. Thus with the (posterior) probability of $\frac{(1-\mu)^2}{1-\mu^2}$ the strong bidder is in fact facing an L-type bidder, whom he would beat with certainty by bidding any $b \geq L$. However, with the probability $\frac{2\mu(1-\mu)}{1-\mu^2}$ the strong bidder's opponent is in fact an H-type, and the probability of beating the H-type by bidding b is $G_H^*(b)$. The expected surplus of the strong bidder from bidding b is then:

$$\Pi_S^*(b)=\left(\frac{2\mu(1-\mu)}{1-\mu^2}G_H^*(b)+\frac{(1-\mu)^2}{1-\mu^2}\right)(v-b)$$

Solving these two equations proceeds exactly analogously to the first-price case, and the derivations are provided in Appendix B, Section 13. The results are summarized below. Firstly, I have:

$$b^* = v - \frac{(1-\mu)}{(1+\mu)}(v-L) = v - \frac{(1-\mu)^2}{1-\mu^2}(v-L) \le \bar{b}$$

13.2. Anglo-Dutch Auction

The only case in which I need to derive an equilibrium bidding distribution for the Anglo-Dutch auction is in the case when the ascending phase terminates at L. So suppose the ascending auction stops at L. Then with probability $\frac{2\mu(1-\mu)}{1-\mu^2}$ the strong bidder is facing an H type weak bidder, and with probability $\frac{(1-\mu)^2}{1-\mu^2}$ he is facing a type L. A low type weak bidder will bid L, and an H-type weak bidder will mix over $[L, b^*]$, for some $b^* \le H$. The profit for the strong bidder will be:

$$\begin{aligned}\Pi_S^*(b) &= \left(\frac{2\mu(1-\mu)}{1-\mu^2}G_H^*(b) + \frac{(1-\mu)^2}{1-\mu^2}\right)(v-b) \\ &= \left(\frac{2\mu}{(1+\mu)}G_H^*(b) + \frac{(1-\mu)}{(1+\mu)}\right)(v-b)\end{aligned}$$

The profit for H-type weak bidder will be:

$$\Pi_H^*(b) = G_H^*(b)(H-b)$$

Using $G_H^*(L) = 0$

$$\Pi_S^*(L) = \frac{(1-\mu)}{(1+\mu)}(v-L) = \Pi_S^*(b)$$

The last equality follows from the fact that each bid played with positive probability must give the same expected payoff in equilibrium. Hence:

$$\begin{aligned}\frac{(1-\mu)}{(1+\mu)}(v-L) &= \left(\frac{2\mu}{(1+\mu)}G_H^*(b) + \frac{(1-\mu)}{(1+\mu)}\right)(v-b) \\ G_H^*(b) &= \frac{(1-\mu)}{2\mu}\left(\frac{b-L}{v-b}\right)\end{aligned}$$

Using $G_H^*(b^*) = 1$:

$$\begin{aligned}\frac{(1-\mu)}{2\mu}\left(\frac{(v-L)}{(v-b^*)} - 1\right) &= 1 \\ b^* &= v - \frac{1-\mu}{1+\mu}(v-L) = v - \frac{(1-\mu)^2}{1-\mu^2}(v-L) < \bar{b}\end{aligned}$$

Similarly, since $G_S(b^*) = 1$:

$$\begin{aligned}\Pi_H^*(b^*) &= (H-b^*) = \Pi_H^*(b) \\ (H-b^*) &= G_S^*(b)(H-b) \\ G_S^*(b) &= \frac{(H-b^*)}{(H-b)} = \frac{\left(H - v + \frac{1-\mu}{1+\mu}(v-L)\right)}{(H-b)}\end{aligned}$$

13.3. Justifying "switch-over" of strong bidder's strategy when $b^* > H$

I now also look for the equilibrium when $b^* > H$. We must have:

$$v > H + \frac{(1-\mu)}{2\mu}(H-L)$$

The profits obtainable from just bidding H are:

$$(v-H)$$

Whereas the mixing strategy delivered:

$$\frac{1-\mu}{1+\mu}(v-L)$$

The difference is thus:

$$\begin{aligned}
\Delta &= (v-H) - \frac{1-\mu}{1+\mu}(v-L) \\
&= \frac{2\mu}{1+\mu}v - H + \frac{1-\mu}{1+\mu}L \\
&> \left(\frac{2\mu}{1+\mu}\right)\left(H + \frac{(1-\mu)}{2\mu}(H-L)\right) - H + \frac{1-\mu}{1+\mu}L = 0
\end{aligned}$$

So in this case also I have my desired conclusion of $\Delta > 0$. That is:

$$\begin{aligned}
(v-H) &> (1-\mu)^2(v-L) \\
when &: b^* > H
\end{aligned}$$

13.4. Revenue in the Anglo-Dutch Auction

13.4.1. When $v \in [0, L]$.

In this case the Anglo-Dutch auction is equivalent to the first-price.

13.4.2. When $v > L$

As in the first-price auction, here I also proceed by considering the expected revenues from sub-cases first, and then aggregate them.

Expected revenue calculations - the sub-cases. The following list of cases is jointly exhaustive (and individually mutually exclusive):

- Two weak bidders are L, strong bidder plays Atom
- Two weak bidders are L, strong bidder mixes
- Exactly one weak bidder is L, one is H, strong bidder plays Atom
- Exactly one weak bidder is L, one is H, strong bidder mixes
- Both weak bidders are H, and strong bidder plays $\min(v, H)$.

Two weak bidders are L, strong bidder plays Atom. Probability of this event is:

$$(1-\mu)^2 \frac{(H-b^*)}{(H-L)}$$

Revenue is :

$$R_{AD}(2L, atom) = L$$

Two weak bidders are L, strong bidder mixes. Probability of this case is:

$$(1-\mu)^2 \left(1 - \frac{(H-b^*)}{(H-L)}\right)$$

The cumulative density of the winning bid is then:

$$\frac{1}{1-G_S^*(L)} \left(G_S^*(b) - G_S^*(L)\right)$$

The relevant density is given by:

$$\frac{1}{1-G_S^*(L)} G_S^{*\prime}(b)$$

Since:

$$G_S^{*\prime}(b) = \frac{(H-b^*)}{(H-b)^2}$$

Substituting in for G_S^* and $G_S^{*\prime}$:

$$\frac{1}{1-\frac{(H-b^*)}{(H-L)}} \frac{(H-b^*)}{(H-b)^2}$$

The expected revenue is:

$$R_{AD}(2L, mix) = \frac{1}{1-\frac{(H-b^*)}{(H-L)}} \int_L^{b^*} t \frac{(H-b^*)}{(H-t)^2} dt$$

Exactly one weak bidder is L, one is H, strong bidder plays Atom. The probability of this event is:

$$2\mu(1-\mu) \frac{(H-b^*)}{(H-L)}$$

The cumulative density of the winning bid is then:

$$\left(\frac{1-\mu}{2\mu}\right) \left(\frac{b-L}{v-b}\right)$$

giving a density of:

$$\left(\frac{1-\mu}{2\mu}\right) \left(\frac{(v-b)+(b-L)}{(v-b)^2}\right) = \left(\frac{1-\mu}{2\mu}\right) \frac{v-L}{(v-b)^2}$$

So the expected revenue is:

$$R_{AD}\left(1L, atom\right) = \left(\frac{1-\mu}{2\mu}\right)\left(v-L\right)\int_L^{b^*} t\frac{1}{\left(v-t\right)^2}dt$$

Exactly one weak bidder is L, one is H, strong bidder mixes. The probability of this event is:

$$2\mu\left(1-\mu\right)\left(1-\frac{\left(H-b^*\right)}{\left(H-L\right)}\right)$$

The cumulative density of the winning bids will be:

$$\frac{G_S^*\left(b\right)-G_S^*\left(L\right)}{1-G_S^*\left(L\right)}G_H^*\left(b\right)$$

The relevant density is then:

$$\frac{1}{1-G_S^*\left(L\right)}\left[G_S^{*\prime}\left(b\right)G_H^*\left(b\right)+G_H^{*\prime}\left(b\right)\left(G_S^*\left(b\right)-G_S^*\left(L\right)\right)\right]$$

$$\begin{aligned} G_H^{*\prime}\left(b\right) &= \left(\frac{1-\mu}{2\mu}\right)\frac{v-L}{\left(v-b\right)^2} \\ G_S^{*\prime}\left(b\right) &= \frac{\left(H-b^*\right)}{\left(H-b\right)^2} \end{aligned}$$

Substituting in the above results gives:

$$\begin{aligned} &\frac{1}{1-\frac{(H-b^*)}{(H-L)}}\left(\frac{1-\mu}{2\mu}\right)\left[\frac{\left(H-b^*\right)}{\left(H-b\right)^2}\left(\frac{b-L}{v-b}\right)+\left(\frac{\left(H-b^*\right)}{\left(H-b\right)}-\frac{\left(H-b^*\right)}{\left(H-L\right)}\right)\frac{v-L}{\left(v-b\right)^2}\right] \\ = &\frac{1}{1-\frac{(H-b^*)}{(H-L)}}\left(\frac{1-\mu}{2\mu}\right)\left(H-b^*\right)\left[\frac{b-L}{\left(H-b\right)^2\left(v-b\right)}+\left(\frac{1}{\left(H-b\right)}-\frac{1}{\left(H-L\right)}\right)\frac{v-L}{\left(v-b\right)^2}\right] \end{aligned}$$

The expected revenue is then:

$$\begin{aligned} R_{AD}\left(1L, mix\right) = &\frac{1}{1-\frac{(H-b^*)}{(H-L)}}\left(\frac{1-\mu}{2\mu}\right)\left(H-b^*\right) * \\ &\int_L^{b^*} t\left[\frac{t-L}{\left(H-t\right)^2\left(v-t\right)}+\left(\frac{1}{\left(H-t\right)}-\frac{1}{\left(H-L\right)}\right)\frac{v-L}{\left(v-t\right)^2}\right]dt \end{aligned}$$

Both weak bidders are H, and strong bidder plays min(v,H). The probability of this event is μ^2, and the revenue in this case is $R_{AD}\left(2H\right) = H$.

Overall expected revenue. Aggregating across the above cases, after appropriate pre-multiplication with probabilities, I obtain:

$$\begin{aligned}
R_{AD} &= (1-\mu)^2 \frac{(H-b^*)}{(H-L)} L \\
&+ (1-\mu)^2 \left(1 - \frac{(H-b^*)}{(H-L)}\right) \frac{1}{1-\frac{(H-b^*)}{(H-L)}} \int_L^{b^*} t \frac{(H-b^*)}{(H-t)^2} dt \\
&+ 2\mu(1-\mu) \frac{(H-b^*)}{(H-L)} \left(\frac{1-\mu}{2\mu}\right)(v-L) \int_L^{b^*} t \frac{1}{(v-t)^2} dt \\
&+ 2\mu(1-\mu) \left(1 - \frac{(H-b^*)}{(H-L)}\right) \frac{1}{1-\frac{(H-b^*)}{(H-L)}} \left(\frac{1-\mu}{2\mu}\right)(H-b^*) * \\
&\quad * \int_L^{b^*} t \left[\frac{t-L}{(H-t)^2(v-t)} + \left(\frac{1}{(H-t)} - \frac{1}{(H-L)}\right)\frac{v-L}{(v-t)^2}\right] dt \\
&+ \mu^2 H
\end{aligned}$$

This simplifies to:

$$\begin{aligned}
R_{AD} &= (1-\mu)^2 (H-b^*) \frac{L}{(H-L)} + \mu^2 H \\
&+ (1-\mu)^2 (H-b^*)(v-L) \int_L^{b^*} t \left[\frac{1}{(H-t)^2(v-t)} + \frac{1}{(H-t)(v-t)^2}\right] dt
\end{aligned}$$

Integrating by parts:

$$\begin{aligned}
R_{AD} &= (1-\mu)^2 (H-b^*) \frac{L}{(H-L)} + \mu^2 H \\
&+ (1-\mu)^2 (H-b^*)(v-L) \left(\left[\frac{t}{(H-t)(v-t)}\right]_L^{b^*} - \int_L^{b^*} \frac{1}{(H-t)(v-t)} dt\right)
\end{aligned}$$

Substituting for boundaries:

$$\begin{aligned}
R_{AD} &= (1-\mu)^2 (H-b^*) \frac{L}{(H-L)} + \mu^2 H \\
&+ (1-\mu)^2 (H-b^*)(v-L) \left(\left[\frac{b^*}{(H-b^*)(v-b^*)}\right] - \left[\frac{L}{(H-L)(v-L)}\right]\right) \\
&- (1-\mu)^2 (H-b^*)(v-L) \int_L^{b^*} \frac{1}{(H-t)(v-t)} dt
\end{aligned}$$

Simplifying and substituting for b^* :

$$\begin{aligned}
R_{AD} &= \mu^2 H \\
&+ (1-\mu)^2 (v-L) \left(\left[\frac{v - \frac{(1-\mu)^2}{1-\mu^2}(v-L)}{\left(\frac{(1-\mu)^2}{1-\mu^2}(v-L)\right)}\right]\right) \\
&- (1-\mu)^2 (H-b^*)(v-L) \int_L^{b^*} \frac{1}{(H-t)(v-t)} dt
\end{aligned}$$

Simplifying again, I finally get:

$$
\begin{aligned}
R_{AD} = \mu^2 H \quad & + \left(1-\mu^2\right) v - (1-\mu)^2 (v-L) \\
& - (1-\mu)^2 (H-b^*) (v-L) \int_L^{b^*} \frac{1}{(H-t)(v-t)} dt
\end{aligned}
$$

14. Appendix E: Derivation of Efficiency, and the Revenue Comparison

14.1. Efficiency in the Ascending Auction

When $v \in [L, H]$, if both weak bidders are low, the winning bidders is the strong bidder, with valuation v. If at least one weak bidder is of H type, then an H-type will win the auction. The efficiency in this case is thus:

$$\text{Eff}_{ASC} = (1-\mu)^2 v + \left(1 - (1-\mu)^2\right) H$$

However, when $v > H$, the strong bidder always wins the ascending auction, and so the winning valuation will be simply v.

14.2. Efficiency in the First-price auction

In this section I use expressions derived in Appendix C, Section 12.4. Efficiency, measured by the expected valuation of the winning bidder, can be calculated as follows. If $\bar{b} > H$ (i.e., $v > v_\beta$) then efficiency is simply v. If $v \in [v_\alpha, v_\beta]$, the efficiency is obtained by the following formula.

$$
\begin{aligned}
\text{Eff}_{FPS} \;=\; & (1-\mu)^2 v \\
& + 2\mu(1-\mu) G_S(L) H \\
& + 2\mu(1-\mu)(1-G_S(L)) \begin{pmatrix} \left(v * \int_L^{\bar{b}} \frac{G_S'(t)}{1-G_S(L)} G_H(t)\, dt\right) \\ + H * \left(1 - \int_L^{\bar{b}} \frac{G_S'(t)}{1-G_S(L)} G_H(t)\, dt\right) \end{pmatrix} \\
& + \mu^2 \left(G_S(L) H + (1-G_S(L)) \begin{pmatrix} \left(v * \int_L^{\bar{b}} \frac{G_S'(t)}{1-G_S(L)} (G_H(t))^2\, dt\right) \\ + H * \left(1 - \int_L^{\bar{b}} \frac{G_S'(t)}{1-G_S(L)} (G_H(t))^2\, dt\right) \end{pmatrix} \right)
\end{aligned}
$$

The individual terms in the above expression contribute to the efficiency measure by the following reasoning. In the case when both weak bidders are L (occurs with probability $(1-\mu)^2$) the strong bidder always wins, and his valuation is v. With probability $2\mu(1-\mu)$, exactly one weak bidder is H. If in this case the strong bidder plays the atom (occurs with probability $G_S(L)$), the H-type weak bidder wins for sure, with value H. However, in the case when the strong bidder mixes (occurs with probability $(1 - G_S(L))$), the strong bidder wins with probability $\int_L^{\bar{b}} \frac{G_S'(t)}{1-G_S(L)} G_H(t)\, dt$, with valuation v, and the H-type weak bidder wins with the residual probability, $1 - \int_L^{\bar{b}} \frac{G_S'(t)}{1-G_S(L)} G_H(t)\, dt$, with value H. The reasoning for the case when there are two H-type weak bidders is precisely analogous to the case with one H-type weak bidder, with the probabilities adjusted accordingly.

14.3. Efficiency in the Anglo-Dutch auction

The reasoning to justify the expressions for efficiency for the Anglo-Dutch auction is precisely analogous to the arguments presented for the first-price case. Thus I just provide a summary of the results. When $v > v_\gamma$, the strong bidder always wins, and the efficiency will be v. For the case when $v \in (L, v_\gamma)$ the efficiency is given by:

$$\begin{aligned}
\mathrm{Eff}_{AD} &= (1-\mu)^2 v \\
&+2\mu(1-\mu) G_S^*(L) H + \\
&+2\mu(1-\mu)(1-G_S^*(L)) \left(\begin{array}{c} \left(v * \int_L^{\bar{b}} \frac{(G_S^*(t))'}{1-G_S^*(L)} G_H^*(t)\, dt \right) \\ +H * \left(1 - \int_L^{\bar{b}} \frac{(G_S^*(t))'}{1-G_S^*(L)} G_H^*(t)\, dt \right) \end{array} \right) \\
&+\mu^2 \max(v, H)
\end{aligned}$$

14.4. Showing the Analytical Revenue Ranking (Proposition 6)

Showing the first part of Proposition 6proceeds as follows. In the case when $v = H$, we have:

$$\bar{b} = H - (1-\mu)^2 (H-L) < H$$

$$b^* = H - \frac{(1-\mu)^2}{1-\mu^2}(H-L) < H$$

$$R_{ASC} = (1-\mu)^2 L + \left(1-(1-\mu)^2\right) H = H - (1-\mu)^2 (H-L)$$

$$\begin{aligned}
R_{FPS} &= H - (1-\mu)^2 (H-L) - \\
&\quad - (1-\mu)(1-\mu)^2 (H-L)(H-L)^{\frac{1}{2}} \int_L^{\bar{b}} \frac{1}{(H-t)(H-t)^{\frac{1}{2}}} dt \\
&= H - (1-\mu)^2 (H-L) - 2(1-\mu)^2 \mu (H-L) < R_{ASC}
\end{aligned}$$

$$\begin{aligned}
R_{AD} &= \mu^2 H + \left(1-\mu^2\right) H - (1-\mu)^2 (H-L) - \\
&\quad - (1-\mu)^2 (H-b^*)(H-L) \int_L^{b^*} \frac{1}{(H-t)(H-t)} dt \\
&= H - (1-\mu)^2 (H-L) - 2\mu \frac{(1-\mu)^3}{1-\mu^2}(H-L) < R_{ASC}
\end{aligned}$$

And also:

$$R_{AD} - R_{FPS} = (1-\mu)^3 (H-L) \left[\frac{2\mu^2}{1-\mu^2} \right] > 0$$

So the revenue ranking in this case is:

$$R_{FPS} < R_{AD} < R_{ASC}$$

This establishes the first statement in Proposition 6. For the second statement, when $H = \bar{b}$, w have:

$$(v - H) = (1-\mu)^2 (v - L)$$

Then the expression for $R_{AD} - R_{FPS}$ becomes:

$$R_{AD} - R_{FPS} = -\mu^2 (1-\mu)^2 (v-L) - (1-\mu)^2 (H-b^*)(v-L) \cdot \\ \cdot \int_L^{b^*} \frac{1}{(H-t)(v-t)} dt < 0$$

So AD does worse than FPS. I can also show that:

$$R_{FPS} = v - (1-\mu)^2 (v-L)$$

$$\begin{aligned} R_{ASC} &= (1-\mu)^2 L + \left(1-(1-\mu)^2\right)\left(v-(1-\mu)^2(v-L)\right) \\ &= v-(1-\mu)^2(v-L) - \mu(2-\mu)(v-L) < R_{FPS} \end{aligned}$$

So we know that in this case:

$$\begin{aligned} R_{AD} &< R_{FPS} \\ R_{ASC} &< R_{FPS} \end{aligned}$$

When $\bar{b} > H$, this means that the first-price auction does not sustain a mixing equilibrium, and always returns revenue of H; both other auctions give lower revenues than this. The ranking between R_{AD} and R_{ASC} is ambiguous. This establishes the second statement in Proposition 6. For the third part, when $b^* = H$, we have:

$$b^* = v - \frac{(1-\mu)^2}{1-\mu^2}(v-L) = H$$

$$\Longrightarrow \bar{b} = v - (1-\mu)^2 \frac{1-\mu^2}{(1-\mu)^2} v - H = \mu^2 v + \left(1-\mu^2\right) H > H$$

So a mixing equilibrium does not exist in a first-price auction. Hence:

$$\begin{aligned} R_{FPS} &= H \\ R_{ASC} &= (1-\mu)^2 L + \left(1-(1-\mu)^2\right) H < H \\ R_{AD} &= \mu^2 H + \left(1-\mu^2\right) v - (1-\mu)^2 (v-L) = H \end{aligned}$$

So revenue ranking is:

$$R_{ASC} < R_{AD} = R_{FPS}$$

When $b^* > H$, neither the first-price or the Anglo-Dutch auction can sustain a mixing equilibrium, and so both auctions return revenues of H; the ascending auction generates less revenue, so long as $\mu < 1$. This concludes the proof of Proposition 6.