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# An Approximation Formula for Basket Option Prices under Local Stochastic Volatility with Jumps: an Application to Commodity Markets

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# An approximation formula for basket option prices under local stochastic volatility with jumps: an application to commodity markets *

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**Abstract**

This paper develops a new approximation formula for pricing basket options in a local-stochastic volatility model with jumps. In particular, the model admits local volatility functions and jump components in not only the underlying asset price processes, but also the volatility processes. To the best of our knowledge, the proposed formula is the first one which achieves an analytical approximation for the basket option prices under this type of the models.

Moreover, in numerical experiments, we provide approximate prices for basket options on the WTI futures and Brent futures based on the parameters through calibration to the plain-vanilla option prices, and confirm the validity of our approximation formula.

## 1 Introduction

The basket options are one of the most popular exotic-type options in the commodity and equity markets. However, it is a tough task to calculate a basket option price with computational speed fast enough for practical purpose, mainly due to the difficulty of the analytical tractability and its high dimensionality. For instance, although the Monte Carlo method is easy to implement, it requires a substantial computational time to obtain an accurate value. Also, the numerical methods for the partial differential equations (PDEs) have been well developed, but it is still very difficult to solve high dimensional PDEs with accuracy and computational speed satisfactory enough in the financial business. To overcome the difficulties, this paper develops a new analytical approximation formula for basket options. In particular, to the best of our knowledge, our approximation formula is the first one which achieves a closed form approximation of basket options under stochastic volatility models with local volatility functions

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and jump components for not only the underlying asset price processes, but also the volatility processes.

There exist a large number of preceding studies on pricing basket options. In the Black-Scholes model, Brigo, Mercurio, Rapisarda and Scotti (2004) applied a moment matching method to approximate basket option prices. Deelstra, Liinev and Vanmaele (2004) derived the lower and upper bounds for basket call option prices with comonotonic approach.

Under a Local volatility model, Takahashi (1999) showed an approximation for basket option prices with an asymptotic expansion technique. Bayer and Laurence (2014) used a heat kernel expansion and the Laplace approximation method to derive very accurate approximate prices of basket options.

In a local volatility jump-diffusion model, Xu and Zheng (2010) derived a forward partial integral differential equation (PIDE) for basket option pricing and approximated its solution. Also, Xu and Zheng (2013) applied the lower bound technique in Rogers and Shi (1995) and the asymptotic expansion method in Kunitomo and Takahashi (2001) to obtain the approximate value of the lower bound of European basket call prices. Moreover, when the local volatility function is time independent, they suggested to have a closed-form expression for their approximation.

Under a local stochastic volatility model, Shiraya and Takahashi (2014) has developed a general pricing method for multi-asset cross currency options which include cross currency options, cross currency basket options and cross currency average options. They also demonstrated that the scheme is able to evaluate options with high dimensional state variables such as 200 dimensions, which is necessary for pricing basket options with 100 underlying assets under stochastic volatility environment. Moreover, in practice, fast calibration is necessary in the option markets relevant for the underlying assets and the currency, which was also achieved in the work.

Models within the class of the so called local stochastic volatility (LSV) model are mainly used in practice: for example, SABR (Hagan, Kumar, Lesniewskie, and Woodward (2002)), ZABR (Andreasen and Huge (2011)), CEV Heston (e.g. Shiraya et al. (2012)) and Quadratic Heston models (e.g. Shiraya et al. (2012)) are well known. Nonetheless, the LSV model is not always enough to fit to a volatility smile and term structure. Hence, some advanced researches investigated a local stochastic volatility with jump model. Among them, Eraker (2004) found that the models with jump components in the underlying price and volatility processes showed better performance in fitting to option prices and the underlying price returns' data simultaneously in stock markets. Pagliarani and Pascucci (2013) derived an analytical approximation of plain-vanilla option prices by applying the adjoint expansion method. However, to the best of our knowledge no works have derived an analytical approximation formula for the option prices under a model which admits a local volatility function and jumps both in the underlying asset price and its volatility processes. This paper develops a formula for pricing basket options under the setting by extending an asymptotic expansion approach. This closed form equation has an advantage in making use of the better calibration to the traded individual options whose underlying assets are included in a basket option's underlying.

In fact, our numerical experiments provide estimates of basket option prices based on the parameters obtained by calibration to the market prices of WTI futures options and Brent futures options. Then, those estimated prices are compared with the prices calculated by Monte Carlo simulations.

An asymptotic expansion approach in finance was initiated by Kunitomo and Takahashi (1992), Yoshida (1992), and Takahashi (1995,1999), which provides us a unified methodology for evaluation of prices and Greeks in general diffusion setting. Recently, the method was further developed to be applied to the forward backward stochastic differential equations (FBSDEs). (See Fujii and Takahashi (2012 a,b,c,d), Takahashi and Yamada (2012, 2013) for the details.)

Although the method was extended to be applied to a jump-diffusion model by Kunitomo and Takahashi (2004) and Takahashi(2007,2009), they concentrated on approximation of only bond prices or/and plain-vanilla option prices under a local volatility jump-diffusion model, and did not derive higher order expansions than the first order for the option pricing. Subsequently, Takahashi and Takehara (2010) found a scheme for pricing plain-vanilla options in a jump-diffusion with stochastic volatility model. However, thanks to a linear structure of the underlying asset price process in their model they separated the jump component with a known characteristic function and then applied the expansion technique developed in the diffusion models. Hence, their scheme can not be applied directly to more general models nor basket option pricing. The current work generalizes these preceding researches in the asymptotic expansion approach.

The organization of the paper is as follows: After the next section briefly describes our model for basket options, Section 3 derives a new approximate pricing formula, and Sections 4 and 5 show numerical examples. Particularly, Section 5 provide approximate prices for basket options on the WTI futures and Brent futures based on the parameters through calibration to the plain vanilla option prices. Section 6 concludes. Appendix shows the derivation of the coefficients in the pricing equation and the conditional expectation formulas necessary for obtaining the main theorem.

## 2 Model

This section shows the model of the underlying asset prices and their volatility processes, which is used for pricing the European type basket options.

In particular, suppose that the filtered probability space $(\Omega, \mathbf{F}, \mathbf{P}, \{\mathbf{F}_t\}_{t\geq 0})$ is given, where $\mathbf{P}$ is an equivalent martingale measure and the filtration satisfies the usual conditions. Then, $(S_t^i)_{t\in[0,T]}$ and $(\sigma_t^i)_{t\in[0,T]}$, $i = 1, \cdots, d$ represent the underlying asset prices and their volatilities for $t \in [0, T]$, respectively. Particularly, let us assume that $S_T^i$ and $\sigma_T^i$ are given by the solutions of the following stochastic integral equations:

$$S_T^i \;=\; s_0^i + \int_0^T \alpha^i S_{t-}^i dt + \int_0^T \phi_{S^i}\left(\sigma_{t-}^i, S_{t-}^i\right) dW_t^{S^i}$$

$$
\begin{aligned}
& +\sum_{l=1}^{n}\left(\sum_{j=1}^{N_{l,T}} h_{S^i,l,j} S^i_{\tau_{j,l}-} - \int_0^T \Lambda_l S^i_{t-}\mathbf{E}[h_{S^i,l,1}]dt\right), \qquad (1)\\
\sigma^i_T \;=\;& \sigma^i_0 + \int_0^T \lambda^i(\theta^i - \sigma^i_{t-})dt + \int_0^T \phi_{\sigma^i}\left(\sigma^i_{t-}\right) dW^{\sigma^i}_t \\
& +\sum_{l=1}^{n}\left(\sum_{j=1}^{N_{l,T}} h_{\sigma^i,l,j} \sigma^i_{\tau_{j,l}-} - \int_0^T \Lambda_l \sigma^i_{t-}\mathbf{E}[h_{\sigma^i,l,1}]dt\right), \qquad (2)
\end{aligned}
$$

where $s^i_0$ and $\sigma^i_0$, $i = 1, \cdots, d$ are given as some constants. The notations are defined as follows:

- $\alpha^i$ $(i = 1, \cdots, d)$ are constants.

  $\lambda^i$ and $\theta^i$ $(i = 1, \cdots, d)$ are nonnegative constants.
- $\phi_{S^i}(x, y)$ and $\phi_{\sigma^i}(x)$ are some functions with appropriate regularity conditions.
- $W^{S^i}$ and $W^{\sigma^i}$, $(i = 1, \cdots, d)$ are correlated Brownian motions.
- Each $N_l$, $(l = 1, \cdots, n)$ is a Poisson process with constant intensity $\Lambda_l$. $N_l$, $l = 1, \cdots, n$ are independent, and also independent of all $W^{S^i}$ and $W^{\sigma^i}$.
- $\tau_{j,l}$ stands for the $j$-th jump time of $N_l$.
- For each $l = 1, \cdots, n$ and $i = 1, \cdots, d$, both $\left(\sum_{j=1}^{N_{l,t}} h_{S^i,l,j}\right)_{t\geq 0}$ and $\left(\sum_{j=1}^{N_{l,t}} h_{\sigma^i,l,j}\right)_{t\geq 0}$ are compound Poisson processes. ($\sum_{j=1}^{N_{l,t}} \equiv 0$ when $N_{l,t} = 0$).
- For each $l$ and $x^i$, $h_{x^i,l,j}$ $(j \in \mathbf{N})$ are independent and identically distributed random variables, where $x^i$ stands for one of $S^i$ and $\sigma^i$ $(i = 1, \cdots, d)$.
  - for the constant jump case, $h_{x^i,l,j} = H_{x^i,l}$ for some constant $H_{x^i,l}$ in all $j$.
  - for the log-normal jump case, $h_{x^i,l,j} = e^{Y_{x^i,l,j}} - 1$, where $Y_{x^i,l,j}$ is a random variable which follows a normal distribution with mean $m_{x^i,l}$ and variance $\gamma^2_{x^i,l}$ that is, $N(m_{x^i,l}, \gamma^2_{x^i,l})$.
- $h_{x^i,l,j}$ and $h_{x^{i'},l',j'}$ $(l \neq l')$ are independent.

  $h_{x^i,l,j}$ and $h_{x^{i'},l',j'}$ $(j \neq j')$ are independent.

  $N_l$ and $h_{x^i,l',j}$ are independent.

  For the same $l$ and $j$, $h_{S^i,l,j}$ and $h_{\sigma^{i'},l,j}$ $(i, i' = 1, \cdots, d)$ are allowed to be dependent, that is $Y_{S^i,l,j}$ and $Y_{\sigma^{i'},l,j}$ $(i, i' = 1, \cdots, d)$ are generally correlated.

**Remark .** *By specifying the functions $\phi_{S^i}$ and $\phi_{\sigma^i}$, we can express various types of local-stochastic volatility models. For example, the model with $\phi_{S^i}(\sigma, S) = (aS^2 + bS + c)\sqrt{\sigma}$ and $\phi_{\sigma^i}(\sigma) = \sqrt{\sigma}$ corresponds to an extension of the Quadratic Heston model. The model with $\phi_{S^i}(\sigma, S) = S^{\beta_S}\sigma$ and $\phi_{\sigma^i}(\sigma) = \sigma$ corresponds to an extended SABR ($\lambda$-SABR) model, and the one with $\phi_{S^i}(\sigma, S) = S^{\beta_S}\sigma$, $\phi_{\sigma^i}(\sigma) = \sigma^{\beta_\sigma}$ and $\lambda = 0$ corresponds to a local volatility on volatility with jumps model.*

## 3 New Pricing formula for Basket Option

In this section, we derive an approximation formula for the basket option price in the following steps.

1. Introduce perturbation parameter $\epsilon$ to the model processes, and expand the processes with respect to $\epsilon$ around $\epsilon = 0$ as in Proposition 3.1.

2. Substitute the expanded processes for the payoff function, and expand the payoff function with respect to $\epsilon$ around $\epsilon = 0$.

3. Take the conditional expectation of each term in the expanded payoff functions to calculate analytically the expectation of the expanded payoff functions.

4. Use Lemma 3.2 and Appendix B to calculate each conditional expectation. In the conditional expectation, each formula in Lemma 3.2 or Appendix B is applied according to the type of the functional form of the integrand, and the calculation results are given in Appendix A.

5. Collecting these terms in Appendix A with the same order of the Hermite polynomials, we obtain the coefficients in Theorem 3.3.

First, we introduce perturbations to the model (1) and (2). That is, for a known parameter $\epsilon \in [0, 1]$ we consider the following stochastic integral equations: for $i = 1, \cdots, d$,

$$
\begin{aligned}
S_T^{i,(\epsilon)} &= s_0^i + \int_0^T \alpha^i S_{t-}^{i,(\epsilon)} dt + \epsilon \int_0^T \phi_{S^i}\left(\sigma_{t-}^{i,(\epsilon)}, S_{t-}^{i,(\epsilon)}\right) dW_t^{S^i} \\
&\quad + \sum_{l=1}^n \left( \sum_{j=1}^{N_{l,T}} h_{S^i,l,j}^{(\epsilon)} S_{\tau_{j,l}-}^{i,(\epsilon)} - \int_0^T \Lambda_l S_{t-}^{i,(\epsilon)} \mathbf{E}[h_{S^i,l,1}^{(\epsilon)}] dt \right), \qquad (3) \\
\sigma_T^{i,(\epsilon)} &= \sigma_0^i + \int_0^T \lambda^i(\theta^i - \sigma_{t-}^{i,(\epsilon)}) dt + \epsilon \int_0^T \phi_{\sigma^i}\left(\sigma_{t-}^{i,(\epsilon)}\right) dW_t^{\sigma^i} \\
&\quad + \sum_{l=1}^n \left( \sum_{j=1}^{N_{l,T}} h_{\sigma^i,l,j}^{(\epsilon)} \sigma_{\tau_{j,l}-}^{i,(\epsilon)} - \int_0^T \Lambda_l \sigma_{t-}^{i,(\epsilon)} \mathbf{E}[h_{\sigma^i,l,1}^{(\epsilon)}] dt \right). \qquad (4)
\end{aligned}
$$

Here, $h_{x^i,l,j}^{(\epsilon)} = \epsilon H_{x^i,l}$ for all $j$ in the constant jump case. $h_{x^i,l,j}^{(\epsilon)} = e^{\epsilon Y_{x^i,l,j}} - 1$, where $\epsilon Y_{x^i,l,j} \sim N(\epsilon m_{x^i,l}, \epsilon^2 \gamma_{x^i,l}^2)$ in the log-normal jump case. Note that $h_{x^i,l,j}^{(0)} = 0$ in the both cases.

We also define the following perturbed model with no jump processes, $S^{i,LSV(\epsilon)}$ and $\sigma^{i,LSV(\epsilon)}$, which will be used for our approximation of the basket option pricing: for $i = 1, \cdots, d$,

$$
S_T^{i,LSV(\epsilon)} = s_0^i + \int_0^T \alpha^i S_t^{i,LSV(\epsilon)} dt + \epsilon \int_0^T \phi_{S^i}\left(\sigma_t^{i,LSV(\epsilon)}, S_t^{i,LSV(\epsilon)}\right) dW_t^{S^i}, \qquad (5)
$$

$$\sigma_T^{i,LSV(\epsilon)} = \sigma_0^i + \int_0^T \lambda^i(\theta^i - \sigma_t^{i,LSV(\epsilon)})dt + \epsilon \int_0^T \phi_{\sigma^i}\left(\sigma_t^{i,LSV(\epsilon)}\right) dW_t^{\sigma^i}. \tag{6}$$

We assume the asymptotic expansions of $S_T^{i,(\epsilon)}$ and $\sigma_T^{i,(\epsilon)}$ around $\epsilon = 0$ as follows:

$$S_T^{i,(\epsilon)} = S_T^{i,(0)} + \epsilon S_T^{i,(1)} + \frac{\epsilon^2}{2!} S_T^{i,(2)} + \cdots, \tag{7}$$

$$\sigma_T^{i,(\epsilon)} = \sigma_T^{i,(0)} + \epsilon \sigma_T^{i,(1)} + \frac{\epsilon^2}{2!} \sigma_T^{i,(2)} + \cdots, \tag{8}$$

$$h_{x^i,l,j}^{(\epsilon)} = h_{x^i,l,j}^{(0)} + \epsilon h_{x^i,l,j}^{(1)} + \frac{\epsilon^2}{2!} h_{x^i,l,j}^{(2)} + \cdots, \tag{9}$$

where $S_t^{i,(\iota)} := \frac{\partial^\iota S_t^{i,(\epsilon)}}{\partial \epsilon^\iota}\Big|_{\epsilon=0}$, $\sigma_t^{i,(\iota)} := \frac{\partial^\iota \sigma_t^{i,(\epsilon)}}{\partial \epsilon^\iota}\Big|_{\epsilon=0}$, $h_{x^i,l,j}^{(\iota)} := \frac{\partial^\iota h_{x^i,l,j}^{(\epsilon)}}{\partial \epsilon^\iota}\Big|_{\epsilon=0}$.

We also suppose that $(W^{S^1}, \cdots, W^{S^d}, W^{\sigma^1}, \cdots, W^{\sigma^d})' = \varrho \cdot Z$ where $\varrho$ is a $2d \times 2d$ correlation matrix, and $Z$ is a $2d$-dimensional (independent) Brownian motion.

Firstly, we consider a simple case with one asset and one jump factor, that is $i = 1$ and $l = 1$ in the above model:

$$\begin{aligned} S_T^{(\epsilon)} &= s_0 + \int_0^T \alpha S_{t-}^{(\epsilon)} dt + \epsilon \int_0^T \phi_{S^i}\left(\sigma_{t-}^{(\epsilon)}, S_{t-}^{(\epsilon)}\right) dZ_t^S \\ &\quad + \sum_{j=1}^{N_T} h_{S,j}^{(\epsilon)} S_{\tau_j-}^{(\epsilon)} - \int_0^T \Lambda S_{t-}^{(\epsilon)} \mathbf{E}[h_{S,1}^{(\epsilon)}]dt, \end{aligned} \tag{10}$$

$$\begin{aligned} \sigma_T^{(\epsilon)} &= \sigma_0 + \int_0^T \lambda(\theta - \sigma_{t-}^{(\epsilon)})dt + \epsilon \int_0^T \phi_{\sigma^i}\left(\sigma_{t-}^{(\epsilon)}\right) dZ_t^\sigma \\ &\quad + \sum_{j=1}^{N_T} h_{\sigma,j}^{(\epsilon)} \sigma_{\tau_j-}^{(\epsilon)} - \int_0^T \Lambda \sigma_{t-}^{(\epsilon)} \mathbf{E}[h_{\sigma,1}^{(\epsilon)}]dt. \end{aligned} \tag{11}$$

We derive $S_T^{(0)}$ and $S_T^{(1)}$ explicitly.

$S_T^{(0)}$ is calculated as follows:

$$\begin{aligned} S_T^{(0)} &= s_0 + \Bigg( \int_0^T \alpha\left(S_{t-}^{(0)} + \epsilon S_{t-}^{(1)} + \cdots\right) dt \\ &\quad + \epsilon \int_0^T \phi_{S^i}\left(\sigma_{t-}^{(0)} + \epsilon \sigma_{t-}^{(1)} + \cdots, S_{t-}^{(0)} + \epsilon S_{t-}^{(1)} + \cdots\right) dZ_t^S \\ &\quad + \sum_{j=1}^{N_T} \left(h_{S,j}^{(0)} + \epsilon h_{S,j}^{(1)} + \cdots\right)\left(S_{\tau_j-}^{(0)} + \epsilon S_{\tau_j-}^{(1)} + \cdots\right) \\ &\quad - \int_0^T \Lambda\left(S_{t-}^{(0)} + \epsilon S_{t-}^{(1)} + \cdots\right) \mathbf{E}[h_{S,1}^{(0)} + \epsilon h_{S,1}^{(1)} + \cdots]dt \Bigg)\Bigg|_{\epsilon=0} \\ &= s_0 + \int_0^T \alpha_{t-} S_{t-}^{(0)} dt. \end{aligned} \tag{12}$$

$S_T^{(0)}$ can be solved as $S_T^{(0)} = e^{\alpha T} s_0$, and $\sigma_T^{(0)} = \theta + (\sigma_0 - \theta)e^{-\lambda T}$ is derived in the same way.

Next, we calculate $S_T^{(1)}$.

$$\begin{aligned}
S_T^{(1)} &= \frac{\partial S_T^{(\epsilon)}}{\partial \epsilon}\Big|_{\epsilon=0} \\
&= \left( \int_0^T \alpha_{t-} \left( S_{t-}^{(1)} + \epsilon S_{t-}^{(2)} + \cdots \right) dt \right. \\
&\quad + \int_0^T \phi_{S^i} \left( \sigma_{t-}^{(0)} + \epsilon \sigma_{t-}^{(1)} + \cdots , S_{t-}^{(0)} + \epsilon S_{t-}^{(1)} + \cdots \right) dZ_t^S \\
&\quad + \epsilon \int_0^T \phi_{S^i} \left( \sigma_{t-}^{(1)} + \epsilon \sigma_{t-}^{(2)} + \cdots , S_{t-}^{(0)} + \epsilon S_{t-}^{(1)} + \cdots \right) dZ_t^S \\
&\quad + \epsilon \int_0^T \phi_{S^i} \left( \sigma_{t-}^{(0)} + \epsilon \sigma_{t-}^{(1)} + \cdots , S_{t-}^{(1)} + \epsilon S_{t-}^{(2)} + \cdots \right) dZ_t^S \\
&\quad + \sum_{j=1}^{N_T} \left( h_{S,j}^{(1)} + \epsilon h_{S,j}^{(2)} + \cdots \right) \left( S_{\tau_j-}^{(0)} + \epsilon S_{\tau_j-}^{(1)} + \cdots \right) \\
&\quad + \sum_{j=1}^{N_T} \left( h_{S,j}^{(0)} + \epsilon h_{S,j}^{(1)} + \cdots \right) \left( S_{\tau_j-}^{(1)} + \epsilon S_{\tau_j-}^{(2)} + \cdots \right) \\
&\quad - \int_0^T \Lambda \left( S_{t-}^{(1)} + \epsilon S_{t-}^{(2)} + \cdots \right) \mathbf{E}[h_{S,1}^{(0)} + \epsilon h_{S,1}^{(1)} + \cdots ] dt \\
&\quad \left. - \int_0^T \Lambda \left( S_{t-}^{(0)} + \epsilon S_{t-}^{(1)} + \cdots \right) \mathbf{E}[h_{S,1}^{(1)} + \epsilon h_{S,1}^{(2)} + \cdots ] dt \right)\Big|_{\epsilon=0} \\
&= \int_0^T \alpha_{t-} S_{t-}^{(1)} dt + \int_0^T \phi_{S^i} \left( \sigma_{t-}^{(0)}, S_{t-}^{(0)} \right) dZ_t^S \\
&\quad + \sum_{j=1}^{N_T} h_{S,j}^{(1)} S_{\tau_j-}^{(0)} - \int_0^T \Lambda S_{t-}^{(0)} \mathbf{E}[h_{S,1}^{(1)}] dt. \qquad (13)
\end{aligned}$$

This equation can be solved by method of variation of constants as:

$$\begin{aligned}
S_T^{(1)} &= \int_0^T e^{\alpha(T-t)} \Phi_S \left( \sigma_{t-}^{(0)}, S_{t-}^{(0)} \right) dZ_t^S + \sum_{j=1}^{N_T} h_{S,j}^{(1)} e^{\alpha T} s_0 \\
&\quad - \Lambda \mathbf{E}[h_{S,1}^{(1)}] e^{\alpha T} s_0 T. \qquad (14)
\end{aligned}$$

$S_T^{(i)}$ $(i = 2, 3, \cdots)$ can be derived in a similar manner.

We explain the multi dimensional case (3), (4), (5), (6) based on these results. For ease of the expressions we introduce the following notations:

- $\Phi_{S^i,j} := \phi_{S^i}(\sigma^i, S^i)(\varrho)_{i,j}$ and $\Phi_{\sigma^i,j} := \phi_{\sigma^i}(\sigma^i)(\varrho)_{d+i,j}$, where $(\varrho)_{i,j}$ denotes the $(i,j)$-element of $\varrho$.

- $\Phi_{S^i} := (\Phi_{S^i,1}, \cdots, \Phi_{S^i,2d})$ and $\Phi_{\sigma^i} := (\Phi_{\sigma^i,1}, \cdots, \Phi_{\sigma^i,2d})$ are $2d$-dimensional vectors.

- $\Phi_S := (\Phi_{S^1}, \cdots, \Phi_{S^d})'$ and $\Phi_\sigma := (\Phi_{\sigma^1}, \cdots, \Phi_{\sigma^d})'$ are $d \times 2d$ matrices.

- We define a operator "$*$" as follows: When $A$ and $B$ are $d \times 2d$ matrices,

$$A * B := \begin{bmatrix} (A)_{1,1}(B)_{1,1} & \cdots & (A)_{1,2d}(B)_{1,2d} \\ \vdots & \ddots & \vdots \\ (A)_{d,1}(B)_{d,1} & \cdots & (A)_{d,2d}(B)_{d,2d} \end{bmatrix}. \tag{15}$$

When $A$ is a $d \times 2d$ matrix and $B$ is a $d$-dimensional vector,

$$A * B = B * A := \begin{bmatrix} (A)_{1,1}(B)_{1} & \cdots & (A)_{1,2d}(B)_{1} \\ \vdots & \ddots & \vdots \\ (A)_{d,1}(B)_{d} & \cdots & (A)_{d,2d}(B)_{d} \end{bmatrix}. \tag{16}$$

When $A$ and $B$ are $d$-dimensional vectors,

$$A * B := \begin{bmatrix} (A)_{1}(B)_{1} \\ \vdots \\ (A)_{d}(B)_{d} \end{bmatrix}. \tag{17}$$

- We also define $\partial_x \Phi_{\hat{x}}$ ($x = S$ or $\sigma$, $\hat{x} = S$ or $\sigma$) as

$$\partial_x \Phi_{\hat{x}} := \begin{bmatrix} \frac{\partial}{\partial x^1}(\Phi_{\hat{x}})_{1,1} & \cdots & \frac{\partial}{\partial x^1}(\Phi_{\hat{x}})_{1,2d} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x^d}(\Phi_{\hat{x}})_{d,1} & \cdots & \frac{\partial}{\partial x^d}(\Phi_{\hat{x}})_{d,2d} \end{bmatrix}, \tag{18}$$

where $(\Phi_{\hat{x}})_{i,j}$ denotes the $(i,j)$-element of the $d \times 2d$ matrix $\Phi_{\hat{x}}$.

- Let us introduce the following notations:
  $S_t = (S_t^1, \cdots, S_t^d)$, $\sigma_t = (\sigma_t^1, \cdots, \sigma_t^d)$,
  $S_t^{(j)} = (S_t^{1,(j)}, \cdots, S_t^{d,(j)})$, $\sigma_t^{(j)} = (\sigma_t^{1,(j)}, \cdots, \sigma_t^{d,(j)})$,
  $h_{S,l,j}^{(i)} = (h_{S^1,l,j}^{(i)}, \cdots, h_{S^d,l,j}^{(i)})$, $h_{\sigma,l,j}^{(i)} = (h_{\sigma^1,l,j}^{(i)}, \cdots, h_{\sigma^d,l,j}^{(i)})$,
  $e^{\alpha t} = (e^{\alpha^1 t}, \cdots, e^{\alpha^d t})$ and $e^{\lambda t} = (e^{\lambda^1 t}, \cdots, e^{\lambda^d t})$.

Based on these preparations, we obtain the next proposition.

**Proposition 3.1.** *1. The coefficients, $S_T^{(i)}$, $h_{x,l,j}^{(i)}$ $(x =, S, \sigma)$, $i = 0, 1, 2$ and $\sigma_T^{(i)}$ $i = 0, 1$ in the expansions (7), (8) and (9) are given as follows:*

$$S_T^{(0)} = e^{\alpha T} * s_0, \tag{19}$$

$$\sigma_T^{(0)} = \theta + (\sigma_0 - \theta) * e^{-\lambda T}, \tag{20}$$

$$h_{x,l,j}^{(0)} = 0, \tag{21}$$

$$\begin{aligned} S_T^{(1)} &= \int_0^T e^{\alpha(T-t)} * \Phi_S\left(\sigma_{t-}^{(0)}, S_{t-}^{(0)}\right) dZ_t \\ &\quad + \sum_{l=1}^{n}\left(\sum_{j=1}^{N_{l,T}} h_{S,l,j}^{(1)} - \Lambda_l T\mathbf{E}\left[h_{S,l,1}^{(1)}\right]\right) * S_T^{(0)}, \end{aligned} \tag{22}$$

$$\begin{aligned} \sigma_T^{(1)} &= \int_0^T e^{-\lambda(T-t)} * \Phi_\sigma\left(\sigma_{t-}^{(0)}\right) dZ_t + \sum_{l=1}^{n}\left(\sum_{j=1}^{N_{l,T}} h_{\sigma,l,j}^{(1)} * e^{-\lambda(T-\tau_{j,l})} * \sigma_{\tau_{j,l}-}^{(0)}\right. \\ &\quad \left. -\Lambda_l \mathbf{E}\left[h_{\sigma,l,1}^{(1)}\right] * e^{-\lambda T} * \int_0^T e^{\lambda t} * \sigma_{t-}^{(0)} dt\right), \end{aligned} \tag{23}$$

$$h_{x,l,j}^{(1)} = H_{x,l} := (H_{x^1,l}, \cdots, H_{x^d,l}), \text{ (for all } j\text{, constant jump case)} \tag{24}$$

$$h_{x,l,j}^{(1)} = Y_{x,l,j} := (Y_{x^1,l,j}, \cdots, Y_{x^d,l,j}), \text{ (log-normal jump case)} \tag{25}$$

$$\begin{aligned} S_T^{(2)} &= 2\int_0^T e^{\alpha(T-t)} * \partial_S \Phi_S\left(\sigma_{t-}^{(0)}, S_{t-}^{(0)}\right) * S_{t-}^{(1)} dZ_t \\ &\quad + 2\int_0^T e^{\alpha(T-t)} * \partial_\sigma \Phi_S\left(\sigma_{t-}^{(0)}, S_{t-}^{(0)}\right) * \sigma_{t-}^{(1)} dZ_t \\ &\quad + \sum_{l=1}^{n}\left(\left\{\sum_{j=1}^{N_{l,T}} h_{S,l,j}^{(2)} - \Lambda_l T\mathbf{E}\left[h_{S,l,1}^{(2)}\right]\right\} * S_T^{(0)}\right. \\ &\quad \left. + 2\sum_{j=1}^{N_{l,T}} h_{S,l,j}^{(1)} * e^{\alpha(T-\tau_{j,l})} * S_{\tau_{j,l}-}^{(1)} - 2\Lambda_l \mathbf{E}\left[h_{S,l,1}^{(1)}\right] * e^{\alpha T} * \int_0^T e^{-\alpha t} * S_{t-}^{(1)} dt\right), \end{aligned} \tag{26}$$

$$h_{x,l,j}^{(2)} = 0 \in \mathbf{R}^{\mathbf{d}}, \text{ (for all } j\text{, constant jump case)} \tag{27}$$

$$h_{x,l,j}^{(2)} = Y_{x,l,j} * Y_{x,l,j}. \text{ (log-normal jump case)} \tag{28}$$

*2. The coefficients, $S_T^{LSV(i)}$ $(i = 1, 2, 3)$ and $\sigma_T^{LSV(i)}$ $(i = 1, 2)$ in the asymptotic expansions of (5) and (6) are given as follows:*

$$S_T^{LSV(1)} = \int_0^T e^{\alpha(T-t)} * \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right) dZ_t, \tag{29}$$

$$\sigma_T^{LSV(1)} = \int_0^T e^{-\lambda(T-t)} * \Phi_\sigma\left(\sigma_t^{(0)}\right) dZ_t, \tag{30}$$

$$
\begin{aligned}
S_T^{LSV(2)} &= 2\int_0^T e^{\alpha(T-t)} * \partial_S \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right) * S_t^{LSV(1)} dZ_t \\
&\quad + 2\int_0^T e^{\alpha(T-t)} * \partial_\sigma \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right) * \sigma_t^{LSV(1)} dZ_t, \qquad (31)\\
\sigma_T^{LSV(2)} &= 2\int_0^T e^{-\lambda(T-t)} * \partial_\sigma \Phi_\sigma\left(\sigma_t^{(0)}\right) * \sigma_t^{LSV(1)} dZ_t \\
&= 2\int_0^T e^{-\lambda(T-t)} * \partial_\sigma \Phi_\sigma\left(\sigma_t^{(0)}\right) * \int_0^t e^{-\lambda(t-u)} * \Phi_\sigma\left(\sigma_u^{(0)}\right) dZ_u dZ_t, \\
& \qquad (32)\\
S_T^{LSV(3)} &= 6\int_0^T e^{\alpha(T-t)} * \partial_S^2 \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right) * (S_t^{LSV(1)}) * (S_t^{LSV(1)}) dZ_t \\
&\quad + 6\int_0^T e^{\alpha(T-t)} * \partial_S \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right) * S_t^{LSV(2)} dZ_t \\
&\quad + 6\int_0^T e^{\alpha(T-t)} * \partial_\sigma^2 \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right) * (\sigma^{LSV(1)}) * (\sigma^{LSV(1)}) dZ_t \\
&\quad + 6\int_0^T e^{\alpha(T-t)} * \partial_\sigma \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right) * \sigma^{LSV(2)} dZ_t. \qquad (33)
\end{aligned}
$$

Next, let us define the payoff of a basket call option with strike price $K$ as

$$
\begin{aligned}
&(g(x) - K)^+ \; (:= \max\{g(x) - K, 0\}), \qquad (34)\\
&g(x) := w \cdot x = \sum_{i=1}^d w_i x^i,
\end{aligned}
$$

where $g(x)$ represents a weighted sum of the underlying asset prices of $x^1, \cdots, x^d$ with the constant (both positive and negative) weights $w_1, \cdots, w_d$. Here, we set $x := (x^1, \cdots, x^d)$ and $w := (w_1, \cdots, w_d)$.

For an approximation of a basket option price, we firstly note that $g\left(S_T^{(\epsilon)}\right)$ is expanded around $\epsilon = 0$ as:

$$
g\left(S_T^{(\epsilon)}\right) = g\left(S_T^{(0)}\right) + \epsilon g\left(S_T^{(1)}\right) + \frac{\epsilon^2}{2} g\left(S_T^{(2)}\right) + \frac{\epsilon^3}{6} g\left(S_T^{(3)}\right) + o(\epsilon^3). \qquad (35)
$$

Then, for a strike price $K = g(S_T^0) - \epsilon y$ for an arbitrary $y \in \mathbf{R}$, the payoff of the call option with maturity $T$ is expanded as follows:

$$
\begin{aligned}
\left(g\left(S_T^{(\epsilon)}\right) - K\right)^+ &= \epsilon\left(\frac{g(S_T^{(\epsilon)}) - g(S_T^{(0)})}{\epsilon} + y\right)^+ \\
&= \epsilon\left(g\left(S_T^{(1)}\right) + \frac{\epsilon}{2} g\left(S_T^{(2)}\right) + \frac{\epsilon^2}{6} g\left(S_T^{(3)}\right) + y + o(\epsilon^2)\right)^+ \\
&= \epsilon\left(g(S_T^{(1)}) + y\right)^+ + \frac{\epsilon^2}{2} 1_{\left\{g(S_T^{(1)}) > -y\right\}} g\left(S_T^{(2)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\epsilon^3\left(\frac{1}{6}1_{\left\{g(S_T^{(1)})>-y\right\}}g\left(S_T^{(3)}\right)+\frac{1}{8}\delta_{\left\{g(S_T^{(1)})=-y\right\}}g\left(S_T^{(2)}\right)^2\right)\\
&+o(\epsilon^3)\\
\approx\ &\epsilon\left(g(S_T^{(1)})+y\right)^+ +\frac{\epsilon^2}{2}1_{\left\{g(S_T^{(1)})>-y\right\}}g\left(S_T^{(2)}\right)\\
&+\epsilon^3\left(\frac{1}{6}1_{\left\{g(S_T^{(1)})>-y\right\}}g\left(S_T^{LSV(3)}\right)+\frac{1}{8}\delta_{\left\{g(S_T^{(1)})=-y\right\}}g\left(S_T^{LSV(2)}\right)^2\right).
\end{aligned}
\tag{36}
$$

We apply $g\left(S_T^{LSV(3)}\right)$ and $g\left(S_T^{LSV(2)}\right)^2$ instead of $g\left(S_T^{(3)}\right)$ and $g\left(S_T^{(2)}\right)^2$, which provides reasonable accuracies with less computational burden in the approximations.

We next note that when the number of jumps is $k_l$ ($l = 1, \cdots, n$), that is on $\{N_l = k_l\} := \{N_{1,T} = k_1, \cdots, N_{n,T} = k_n\}$, $S_T^{(1)}$ in the equation (22) becomes

$$
\xi_{\{k_l\}} + \hat{S}_T, \tag{37}
$$

where

$$
\xi_{\{k_l\}} := \sum_{l=1}^{n}(k_l - \Lambda_l T)m_{S,l} * e^{\alpha T} * s_0, \tag{38}
$$

(constant jump)

$$
\hat{S}_T := \int_0^T e^{\alpha(T-t)} * \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right)dZ_t, \tag{39}
$$

(log-normal jump)

$$
\hat{S}_T := \int_0^T e^{\alpha(T-t)} * \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right)dZ_t + \sum_{l=1}^{n}\left(\sum_{j=1}^{k_l}\gamma_{S,l} * \zeta_{S,j,l} * e^{\alpha T} * s_0\right). \tag{40}
$$

Here, we use the following notations:

- $\gamma_{S,l} = (\gamma_{S^1,l}, \cdots, \gamma_{S^d,l})$ and $s_0 = (s_0^1, \cdots, s_0^d)$
- $\zeta_{S,j,l} = (\zeta_{S^1,j,l}, \cdots, \zeta_{S^d,j,l})$ and $\zeta_{\sigma,j,l} = (\zeta_{\sigma^1,j,l}, \cdots, \zeta_{\sigma^d,j,l})$ are vectors of random variables, where $\zeta_{S^i,j,l}$ and $\zeta_{\sigma^i,j,l}$ follow $N(0,1)$, that is the standard normal distribution.

  ($\vartheta$ is defined to be the $2d \times 2d$ correlation matrix among $\zeta_{S^i,j,l}$ and $\zeta_{\sigma^i,j,l}$, $i = 1, \cdots, d$, though it does not explicitly appear here.)

We remark that the distribution of $g(\hat{S}_T)$ is $N\left(0, \Sigma_T^{\{k_l\}}\right)$, that is the normal distribution with mean zero and variance $\Sigma_T^{\{k_l\}}$ whose density function is expressed as

$$
n(x; 0, \Sigma_T^{\{k_l\}}) := \frac{1}{\sqrt{2\pi\Sigma_T^{\{k_l\}}}}\exp\left\{\frac{-x^2}{2\Sigma_T^{\{k_l\}}}\right\}. \tag{41}
$$

Here, $\Sigma_T^{\{k_l\}}$ is defined as follows: (constant jump)

$$\Sigma_T^{\{k_l\}} := \int_0^T \left(w * e^{\alpha(T-t)} * \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right)\right)^\top \left(w * e^{\alpha(T-t)} * \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right)\right) dt, \tag{42}$$

(log-normal jump)

$$\begin{aligned}\Sigma_T^{\{k_l\}} := &\int_0^T \left(w * e^{\alpha(T-t)} * \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right)\right)^\top \left(w * e^{\alpha(T-t)} * \Phi_S\left(\sigma_t^{(0)}, S_t^{(0)}\right)\right) dt \\ &+ \sum_{l=1}^n k_l (w * \gamma_{S,l} * e^{\alpha T} * s_0)^\top \vartheta_{\zeta_{S,l}} (w * \gamma_{S,l} * e^{\alpha T} * s_0),\end{aligned} \tag{43}$$

where $\vartheta_{\zeta_{S,l}}$ stands for the correlation matrix of $\zeta_{S,j,l} = (\zeta_{S^1,j,l}, \cdots, \zeta_{S^d,j,l})$, and $x^\top$ denotes the transpose of $x$.

Next, we define

$$\eta_2(x, \{k_l\}) = \mathbf{E}\left[g\left(S_T^{(2)}\right)\Big| g(\hat{S}_T) = x, \{N_l = k_l\}\right], \tag{44}$$

$$\eta_3(x, \{k_l\}) = \mathbf{E}\left[g\left(S_T^{LSV(3)}\right)\Big| g(\hat{S}_T) = x, \{N_l = k_l\}\right], \tag{45}$$

$$\eta_{22}(x, \{k_l\}) = \mathbf{E}\left[g\left(S_T^{LSV(2)}\right)^2\Big| g(\hat{S}_T) = x, \{N_l = k_l\}\right]. \tag{46}$$

With those preparations, we approximate the expectation of the basket call payoff under an equivalent martingale measure in the following way:

$$\begin{aligned}&\mathbf{E}\left[\left(g\left(S_T^{(\epsilon)}\right) - K\right)^+\right] \\ &\approx \epsilon \mathbf{E}\left[\mathbf{E}\left[\left(g(S_T^{(1)}) + y\right)^+ \Big| g(\hat{S}_T) = x, \{N_l = k_l\}\right]\right] \\ &+ \frac{\epsilon^2}{2}\mathbf{E}\left[\mathbf{E}\left[1_{\left\{g(S_T^{(1)}) > -y\right\}} g\left(S_T^{(2)}\right) \Big| g(\hat{S}_T) = x, \{N_l = k_l\}\right]\right] \\ &+ \frac{\epsilon^3}{6}\mathbf{E}\left[\mathbf{E}\left[1_{\left\{g(S_T^{(1)}) > -y\right\}} g\left(S_T^{LSV(3)}\right) \Big| g(\hat{S}_T) = x, \{N_l = k_l\}\right]\right] \\ &+ \frac{\epsilon^3}{8}\mathbf{E}\left[\mathbf{E}\left[\delta_{\left\{g(S_T^{(1)}) = -y\right\}} g\left(S_T^{LSV(2)}\right)^2 \Big| g(\hat{S}_T) = x, \{N_l = k_l\}\right]\right].\end{aligned} \tag{47}$$

We also note that the probability of $\{N_l = k_l\} := \{N_{1,T} = k_1, \cdots, N_{n,T} = k_n\}$ is expressed as

$$p_{\{k_l\}} := \prod_{l=1}^n \frac{(\Lambda_l T)^{k_l} e^{-\Lambda_l T}}{k_l!}, \tag{48}$$

which is the product of the $k_l$ times of the jump probabilities of $N_{l,T}$ $(l = 1, \cdots, n)$, that is $\prod_{l=1}^n P(\{N_{l,T} = k_l\})$, thanks to the independence of $N_{l,T}$ $(l = 1, \cdots, n)$.

Then, we calculate the coefficients of $\epsilon$, $\frac{\epsilon^2}{2}$, $\frac{\epsilon^3}{6}$ and $\frac{\epsilon^3}{8}$ on the right hand of (47) as follows:

The coefficient of $\epsilon$ is given by:

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{E}\left[\left(g\left(S_T^{(1)}\right)+y\right)^+\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right]\right]\\
&=\sum_{k=0}^{\infty}\sum_{\sum_{l=1}^n k_l=k} p_{\{k_l\}}\int_{-(g(\xi_{\{k_l\}})+y)}^{\infty}\left(x+g(\xi_{\{k_l\}})+y\right)n(x;0,\Sigma_T^{\{k_l\}})dx, \qquad (49)
\end{aligned}$$

where the formula 1 in Lemma 3.2 is used to calculate the conditional expectation of the second term on the right-hand side of the equation (22) for the constant jump case, and the formula 2 in Lemma 3.2 is used to calculate the conditional expectation of the second term of the equation (22) for the log-normal jump case.

The following calculations for conditional expectations also use the formulas in Lemma 3.2 or/and Appendix B.

The coefficient of $\frac{\epsilon^2}{2}$ is given by:

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{E}\left[1_{\left\{g(S_T^{(1)})>-y\right\}}g\left(S_T^{(2)}\right)\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right]\right]\\
&=\sum_{k=0}^{\infty}\sum_{\sum_{l=1}^n k_l=k} p_{\{k_l\}}\int_{-(g(\xi_{\{k_l\}})+y)}^{\infty}\eta_2(x,\{k_l\})n(x;0,\Sigma_T^{\{k_l\}})dx, \qquad (50)
\end{aligned}$$

the coefficient of $\frac{\epsilon^3}{6}$ is given by:

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{E}\left[1_{\left\{g(S_T^{(1)})>-y\right\}}g\left(S_T^{LSV(3)}\right)\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right]\right]\\
&=\sum_{k=0}^{\infty}\sum_{\sum_{l=1}^n k_l=k} p_{\{k_l\}}\int_{-(g(\xi_{\{k_l\}})+y)}^{\infty}\eta_3(x,\{k_l\})n(x;0,\Sigma_T^{\{k_l\}})dx, \qquad (51)
\end{aligned}$$

and the coefficient of $\frac{\epsilon^3}{8}$ is given by:

$$\begin{aligned}
&\mathbf{E}\left[\mathbf{E}\left[\delta_{\left\{g(S_T^{(1)})=-y\right\}}g\left(S_T^{LSV(2)}\right)^2\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right]\right]\\
&=\sum_{k=0}^{\infty}\sum_{\sum_{l=1}^n k_l=k} p_{\{k_l\}}\eta_{22}(-(g(\xi_{\{k_l\}})+y),\{k_l\})n(-(g(\xi_{\{k_l\}})+y);0,\Sigma_T^{\{k_l\}}). \qquad (52)
\end{aligned}$$

Then, the initial value, $C(K,T)$ of the basket call option with maturity $T$ and strike $K$ is expanded around $\epsilon=0$ as follows:

$$\begin{aligned}
C(K,T) &= \mathbf{E}\left[\left(g\left(S_T^{(\epsilon)}\right)-K\right)^+\right]\\
&\approx \sum_{k=0}^{\infty}\sum_{\sum_{l=1}^n k_l=k} p_{\{k_l\}}e^{-rT}\Bigg\{\epsilon\int_{-y_{\{k_l\}}}^{\infty}\left(x+y_{\{k_l\}}\right)n(x;0,\Sigma_T^{\{k_l\}})dx
\end{aligned}$$

$$
\begin{aligned}
&+\epsilon^2 \int_{-y_{\{k_l\}}}^{\infty} \eta_2(x,\{k_l\}) n(x;0,\Sigma_T^{\{k_l\}}) dx \\
&+\epsilon^3 \int_{-y_{\{k_l\}}}^{\infty} \eta_3(x,\{k_l\}) n(x;0,\Sigma_T^{\{k_l\}}) dx \\
&\left. +\epsilon^3 \eta_{22}(-y_{\{k_l\}},\{k_l\}) n(-y_{\{k_l\}};0,\Sigma_T^{\{k_l\}}) \right\}, \qquad (53)
\end{aligned}
$$

where $y_{\{k_l\}} := g(\xi_{\{k_l\}}) + y$, and $r$ is a constant risk-free rate.

In order to evaluate $\eta_2(x,\{k_l\})$, the conditional expectations defined in (44), we prepare the following lemma. To evaluate $\eta_3(x,\{k_l\})$ and $\eta_{22}(-y_{\{k_l\}},\{k_l\})$ defined in (45) and (46), respectively, we apply the conditional expectation formulas for the Wiener-Itô integrals listed in Appendix B.

**Lemma 3.2.** *We suppose the following:*

- *$W$ is a $d$-dimensional Brownian motion.*
- *Each $N_l$, ( $l = 1,\cdots,n$) is a Poisson process with intensity $\Lambda_l$ and they are independent. $\tau_{j,l}$ stands for the time of the $j$-th jump in $N_l$.*
- *$W$ and $N_l$ are independent.*
- *$X_{j,l} = \left(X_{j,l}^{(1)},\cdots,X_{j,l}^{(d)}\right)$, ($j = 1,\cdots,$ $l = 1,\cdots,n$) follows a $d$-dimensional normal distribution with mean $0$ and variance-covariance matrix $\Theta_{X,l}$ whose diagonal elements are $1$, that is each variance is $1$.*
- *$X_{j,l}$ and $X_{j',l'}$ are independent for $j \neq j'$ or $l \neq l'$.*
- *$X_{j,l}$ are independent of $W$ and $N_l$.*
- *Each $f_{1,l}$ is a $d$-dimensional vector in $\mathbf{R^d}$.*
- *$f_2(t)$, $g_{1,l}(t)$, $g_2(t)$ and $g_{2,l}(t)$ ($l = 1,\cdots,n$) are $\mathbf{R}_+ \to \mathbf{R^d}$ deterministic functions and are integrable with respect to $t$ in the formulas below.*
- *For the notational convenience, $f_2(t)$, $g_{1,l}(t)$, $g_2(t)$ and $g_{2,l}(t)$ are expressed as $f_{2,t}$, $g_{1,l,t}$, $g_{2,t}$ and $g_{2,l,t}$, respectively.*
- *We define $\hat{Y}_T$ and $\Sigma_{\hat{Y}_T}^{\{k_l\}}$ as follows:*

$$
\hat{Y}_T := \int_0^T f_{2,t} \cdot dW_t + \sum_{l=1}^{n} \sum_{j=1}^{N_{l,T}} f_{1,l} \cdot X_{j,l}, \qquad (54)
$$

$$
\Sigma_{\hat{Y}_T}^{\{k_l\}} := \int_0^T |f_{2,t}|^2 dt + \sum_{l=1}^{n} k_l \; f_{1,l}^\top \; \Theta_{X,l} \; f_{1,l}, \qquad (55)
$$

*where $x \cdot y$ stands for the inner product of $x$ and $y$ in $\mathbf{R^d}$, and $x^\top$ denotes the transpose of $x$.*

- *We define $I$ as $I = (1, \cdots, 1)$.*

*Then, we have the following formulas 1. - 13. The proof will be given upon request.*

1.
$$\mathbf{E}\left[\sum_{l=1}^{n}\sum_{j=1}^{N_{T,l}} g_{1,l,\tau_{j,l}-}\cdot I\Big|\hat{Y}=y,\{N_{T,l}=k_l\}\right] = \sum_{l=1}^{n}\frac{k_l}{T}\int_0^T g_{1,l,t}\cdot I dt, \tag{56}$$

2.
$$\mathbf{E}\left[\sum_{l=1}^{n}\sum_{j=1}^{N_{T,l}} g_{1,l,\tau_{j,l}-}\cdot X_j\Big|\hat{Y}=y,\{N_{T,l}=k_l\}\right]$$
$$= \sum_{l=1}^{n}\frac{k_l}{T}\int_0^T g_{1,l,t}^{\top}\Theta_{X,l}f_{1,l}dt\frac{H_1\left(y,\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)}{\Sigma_{\hat{Y}_T}^{\{k_l\}}}, \tag{57}$$

3.
$$\mathbf{E}\left[\int_0^T g_{2,t}\left(\sum_{l=1}^{n}\sum_{j=1}^{N_{t,l}} g_{1,l,\tau_{j,l}-}\cdot I\right)\cdot dW_t\Big|\hat{Y}=y,\{N_{T,l}=k_l\}\right]$$
$$= \frac{k_l}{T}\int_0^T g_{2,t}\cdot f_{2,t}\sum_{l=1}^{n}\int_0^t g_{1,l,s}\cdot I dsdt\frac{H_1\left(y,\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)}{\Sigma_{\hat{Y}_T}^{\{k_l\}}}, \tag{58}$$

4.
$$\mathbf{E}\left[\int_0^T g_{2,t}\left(\sum_{l=1}^{n}\sum_{j=1}^{N_{t,l}} g_{1,l,\tau_{j,l}-}\cdot X_j\right)\cdot dW_t\Big|\hat{Y}=y,\{N_{T,l}=k_l\}\right]$$
$$= \sum_{l=1}^{n}\frac{k_l}{T}\int_0^T g_{2,t}\cdot f_{2,t}\int_0^t g_{1,l,s}^{\top}\Theta_{X,l}f_{1,l}dsdt\frac{H_2\left(y,\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)}{\left(\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)^2}, \tag{59}$$

5.
$$\mathbf{E}\left[\int_0^T g_{2,t}\cdot I\sum_{l=1}^{n}\sum_{j=1}^{N_{t,l}} g_{1,l,\tau_{j,l}-}\cdot I dt\Big|\hat{Y}=y,\{N_{T,l}=k_l\}\right]$$
$$= \sum_{l=1}^{n}\frac{k_l}{T}\int_0^T g_{2,t}\cdot I\int_0^t g_{1,l,s}\cdot I dsdt, \tag{60}$$

6.
$$\mathbf{E}\left[\int_0^T g_{2,t}\cdot I\sum_{l=1}^{n}\sum_{j=1}^{N_{t,l}} g_{1,l,\tau_{j,l}-}\cdot X_j dt\Big|\hat{Y}=y,\{N_{T,l}=k_l\}\right]$$
$$= \sum_{l=1}^{n}\frac{k_l}{T}\int_0^T g_{2,t}\cdot I\int_0^t g_{1,l,s}^{\top}\Theta_{X,l}f_{1,l}dsdt\frac{H_1\left(y,\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)}{\Sigma_{\hat{Y}_T}^{\{k_l\}}}, \tag{61}$$

7.

$$\mathbf{E}\left[\sum_{l=1}^{n}\sum_{j=1}^{N_{T,l}} g_{1,l,\tau_{j,l}-}\cdot I\int_0^{\tau_{j,l}-} g_{2,t}\cdot dW_t\Big|\hat{Y}=y,\{N_{T,l}=k_l\}\right]$$

$$=\sum_{l=1}^{n}\frac{k_l}{T}\int_0^T g_{1,l,t}\cdot Idt\int_0^T g_{2,t}\cdot f_{2,t}dt\frac{H_1\left(y,\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)}{\Sigma_{\hat{Y}_T}^{\{k_l\}}}$$

$$-\sum_{l=1}^{n}\frac{k_l}{T}\int_0^T\int_0^t g_{1,l,s}\cdot Idsg_{2,t}\cdot f_{2,t}dt\frac{H_1\left(y,\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)}{\Sigma_{\hat{Y}_T}^{\{k_l\}}}$$

$$=\sum_{l=1}^{n}\frac{k_l}{T}\left(\int_0^T g_{1,l,t}\cdot I\int_0^t g_{2,s}\cdot f_{2,s}dsdt\right)\frac{H_1\left(y,\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)}{\Sigma_{\hat{Y}_T}^{\{k_l\}}}, \tag{62}$$

8.

$$\mathbf{E}\left[\sum_{l=1}^{n}\sum_{j=1}^{N_{T,l}} g_{1,l,\tau_{j,l}-}\left(\int_0^{\tau_{j,l}-} g_{2,t}\cdot dW_t\right)\cdot X_j\Big|\hat{Y}=y,\{N_{T,l}=k_l\}\right]$$

$$=\sum_{l=1}^{n}\frac{k_l}{T}\int_0^T g_{1,l,t}^{\top}\Theta_{X,l}f_{1,l}dt\int_0^T g_{2,t}\cdot f_{2,t}dt\frac{H_2\left(y,\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)}{\left(\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)^2}$$

$$-\sum_{l=1}^{n}\frac{k_l}{T}\int_0^T\int_0^t g_{1,l,s}^{\top}\Theta_{X,l}f_{1,l}dsg_{2,t}\cdot f_{2,t}dt\frac{H_2\left(y,\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)}{\left(\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)^2}$$

$$=\sum_{l=1}^{n}\frac{k_l}{T}\left(\int_0^T g_{1,l,t}^{\top}\Theta_{X,l}f_{1,l}\int_0^t g_{2,s}\cdot f_{2,s}dsdt\right)\frac{H_2\left(y,\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)}{\left(\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)^2}, \tag{63}$$

9.

$$\mathbf{E}\left[\sum_{l=1}^{n}\sum_{j=1}^{N_{T,l}}\left(g_{1,l,\tau_{j,l}-}\cdot X_j\right)\left(g_{2,l,\tau_{j,l}-}\cdot X_j\right)\Big|\hat{Y}=y,\{N_{T,l}=k_l\}\right]$$

$$=\sum_{l=1}^{n}\frac{k_l}{T}\left(\int_0^T g_{1,l,t}\Theta_{X,l}f_{1,l}^{\top}\cdot g_{2,l,t}\Theta_{X,l}f_{1,l}dt\frac{H_2\left(y,\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)}{\left(\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)^2}+\int_0^T g_{1,l,t}\cdot g_{2,l,t}dt\right), \tag{64}$$

10.

$$\mathbf{E}\left[\sum_{l=1}^{n}\sum_{j=2}^{N_{T,l}} g_{1,l,\tau_{j,l}-}\cdot I\sum_{L=1}^{n}\sum_{J=1}^{N_{\tau_{j,l}-,L}} g_{2,L,\tau_{J,L}-}\cdot I\Big|\hat{Y}=y,\{N_{T,l}=k_l\}\right]$$

$$
=\sum_{l=1}^{n}\frac{k_l(k_l-1)}{T^2}\int_0^T g_{1,l,t}\cdot I\sum_{L=1}^{n}\int_0^t g_{2,L,s}\cdot Idsdt, \tag{65}
$$

11.

$$
\mathbf{E}\left[\sum_{l=1}^{n}\sum_{j=2}^{N_{T,l}} g_{1,l,\tau_{j,l}-}\cdot I\sum_{L=1}^{n}\sum_{J=1}^{N_{\tau_{j,l}-,L}} g_{2,L,\tau_{J,L}-}\cdot X_J\middle|\hat{Y}=y,\{N_{T,l}=k_l\}\right]
$$

$$
=\sum_{l=1}^{n}\frac{k_l(k_l-1)}{T^2}\int_0^T g_{1,l,t}\cdot I\sum_{L=1}^{n}\int_0^t g_{2,L,s}^{\top}\Theta_{X,L}f_{1,L}dsdt\frac{H_1\left(y,\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)}{\Sigma_{\hat{Y}_T}^{\{k_l\}}}, \tag{66}
$$

12.

$$
\mathbf{E}\left[\sum_{l=1}^{n}\sum_{j=2}^{N_{T,l}} g_{1,l,\tau_{j,l}-}\cdot I\sum_{L=1}^{n}\sum_{J=1}^{N_{\tau_{j,l}-,L}} g_{2,L,\tau_{J,L}-}\cdot X_j\middle|\hat{Y}=y,\{N_{T,l}=k_l\}\right]
$$

$$
=\sum_{l=1}^{n}\frac{k_l(k_l-1)}{T^2}\int_0^T g_{1,l,t}^{\top}\Theta_{X,l}f_{1,l}\sum_{L=1}^{n}\int_0^t g_{2,L,s}\cdot Idsdt\frac{H_1\left(y,\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)}{\Sigma_{\hat{Y}_T}^{\{k_l\}}}, \tag{67}
$$

13.

$$
\mathbf{E}\left[\sum_{l=1}^{n}\sum_{j=2}^{N_{T,l}} g_{1,l,\tau_{j,l}-}\left(\sum_{L=1}^{n}\sum_{J=1}^{N_{\tau_{j,l}-,L}} g_{2,L,\tau_{J,L}-}\cdot X_J\right)\cdot X_j\middle|\hat{Y}=y,\{N_{T,l}=k_l\}\right]
$$

$$
=\sum_{l=1}^{n}\frac{k_l(k_l-1)}{T^2}\int_0^T g_{1,l,t}^{\top}\Theta_{X,l}f_{1,l}\sum_{L=1}^{n}\int_0^t g_{2,L,s}^{\top}\Theta_{X,L}f_{1,L}dsdt\frac{H_2\left(y,\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)}{\left(\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)^2}, \tag{68}
$$

*where* $H_k\left(x;\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)$ *denotes the* $k$*-th order Hermite polynomial. Particularly,* $H_1\left(x;\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)=x$*,* $H_2\left(x;\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)=x^2-\Sigma_{\hat{Y}_T}^{\{k_l\}}$ *and* $H_4\left(x;\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)=x^4-6\Sigma_{\hat{Y}_T}^{\{k_l\}}x^2+3\left(\Sigma_{\hat{Y}_T}^{\{k_l\}}\right)^2$*.*

Applying the above lemma and the conditional expectation formulas in Shiraya and Takahashi (2014) which are listed in Appendix B, we obtain an approximate pricing formula for a basket call option with $\epsilon=1$. The formula for a basket put option is easily obtained through the put-call parity.

**Theorem 3.3.** *An approximation formula for the initial value* $C(K,T)$ *of a basket call option with maturity* $T$ *and strike price* $K$ *is given by the following equation:*

$$
\sum_{k=0}^{\infty}\sum_{\sum_{l=1}^{n}k_l=k} p_{\{k_l\}}e^{-rT}\left\{(y_{k_l}+C_{1,k_l})N\left(\frac{y_{k_l}}{\sqrt{\Sigma_T^{\{k_l\}}}}\right)+\left(C_{2,k_l}\Sigma_T^{\{k_l\}}+C_{3,k_l}\frac{H_1\left(y_{k_l};\Sigma_T^{\{k_l\}}\right)}{\Sigma_T^{\{k_l\}}}\right.\right.
$$

$$
\left.\left.+C_{4,k_l}\frac{H_2\left(y_{k_l};\Sigma_T^{\{k_l\}}\right)}{\left(\Sigma_T^{\{k_l\}}\right)^2}+C_{5,k_l}\frac{H_4\left(y_{k_l};\Sigma_T^{\{k_l\}}\right)}{\left(\Sigma_T^{\{k_l\}}\right)^4}+C_{6,k_l}\right)n(y_{k_l};0,\Sigma_T^{\{k_l\}})\right\}, \tag{69}
$$

*where* $p_{\{k_l\}} = \prod_{l=1}^{n} \frac{(\Lambda_l T)^{k_l} e^{-\Lambda_l T}}{k_l!}$, *r is a constant risk-free rate,* $y = g(S_T^{(0)}) - K$, $y_{\{k_l\}} = g(\xi_{\{k_l\}}) + y$, $N(x)$ *denotes the standard normal distribution function and* $n(x; 0, \Sigma) = \frac{1}{\sqrt{2\pi\Sigma}} \exp\left(\frac{-x^2}{2\Sigma}\right)$. *Here,* $\Sigma_T^{\{k_l\}}$ *is given by (43), and* $\xi_{\{k_l\}}$ *is defined by (38). The coefficients* $C_{1,k_l}, \cdots, C_{6,k_l}$ *are some constants. The derivation of the coefficients* $C_{1,k_l}, \cdots, C_{6,k_l}$ *is shown in Appendix A. Moreover,* $H_k\left(x; \Sigma_T^{\{k_l\}}\right)$ *denotes the k-th order Hermite polynomial: particularly,* $H_1\left(x; \Sigma_T^{\{k_l\}}\right) = x$, $H_2\left(x; \Sigma_T^{\{k_l\}}\right) = x^2 - \Sigma_T^{\{k_l\}}$ *and* $H_4\left(x; \Sigma_T^{\{k_l\}}\right) = x^4 - 6\Sigma_T^{\{k_l\}} x^2 + 3\left(\Sigma_T^{\{k_l\}}\right)^2$.

# 4 Numerical Examples

This section shows concrete numerical examples based on our method developed in the previous section.

## 4.1 Setup

We apply the following model for numerical experiments under the risk-neutral probability measure: each underlying asset price process has a CEV (constant elasticity of variance)-type diffusion term with compound Poisson jump component, and each volatility process has a CEV-type diffusion term with mean reversion drift and compound Poisson jump component:

$$\begin{aligned} S_T^i &= S_0^i + \int_0^T \alpha^i S_t^i dt + \int_0^T \sigma_t^i (S_t^i)^{\beta_{S^i}} dW_t^{S^i} \\ &\quad + \sum_{l=1}^{n} \left( \sum_{j=1}^{N_{l,T}} h_{S^i,l,j} S_{\tau_{j,l}-}^i - \int_0^T \Lambda_l S_{t-}^i \mathbf{E}[h_{S^i,l,1}] dt \right), \end{aligned} \tag{70}$$

$$\begin{aligned} \sigma_T^i &= \sigma_0^i + \int_0^T \lambda^i (\theta^i - \sigma_t^i) dt + \int_0^T \nu^i (\sigma_t^i)^{\beta_{\sigma^i}} dW_t^{\sigma^i} \\ &\quad + \sum_{l=1}^{n} \left( \sum_{j=1}^{N_{l,T}} h_{\sigma^i,l,j} \sigma_{\tau_{j,l}-}^i - \int_0^T \Lambda_l \sigma_{t-}^i \mathbf{E}[h_{\sigma^i,l,1}] dt \right), \end{aligned} \tag{71}$$

where the jump size $h_{x^i,l,j}$ is given by $h_{x^i,l,j} = H_{x^i,l}$ for all $j$ with a constant $H_{x^i,l}$ in the constant jump case, and by $h_{x^i,l,j} = e^{Y_{x^i,l,j}} - 1$ with $Y_{x^i,l,j}$ following a normal distribution $N(m_{x^i,l}, \gamma_{x^i,l}^2)$ for all $j$ in the log-normal jump case.

Applying our approximate formula we calculate the basket call options whose number of the underlying asset is five in the basket. For illustrative purpose we only consider a systematic jump case, that is all the jumps of the underlying asset prices and their volatilities occur at the same time (i.e. $n = 1$ and $(\vartheta)_{x^i,y^j} = 1$ where $\vartheta$ denotes the $10 \times 10$ correlation matrix among $\zeta_{S^i,j,l}$ and $\zeta_{\sigma^i,j,l}$ $(i = 1, \cdots, 5)$), and the intensity parameter $\Lambda$ is fixed as 1, though we are able to treat more general cases. The base parameters in the asset price and their volatility processes are the same among all the assets, which are listed in the following tables (Table 1 and Table 2).

Table 1: Common Parameters

| $S_0^i$ | $\sigma_0^i$ | $\alpha^i$ | $\beta_{S^i}$ | $\beta_{\sigma^i}$ | $\lambda_i$ | $\theta_i$ | $\nu_i$ | $w_i$ | $\Lambda$ | $T$ | $n$ | $m_{S_i}$ | $\gamma_{S_i}$ | $H_{\sigma^i}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 100 | 2 | 0 | 0.5 | 0.5 | 1 | 2 | 1 | 0.2 | 1 | 1 | 1 | -0.1 | 0.15 | 0.2 |

Table 2: Correlations

| | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ | $\sigma_5$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $S_1$ | 1 | 0.5 | 0.5 | 0.5 | 0.5 | -0.5 | -0.5 | -0.5 | -0.5 | -0.5 |
| $S_2$ | 0.5 | 1 | 0.5 | 0.5 | 0.5 | -0.5 | -0.5 | -0.5 | -0.5 | -0.5 |
| $S_3$ | 0.5 | 0.5 | 1 | 0.5 | 0.5 | -0.5 | -0.5 | -0.5 | -0.5 | -0.5 |
| $S_4$ | 0.5 | 0.5 | 0.5 | 1 | 0.5 | -0.5 | -0.5 | -0.5 | -0.5 | -0.5 |
| $S_5$ | 0.5 | 0.5 | 0.5 | 0.5 | 1 | -0.5 | -0.5 | -0.5 | -0.5 | -0.5 |
| $\sigma_1$ | -0.5 | -0.5 | -0.5 | -0.5 | -0.5 | 1 | 0.5 | 0.5 | 0.5 | 0.5 |
| $\sigma_2$ | -0.5 | -0.5 | -0.5 | -0.5 | -0.5 | 0.5 | 1 | 0.5 | 0.5 | 0.5 |
| $\sigma_3$ | -0.5 | -0.5 | -0.5 | -0.5 | -0.5 | 0.5 | 0.5 | 1 | 0.5 | 0.5 |
| $\sigma_4$ | -0.5 | -0.5 | -0.5 | -0.5 | -0.5 | 0.5 | 0.5 | 0.5 | 1 | 0.5 |
| $\sigma_5$ | -0.5 | -0.5 | -0.5 | -0.5 | -0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 1 |

## 4.2 Numerical Results

Table 3 shows the results for the numerical experiment with the benchmarks computed by Monte Carlo simulations, where the number of the time steps is 128 and the number of trials is 1 million with antithetic variables in computation of each benchmark.

We provide a sensitivity analysis to examine how the approximation errors by our formula change with changes in the model parameters. In particular, we compare the approximation errors for basket call option prices with different parameters. "High" or "Long" means the twice value of the base parameter given in Table 1, and "Low" or "Short" means the half value of the base parameter, except for the correlation parameter. AE means the asymptotic expansion method, and MC means the Monte Carlo method.

| Strikes | AE 80 | 90 | 100 | 110 | 120 | MC 80 | 90 | 100 | 110 | 120 | Diff 80 | 90 | 100 | 110 | 120 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Base Parameters | 22.62 | 15.13 | 9.04 | 4.61 | 1.91 | 22.67 | 15.15 | 9.03 | 4.59 | 1.92 | -0.05 | -0.02 | 0.01 | 0.02 | -0.01 |
| $\|\rho\| = 0.99$ | 23.74 | 16.43 | 10.37 | 5.72 | 2.56 | 23.78 | 16.45 | 10.36 | 5.71 | 2.60 | -0.04 | -0.02 | 0.01 | 0.01 | -0.03 |
| $\|\rho\| = 0.75$ | 23.19 | 15.81 | 9.75 | 5.21 | 2.25 | 23.23 | 15.83 | 9.74 | 5.19 | 2.28 | -0.04 | -0.02 | 0.01 | 0.01 | -0.02 |
| $\|\rho\| = 0.25$ | 22.09 | 14.43 | 8.25 | 3.93 | 1.53 | 22.15 | 14.45 | 8.25 | 3.91 | 1.54 | -0.06 | -0.03 | 0.00 | 0.01 | -0.01 |
| $\|\rho\| = 0$ | 21.67 | 13.77 | 7.34 | 3.09 | 1.11 | 21.72 | 13.80 | 7.36 | 3.09 | 1.12 | -0.05 | -0.03 | -0.02 | 0.00 | -0.01 |
| $\sigma_0$ High | 24.70 | 17.80 | 12.08 | 7.62 | 4.40 | 24.74 | 17.81 | 12.08 | 7.62 | 4.43 | -0.04 | -0.01 | 0.00 | 0.00 | -0.03 |
| $\sigma_0$ Low | 21.92 | 14.14 | 7.81 | 3.37 | 1.06 | 21.98 | 14.16 | 7.81 | 3.34 | 1.08 | -0.06 | -0.02 | 0.01 | 0.03 | -0.02 |
| $lambda$ High | 22.52 | 15.05 | 9.01 | 4.65 | 2.00 | 22.58 | 15.08 | 9.01 | 4.64 | 2.00 | -0.06 | -0.03 | 0.00 | 0.02 | 0.00 |
| $lambda$ Low | 22.70 | 15.20 | 9.07 | 4.59 | 1.85 | 22.74 | 15.20 | 9.06 | 4.57 | 1.87 | -0.04 | -0.01 | 0.02 | 0.02 | -0.02 |
| $\theta$ High | 23.75 | 16.62 | 10.76 | 6.32 | 3.29 | 23.79 | 16.62 | 10.74 | 6.30 | 3.29 | -0.03 | -0.01 | 0.01 | 0.01 | 0.00 |
| $\theta$ Low | 22.19 | 14.54 | 8.32 | 3.89 | 1.38 | 22.25 | 14.56 | 8.32 | 3.87 | 1.40 | -0.06 | -0.03 | 0.00 | 0.02 | -0.02 |
| $\nu$ High | 23.18 | 15.65 | 9.41 | 4.71 | 1.75 | 23.08 | 15.54 | 9.31 | 4.66 | 1.81 | 0.11 | 0.11 | 0.10 | 0.05 | -0.06 |
| $\nu$ Low | 22.44 | 14.94 | 8.92 | 4.63 | 2.05 | 22.50 | 14.98 | 8.93 | 4.62 | 2.05 | -0.06 | -0.03 | -0.01 | 0.01 | 0.01 |
| $m_S$ High | 23.88 | 16.81 | 10.92 | 6.34 | 3.09 | 23.98 | 16.86 | 10.92 | 6.29 | 3.02 | -0.10 | -0.05 | 0.00 | 0.05 | 0.07 |
| $m_S$ Low | 22.23 | 14.60 | 8.50 | 4.25 | 1.84 | 22.25 | 14.61 | 8.50 | 4.26 | 1.90 | -0.02 | -0.01 | 0.00 | -0.02 | -0.06 |
| $\gamma_S$ High | 24.82 | 17.96 | 12.28 | 8.00 | 5.14 | 25.04 | 18.14 | 12.41 | 8.09 | 5.28 | -0.22 | -0.18 | -0.12 | -0.09 | -0.14 |
| $\gamma_S$ Low | 21.92 | 14.10 | 7.83 | 3.48 | 1.07 | 21.92 | 14.08 | 7.81 | 3.47 | 1.09 | 0.00 | 0.02 | 0.03 | 0.01 | -0.02 |
| $H_\sigma$ High | 22.67 | 15.14 | 9.00 | 4.53 | 1.83 | 22.74 | 15.17 | 9.00 | 4.52 | 1.86 | -0.07 | -0.03 | 0.00 | 0.01 | -0.03 |
| $H_\sigma$ Low | 22.60 | 15.12 | 9.07 | 4.66 | 1.95 | 22.65 | 15.15 | 9.06 | 4.64 | 1.96 | -0.05 | -0.02 | 0.01 | 0.01 | -0.01 |
| $\Lambda$ High | 23.95 | 16.98 | 11.26 | 6.88 | 3.81 | 24.01 | 17.00 | 11.25 | 6.84 | 3.77 | -0.06 | -0.02 | 0.01 | 0.04 | 0.03 |
| $\Lambda$ Low | 21.92 | 14.07 | 7.73 | 3.34 | 1.03 | 21.94 | 14.07 | 7.72 | 3.34 | 1.06 | -0.02 | -0.01 | 0.01 | 0.00 | -0.04 |
| Maturity Long | 25.25 | 18.54 | 12.94 | 8.51 | 5.21 | 25.29 | 18.53 | 12.91 | 8.47 | 5.20 | -0.04 | 0.00 | 0.03 | 0.03 | 0.02 |
| Maturity Short | 21.14 | 12.86 | 6.22 | 2.05 | 0.45 | 21.19 | 12.89 | 6.22 | 2.05 | 0.48 | -0.05 | -0.03 | 0.00 | 0.00 | -0.03 |

Table 3: AE vs MC

It is observed that the approximation errors become large when the jump size parameters such as the standard deviation of the price's jump size $\gamma_S$ and the volatility jump size $H_\sigma$, and the volatility on the volatility parameter $\nu$ are large.

However, in most of the cases, our approximation formula works quite well. While in terms of the computational time our analytical method is obviously much faster than the Monte Carlo simulations with 128 time steps and one million trials.

# 5 WTI - Brent basket options

This section presents numerical examples for pricing WTI - Brent basket options based on our approximation scheme with the parameters obtained by calibration to the actual futures option prices. In particular, we use the following model under the risk-neutral probability measure, where each underlying asset price process has a CEV (constant elasticity of variance)-type diffusion term with compound Poisson jump component and each volatility process follows a log-normal model: for $i = 1, 2$,

$$
\begin{aligned}
S_T^i &= S_0^i + \int_0^T \sigma_t^i (S_t^i)^{\beta_i} dW_t^{S^i} + \sum_{l=1}^{n} \left( \sum_{j=1}^{N_{l,T}} h_{S^i,l,j} S^i_{\tau_{j,l}-} - \int_0^T \Lambda_l S^i_{t-} \mathbf{E}[h_{S^i,l,1}] dt \right), \\
\sigma_T^i &= \sigma_0^i + \int_0^T \nu^i \sigma_t^i dW_t^{\sigma^i},
\end{aligned}
$$

where the jump size in the futures price process is log-normally distributed, that is $h_{S^i,l,j} = e^{Y_{S^i,l,j}} - 1$ with $Y_{S^i,l,j}$ following a normal distribution $N(m_{S^i,l}, \gamma^2_{S^i,l})$ for all $j$.

Applying our approximate formula to this model, we calculate the basket options on WTI futures and Brent futures. For simplicity, we only consider a systematic jump case, that is all the jumps of the underlying asset prices (i.e. $n = 1$ and $(\vartheta)_{S^i,S^j} = 1$ where $\vartheta$ denotes the $2 \times 2$ correlation matrix among $\zeta_{S^i,j,l}$ $(i = 1, 2)$).

We set the calculation date for basket option prices on March 31, 2015. In Table 4, we report the target basket prices with their underlying futures prices on the date, the terms to maturities and the relevant (risk-free) interest rates.

| | | Asset Price | Maturity | Risk Free Rate |
|---|---|---|---|---|
| JUN15 | WTI | 49.34 | 0.121 | 0.3% |
| | Brent | 56.21 | 0.112 | 0.3% |
| | basket | 52.78 | 0.112 | 0.3% |
| DEC15 | WTI | 54.87 | 0.633 | 0.4% |
| | Brent | 60.84 | 0.614 | 0.4% |
| | basket | 57.86 | 0.614 | 0.4% |

Table 4: Asset Price, Maturity and Risk Free Rate

We firstly need to obtain the model parameters through calibration to the relevant option prices of WTI futures and those of Brent futures. In the jump component, the intensity parameter $\Lambda$ is fixed as 1. The other jump parameters are assumed to take common values for the two relevant futures price processes used for the calculation of a basket option price.

For computational efficiency, the settlement prices of American options are transformed to those of the European options before calibration: More precisely, after an implied volatility of each American option price is estimated under a binomial version of the Black-Scholes model, the corresponding European option price is computed. Hereafter, this European option price is called the “transformed CME” or “transformed ICE” option price. Then, calibration is implemented against the “transformed CME” or “transformed ICE” option prices with different strikes simultaneously, where out-of-the-money (OTM) prices are used for the calibration; for JUN15 futures options, the strikes of the options range usd 35 to usd 75 with every five dollars, and for DEC15 futures options, those of the options range usd 40 to usd 80 with every five dollars.

Moreover, the correlations between the futures prices and their volatilities are assumed to take common values for the two relevant futures, which are used to calculate a basket option. These correlations are obtained by calibration to the market futures option prices, which are shown in the $\rho$-column of Tables 5 and 6.

The correlations between the two futures price processes are estimated by the past three-month’s historical data of the futures prices. The correlations between the corresponding volatility processes of the two futures prices are assumed to be the same as the correlations of the futures prices. Then, we obtain the following estimates: the correlation between the WTI and Brent futures prices for JUN15 is 0.955, and the correlation for DEC15 is 0.975.

Given the above assumptions in the calibration, we compare the following two specifications of the model:

(i) SABR type Local stochastic volatility model without jump (LSV model)

(ii) SABR type Local stochastic volatility with log-normal jumps in the futures prices model (LSV jump diffusion model)

The parameters obtained by calibration to the JUN 15 option prices of WTI and Brent futures are shown in Table 5.

| | | $\sigma(0)$ | $\beta$ | $\nu$ | $\rho$ | $m_S$ | $\gamma_S$ |
|---|---|---|---|---|---|---|---|
| LSV | WTI | 24.83 | 0.001 | 121.1% | -0.224 | - | - |
| | Brent | 7.62 | 0.308 | 152.2% | -0.224 | - | - |
| LSV jump | WTI | 17.47 | 0.064 | 62.6% | -0.541 | -0.123 | 0.285 |
| | Brent | 2.02 | 0.609 | 84.2% | -0.541 | -0.123 | 0.285 |

Table 5: Parameters on JUN15

The parameters obtained by calibration to the DEC 15 option prices of WTI and Brent futures are shown in Table 6.

| | | $\sigma(0)$ | $\beta$ | $\nu$ | $\rho$ | $m_S$ | $\gamma_S$ |
|---|---|---|---|---|---|---|---|
| LSV | WTI | 3.31 | 0.448 | 62.8% | -0.275 | - | - |
| | Brent | 2.52 | 0.521 | 59.6% | -0.275 | - | - |
| LSV jump | WTI | 10.01 | 0.123 | 42.6% | -0.631 | -0.036 | 0.231 |
| | Brent | 0.98 | 0.699 | 70.7% | -0.631 | -0.036 | 0.231 |

Table 6: Parameters on DEC15

The results for the calibration to the JUN 15 option prices of WTI and Brent futures are shown in Tables 7 and 8.

| Strike | 35 | 40 | 45 | 50 | 55 | 60 | 65 | 70 | 75 |
|---|---|---|---|---|---|---|---|---|---|
| Transformed CME | 14.63 | 10.15 | 6.21 | 3.17 | 1.30 | 0.45 | 0.16 | 0.08 | 0.05 |
| LSV | 14.66 | 10.14 | 6.18 | 3.17 | 1.32 | 0.45 | 0.14 | 0.04 | 0.01 |
| LSV jump | 14.65 | 10.14 | 6.20 | 3.17 | 1.30 | 0.45 | 0.17 | 0.08 | 0.05 |
| Diff (LSV) | 0.04 | -0.00 | -0.03 | 0.00 | 0.02 | 0.00 | -0.02 | -0.04 | -0.04 |
| Diff (LSV jump) | 0.02 | -0.00 | -0.01 | 0.00 | 0.00 | 0.00 | 0.01 | 0.00 | -0.01 |

Table 7: JUN15 WTI

| Strike | 35 | 40 | 45 | 50 | 55 | 60 | 65 | 70 | 75 |
|---|---|---|---|---|---|---|---|---|---|
| Transformed ICE | 21.28 | 16.45 | 11.82 | 7.64 | 4.25 | 1.96 | 0.78 | 0.30 | 0.14 |
| LSV | 21.29 | 16.46 | 11.83 | 7.63 | 4.24 | 1.98 | 0.79 | 0.29 | 0.10 |
| LSV jump | 21.27 | 16.44 | 11.82 | 7.64 | 4.25 | 1.97 | 0.77 | 0.30 | 0.14 |
| Diff (LSV) | 0.01 | 0.01 | 0.01 | -0.01 | -0.02 | 0.02 | 0.01 | -0.01 | -0.04 |
| Diff (LSV jump) | -0.01 | -0.00 | 0.01 | -0.00 | -0.00 | 0.01 | -0.01 | -0.00 | -0.00 |

Table 8: JUN15 Brent

The results for the calibration to the DEC 15 option prices of WTI and Brent futures are shown in Tables 9 and 10.

| Strike | 40 | 45 | 50 | 55 | 60 | 65 | 70 | 75 | 80 |
|---|---|---|---|---|---|---|---|---|---|
| Transformed CME | 16.45 | 12.57 | 9.15 | 6.34 | 4.15 | 2.55 | 1.57 | 0.93 | 0.59 |
| LSV | 16.49 | 12.55 | 9.12 | 6.31 | 4.15 | 2.61 | 1.58 | 0.93 | 0.53 |
| LSV jump | 16.45 | 12.56 | 9.15 | 6.33 | 4.14 | 2.57 | 1.55 | 0.94 | 0.59 |
| Diff (LSV) | 0.04 | -0.02 | -0.02 | -0.03 | 0.00 | 0.06 | 0.01 | -0.00 | -0.06 |
| Diff (LSV jump) | 0.01 | -0.01 | 0.01 | -0.01 | -0.01 | 0.02 | -0.02 | 0.01 | -0.00 |

Table 9: DEC15 WTI

| Strike | 40 | 45 | 50 | 55 | 60 | 65 | 70 | 75 | 80 |
|---|---|---|---|---|---|---|---|---|---|
| Transformed ICE | 21.75 | 17.47 | 13.55 | 10.10 | 7.18 | 4.87 | 3.16 | 2.02 | 1.30 |
| LSV | 21.79 | 17.47 | 13.52 | 10.06 | 7.17 | 4.89 | 3.21 | 2.03 | 1.24 |
| LSV jump | 21.74 | 17.47 | 13.56 | 10.09 | 7.17 | 4.86 | 3.17 | 2.02 | 1.29 |
| Diff (LSV) | 0.04 | 0.00 | -0.02 | -0.04 | -0.01 | 0.03 | 0.05 | 0.01 | -0.06 |
| Diff (LSV jump) | -0.01 | 0.00 | 0.01 | -0.00 | -0.00 | -0.00 | 0.01 | -0.00 | -0.01 |

Table 10: DEC15 Brent

We can observe that the LSV jump diffusion model gives much better fitting to the futures options than the LSV model.

Figures 1 and 2 show the implied volatilities of DEC 15 WTI and Brent futures.

Figure 1: Implied volatilities of DEC 15 WTI futures

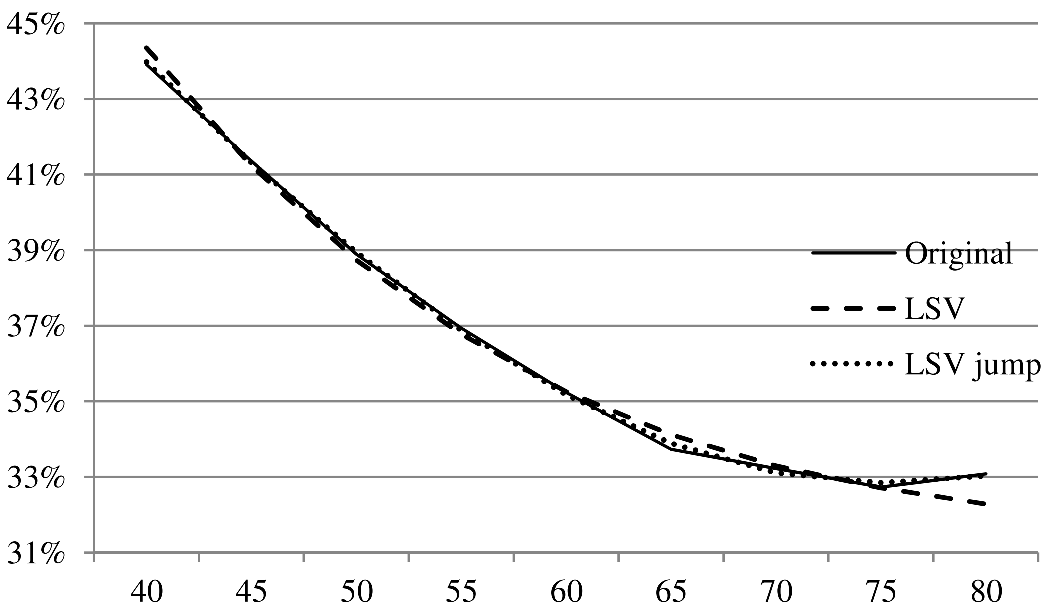
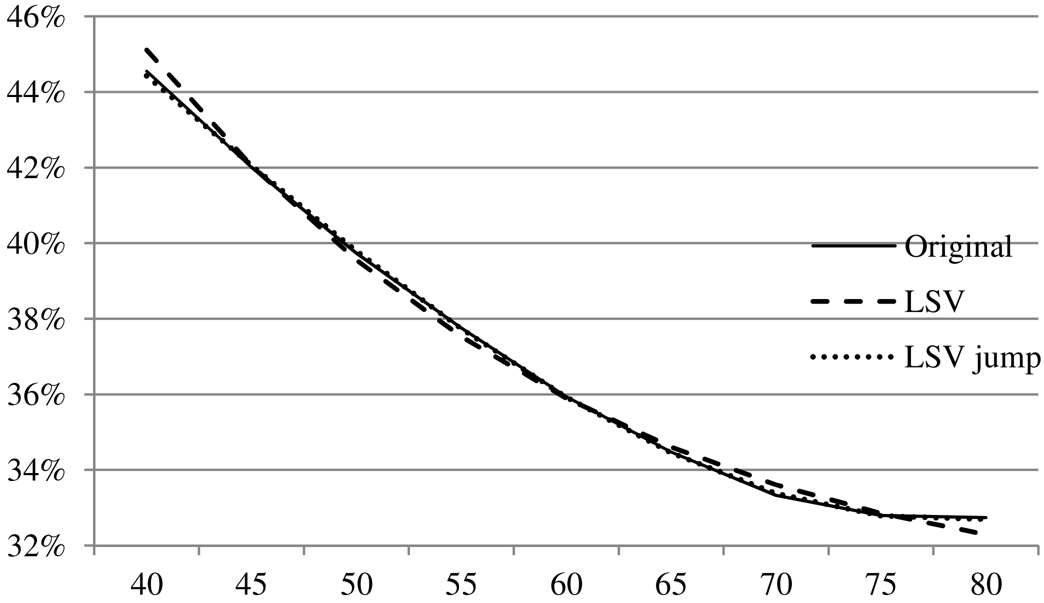


Figure 2: Implied volatilities of DEC 15 Brent futures




Especially in OTM, LSV model is not able to duplicate the implied volatilities.

Using the parameters obtained through the calibration, Tables 11 and 12 show the comparison of the basket option prices given by our approximation (AE), and Monte Carlo simulations (MC) in the LSV Jump diffusion model.

| Strike | 40 | 45 | 50 | 55 | 60 | 65 |
|---|---|---|---|---|---|---|
| AE | 0.41 | 1.02 | 2.30 | 2.41 | 0.93 | 0.33 |
| MC | 0.44 | 1.04 | 2.31 | 2.41 | 0.94 | 0.34 |
| Diff | -0.03 | -0.02 | -0.01 | 0.00 | -0.01 | -0.01 |

Table 11: JUN15 basket option price

| Strike | 40 | 45 | 50 | 55 | 60 | 65 |
|---|---|---|---|---|---|---|
| AE | 2.07 | 3.36 | 5.18 | 5.46 | 3.53 | 2.20 |
| MC | 2.08 | 3.36 | 5.17 | 5.45 | 3.54 | 2.24 |
| Diff | -0.01 | 0.00 | 0.01 | 0.01 | -0.01 | -0.04 |

Table 12: DEC15 basket option price

These results show our approximation formula works well.

## 6 Conclusion

We have derived a new approximation formula for basket option pricing in a model with local-stochastic volatility and jumps. In particular, our model admits a local volatility function and jumps in both the underlying asset price and its volatility processes. Thanks to the closed form formula the computational speed of the method is much faster than the other numerical schemes.

Moreover, in numerical experiments, we firstly calibrate the model to the options on the WTI and Brent futures by applying our approximation formula for the plain-vanilla option. Then, by using the calibrated parameters, we approximate the prices of the basket options on the WTI and the Brent futures and compare those with the benchmark prices obtained by the Monte Carlo method, which has demonstrated the effectiveness of our approximation scheme.

We also note that the higher order expansions can be derived in the similar manner, which is expected to provide more precise approximations as in the diffusion cases in Shiraya, Takahashi and Toda (2012) and Takahashi, Takehara and Toda (2012).

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# A Derivation of Coefficients

This section derives the coefficients, $C_{i,k_l}$, $(i = 1, \cdots, 6)$ in the expansion formula (69) in Theorem 3.3 under a log-normal jump case. A constant jump case is obtained in a similar way. In the following we omit some notations for simplicity.

Firstly, let us show the expressions of $g(S_T^{(1)})$ and $g\left(\frac{1}{2!}S_T^{(2)}\right)$:

$$
\begin{aligned}
g(S_T^{(1)}) &= g\left(\int_0^T e^{\alpha(T-t)} * \Phi_S dZ_t\right) && (72)\\
&\quad + g\left(\sum_{l=1}^{n}\sum_{j=1}^{N_{l,T}} e^{\alpha(T-\tau_{j,l})} * S^{(0)}_{\tau_{j,l}-} * h^{(1)}_{S,l,j}\right) && (73)\\
&\quad - g\left(\sum_{l=1}^{n} \Lambda_l \mathbf{E}[h^{(1)}_{S,l,1}] * e^{\alpha T} * \int_0^T e^{-\alpha t} * S^{(0)}_{t-} dt\right), && (74)\\
g\left(\frac{1}{2!}S_T^{(2)}\right) &= g\left(\int_0^T e^{\alpha(T-t)} * \partial_S\Phi_S * \int_0^t e^{\alpha(t-u)} * \Phi_S dZ_u dZ_t\right) && (75)\\
&\quad + g\left(\int_0^T e^{\alpha(T-t)} * \partial_S\Phi_S * \sum_{l=1}^{n}\sum_{j=1}^{N_{l,t}} h^{(1)}_{S,l,j} * e^{\alpha(t-\tau_{j,l})} * S^{(0)}_{\tau_{j,l}-} dZ_t\right) && (76)\\
&\quad - g\left(\int_0^T e^{\alpha(T-t)} * \partial_S\Phi_S * \sum_{l=1}^{n} \Lambda_l \mathbf{E}[h^{(1)}_{S,l,1}] * \int_0^t e^{\alpha(t-u)} * S^{(0)}_{u-} du dZ_t\right) && (77)
\end{aligned}
$$

$$+g\left(\int_0^T e^{\alpha(T-t)} * \partial_\sigma \Phi_S * \int_0^t e^{-\lambda(t-u)} * \Phi_\sigma dZ_u dZ_t\right) \tag{78}$$

$$+g\left(\int_0^T e^{\alpha(T-t)} * \partial_\sigma \Phi_S * \sum_{l=1}^{n}\sum_{j=1}^{N_{l,t}} h^{(1)}_{\sigma,l,j} * e^{-\lambda(t-\tau_{j,l})} * \sigma^{(0)}_{\tau_{j,l}-} dZ_t\right) \tag{79}$$

$$-g\left(\int_0^T e^{\alpha(T-t)} * \partial_\sigma \Phi_S * \sum_{l=1}^{n} \Lambda_l \mathbf{E}[h^{(1)}_{\sigma,l,1}] * e^{-\lambda t} * \int_0^t e^{\lambda u-} * \sigma^{(0)}_{u-} du dZ_t\right) \tag{80}$$

$$+g\left(\sum_{l=1}^{n}\sum_{j=1}^{N_{l,T}} h^{(2)}_{S,l,j} * e^{\alpha(T-\tau_{j,l})} * S^{(0)}_{\tau_{j,l}-}\right) \tag{81}$$

$$-g\left(\sum_{l=1}^{n} \Lambda_l \mathbf{E}[h^{(2)}_{S,l,1}] * e^{\alpha T} * \int_0^T e^{-\alpha t} * S^{(0)}_{t-} dt\right) \tag{82}$$

$$+g\left(\sum_{l=1}^{n}\sum_{j=1}^{N_{l,T}} h^{(1)}_{S,l,j} * e^{\alpha(T-\tau_{j,l})} * \int_0^{\tau_{j,l}} e^{\alpha(\tau_{j,l}-u)} * \Phi_S dZ_u\right) \tag{83}$$

$$+g\left(\sum_{l=1}^{n}\sum_{j=1}^{N_{l,T}} h^{(1)}_{S,l,j} * e^{\alpha(T-\tau_{j,l})} * \sum_{L=1}^{n}\sum_{m=1}^{N_{L,\tau_{j,l}}} h^{(1)}_{S,L,m} * e^{\alpha(\tau_{j,l}-\tau_{m,L})} * S^{(0)}_{\tau_{m,L}-}\right) \tag{84}$$

$$-g\left(\sum_{l=1}^{n}\sum_{j=1}^{N_{l,T}} h^{(1)}_{S,l,j} * e^{\alpha(T-\tau_{j,l})} * \sum_{L=1}^{n} \Lambda_L \mathbf{E}[h^{(1)}_{S,L,1}] * e^{\alpha\tau_{j,l}} * \int_0^{\tau_{j,l}-} e^{-\alpha u} * S^{(0)}_{u-} du\right) \tag{85}$$

$$+g\left(\int_0^T \sum_{l=1}^{n} \Lambda_l \mathbf{E}[h^{(1)}_{S,l,1}] * e^{\alpha(T-t)} * \int_0^t e^{\alpha(t-u)} * \Phi_S dZ_u dt\right) \tag{86}$$

$$-g\left(\int_0^T \sum_{l=1}^{n} \Lambda_l \mathbf{E}[h^{(1)}_{S,l,1}] * e^{\alpha(T-t)} * \sum_{L=1}^{n}\sum_{m=1}^{N_{L,t}} h^{(1)}_{S,L,m} * e^{\alpha(t-\tau_{m,L})} * S^{(0)}_{\tau_{m,L}-} dt\right) \tag{87}$$

$$+g\left(\int_0^T \sum_{l=1}^{n} \Lambda_l \mathbf{E}[h^{(1)}_{S,l,1}] * e^{\alpha(T-t)} * \sum_{L=1}^{n} \Lambda_L \mathbf{E}[h^{(1)}_{S,L,1}] * e^{\alpha t} * \int_0^t e^{-\alpha u} * S^{(0)}_{u-} du dt\right), \tag{88}$$

where

$$\mathbf{E}\left[h^{(\epsilon)}_{x^i,l,1}\right] = \mathbf{E}\left[e^{\epsilon Y_{x^i,l,1}} - 1\right] = e^{\epsilon m_{x^i,l} + \frac{1}{2}\epsilon^2\gamma^2_{x^i,l}} - 1, \tag{89}$$

$$\mathbf{E}\left[h^{(0)}_{x^i,l,1}\right] = \mathbf{E}\left[h^{(\epsilon)}_{x^i,l,1}\right]\Big|_{\epsilon=0} = 1 - 1 = 0, \tag{90}$$

$$\mathbf{E}\left[h^{(1)}_{x^i,l,1}\right] = \partial_\epsilon \mathbf{E}\left[h^{(\epsilon)}_{x^i,l,1}\right]\Big|_{\epsilon=0} = (m_{x^i,l} + \epsilon\gamma^2_{x^i,l}) e^{\epsilon m_{x^i,l} + \frac{1}{2}\epsilon^2\gamma^2_{x^i,l}}\Big|_{\epsilon=0} = m_{x^i,l}, \tag{91}$$

$$\begin{aligned}\mathbf{E}\left[h^{(2)}_{x^i,l,1}\right] &= \partial^2_\epsilon \mathbf{E}\left[h^{(\epsilon)}_{x^i,l,1}\right]\Big|_{\epsilon=0} = \left(\gamma^2_{x^i,l} e^{\epsilon m_{x^i,l} + \frac{1}{2}\epsilon^2\gamma^2_{x^i,l}} + (m_{x^i,l} + \epsilon\gamma^2_{x^i,l})^2 e^{\epsilon m_{x^i,l} + \frac{1}{2}\epsilon^2\gamma^2_{x^i,l}}\right)\Big|_{\epsilon=0} \\ &= m^2_{x^i,l} + \gamma^2_{x^i,l}.\end{aligned} \tag{92}$$

Next, we define the expression $F(X)$ as

$$F(X) := \mathbf{E}\left[X|g(\hat{S}_T)=x,\{N_l=k_l\}\right] = g\left(\mathbf{E}\left[X|\hat{S}_T=x,\{N_l=k_l\}\right]\right), \tag{93}$$

where $X$ stands for the expression in the equation number $(X)$. We also define $\Sigma_T^{\{k_l\}}$ as

$$\begin{aligned}\Sigma_T^{\{k_l\}} &:= \int_0^T \sum_{i=1}^d w_i e^{\alpha^i(T-t)}\Phi_{S^i}\sum_{I=1}^d w_I e^{\alpha^I(T-t)}\Phi'_{S^I}dt \\ &\quad+\sum_{l=1}^n k_l \sum_{i=1}^d w_i\gamma_{S^i,l}e^{\alpha^i T}s_0^i \sum_{I=1}^d w_I \vartheta_{S^i,S^I}\gamma_{S^I,l}e^{\alpha^I T}s_0^I. \end{aligned} \tag{94}$$

Then, we obtain the following calculations by using Lemma 3.2 and Appendix B.

$$\begin{aligned}
F(g(S_T^{(1)})) &= F(72)+F(73)+F(74) \\
&= (102)+(103)+(104), \qquad (95)\\
F(g(S_T^{(2)})) &= F(75)+F(76)+F(77)+F(78)+F(79)+F(80)+F(81) \\
&\quad +F(82)+F(83)+F(84)+F(85)+F(86)+F(87)+F(88) \\
&= F(75)+F(106)+F(107)+F(77)+F(78)+F(79)+F(80) \\
&\quad +F(114)+F(115)+F(116)+F(82)+F(121)+F(122) \\
&\quad +F(125)+F(126)+F(127)+F(128)+F(85)+F(86) \\
&\quad +F(135)+F(136)+F(88) \\
&= (105)+(108)+(109)+(110)+(111)+(112)+(113) \\
&\quad +(117)+(118)+(119)+(120)+(123)+(124) \\
&\quad +(129)+(130)+(131)+(132)+(133)+(134) \\
&\quad +(137)+(138)+(139), \qquad (96)
\end{aligned}$$

where (102)-(139) stand for the equation numbers listed below. Moreover, $F(75) = (105)$, $F(77) = (110)$, $F(78) = (111)$, $F(79) = (112)$, $F(80) = (113)$, $F(82) = (120)$, $F(85) = (133)$, $F(86) = (134)$, $F(88) = (139)$, and

$$\begin{aligned}
F(76) &= F(106)+F(107) = (108)+(109), \qquad (97)\\
F(81) &= F(114)+F(115)+F(116) = (117)+(118)+(119), \qquad (98)\\
F(83) &= F(121)+F(122) = (123)+(124), \qquad (99)\\
F(84) &= F(125)+F(126)+F(127)+F(128) \\
&= (129)+(130)+(131)+(132), \qquad (100)\\
F(87) &= F(135)+F(136) = (137)+(138). \qquad (101)
\end{aligned}$$

$H_k\left(x;\Sigma_T^{\{k_l\}}\right)$ stands for the $k$-th order Hermite polynomial.

$$F(72) = \mathbf{E}\left[g\left(\int_0^T e^{\alpha(T-t)} * \Phi_S dZ_t\right)\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right]$$

$$= \sum_{i=1}^d w_i \int_0^T e^{\alpha^i(T-t)}\Phi_{S^i}\sum_{I=1}^d w_I e^{\alpha^I(T-t)}\Phi'_{S^I}dt\frac{H_1(x,\Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}}, \quad (102)$$

$$F(73) = \mathbf{E}\left[g\left(\sum_{l=1}^n\sum_{j=1}^{N_{l,T}} h_{S,l,j}^{(1)} * e^{\alpha(T-\tau_{j,l}-)} * S^{(0)}_{\tau_{j,l}-}\right)\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right]$$

$$= \sum_{i=1}^d w_i\sum_{l=1}^n e^{\alpha^i T}s_0^i\mathbf{E}\left[k_l Y_{S^i,l,1}\Big|g(\hat{S}_T)=x\right]$$

$$= \sum_{i=1}^d w_i\sum_{l=1}^n e^{\alpha^i T}s_0^i\mathbf{E}\left[k_l\left(m_{S^i,l}+\gamma_{S^i,l}\zeta_{S,1,l}\right)\Big|g(\hat{S}_T)=x\right]$$

$$= \sum_{i=1}^d w_i\sum_{l=1}^n e^{\alpha^i T}s_0^i k_l\left(m_{S^i,l}+\gamma_{S^i,l}\sum_{I=1}^d w_I\vartheta_{S^i,S^I}\gamma_{S^I,l}e^{\alpha^I T}s_0^I\frac{H_1(x,\Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}}\right), \quad (103)$$

$$F(74) = \mathbf{E}\left[g\left(\sum_{l=1}^n\Lambda_l\mathbf{E}[h^{(1)}_{S,l,1}] * e^{\alpha T} * \int_0^T e^{-\alpha t} * S^{(0)}_{t-}dt\right)\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right]$$

$$= \sum_{i=1}^d w_i\sum_{l=1}^n\Lambda_l m_{S^i,l}e^{\alpha^i T}Ts_0^i, \quad (104)$$

$$F(75) = \mathbf{E}\left[g\left(\int_0^T e^{\alpha(T-t)} * \partial_S\Phi_S * \int_0^t e^{\alpha(t-u)} * \Phi_S dZ_u dZ_t\right)\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right]$$

$$= \sum_{i=1}^d w_i\int_0^T e^{\alpha^i(T-t)}\partial_{S^i}\Phi_{S^i}\sum_{I=1}^d w_I e^{\alpha^I(T-t)}\Phi'_{S^I}\int_0^t e^{\alpha^i(t-u)}\Phi_{S^i}\sum_{J=1}^d w_J e^{\alpha^J(t-u)}\Phi'_{S^J}dudt\frac{H_2(x,\Sigma_T^{\{k_l\}})}{(\Sigma_T^{\{k_l\}})^2}, \quad (105)$$

$$F(76) = \mathbf{E}\left[g\left(\int_0^T e^{\alpha(T-t)} * \partial_S\Phi_S * \sum_{l=1}^n\sum_{j=1}^{N_{l,t}} h^{(1)}_{S,l,j} * e^{\alpha(t-\tau_{j,l})} * S^{(0)}_{\tau_{j,l}-}dZ_t\right)\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right]$$

$$= \mathbf{E}\left[g\left(\int_0^T e^{\alpha(T-t)} * \partial_S\Phi_S * \sum_{l=1}^n\sum_{j=1}^{N_{l,t}} (m_{S,l}+\gamma_{S,l} * \zeta_{S,j,l}) * e^{\alpha(t-\tau_{j,l})} * S^{(0)}_{\tau_{j,l}-}dZ_t\right)\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right]$$

$$= \mathbf{E}\left[g\left(\int_0^T e^{\alpha(T-t)} * \partial_S\Phi_S * \sum_{l=1}^n\sum_{j=1}^{N_{l,t}} m_{S,l} * e^{\alpha(t-\tau_{j,l})} * S^{(0)}_{\tau_{j,l}-}dZ_t\right)\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right] \quad (106)$$

$$+\mathbf{E}\left[g\left(\int_0^T e^{\alpha(T-t)} * \partial_S\Phi_S * \sum_{l=1}^n\sum_{j=1}^{N_{l,t}} \gamma_{S,l} * \zeta_{S,j,l} * e^{\alpha(t-\tau_{j,l})} * S^{(0)}_{\tau_{j,l}-}dZ_t\right)\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right], \quad (107)$$

$$F(106) = \sum_{i=1}^d w_i e^{\alpha^i T}\int_0^T\partial_{S^i}\Phi_{S^i}\sum_{l=1}^n\frac{k_l t}{T}m_{S^i,l}s_0^i\sum_{I=1}^d w_I e^{\alpha^I(T-t)}\Phi'_{S^I}dt\frac{H_1(x,\Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}}, \quad (108)$$

$$F(107) = \mathbf{E}\left[\sum_{i=1}^d w_i\int_0^T e^{\alpha^i(T-t)}\partial_{S^i}\Phi_{S^i}\sum_{I=1}^d w_I e^{\alpha^I(T-t)}\Phi'_{S^I}\right.$$
$$\left.\sum_{l=1}^n\sum_{j=1}^{N_{l,t}|_{N_{l,T}=k_l}}\gamma_{S^i,l}e^{\alpha^i(t-\tau_{j,l})}S^{i,(0)}_{\tau_{j,l}-}\sum_{J=1}^d w_J\vartheta_{S^i,S^J}\gamma_{S^J,l}e^{\alpha^J T}s_0^J dt\frac{H_2(x,\Sigma_T^{\{k_l\}})}{(\Sigma_T^{\{k_l\}})^2}\Big|\{N_l=k_l\}\right]$$

$$= \sum_{i=1}^d w_i\int_0^T e^{\alpha^i T}\partial_{S^i}\Phi_{S^i}\sum_{I=1}^d w_I e^{\alpha^I T}\Phi'_{S^I}\sum_{l=1}^n\frac{k_l t}{T}\gamma_{S^i,l}s_0^i\sum_{J=1}^d w_J\vartheta_{S^i,S^J}\gamma_{S^J,l}e^{\alpha^J T}s_0^J dt\frac{H_2(x,\Sigma_T^{\{k_l\}})}{(\Sigma_T^{\{k_l\}})^2}, \quad (109)$$

$$F(77) = \mathbf{E}\left[g\left(\int_0^T e^{\alpha(T-t)} * \partial_S\Phi_S * \sum_{l=1}^n\Lambda_l * \mathbf{E}[h^{(1)}_{S,l,1}] * \int_0^t e^{\alpha(t-u)} * S^{(0)}_{u-}dudZ_t\right)\Big|g(\hat{S}_T)=x,\{N_l=k_l\}\right]$$

$$
\begin{aligned}
&= \sum_{i=1}^{d} w_i \int_0^T e^{\alpha^i T} \partial_{S^i}\Phi_{S^i} \sum_{I=1}^{d} w_I e^{\alpha^I (T-t)} \Phi'_{S^I} \sum_{l=1}^{n} \Lambda_l m_{S^i,l} t s_0^i dt \frac{H_1(x,\Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}}, && (110)\\
F(78) &= \mathbf{E}\left[ g\left( \int_0^T e^{\alpha(T-t)} * \partial_\sigma \Phi_S * \int_0^t e^{-\lambda(t-u)} * \Phi_\sigma dZ_u dZ_t \right) \Big| g(\hat{S}_T) = x, \{N_l = k_l\} \right] \\
&= \sum_{i=1}^{d} w_i \int_0^T e^{\alpha^i (T-t)} \partial_{\sigma^i}\Phi_{S^i} \sum_{I=1}^{d} w_I e^{\alpha^I (T-t)} \Phi'_{S^I} \int_0^t e^{-\lambda^i (t-u)} \Phi_{\sigma^i} \sum_{J=1}^{d} w_J e^{\alpha^J (t-u)} \Phi'_{S^J} dudt \frac{H_2(x,\Sigma_T^{\{k_l\}})}{(\Sigma_T^{\{k_l\}})^2}, && (111)\\
F(79) &= \mathbf{E}\left[ g\left( \int_0^T e^{\alpha(T-t)} * \partial_\sigma \Phi_S * \sum_{l=1}^{n} \sum_{j=1}^{N_{l,t}} h_{\sigma,l,j}^{(1)} * e^{-\lambda(t-\tau_{j,l})} * \sigma_{\tau_{j,l}-}^{(0)} dZ_t \right) \Big| g(\hat{S}_T) = x, \{N_l = k_l\} \right] \\
&= \sum_{i=1}^{d} w_i \int_0^T e^{\alpha^i (T-t)} \partial_{\sigma^i}\Phi_{S^i} \sum_{I=1}^{d} w_I e^{\alpha^I (T-t)} \Phi'_{S^I} \sum_{l=1}^{n} \int_0^t \frac{k_l}{T} m_{\sigma^i,l} e^{-\lambda^i (t-u)} \sigma_u^{i,(0)} dudt \frac{H_1(x,\Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}} \\
&\quad + \sum_{i=1}^{d} w_i \int_0^T e^{\alpha^i (T-t)} \partial_{\sigma^i}\Phi_{S^i} \sum_{I=1}^{d} w_I e^{\alpha^I (T-t)} \Phi'_{S^I} \\
&\quad \times \sum_{l=1}^{n} \int_0^t \frac{k_l}{T} \gamma_{\sigma^i,l} e^{-\lambda^i (t-u)} \sigma_u^{i,(0)} \sum_{J=1}^{d} w_J \vartheta_{S^i,\sigma^J} \gamma_{S^J,l} e^{\alpha^J T} s_0^J dudt \frac{H_2(x,\Sigma_T^{\{k_l\}})}{(\Sigma_T^{\{k_l\}})^2}, && (112)\\
F(80) &= \mathbf{E}\left[ g\left( \int_0^T e^{\alpha(T-t)} * \partial_\sigma \Phi_S * \sum_{l=1}^{n} \Lambda_l * \mathbf{E}[h_{\sigma,l,1}^{(1)}] * e^{-\lambda t} * \int_0^t e^{\lambda u} * \sigma_{u-}^{(0)} du dZ_t \right) \Big| \{N_l = k_l\} \right] \\
&= \sum_{i=1}^{d} w_i \int_0^T e^{\alpha^i (T-t)} \partial_{\sigma^i}\Phi_{S^i} \sum_{I=1}^{d} w_I e^{\alpha^I (T-t)} \Phi'_{S^I} \sum_{l=1}^{n} \Lambda_l m_{\sigma^i,l} \int_0^t e^{-\lambda^i (t-u)} \sigma_{u-}^{i,(0)} dudt \frac{H_1(x,\Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}}, && (113)\\
F(81) &= \mathbf{E}\left[ g\left( \sum_{l=1}^{n} \sum_{j=1}^{N_{l,T}} h_{S,l,j}^{(2)} * e^{\alpha(T-\tau_{j,l})} * S_{\tau_{j,l}-}^{(0)} \right) \Big| g(\hat{S}_T) = x, \{N_l = k_l\} \right] \\
&= \mathbf{E}\left[ g\left( \sum_{l=1}^{n} \sum_{j=1}^{N_{l,T}} m_{S,l} * m_{S,l} * e^{\alpha(T-\tau_{j,l})} * S_{\tau_{j,l}-}^{(0)} \right) \Big| g(\hat{S}_T) = x, \{N_l = k_l\} \right] && (114)\\
&\quad + \mathbf{E}\left[ g\left( \sum_{l=1}^{n} \sum_{j=1}^{N_{l,T}} 2 m_{S,l} * \gamma_{S,l} * \zeta_{S,j,l} * e^{\alpha(T-\tau_{j,l})} * S_{\tau_{j,l}-}^{(0)} \right) \Big| g(\hat{S}_T) = x, \{N_l = k_l\} \right] && (115)\\
&\quad + \mathbf{E}\left[ g\left( \sum_{l=1}^{n} \sum_{j=1}^{N_{l,T}} \gamma_{S,l} * \gamma_{S,l} * \zeta_{S,j,l} * \zeta_{S,j,l} * e^{\alpha(T-\tau_{j,l})} * S_{\tau_{j,l}-}^{(0)} \right) \Big| g(\hat{S}_T) = x, \{N_l = k_l\} \right], && (116)\\
F(114) &= \sum_{i=1}^{d} w_i \sum_{l=1}^{n} k_l m_{S^i,l}^2 e^{\alpha^i T} s_0^i, && (117)\\
F(115) &= \sum_{i=1}^{d} w_i \sum_{l=1}^{n} k_l 2 m_{S^i,l} \gamma_{S^i,l} e^{\alpha^i T} s_0^i \sum_{I=1}^{d} w_I \vartheta_{S^i,S^I} \gamma_{S^I,l} e^{\alpha^I T} s_0^I \frac{H_1(x,\Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}}, && (118)\\
F(116) &= \sum_{i=1}^{d} w_i \sum_{l=1}^{n} k_l (\gamma_{S^i,l})^2 e^{2\alpha^i T} (s_0^i)^2 + \sum_{i=1}^{d} w_i \sum_{l=1}^{n} k_l \left( \gamma_{S^i,l} e^{\alpha^i T} s_0^i \sum_{I=1}^{d} w_I \vartheta_{S^i,S^I} \gamma_{S^I,l} e^{\alpha^I T} s_0^I \right)^2 \frac{H_2(x,\Sigma_T^{\{k_l\}})}{(\Sigma_T^{\{k_l\}})^2}, && (119)\\
F(82) &= \mathbf{E}\left[ g\left( \sum_{l=1}^{n} \Lambda_l * \mathbf{E}[h_{S,l,1}^{(2)}] * e^{\alpha T} * \int_0^T e^{-\alpha t} * S_{t-}^{(0)} dt \right) \Big| g(\hat{S}_T) = x, \{N_l = k_l\} \right] \\
&= \sum_{i=1}^{d} w_i \sum_{l=1}^{n} \Lambda_l (m_{S^i,l}^2 + \gamma_{S^i,l}^2) e^{\alpha^i T} T s_0^i, && (120)
\end{aligned}
$$

$$\begin{aligned}
F(83) &= \mathbf{E}\left[g\left(\sum_{l=1}^{n}\sum_{j=1}^{N_{l,T}} m_{S,l} * e^{\alpha(T-\tau_{j,l})} * \int_0^{\tau_{j,l}} e^{\alpha(\tau_{j,l}-u)} * \Phi_S dZ_u\right)\Big| g(\hat{S}_T)=x, \{N_l=k_l\}\right] && (121)\\
&+\mathbf{E}\left[g\left(\sum_{l=1}^{n}\sum_{j=1}^{N_{l,T}} \gamma_{S,l} * \zeta_{S,j,l} * e^{\alpha(T-\tau_{j,l})} * \int_0^{\tau_{j,l}} e^{\alpha(\tau_{j,l}-u)} * \Phi_S dZ_u\right)\Big| g(\hat{S}_T)=x, \{N_l=k_l\}\right], && (122)\\
F(121) &= \sum_{i=1}^{d} w_i \sum_{l=1}^{n} m_{S^i,l} T \int_0^T e^{\alpha^i(T-t)}\Phi_{S^i} \sum_{I=1}^{d} w_I e^{\alpha^I(T-t)}\Phi'_{S^I}\,dt \frac{k_l}{T}\frac{H_1(x,\Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}}\\
&-\sum_{i=1}^{d} w_i \sum_{l=1}^{n} m_{S^i,l} \int_0^T t e^{\alpha^i(T-t)}\Phi_{S^i} \sum_{I=1}^{d} w_I e^{\alpha^I(T-t)}\Phi'_{S^i}\,dt \frac{k_l}{T}\frac{H_1(x,\Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}}, && (123)\\
F(122) &= \sum_{i=1}^{d} w_i \sum_{l=1}^{n} \gamma_{S^i,l} \sum_{I=1}^{d} T w_I \vartheta_{S^i,S^I}\gamma_{S^I,l} e^{\alpha^I T} s_0^I \sum_{J=1}^{d}\int_0^T e^{\alpha^i(T-t)}\Phi_{S^i} w_J e^{\alpha^J(T-t)}\Phi'_{S^J}\,dt \frac{k_l}{T}\frac{H_2(x,\Sigma_T^{\{k_l\}})}{(\Sigma_T^{\{k_l\}})^2}\\
&-\sum_{i=1}^{d} w_i \sum_{l=1}^{n} \gamma_{S^i,l} \sum_{I=1}^{d}\int_0^T t w_I \vartheta_{S^i,S^I}\gamma_{S^I,l} e^{\alpha^I t} s_0^I \sum_{J=1}^{d} e^{\alpha^i(T-t)}\Phi_{S^i} w_J e^{\alpha^J(T-t)}\Phi'_{S^J}\,dt \frac{k_l}{T}\frac{H_2(x,\Sigma_T^{\{k_l\}})}{(\Sigma_T^{\{k_l\}})^2}, && (124)\\
F(84) &= \mathbf{E}\left[g\left(e^{\alpha T} * s_0 * \sum_{l=1}^{n}\sum_{j_l=1}^{k_l} h^{(1)}_{S,l,j_l} * \sum_{L=1}^{n}\sum_{I=1}^{j_L-1} h^{(1)}_{S,L,I}\right)\Big| g(\hat{S}_T)=x\right]\\
&= \mathbf{E}\left[g\left(e^{\alpha T} * s_0 * \sum_{l=1}^{n}\sum_{j_l=1}^{k_l} (m_{S,l}+\gamma_{S,l}*\zeta_{S,j_l,l}) * \sum_{L=1}^{n}\sum_{I=1}^{j_L-1} (m_{S,L}+\gamma_{S,L}*\zeta_{S,I,L})\right)\Big| g(\hat{S}_T)=x\right]\\
&= \mathbf{E}\left[g\left(e^{\alpha T} * s_0 * \sum_{l=1}^{n}\sum_{j_l=1}^{k_l}\sum_{L=1}^{n}\sum_{I=1}^{j_L-1} m_{S,l} * m_{S,L}\right)\Big| g(\hat{S}_T)=x\right] && (125)\\
&+\mathbf{E}\left[g\left(e^{\alpha T} * s_0 * \sum_{l=1}^{n}\sum_{j_l=1}^{k_l}\sum_{L=1}^{n}\sum_{I=1}^{j_L-1} m_{S,l} * \gamma_{S,L} * \zeta_{S,I,L}\right)\Big| g(\hat{S}_T)=x\right] && (126)\\
&+\mathbf{E}\left[g\left(e^{\alpha T} * s_0 * \sum_{l=1}^{n}\sum_{j_l=1}^{k_l}\sum_{L=1}^{n}\sum_{I=1}^{j_L-1} \gamma_{S,l} * \zeta_{S,j_l,l} * m_{S,L}\right)\Big| g(\hat{S}_T)=x\right] && (127)\\
&+\mathbf{E}\left[g\left(e^{\alpha T} * s_0 * \sum_{l=1}^{n}\sum_{j_l=1}^{k_l}\sum_{L=1}^{n}\sum_{I=1}^{j_L-1} \gamma_{S,l} * \zeta_{S,j_l,l} * \gamma_{S,L} * \zeta_{S,I,L}\right)\Big| g(\hat{S}_T)=x\right], && (128)\\
F(125) &= \sum_{i=1}^{d} w_i e^{\alpha^i T} s_0^i \sum_{l=1}^{n}\sum_{j_l=1}^{k_l}\sum_{L=1}^{n}\sum_{I=1}^{j_L-1} m_{S^i,l} m_{S^i,L}, && (129)\\
F(126) &= \sum_{i=1}^{d} w_i e^{\alpha^i T} s_0^i \sum_{l=1}^{n}\sum_{j_l=1}^{k_l}\sum_{L=1}^{n}\sum_{I=1}^{j_L-1} m_{S^i,l}\mathbf{E}\left[\gamma_{S^i,L}\zeta_{S,I,L}\Big| g(\hat{S}_T)=x\right]\\
&= \sum_{i=1}^{d} w_i e^{\alpha^i T} s_0^i \sum_{l=1}^{n}\sum_{j_l=1}^{k_l}\sum_{L=1}^{n}\sum_{I=1}^{j_L-1} m_{S^i,l}\gamma_{S^i,L} \sum_{J=1}^{d} w_J \vartheta_{S^i,S^J}\gamma_{S^J,L} e^{\alpha^J T} s_0^J \frac{H_1(x,\Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}}, && (130)\\
F(127) &= \sum_{i=1}^{d} w_i e^{\alpha^i T} s_0^i \sum_{l=1}^{n}\sum_{j_l=1}^{k_l}\sum_{L=1}^{n}\sum_{I=1}^{j_L-1} m_{S^i,L}\mathbf{E}\left[\gamma_{S^i,l}\zeta_{S,j_l,l}\Big| g(\hat{S}_T)=x\right]\\
&= \sum_{i=1}^{d} w_i e^{\alpha^i T} s_0^i \sum_{l=1}^{n}\sum_{j_l=1}^{k_l}\sum_{L=1}^{n}\sum_{I=1}^{j_L-1} m_{S^i,L}\gamma_{S^i,l} \sum_{J=1}^{d} w_J \vartheta_{S^i,S^J}\gamma_{S^J,l} e^{\alpha^J T} s_0^J \frac{H_1(x,\Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}}, && (131)
\end{aligned}$$

$$
\begin{aligned}
F(128) &= \sum_{i=1}^{d} w_i e^{\alpha^i T} s_0^i \sum_{l=1}^{n} \sum_{j_l=1}^{k_l} \sum_{L=1}^{n} \sum_{I=1}^{j_L-1} \mathbf{E}\left[\gamma_{S^i,l}\zeta_{S,j_l,l}\gamma_{S^i,L}\zeta_{S,I,L}\Big| g(\hat{S}_T)=x\right] \\
&= \sum_{i=1}^{d} w_i e^{\alpha^i T} s_0^i \sum_{l=1}^{n} \sum_{j_l=1}^{k_l} \sum_{L=1}^{n} \sum_{I=1}^{j_L-1} \gamma_{S^i,l} \sum_{\iota=1}^{d} w_\iota \vartheta_{S^i,S^\iota}\gamma_{S^\iota,l} e^{\alpha^\iota T} s_0^\iota \\
&\quad \times \gamma_{S^i,L} \sum_{J=1}^{d} w_J \vartheta_{S^i,S^J}\gamma_{S^J,L} e^{\alpha^J T} s_0^J \frac{H_2(x,\Sigma_T^{\{k_l\}})}{(\Sigma_T^{\{k_l\}})^2}, \qquad (132)\\
F(85) &= \mathbf{E}\left[g\left(\sum_{l=1}^{n}\sum_{j=1}^{N_{l,T}} h^{(1)}_{S,l,j} * e^{\alpha(T-\tau_{j,l})} * \sum_{L=1}^{n}\Lambda_L * \mathbf{E}[h^{(1)}_{S,L,1}] * \int_0^{\tau_{j,l}} e^{\alpha(\tau_{j,l}-u)} * S^{(0)}_{u-} du\right)\Big| g(\hat{S}_T)=x, \{N_l=k_l\}\right] \\
&= \mathbf{E}\left[g\left(\sum_{l=1}^{n}\sum_{j=1}^{N_{l,T}} (m_{S,l}+\gamma_{S,l}*\zeta_{S,j,l}) * e^{\alpha T} * \sum_{L=1}^{n}\Lambda_L * m_{S,l} * \int_0^{\tau_{j,l}} e^{-\alpha u} * S^{(0)}_{u-} du\right)\Big| g(\hat{S}_T)=x, \{N_l=k_l\}\right] \\
&= \sum_{i=1}^{d} w_i \sum_{l=1}^{n} m_{S^i,l} e^{\alpha^i T} \sum_{L=1}^{n} \Lambda_L m_{S^i,l} s_0^i \frac{k_l T}{2} \\
&\quad + \sum_{i=1}^{d} w_i \sum_{l=1}^{n} \gamma_{S^i,l} e^{\alpha^i T} \sum_{I=1}^{d} w_I \vartheta_{S^i,S^I}\gamma_{S^I,l} e^{\alpha^I T} s_0^I \sum_{L=1}^{n}\Lambda_L m_{S^i,l} s_0^i \frac{k_l T}{2}\frac{H_1(x,\Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}}, \qquad (133)\\
F(86) &= \mathbf{E}\left[g\left(\int_0^T \sum_{l=1}^{n}\Lambda_l * \mathbf{E}[h^{(1)}_{S,l,1}] * e^{\alpha(T-t)} * \int_0^t e^{\alpha(t-u)} * \Phi_S dZ_u dt\right)\Big| g(\hat{S}_T)=x, \{N_l=k_l\}\right] \\
&= \mathbf{E}\left[g\left(\sum_{l=1}^{n}\int_0^T \Lambda_l * \mathbf{E}[h^{(1)}_{S,l,1}] * e^{\alpha T} dt * \int_0^T e^{-\alpha t} * \Phi_S dZ_t\right.\right. \\
&\quad \left.\left. - \sum_{l=1}^{n}\int_0^T\int_0^t \Lambda_l * \mathbf{E}[h^{(1)}_{S,l,1}] * e^{\alpha T} du * e^{-\alpha t} * \Phi_S dZ_t\right)\Big| g(\hat{S}_T)=x, \{N_l=k_l\}\right] \\
&= \sum_{i=1}^{d} w_i T \sum_{l=1}^{n}\Lambda_l m_{S^i,l,j} e^{\alpha^i T}\int_0^T e^{-\alpha^i t}\Phi_{S^i}\sum_{I=1}^{d} w_I e^{\alpha^I(T-t)}\Phi'_{S^I} dt \frac{H_1(x,\Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}} \\
&\quad - \sum_{i=1}^{d} w_i \int_0^T t \sum_{l=1}^{n}\Lambda_l m_{S^i,l,j} e^{\alpha^i(T-t)}\Phi_{S^i}\sum_{I=1}^{d} w_I e^{\alpha^I(T-t)}\Phi'_{S^I} dt \frac{H_1(x,\Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}}, \qquad (134)\\
F(87) &= \mathbf{E}\left[g\left(\int_0^T \sum_{l=1}^{n}\Lambda_l * m_{S,l} * e^{\alpha T} * \sum_{L=1}^{n}\sum_{I=1}^{N_{L,t}} m_{S,L} * s_0 dt\right)\Big| g(\hat{S}_T)=x, \{N_l=k_l\}\right] \qquad (135)\\
&\quad + \mathbf{E}\left[g\left(\int_0^T \sum_{l=1}^{n}\Lambda_l * m_{S^i,l} * e^{\alpha^i T} * \sum_{L=1}^{n}\sum_{I=1}^{N_{L,t}} \gamma_{S,L}*\zeta_{S,j,L} * s_0 dt\right)\Big| g(\hat{S}_T)=x, \{N_l=k_l\}\right], \qquad (136)\\
F(135) &= \sum_{i=1}^{d} w_i \int_0^T \sum_{l=1}^{n}\Lambda_l m_{S^i,l} e^{\alpha^i T} s_0^i * \sum_{L=1}^{n} m_{S^i,L}\int_0^t \frac{k_l}{T} du dt \\
&= \sum_{i=1}^{d} w_i \sum_{l=1}^{n}\Lambda_l m_{S^i,l} e^{\alpha^i T} s_0^i \sum_{L=1}^{n} m_{S^i,l}\frac{k_l T}{2}, \qquad (137)\\
F(136) &= \mathbf{E}\left[g\left(\int_0^T \sum_{l=1}^{n}\Lambda_l * m_{S,l} * e^{\alpha T} * \sum_{L=1}^{n}\sum_{I=1}^{N_{L,t}|_{N_{L,T}=k_L}} \mathbf{E}\left[\gamma_{S,L}*\zeta_{S,j,L}\Big| g(\hat{S}_T)=x\right] s_0 dt\right)\Big| g(\hat{S}_T)=x, \{N_l=k_l\}\right] \\
&= \mathbf{E}\Bigg[g\Bigg(s_0\int_0^T \sum_{l=1}^{n}\Lambda_l * m_{S,l} * e^{\alpha T}
\end{aligned}
$$

$$
\begin{aligned}
&\quad * \sum_{L=1}^{n} \sum_{I=1}^{N_{L,t}|N_{L,T}=k_L} \gamma_{S,L} \sum_{J=1}^{d} w_J \vartheta_{S^i,S^J} \gamma_{S^J,L} e^{\alpha^J T} s_0^J \frac{H_1(x, \Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}} dt \Big) \Big| g(\hat{S}_T) = x, \{N_l = k_l\} \Big] \\
&= \sum_{i=1}^{d} w_i s_0^i \int_0^T \sum_{l=1}^{n} \Lambda_l m_{S^i,l} e^{\alpha^i T} \sum_{L=1}^{n} \sum_{I=1}^{k_L} \gamma_{S^i,L} \sum_{J=1}^{d} w_J \vartheta_{S^i,S^J} \gamma_{S^J,L} e^{\alpha^J T} s_0^J \frac{t}{T} \frac{H_1(x, \Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}} dt \\
&= \sum_{i=1}^{d} w_i s_0^i \sum_{l=1}^{n} \Lambda_l m_{S^i,l} e^{\alpha^i T} \sum_{L=1}^{n} \gamma_{S^i,L} \sum_{J=1}^{d} w_J \vartheta_{S^i,S^J} \gamma_{S^J,L} e^{\alpha^J T} s_0^J \frac{k_l T}{2} \frac{H_1(x, \Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}}, \qquad (138)
\end{aligned}
$$

$$
\begin{aligned}
F(88) &= \sum_{i=1}^{d} w_i \int_0^T \sum_{l=1}^{n} \Lambda_l m_{S^i,l} e^{\alpha^i T} \sum_{L=1}^{n} \Lambda_L m_{S^i,L} t s_0^i dt \\
&= \sum_{i=1}^{d} w_i \sum_{l=1}^{n} \Lambda_l m_{S^i,l} e^{\alpha^i T} \sum_{L=1}^{n} \Lambda_L m_{S^i,L} s_0^i \frac{T^2}{2}. \qquad (139)
\end{aligned}
$$

Next, let us show the expression of $g\left(\frac{1}{2!}S_T^{LSV(2)}\right)^2$ by applying Appendix B.

$$
\begin{aligned}
g\left(\frac{1}{2!}S_T^{LSV(2)}\right)^2 &= g\Bigg(\int_0^T e^{\alpha(T-t)} * \partial_S \Phi_S * \int_0^t e^{\alpha(t-u)} * \Phi_S dZ_u dZ_t \\
&\quad + \int_0^T e^{\alpha(T-t)} * \partial_\sigma \Phi_S * \int_0^t e^{-\lambda(t-u)} * \Phi_\sigma dZ_u dZ_t \Bigg)^2 \\
&= g\left(\int_0^T e^{\alpha(T-t)} * \partial_S \Phi_S * \int_0^t e^{\alpha(t-u)} * \Phi_S dZ_u dZ_t\right)^2 \qquad (140) \\
&\quad +2g\Bigg(\left(\int_0^T e^{\alpha(T-t)} * \partial_S \Phi_S * \int_0^t e^{\alpha(t-u)} * \Phi_S dZ_u dZ_t\right) \\
&\quad \times \left(\int_0^T e^{\alpha(T-t)} * \partial_\sigma \Phi_S * \int_0^t e^{-\lambda(t-u)} * \Phi_\sigma dZ_u dZ_t\right)\Bigg) \qquad (141) \\
&\quad +g\left(\int_0^T e^{\alpha(T-t)} * \partial_\sigma \Phi_S * \int_0^t e^{-\lambda(t-u)} * \Phi_\sigma dZ_u dZ_t\right)^2 \qquad (142) \\
&= F(140) + 2F(141) + F(142). \qquad (143)
\end{aligned}
$$

Then, we obtain the expressions of $F(M)$ for $M = 140, 141, 142$ as follows:

$$
\begin{aligned}
F(M) &= \sum_{i=1}^{d} w_i \sum_{I=1}^{d} w_I \left(\int_0^T q'_{M,3t,i} q_{M,1t,i} \int_0^t q'_{M,2s,i} q_{M,1s,i} ds dt\right) \\
&\quad \left(\int_0^T q'_{M,5r,I} q_{M,1r,I} \int_0^r q'_{M,4u,I} q_{M,1u,I} du dr\right) \frac{H_4(x; \Sigma_T^{\{k_l\}})}{(\Sigma_T^{\{k_l\}})^4} \\
&\quad + \sum_{i=1}^{d} w_i \sum_{I=1}^{d} w_I \Bigg\{ \int_0^T q'_{M,3t,i} q_{M,1t,i} \int_0^t q'_{M,5r,I} q_{M,1r,I} \int_0^r q'_{M,2u,i} q_{M,4u,I} du dr dt \\
&\quad + \int_0^T q'_{M,5t,I} q_{M,1t,I} \int_0^t q'_{M,3r,i} q_{M,1r,i} \int_0^r q'_{M,2u,i} q_{M,4u,I} du dr dt \\
&\quad + \int_0^T q'_{M,3t,i} q_{M,1t,i} \int_0^t q'_{M,2r,i} q_{M,5r,I} \int_0^r q'_{M,4u,I} q_{M,1u,I} du dr dt
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^T q'_{M,3t,i}q_{M,5t,I}\left(\int_0^t q'_{M,2s,i}q_{1s}ds\right)\left(\int_0^t q'_{M,4u,I}q_{M,1u,I}du\right)dt \\
&\left.+\int_0^T q'_{M,5r,I}q_{M,1r,I}\int_0^r q'_{M,3u,i}q_{M,4u,I}\int_0^u q'_{M,2s,i}q_{M,1s,i}dsdudr\right\}\frac{H_2(x;\Sigma_T^{\{k_l\}})}{(\Sigma_T^{\{k_l\}})^2} \\
&+\sum_{i=1}^d w_i\sum_{I=1}^d w_I\int_0^T\int_0^t q'_{M,2u,i}q_{M,4u,I}duq'_{M,3t,i}q_{M,5t,I}dt, \qquad (144)
\end{aligned}
$$

where

$$
\begin{aligned}
q'_{140,1t,i} &= \sum_{i=1}^d w_i\Phi_{S^i}, & (145)\\
q'_{140,2t,i} &= e^{-\alpha^i t}*\Phi_{S^i}, & (146)\\
q'_{140,3t,i} &= e^{\alpha^i T}*\partial_{S^i}\Phi_{S^i}, & (147)\\
q'_{140,4t,i} &= e^{-\alpha^i t}*\Phi_{S^i}, & (148)\\
q'_{140,5t,i} &= e^{\alpha^i T}*\partial_{S^i}\Phi_{S^i}, & (149)\\
q'_{141,1t,i} &= \sum_{i=1}^d w_i\Phi_{S^i}, & (150)\\
q'_{141,2t,i} &= e^{-\alpha^i t}*\Phi_{S^i}, & (151)\\
q'_{141,3t,i} &= e^{\alpha^i T}*\partial_{S^i}\Phi_{S^i}, & (152)\\
q'_{141,4t,i} &= e^{\lambda^i u}*\Phi_{\sigma^i}, & (153)\\
q'_{141,5t,i} &= e^{\alpha^i(T-t)-\lambda^i t}*\partial_{\sigma^i}\Phi_{S^i}, & (154)\\
q'_{142,1t,i} &= \sum_{i=1}^d w_i\Phi_{S^i}, & (155)\\
q'_{142,2t,i} &= e^{\lambda^i u}*\Phi_{\sigma^i}, & (156)\\
q'_{142,3t,i} &= e^{\alpha^i(T-t)-\lambda^i t}*\partial_{\sigma^i}\Phi_{S^i}, & (157)\\
q'_{142,4t,i} &= e^{\lambda^i u}*\Phi_{\sigma}, & (158)\\
q'_{142,5t,i} &= e^{\alpha^i(T-t)-\lambda^i t}*\partial_{\sigma^i}\Phi_{S^i}. & (159)
\end{aligned}
$$

Then, we show the expressions of $g\left(\frac{1}{3!}S_T^{LSV(3)}\right)$ by applying Appendix B.

$$
\begin{aligned}
g\left(\frac{1}{3!}S_T^{LSV(3)}\right) &= g\left(\frac{1}{2}\int_0^T e^{\alpha(T-t)}*\partial_S^2\Phi_S*\left(\int_0^t e^{\alpha(t-u)}*\Phi_S dZ_u\right)^2 dZ_t\right) & (160)\\
&+g\left(\int_0^T e^{\alpha(T-t)}*\partial_S\Phi_S\right.\\
&\left.*\int_0^t e^{\alpha(t-u)}*\partial_S\Phi_S*\int_0^u e^{\alpha(u-v)}*\Phi_S dZ_v dZ_u dZ_t\right) & (161)
\end{aligned}
$$

$$
\begin{aligned}
&+g\Bigg(\int_0^T e^{\alpha(T-t)} * \partial_S \Phi_S \\
&* \int_0^t e^{\alpha(t-u)} * \partial_\sigma \Phi_S * \int_0^u e^{-\lambda(u-v)} * \Phi_\sigma dZ_v dZ_u dZ_t\Bigg) && (162)\\
&+g\left(\frac{1}{2}\int_0^T e^{\alpha(T-t)} * \partial_\sigma^2 \Phi_S * \left(\int_0^t e^{-\lambda(t-u)} * \Phi_\sigma dZ_u\right)^2 dZ_t\right) && (163)\\
&+g\Bigg(\int_0^T e^{\alpha(T-t)} * \partial_\sigma \Phi_S \\
&* \int_0^t e^{-\lambda(t-u)} * \partial_\sigma \Phi_\sigma * \int_0^u e^{-\lambda(u-v)} * \Phi_\sigma dZ_v dZ_u dZ_t\Bigg) && (164)\\
= \;& \frac{1}{2}F(160) + F(161) + F(162) + \frac{1}{2}F(163) + F(164). && (165)
\end{aligned}
$$

We obtain the expressions of $F(M)$ for $M = 161, 162, 164$:

$$
F(M) = \sum_{i=1}^d w_i \left(\int_0^T q'_{M,4t,i} q_{M,1t,i} \int_0^t q'_{M,3s,i} q_{M,1s,i} \int_0^s q'_{M,2u,i} q_{M,1u,i} du ds dt\right) \frac{H_3(x;\Sigma_T^{\{k_l\}})}{(\Sigma_T^{\{k_l\}})^3}, \tag{166}
$$

where

$$
\begin{aligned}
q'_{161,1t,i} &= \sum_{i=1}^d w_i \Phi_{S^i}, && (167)\\
q'_{161,2t,i} &= e^{-\alpha^i t} * \Phi_{S^i}, && (168)\\
q'_{161,3t,i} &= \partial_{S^i} \Phi_{S^i}, && (169)\\
q'_{161,4t,i} &= e^{\alpha^i T} * \partial_{S^i} \Phi_{S^i}, && (170)\\
q'_{162,1t,i} &= \sum_{i=1}^d w_i \Phi_{S^i}, && (171)\\
q'_{162,2t,i} &= e^{\lambda^i t} * \Phi_{\sigma^i}, && (172)\\
q'_{162,3t,i} &= e^{-\alpha^i t - \lambda^i t} * \partial_{\sigma^i} \Phi_{S^i}, && (173)\\
q'_{162,4t,i} &= e^{\alpha^i T} * \partial_{S^i} \Phi_{S^i}, && (174)\\
q'_{164,1t,i} &= \sum_{i=1}^d w_i \Phi_{S^i}, && (175)\\
q'_{164,2t,i} &= e^{\lambda^i t} * \Phi_{\sigma^i}, && (176)\\
q'_{164,3t,i} &= \partial_{\sigma^i} \Phi_{\sigma^i}, && (177)\\
q'_{164,4t,i} &= e^{\alpha^i (T-t) - \lambda^i t} * \partial_{\sigma^i} \Phi_{S^i}. && (178)
\end{aligned}
$$

We also have the expressions of $F(M)$ for $M = 160, 163$ as follows:

$$
F(M) = \sum_{i=1}^d w_i \left\{\int_0^T \left(\int_0^t q'_{M,2u,i} q_{M,1u,i} du\right) \left(\int_0^t q'_{M,3s,i} q_{M,1s,i} ds\right) q'_{M,4t,i} q_{M,1t,i} dt\right\} \frac{H_3(x;\Sigma_T^{\{k_l\}})}{(\Sigma_T^{\{k_l\}})^3}
$$

$$+\left(\int_0^T\int_0^t q'_{M,2u,i}q_{M,3u,i}duq'_{M,4t,i}q_{M,1t,i}dt\right)\frac{H_1(x;\Sigma_T^{\{k_l\}})}{\Sigma_T^{\{k_l\}}},\tag{179}$$

where

$$q'_{160,1t,i} = \sum_{i=1}^{d} w_i\Phi_{S^i},\tag{180}$$

$$q'_{160,2t,i} = q_{160,3t,i} = e^{-\alpha^i t} * \Phi_{S^i},\tag{181}$$

$$q'_{160,4t,i} = e^{\alpha^i(T+t)} * \partial^2_{S^i}\Phi_{S^i},\tag{182}$$

$$q'_{163,1t,i} = \sum_{i=1}^{d} w_i\Phi_{S^i},\tag{183}$$

$$q'_{163,2t,i} = q_{163,3t,i} = e^{\lambda^i u} * \Phi_{\sigma^i},\tag{184}$$

$$q'_{163,4t,i} = e^{\alpha^i(T-t)-2\lambda^i t} * \partial^2_{\sigma^i}\Phi_{S^i}.\tag{185}$$

Collecting these terms with integrating them with respect to $x$ from $-y_{k_l}$ to $\infty$ and collecting the terms with the same order of $x$ in Helmite polynomials $H_k\left(x;\Sigma_T^{\{k_l\}}\right)$ $(k = 1, 2, 3, 4)$, we obtain the coefficients $C_{i,k_l}$, $(i = 1, \cdots, 6)$ in (69).

# B Conditional Expectation Formulas for the Wiener-Itô Integrals

This appendix summarizes conditional expectation formulas for explicit computation of the asymptotic expansions up to the third order.

In the following, $W$ is a $d$-dimensional Brownian motion and $q_i = (\hat{q}_{i1}, \cdots, \hat{q}_{id})'$ where $\hat{q}_i \in L^2[0,T]$, $i = 1, 2, ..., 5$ and $x'$ denotes the transpose of $x$. $H_n(x;\Sigma)$ denotes the Hermite polynomial of degree $n$ and $\Sigma = \int_0^T |q_{1t}|^2dt$. For the derivation and more general results, see Section 3 in Takahashi, Takehara and Toda (2009).

1.

$$\mathbf{E}\left[\int_0^T q'_{2t}dW_t|\int_0^T q'_{1v}dW_v = x\right] = \left(\int_0^T q'_{2t}q_{1t}dt\right)\frac{H_1(x;\Sigma)}{\Sigma}.\tag{186}$$

2.

$$\mathbf{E}\left[\int_0^T\int_0^t q'_{2u}dW_uq'_{3t}dW_t|\int_0^T q'_{1v}dW_v = x\right] =$$
$$\left(\int_0^T\int_0^t q'_{2u}q_{1u}duq'_{3t}q_{1t}dt\right)\frac{H_2(x;\Sigma)}{\Sigma^2}.\tag{187}$$

3.

$$\mathbf{E}\left[\left(\int_0^T q'_{2u}dW_u\right)\left(\int_0^T q'_{3s}dW_s\right)|\int_0^T q'_{1v}dW_v = x\right] =$$

$$\left(\int_0^T q'_{2u} q_{1u} du\right)\left(\int_0^T q'_{3s} q_{1s} ds\right)\frac{H_2(x;\Sigma)}{\Sigma^2}$$
$$+\int_0^T q'_{2t} q_{3t} dt. \tag{188}$$

4.

$$\mathbf{E}\left[\int_0^T \int_0^t \int_0^s q'_{2u} dW_u q'_{3s} dW_s q'_{4t} dW_t | \int_0^T q'_{1v} dW_v = x\right] =$$
$$\left(\int_0^T q'_{4t} q_{1t} \int_0^t q'_{3s} q_{1s} \int_0^s q'_{2u} q_{1u} du ds dt\right)\frac{H_3(x;\Sigma)}{\Sigma^3}. \tag{189}$$

5.

$$\mathbf{E}\left[\int_0^T \left(\int_0^t q'_{2u} dW_u\right)\left(\int_0^t q'_{3s} dW_s\right) q'_{4t} dW_t | \int_0^T q'_{1v} dW_v = x\right] =$$
$$\left\{\int_0^T \left(\int_0^t q'_{2u} q_{1u} du\right)\left(\int_0^t q'_{3s} q_{1s} ds\right) q'_{4t} q_{1t} dt\right\}\frac{H_3(x;\Sigma)}{\Sigma^3}$$
$$+\left(\int_0^T \int_0^t q'_{2u} q_{3u} du q'_{4t} q_{1t} dt\right)\frac{H_1(x;\Sigma)}{\Sigma}. \tag{190}$$

6.

$$\mathbf{E}\left[\left(\int_0^T \int_0^t q'_{2s} dW_s q'_{3t} dW_t\right)\left(\int_0^T \int_0^r q'_{4u} dW_u q'_{5r} dW_r\right) | \int_0^T q'_{1v} dW_v = x\right] =$$
$$\left(\int_0^T q'_{3t} q_{1t} \int_0^t q'_{2s} q_{1s} ds dt\right)\left(\int_0^T q'_{5r} q_{1r} \int_0^r q'_{4u} q_{1u} du dr\right)\frac{H_4(x;\Sigma)}{\Sigma^4}$$
$$+\left\{\int_0^T q'_{3t} q_{1t} \int_0^t q'_{5r} q_{1r} \int_0^r q'_{2u} q_{4u} du dr dt + \int_0^T q'_{5t} q_{1t} \int_0^t q'_{3r} q_{1r} \int_0^r q'_{2u} q_{4u} du dr dt\right.$$
$$+\int_0^T q'_{3t} q_{1t} \int_0^t q'_{2r} q_{5r} \int_0^r q'_{4u} q_{1u} du dr dt + \int_0^T q'_{3t} q_{5t}\left(\int_0^t q'_{2s} q_{1s} ds\right)\left(\int_0^t q'_{4u} q_{1u} du\right) dt$$
$$\left.+\int_0^T q'_{5r} q_{1r} \int_0^r q'_{3u} q_{4u} \int_0^u q'_{2s} q_{1s} ds du dr\right\}\frac{H_2(x;\Sigma)}{\Sigma^2}$$
$$+\int_0^T \int_0^t q'_{2u} q_{4u} du q'_{3t} q_{5t} dt. \tag{191}$$