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Perturbative Expansion Technique for Non-linear FBSDEs with Interacting Particle Method *

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Abstract

In this paper, we propose an efficient Monte Carlo implementation of a non-linear FBSDE as a system of interacting particles inspired by the idea of the branching diffusion method of McKean. It will be particularly useful to investigate large and complex systems, and hence it is a good complement of our previous work presenting an analytical perturbation procedure for generic non-linear FBSDEs. There appear multiple species of particles, where the first one follows the diffusion of the original underlying state, and the others the Malliavin derivatives with a grading structure. The number of branching points are capped by the order of perturbation, which is expected to make the scheme less numerically intensive. The proposed method can be applied to semi-linear problems, such as American Options, Credit and Funding Value Adjustments, and even fully non-linear issues, such as the optimal portfolio problems in incomplete and/or constrained markets.

Keywords : BSDE, FBSDE, Asymptotic Expansion, Malliavin Derivative, interacting particle method, branching diffusion

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1 Introduction

The forward backward stochastic differential equations (FBSDEs) were first introduced by Bismut (1973) [2], and then later extended by Pardoux and Peng (1990) [29] for general non-linear cases. They were found particularly relevant for optimal portfolio and indifference pricing issues in incomplete and/or constrained markets. Their financial applications are discussed in, for example, El Karoui, Peng and Quenez (1997a) [11], Ma and Yong (2000) [25] and a book edited by Carmona (2009) [6]. There now exist large number of literature studying Monte Carlo simulation techniques to solve FBSDEs. Since new works are constantly appearing in this research area, it is impossible to mention every relevant work. Just to name but a few, one can find important examples in Bouchard & Touzi (2004) [5], Bender & Denk (2007) [1], Gobet et al. (2005) [16] and Gobet & Lemor (2010) [17].

After the financial crisis in 2008, the researches on FBSDEs have gained significant traction not only from academia but also from financial practitioners. This is because that they inevitably encounter the problems of FBSDEs when they try to handle credit risk, collateralization, funding and regulatory costs. These new developments in the market have given rise to a formidable problem for the existing Monte Carlo methods which can solve the problem only backwardly relying on the least-square regression method (LSM). In fact, it is practically infeasible to handle real financial problems which typically require very high dimensional modeling with a rather long time horizon. The problem and the necessity of a fully forward-looking technique not relying on the LSM are emphasized also in the recent book of counterparty risk by Crépey et al. (2014) [7].

Fujii & Takahashi (2012a) [12] proposed an analytical perturbation technique for generic non-linear FBSDEs. It was shown that a non-linear FBSDE can be decomposed into a series of linear and decoupled ones by treating the non-linear parts of a driver as perturbations with respect to an appropriate linear decoupled system. In particular, it allows analytical explicit expressions for the backward components with the help of the asymptotic expansion technique (See, for example [30, 24, 34, 32].). The control variables (i.e., the coefficients of the Brownian motions) were shown to be obtained by considering dynamics of the stochastic flows, which denote Malliavin derivatives of the underlying state processes. A rigorous justification for a decoupled FBSDE with a smooth driver was recently given by Takahashi & Yamada (2013) [33]. An explicit example of its application can be found in Fujii & Takahashi (2012b) [13] for an incomplete market with stochastic volatility.

Thanks to the linearity of the expanded system, the method [12] allows to solve the problem in a forward-looking manner, which can provide one possible solution to the above mentioned problem relevant for the current financial market. The remaining issue is that it is impractical to carry out complicated analytic calculation for conditional expectations in realist setups. Although we can also use standard Monte Carlo simulation in estimating conditional expectations, it requires nested simulations at each point of time, which makes it too time-consuming for the practical use. In order to overcome the problem, we propose the idea of particle representation in this paper, inspired by the branching diffusion method used in McKean (1975) [28]. There, the convoluted expectation is compressed into a usual one by introducing an intensity of the particle interaction. McKean applied the method to solve a particular type of semi-linear PDE, where a single particle splits into two at

each interaction time and creates a cascade of the identical particles. The similar idea was also adopted by Labordère (2012) [19] for semi-linear problems, and in particular, for CVA calculation.

In this paper, we combine the idea of particle representation and the perturbation technique developed in the previous work [12]. We provide a straightforward simulation scheme to solve fully-nonlinear decoupled as well as coupled FBSDEs at each order of perturbative approximation. It includes, of course, a semi-linear problem as a special case. In contrast to the direct application of branching diffusion method, the number of branching points are capped by the order of perturbative expansion, which is due to the linearity of the decomposed FBSDE system. This property is expected to make Monte Carlo simulation less numerically intensive.

The sole purpose of the current paper is providing a simple explanation of the methodology and does not include concrete numerical examples. After publishing the first working paper version in 2012, there already appeared various numerical applications using our particle method: Crépey & Song (2014) [8] and Shiraya & Takahashi (2014) [31] for counterparty risk modeling, Fujii et al. (2014) [14] for the pricing of American options. See also Haramishi (2013a,b) [18] comparing our method with that of [19] for an inverse problem of derivative pricing, which aims to obtain appropriate parameters for the contract yielding a target fair price with the counterparty risk taken into account.

Remark

The analysis of branching Markov process and related problems in semi-linear PDEs has a long history. Some of the well-known works are Fujita (1966) [15], Ikeda, Nagasawa & Watanabe (1965,1966,1968) [20, 21, 22], Ikeda et.al. (1966,1967) [23] and Nagasawa & Sirao (1969) [27]. For recent developments and reviews of the particle methods, see [9, 10] for example.

2 Setup

We first consider generic decoupled non-linear FBSDEs. Let us use the same setup assumed in the work [12]. The probability space is taken as (Ω, \mathcal{F}, P) and $T \in (0, \infty)$ denotes some fixed time horizon. $W_t = (W_t^1, \dots, W_t^r)^*$, $0 \leq t \leq T$ is \mathbb{R}^r -valued Brownian motion defined on (Ω, \mathcal{F}, P) , and $(\mathcal{F}_t)_{\{0 \leq t \leq T\}}$ stands for P-augmented natural filtration generated by the Brownian motion.

We consider the following forward-backward stochastic differential equation (FBSDE)

$$dV_s = -f(X_s, V_s, Z_s)ds + Z_s \cdot dW_s \quad (2.1)$$

$$V_T = \Psi(X_T) \quad (2.2)$$

where the process V takes the value in \mathbb{R} , and that of X in \mathbb{R}^d , which is assumed to follow a generic Markovian forward SDE

$$dX_s = \gamma_0(X_s)ds + \gamma(X_s) \cdot dW_s. \quad (2.3)$$

$f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^r \rightarrow \mathbb{R}$, $\gamma_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times r}$ are assumed to be smooth

functions. $\Psi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a function to give the terminal condition for V . Here, we absorbed an explicit dependence on time to X by allowing some of its components can be a time itself. “.” in front of the dW represents the summation for the components of r -dimensional Brownian motion. The following approximation procedures can be applied in the same way also in the presence of coupon payments. Throughout this paper, we are going to assume that the appropriate regularity conditions for the introduced functions are satisfied to guarantee the existence of well-defined solution.

Let us fix the initial time as t . We denote the Malliavin derivative of X_u ($u \geq t$) at time t as $(\mathcal{D}_t X_u)$ taking value in $\mathbb{R}^{r \times d}$. Its dynamics in terms of the future time u is specified by the well-known stochastic flow:

$$\begin{aligned} d(Y_{t,u})_j^i &= \partial_k \gamma_0^i(X_u)(Y_{tu})_j^k du + \partial_k \gamma_a^i(X_u)(Y_{tu})_j^k dW_u^a \\ (Y_{t,t})_j^i &= \delta_j^i \end{aligned} \quad (2.4)$$

where ∂_k denotes the differential with respect to the k -th component of X , and δ_j^i denotes Kronecker delta. Here, i and j run through $\{1, \dots, d\}$ and $\{1, \dots, r\}$ for a . Throughout the paper, we adopt Einstein notation which assumes the summation of all the paired indexes. Using the known chain rule of Malliavin derivative, one sees

$$(\mathcal{D}_t X_u^i) = \int_t^u \partial_k \gamma_0^i(X_s)(\mathcal{D}_t X_s^k) ds + \int_t^u \partial_k \gamma^i(X_s)(\mathcal{D}_t X_s^k) \cdot dW_s + \gamma^i(X_t) \quad (2.5)$$

and hence it satisfies

$$(\mathcal{D}_t X_u^i)_a = (Y_{t,u})_j^i \gamma_a^j(X_t) = (Y_{t,u} \gamma(X_t))_a^i \quad (2.6)$$

where “ a ” is the index of r -dimensional Brownian motion.

3 Expansion into a series of Linear FBSDE System

Following the perturbative method proposed in [12], let us introduce the perturbation parameter ϵ and then write the equation as

$$\begin{cases} dV_s^{(\epsilon)} = -\epsilon f(X_s, V_s^{(\epsilon)}, Z_s^{(\epsilon)}) ds + Z_s^{(\epsilon)} \cdot dW_s \\ V_T^{(\epsilon)} = \Psi(X_T) \end{cases} \quad (3.1)$$

where $\epsilon = 1$ corresponds to the original model ¹. We suppose that the solution can be expanded in a power series of ϵ :

$$V_t^{(\epsilon)} = V_t^{(0)} + \epsilon V_t^{(1)} + \epsilon^2 V_t^{(2)} + \epsilon^3 V_t^{(3)} + \dots \quad (3.2)$$

$$Z_t^{(\epsilon)} = Z_t^{(0)} + \epsilon Z_t^{(1)} + \epsilon^2 Z_t^{(2)} + \epsilon^3 Z_t^{(3)} + \dots \quad (3.3)$$

If the non-linearity is sub-dominant, one can expect to obtain reasonable approximation of the original system by putting $\epsilon = 1$ at the end of calculation.

¹It is possible to extract the linear term from the driver and treat separately. Here, we simply leave it in a driver, or work in a “discounted” base to remove linear term in V .

The dynamics of each pair $(V^{(i)}, Z^{(i)})$ can be easily derived as follows:

Zero-th order

$$\begin{cases} dV_s^{(0)} = Z_s^{(0)} \cdot dW_s \\ V_T^{(0)} = \Psi(X_T) \end{cases} \quad (3.4)$$

First order

$$\begin{cases} dV_s^{(1)} = -f(X_s, V_s^{(0)}, Z_s^{(0)})ds + Z_s^{(1)} \cdot dW_s \\ V_T^{(1)} = 0 \end{cases} \quad (3.5)$$

Second order

$$\begin{cases} dV_s^{(2)} = -\left\{ V_s^{(1)} \frac{\partial}{\partial v} + (Z_s^{a(1)}) \frac{\partial}{\partial z^a} \right\} f(X_s, V_s^{(0)}, Z_s^{(0)})ds + Z_s^{(2)} \cdot dW_s \\ V_T^{(2)} = 0 \end{cases} \quad (3.6)$$

Third order

$$\begin{cases} dV_s^{(3)} = -\left\{ V_s^{(2)} \frac{\partial}{\partial v} + Z_s^{2(a)} \frac{\partial}{\partial z^a} + \frac{1}{2} (V_s^{(1)})^2 \frac{\partial^2}{\partial v^2} + V_s^{(1)} Z_s^{a(1)} \frac{\partial^2}{\partial v \partial z^a} \right. \\ \quad \left. + \frac{1}{2} Z_s^{a(1)} Z_s^{b(1)} \frac{\partial^2}{\partial z^a \partial z^b} \right\} f(X_s, V_s^{(0)}, Z_s^{(0)})ds + Z_s^{(3)} \cdot dW_s \\ V_T^{(3)} = 0 \end{cases} \quad (3.7)$$

One can continue to an arbitrary higher order in the same way.

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Note that the higher order backward components $(V^{(n)}, Z^{(n)})_{\{n \geq 1\}}$ are always outside of the non-linear function f . This property arises naturally due to the very nature of perturbation. As we shall see, this is crucial to suppress the number of particles in the numerical simulation.

4 Interacting Particle Interpretation

Let us fix the initial time t and set $X_t = x_t$.

4.1 ϵ -0th Order

For the zeroth order, it is easy to see

$$V_t^{(0)} = \mathbb{E} \left[\Psi(X_T) \middle| \mathcal{F}_t \right] \quad (4.1)$$

$$\begin{aligned} Z_t^{a(0)} &= \mathbb{E} \left[\partial_i \Psi(X_T) (\mathcal{D}_t^a X_T^i) \middle| \mathcal{F}_t \right] \\ &= \mathbb{E} \left[\partial_i \Psi(X_T) (Y_{t,T} \gamma(X_t))_a^i \middle| \mathcal{F}_t \right] \end{aligned} \quad (4.2)$$

It is clear that they can be evaluated by standard Monte Carlo simulation. However, for their use in higher order approximation, it is crucial to obtain explicit approximate

expressions for these two quantities. As proposed in [12], we use asymptotic expansion technique [30, 24, 34, 32] for this purpose. When Ψ is a smooth function, it is quite straightforward. Even if Ψ is not a smooth function, such as an option payoff, one can obtain explicit expressions of $(V^{(0)}, Z^{(0)})$ in terms of X_t , too. This is because, one can derive an approximate joint transition density of general diffusion processes by the asymptotic expansion². In the following, let us suppose that we have obtained the solutions up to a given order of asymptotic expansion, and write each of them as a function of x_t :

$$\begin{cases} V_t^{(0)} = v^{(0)}(x_t) \\ Z_t^{(0)} = z^{(0)}(x_t) \end{cases} \quad (4.3)$$

4.2 ϵ -1st Order

Since the BSDE is linear, we can integrate as before. Here, let us first consider the evaluation of $V_t^{(1)}$.

$$\begin{aligned} V_t^{(1)} &= \int_t^T \mathbb{E} \left[f(X_u, V_u^{(0)}, Z_u^{(0)}) \middle| \mathcal{F}_t \right] du \\ &= \int_t^T \mathbb{E} \left[f \left(X_u, v^{(0)}(X_u), z^{(0)}(X_u) \right) \middle| \mathcal{F}_t \right] du \end{aligned} \quad (4.4)$$

Although it is possible to carry out standard Monte Carlo simulation for every time $u \in [t, T]$ and integrate to obtain the $V_t^{(1)}$, the time integration becomes numerically quite heavy. In fact, it will soon become infeasible for ϵ higher order terms that include multi-dimensional integration of time. We now introduce particle interpretation by McKean [28] developed for the study of semilinear PDEs:

Proposition 1 *The $V_t^{(1)}$ in (4.4) can be equivalently expressed as*

$$V_t^{(1)} = \mathbf{1}_{\{\tau > t\}} \mathbb{E} \left[\mathbf{1}_{\{\tau < T\}} \hat{f}_t \left(X_\tau, v^{(0)}(X_\tau), z^{(0)}(X_\tau) \right) \middle| \mathcal{F}_t \right] \quad (4.5)$$

Here τ is the time of interaction which is drawn independently from Poisson distribution with an arbitrary deterministic positive intensity process λ_t . It can be a positive constant for the simplest case.³ \hat{f} is defined as

$$\hat{f}_t(x, v^{(0)}(x), z^{(0)}(x)) := \frac{1}{\lambda_s} e^{\int_t^s \lambda_u du} f(x, v^{(0)}(x), z^{(0)}(x)) . \quad (4.6)$$

Proof: Define the new process for $(s > t)$:

$$\hat{V}_{t,s}^{(1)} := e^{\int_t^s \lambda_u du} V_s^{(1)} \quad (4.7)$$

²We intend to use the result of asymptotic expansion only for higher order approximations.

³It is not difficult to make it a stochastic process.

then its dynamics is given by

$$\begin{aligned} d\hat{V}_{t,s}^{(1)} &= e^{\int_t^s \lambda_u du} \left\{ \lambda_s V_s^{(1)} ds - f(X_s, v^{(0)}(X_s), z^{(0)}(X_s)) ds + Z_s^{(1)} \cdot dW_s \right\} \\ &= \lambda_s \hat{V}_{t,s}^{(1)} ds - \lambda_s \hat{f}_t(X_s, v^{(0)}(X_s), z^{(0)}(X_s)) ds + e^{\int_t^s \lambda_u du} Z_s^{(1)} \cdot dW_s. \end{aligned} \quad (4.8)$$

Since we have $\hat{V}_{t,t}^{(1)} = V_t^{(1)}$, one can easily see the following relation holds:

$$V_t^{(1)} = \int_t^T \mathbb{E} \left[e^{-\int_t^u \lambda_s ds} \lambda_u \hat{f}_t(X_u, v^{(0)}(X_u), z^{(0)}(X_u)) \middle| \mathcal{F}_t \right] du \quad (4.9)$$

It is clear for those familiar with credit risk modeling [3, 4], it is nothing but the present value of default payment where the default intensity is λ with the default payoff at s ($> t$) given by $\hat{f}_t(X_s, v^{(0)}(X_s), z^{(0)}(X_s))$. Thus, it is clear that (4.9) is equivalent to (4.5). ■

Now, let us consider the martingale component $Z^{(1)}$. It can be expressed as

$$Z_t^{(1)} = \int_t^T \mathbb{E} \left[\mathcal{D}_t f(X_u, v^{(0)}(X_u), z^{(0)}(X_u)) \middle| \mathcal{F}_t \right] du \quad (4.10)$$

We perform the similar transformation for $Z^{(1)}$ to make it easier to interpret in the interacting particle model. Firstly, let us observe that the dynamics of Malliavin derivative of $V^{(1)}$ follows

$$\begin{aligned} d(\mathcal{D}_t V_s^{(1)}) &= -(\mathcal{D}_t X_s^i) \left\{ \partial_i + \partial_i v^{(0)}(X_s) \partial_v + \partial_i z^{a(0)}(X_s) \partial_{z^a} \right\} f(X_s, v^{(0)}(X_s), z^{(0)}(X_s)) ds \\ &\quad + (\mathcal{D}_t Z_s^{(1)}) \cdot dW_s \end{aligned} \quad (4.11)$$

$$\mathcal{D}_t V_t^{(1)} = Z_t^{(1)} \quad (4.12)$$

For lighten the notation, let us introduce a derivative operator

$$\nabla_i(x, v^{(0)}, z^{(0)}) := \partial_i + \partial_i v^{(0)}(x) \partial_v + \partial_i z^{a(0)}(x) \partial_{z^a} \quad (4.13)$$

and also

$$f(x, v^{(0)}, z^{(0)}) := f(x, v^{(0)}(x), z^{(0)}(x)) \quad (4.14)$$

Now, we can write Eq. (4.12) as

$$d(\mathcal{D}_t V_s^{(1)}) = -(\mathcal{D}_t X_s^i) \nabla_i(X_s, v^{(0)}, z^{(0)}) f(X_s, v^{(0)}, z^{(0)}) ds + (\mathcal{D}_t Z_s^{(1)}) \cdot dW_s$$

Define, for ($s > t$),

$$\widehat{\mathcal{D}_t V_s^{(1)}} := e^{\int_t^s \lambda_u du} (\mathcal{D}_t V_s^{(1)}) \quad (4.15)$$

then its dynamics can be written as

$$\begin{aligned}
d(\widehat{\mathcal{D}_t V_s^{(1)}}) &= e^{\int_t^s \lambda_u du} \left\{ \lambda_s (\mathcal{D}_t V_s^{(1)}) ds - (\mathcal{D}_t X_s^i) \nabla_i (X_s, v^{(0)}, z^{(0)}) f(X_s, v^{(0)}, z^{(0)}) ds \right. \\
&\quad \left. + \mathcal{D}_t Z_s^{(0)} \cdot dW_s \right\} \\
&= \lambda_s (\widehat{\mathcal{D}_t V_s^{(1)}}) ds - \lambda_s (\mathcal{D}_t X_s^i) \nabla_i (X_s, v^{(0)}, z^{(0)}) \hat{f}_t(X_s, v^{(0)}, z^{(0)}) ds \\
&\quad + e^{\int_t^s \lambda_u du} (\mathcal{D}_t Z_s^{(0)}) \cdot dW_s
\end{aligned} \tag{4.16}$$

We have

$$\widehat{\mathcal{D}_t V_t^{(1)}} = Z_t^{(1)} \tag{4.17}$$

and hence

$$Z_t^{(1)} = \int_t^T \mathbb{E} \left[e^{-\int_t^u \lambda_s ds} \lambda_s (\mathcal{D}_t X_u^i) \nabla_i (X_u, v^{(0)}, z^{(0)}) \hat{f}_t(X_u, v^{(0)}, z^{(0)}) \Big| \mathcal{F}_t \right] \tag{4.18}$$

Thus, following the same argument of the proposition 1, we can conclude:

Proposition 2 $Z_t^{(1)}$ in (4.10) is equivalently expressed as

$$Z_t^{(1)} = \mathbf{1}_{\{\tau > t\}} \mathbb{E} \left[\mathbf{1}_{\{\tau < T\}} (Y_{t,\tau} \gamma(X_t))_a^i \nabla_i (X_\tau, v^{(0)}, z^{(0)}) \hat{f}_t(X_\tau, v^{(0)}, z^{(0)}) \Big| \mathcal{F}_t \right] \tag{4.19}$$

where the definitions of random time τ and the intensity process λ are the same as those in proposition 1.

As we shall see later, interpreting (X, Y) as a pair of particles allows an efficient Monte Carlo implementation. For the evaluation of $Z^{(1)}$ for example, one can consider it as a system of two particles (X, Y) , which have the intensity λ of the interaction that produces

$$(Y_{t,\tau} \gamma(X_t))_a^i \nabla_i (X_\tau, v^{(0)}, z^{(0)}) \hat{f}_t(X_\tau, v^{(0)}, z^{(0)}) \tag{4.20}$$

at the interaction point and annihilate altogether. For $V^{(1)}$, the interpretation is much simpler. A single particle X with the decay rate of λ leaves \hat{f} at its decay point and vanishes.

4.3 ϵ -2nd Order

For the ϵ -2nd order, one can observe that

$$V_t^{(2)} = \int_t^T \mathbb{E} \left[\left(V_u^{(1)} \partial_v + Z_u^{(1)} \partial_{z^a} \right) f(X_u, v^{(0)}, z^{(0)}) \Big| \mathcal{F}_t \right] du \tag{4.21}$$

$$Z_t^{(2)} = \int_t^T \mathbb{E} \left[\mathcal{D}_t \left\{ \left(V_u^{(1)} \partial_v + Z_u^{(1)} \partial_{z^a} \right) f(X_u, v^{(0)}, z^{(0)}) \right\} \Big| \mathcal{F}_t \right] du \tag{4.22}$$

solve the BSDE (3.6). Its particle interpretation is available by the similar transformation.

Firstly, for ($s > t$), let us define

$$\hat{V}_{t,s}^{(2)} := e^{\int_t^s \lambda_u du} V_s^{(2)} \quad (4.23)$$

with some appropriate intensity process λ . Then it follows

$$\begin{aligned} d\hat{V}_{t,s}^{(2)} &= \lambda_s \hat{V}_{t,s}^{(2)} ds - \lambda_s (V_s^{(1)} \partial_v + Z_s^{a(1)} \partial_{z^a}) \hat{f}_t(X_s, v^{(0)}, z^{(0)}) ds \\ &\quad + e^{\int_t^s \lambda_u du} Z_s^{(2)} \cdot dW_s \end{aligned} \quad (4.24)$$

Observing that $\hat{V}_{t,t}^{(2)} = V_t^{(2)}$, one can confirm that

$$V_t^{(2)} = \mathbf{1}_{\{\tau_1 > t\}} \mathbb{E} \left[\mathbf{1}_{\{\tau_1 < T\}} \left(V_{\tau_1}^{(1)} \partial_v + Z_{\tau_1}^{a(1)} \partial_{z^a} \right) \hat{f}_t(X_{\tau_1}, v^{(0)}, z^{(0)}) \middle| \mathcal{F}_t \right] \quad (4.25)$$

where τ_1 is the random interaction time with intensity λ . Now, using the tower property of conditional expectations, one can conclude that

Proposition 3 $V_t^{(2)}$ in (4.21) is equivalently expressed as

$$\begin{aligned} V_t^{(2)} &= \mathbf{1}_{\{\tau_1 > t\}} \mathbb{E} \left[\mathbf{1}_{\{\tau_1 < \tau_2 < T\}} (\partial_v \hat{f}_{t,\tau_1}) \hat{f}_{\tau_1,\tau_2} \middle| \mathcal{F}_t \right] \\ &\quad + \mathbf{1}_{\{\tau_1 > t\}} \mathbb{E} \left[\mathbf{1}_{\{\tau_1 < \tau_2 < T\}} \partial_{z^a} \hat{f}_{t,\tau_1} \left(Y_{\tau_1,\tau_2} \gamma_{\tau_1} \right)_a^i \nabla_{i,\tau_2} \hat{f}_{\tau_1,\tau_2} \middle| \mathcal{F}_t \right] \end{aligned} \quad (4.26)$$

where we have defined

$$\begin{aligned} \hat{f}_{t,s} &:= \hat{f}_t(X_s, v^{(0)}(X_s), z^{(0)}(X_s)) \\ \nabla_{i,s} &:= \nabla_i(X_s, v^{(0)}(X_s), z^{(0)}(X_s)) \\ \gamma_t &:= \gamma(X_t) \end{aligned} \quad (4.27)$$

and τ_1 and τ_2 are the two interaction times randomly drawn with intensity λ .

A particle interpretation for the first term is quite simple. A particle X starts at t follows the diffusion (2.3) with (self) interaction intensity λ . For the first interaction time τ_1 , it yields $\partial_v \hat{f}_{t,\tau_1}$ and at the 2nd interaction time τ_2 it yields \hat{f}_{τ_1,τ_2} and decays away. The expectation value can be evaluated by preparing a large number of particles X starting from the same point and obeying the same physical law but spend independent lives. For the second term, the interpretation is more interesting. A particle X starts at time t and follows the diffusion (2.3) with interaction intensity λ . At the first interaction time τ_1 , it yields $\partial_{z^a} \hat{f}_{t,\tau_1}$ and at the same time bears a new particle Y . After τ_1 , the two particles (X, Y) follow the diffusions (2.3) and (2.4), respectively. They have interaction intensity λ , and at the second interaction point τ_2 they yield $(Y_{\tau_1,\tau_2} \gamma(X_{\tau_2})) \nabla_{\tau_2} \hat{f}_{\tau_1,\tau_2}$ and annihilate altogether. As in the first example, the expectation can be calculated by preparing a large number of particle X at the same starting point.

Remark: Note that, if we simply use Eqs. (4.4, 4.21) and the tower property, we have to handle a two-dimensional time integration. It makes naive implementation of Monte Carlo simulation numerically too heavy. In our particle interpretation, this problem is solved by introducing random interaction times with some intensity λ . One can choose appropriate

size of intensity that produces enough amount of events for Monte Carlo simulation.

We now consider an interacting particle interpretation of $Z^{(2)}$. For the evaluation of $Z^{(2)}$, we need to define the second order stochastic flow for ($t < s < u$):

$$(\Gamma_{t,s,u})_{jk}^i := \frac{\partial^2}{\partial x_t^j \partial x_s^k} X_u^i = \frac{\partial}{\partial x_t^j} (Y_{s,u})_k^i \quad (4.28)$$

Since we have

$$(Y_{s,u})_k^i = \delta_k^i + \int_s^u (Y_{s,v})_k^l (\partial_l \gamma_0^i(X_v) dv + \partial_l \gamma^i(X_v) \cdot dW_v) \quad (4.29)$$

it is easy to see that

$$\begin{aligned} (\Gamma_{t,s,u})_{j,k}^i &= \int_s^u (\Gamma_{t,s,v})_{j,k}^l (\partial_l \gamma_0^i(X_v) dv + \partial_l \gamma^i(X_v) \cdot dW_v) \\ &\quad + \int_s^u (Y_{t,v})_j^m (Y_{s,v})_k^l (\partial_{lm} \gamma_0^i(X_v) dv + \partial_{lm} \gamma^i(X_v) \cdot dW_v) \end{aligned} \quad (4.30)$$

Note that we have $\Gamma_{t,s,s} = 0$, regardless of time s ($> t$). Using the second order stochastic flow, the Malliavin derivative of Y can be written as

$$\mathcal{D}_t^\alpha (Y_{s,v})_k^i = (\Gamma_{t,s,v})_{j,k}^i (\gamma^j(X_t))_a = (\Gamma_{t,s,v} \gamma(X_t))_{k,a}^i \quad (4.31)$$

Consider the process of Malliavin derivative $\mathcal{D}_t V_s^{(2)}$. One can write its dynamics for ($t < s$) as

$$\begin{aligned} d(\mathcal{D}_t V_s^{(2)}) &= -\left((\mathcal{D}_t V_s^{(1)}) \partial_v + (\mathcal{D}_t Z_s^{a(1)}) \partial_{z^a} \right) f(X_s, v^{(0)}, z^{(0)}) ds \\ &\quad - (\mathcal{D}_t X_s^i) \left\{ V_s^{(1)} \nabla_{i,s} (\partial_v f(X_s, v^{(0)}, z^{(0)})) + (Z_s^{a(1)}) \nabla_{i,s} (\partial_{z^a} f(X_s, v^{(0)}, z^{(0)})) \right\} ds \\ &\quad + \mathcal{D}_t Z_s^{(2)} \cdot dW_s \end{aligned} \quad (4.32)$$

$$\mathcal{D}_t V_t^{(2)} = Z_t^{(2)} \quad (4.33)$$

As before, we define

$$\widehat{\mathcal{D}_t V_s^{(2)}} := e^{\int_t^s \lambda_u du} (\mathcal{D}_t V_s^{(2)}) \quad (4.34)$$

then its dynamics satisfies the following SDE:

$$\begin{aligned} d(\widehat{\mathcal{D}_t V_s^{(2)}}) &= \lambda_s (\widehat{\mathcal{D}_t V_s^{(2)}}) ds - \lambda_s \left[(\mathcal{D}_t X_s^i) \left(V_s^{(1)} \nabla_{i,s} (\partial_v \hat{f}_{t,s}) + (Z_s^{a(1)}) \nabla_{i,s} (\partial_{z^a} \hat{f}_{t,s}) \right) \right. \\ &\quad \left. + \left((\mathcal{D}_t V_s^{(1)}) \partial_v + (\mathcal{D}_t Z_s^{a(1)}) \partial_{z^a} \right) \hat{f}_{t,s} \right] ds + e^{\int_t^s \lambda_u du} \mathcal{D}_t Z_s^{(2)} \cdot dW_s \end{aligned} \quad (4.35)$$

$$(\widehat{\mathcal{D}_t V_t^{(2)}}) = Z_t^{(2)} \quad (4.36)$$

Then, the same arguments leads to

$$Z_t^{(2)} = \mathbf{1}_{\{\tau_1 > t\}} \mathbb{E} \left[\mathbf{1}_{\{\tau_1 < T\}} (\mathcal{D}_t X_{\tau_1}^i) \left(V_{\tau_1}^{(1)} \nabla_{i, \tau_1} (\partial_v \hat{f}_{t, \tau_1}) + (Z_{\tau_1}^{a(1)}) \nabla_{i, \tau_1} (\partial_{z^a} \hat{f}_{t, \tau_1}) \right) \right. \\ \left. + \mathbf{1}_{\{\tau_1 < T\}} \left((\mathcal{D}_t V_{\tau_1}^{(1)}) \partial_v + (\mathcal{D}_t Z_{\tau_1}^{a(1)}) \partial_{z^a} \right) \hat{f}_{t, \tau_1} \middle| \mathcal{F}_t \right] \quad (4.37)$$

using the random interaction time τ_1 .

Proposition 4 $Z_t^{(2)}$ in (4.22) is equivalently expressed as

$$Z_t^{a(2)} = \mathbf{1}_{\{\tau_1 > t\}} \mathbb{E} \left[\mathbf{1}_{\{\tau_1 < \tau_2 < T\}} (Y_{t, \tau_1} \gamma_t)_a^i \nabla_{i, \tau_1} (\partial_v \hat{f}_{t, \tau_1}) \hat{f}_{\tau_1, \tau_2} \right. \\ + \mathbf{1}_{\{\tau_1 < \tau_2 < T\}} (Y_{t, \tau_1} \gamma_t)_a^i \nabla_{i, \tau_1} (\partial_{z^b} \hat{f}_{t, \tau_1}) (Y_{\tau_1, \tau_2} \gamma_{\tau_1})_b^j \nabla_{j, \tau_2} \hat{f}_{\tau_1, \tau_2} \\ + \mathbf{1}_{\{\tau_1 < \tau_2 < T\}} (\partial_v \hat{f}_{t, \tau_1}) (Y_{t, \tau_2} \gamma_t)_a^i \nabla_{i, \tau_2} \hat{f}_{\tau_1, \tau_2} \\ + \mathbf{1}_{\{\tau_1 < \tau_2 < T\}} (\partial_{z^b} \hat{f}_{t, \tau_1}) (\gamma_{\tau_1})_b^j (\Gamma_{t, \tau_1, \tau_2} \gamma_t)_{j, a}^i \nabla_{i, \tau_2} \hat{f}_{\tau_1, \tau_2} \\ + \mathbf{1}_{\{\tau_1 < \tau_2 < T\}} (\partial_{z^b} \hat{f}_{t, \tau_1}) (Y_{t, \tau_1} \gamma_t)_a^j (\partial_j \gamma_{\tau_1})_b^k (Y_{\tau_1, \tau_2})_k^i \nabla_{i, \tau_2} \hat{f}_{\tau_1, \tau_2} \\ \left. + \mathbf{1}_{\{\tau_1 < \tau_2 < T\}} (\partial_{z^b} \hat{f}_{t, \tau_1}) (Y_{t, \tau_2} \gamma_t)_a^j (Y_{\tau_1, \tau_2} \gamma_{\tau_1})_b^i \nabla_{j, \tau_2} (\nabla_{i, \tau_2} \hat{f}_{\tau_1, \tau_2}) \middle| \mathcal{F}_t \right]. \quad (4.38)$$

where τ_1 and τ_2 are sequential interaction times with intensity λ .

Proof: It can be shown straightforwardly by using the tower property of conditional expectations and commutativity between the indicator functions and the Malliavin derivative due to the independence of λ . ■

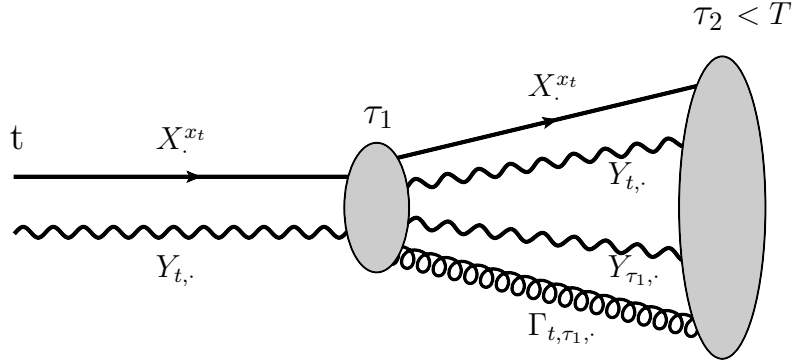


Figure 1: A particle interpretation for $Z_t^{(2)}$.

Despite the apparent complexity, required numerical procedures for the evaluation of $Z^{(2)}$ is, in fact, quite simple. We provide a Feynman diagram for the particle interpretation in Figure 1. At the first stage, there are two particles of (X, Y) with initial values $(x_t, \{\delta_j^i\})$, which survive until the second interaction time $\tau_2 (< T)$. At the first interaction at τ_1 , two additional particles $(Y_{\tau_1}, \Gamma_{t, \tau_1})$ are created. Each interaction occurs randomly with intensity λ . Note that we already know the initial values of the new particles regardless of the interaction time, which makes numerical simulations possible to carry out. What one has to do is to store the information of τ_1 and τ_2 and the values of the particles at these times. Then, all the ingredients in expectations can be calculated. Simply repeating independent experiments and taking average will give the desired values.

4.4 ϵ -3rd Order: $V^{(3)}$

In the similar fashion, we can proceed to higher order. As before, by considering the dynamics of

$$\hat{V}_{t,s}^{(3)} := e^{\int_t^s \lambda_u du} V_s^{(3)} \quad (4.39)$$

one can observe that

$$\begin{aligned} V_t^{(3)} = \mathbf{1}_{\{\tau_1 > t\}} \mathbb{E} \left[\mathbf{1}_{\{\tau_1 < T\}} \left(V_{\tau_1}^{(2)} \partial_v + Z_{\tau_1}^{a(2)} \partial_{z^a} + \frac{1}{2} (V_{\tau_1}^{(1)})^2 \partial_v^2 \right. \right. \\ \left. \left. + V_{\tau_1}^{(1)} Z_{\tau_1}^{a(1)} \partial_v \partial_{z^a} + \frac{1}{2} Z_{\tau_1}^{a(1)} Z_{\tau_1}^{b(1)} \partial_{z^a z^b} \right) \hat{f}_{t,\tau_1} \middle| \mathcal{F}_t \right]. \quad (4.40) \end{aligned}$$

It can be written in terms of the fundamental variables simply applying tower property.

Proposition 5 $V_t^{(3)}$ can be expressed as

$$\begin{aligned} V_t^{(3)} = & \mathbf{1}_{\{\tau_1 > t\}} \mathbb{E} \left[\mathbf{1}_{\{\tau_1 < \tau_2 < \tau_3 < T\}} (\partial_v \hat{f}_{t,\tau_1}) \left\{ (\partial_v \hat{f}_{\tau_1,\tau_2}) \hat{f}_{\tau_2,\tau_3} + (\partial_{z^a} \hat{f}_{\tau_1,\tau_2}) (Y_{\tau_2,\tau_3} \gamma_{\tau_2})_a^i \nabla_{i,\tau_3} \hat{f}_{\tau_2,\tau_3} \right\} \right. \\ & + \mathbf{1}_{\{\tau_1 < \tau_2 < \tau_3 < T\}} (\partial_{z^a} \hat{f}_{t,\tau_1}) \left\{ (Y_{\tau_1,\tau_2} \gamma_{\tau_1})_a^i \nabla_{i,\tau_2} (\partial_v \hat{f}_{\tau_1,\tau_2}) \hat{f}_{\tau_2,\tau_3} \right. \\ & + (Y_{\tau_1,\tau_2} \gamma_{\tau_1})_a^i \nabla_{i,\tau_2} (\partial_{z^b} \hat{f}_{\tau_1,\tau_2}) (Y_{\tau_2,\tau_3} \gamma_{\tau_2})_b^j \nabla_{j,\tau_3} \hat{f}_{\tau_2,\tau_3} \\ & + (\partial_{z^b} \hat{f}_{\tau_1,\tau_2}) (\gamma_{\tau_2})_b^j (\Gamma_{\tau_1,\tau_2,\tau_3} \gamma_{\tau_1})_{j,a}^i \nabla_{i,\tau_3} \hat{f}_{\tau_2,\tau_3} \\ & + (\partial_{z^b} \hat{f}_{\tau_1,\tau_2}) (Y_{\tau_1,\tau_2} \gamma_{\tau_1})_a^j (\partial_j \gamma_{\tau_2})_b^k (Y_{\tau_2,\tau_3})_k^i \nabla_{i,\tau_3} \hat{f}_{\tau_2,\tau_3} \\ & \left. \left. + (\partial_{z^b} \hat{f}_{\tau_1,\tau_2}) (Y_{\tau_1,\tau_3} \gamma_{\tau_1})_a^j (Y_{\tau_2,\tau_3} \gamma_{\tau_2})_b^i \nabla_{j,\tau_3} (\nabla_{i,\tau_3} \hat{f}_{\tau_2,\tau_3}) \right\} \middle| \mathcal{F}_t \right] \quad (4.41) \\ + & \mathbf{1}_{\{\tau_1 > t\}} \mathbb{E} \left[\mathbf{1}_{\{\tau_1 < T\}} \frac{1}{2} (\partial_v^2 \hat{f}_{t,\tau_1}) \prod_{p=1}^2 \left(\mathbf{1}_{\{\tau_1 < \tau_2^p < T\}} \hat{f}_{\tau_1,\tau_2^p} \right) \right. \\ & + \mathbf{1}_{\{\tau_1 < T\}} (\partial_v \partial_{z^a} \hat{f}_{t,\tau_1}) \left(\mathbf{1}_{\{\tau_1 < \tau_2^p < T\}} \hat{f}_{\tau_1,\tau_2^p} \right)^{p=1} \left(\mathbf{1}_{\{\tau_1 < \tau_2^p < T\}} (Y_{\tau_1,\tau_2^p} \gamma_{\tau_1})_a^i \nabla_{i,\tau_2^p} \hat{f}_{\tau_1,\tau_2^p} \right)^{p=2} \\ & + \mathbf{1}_{\{\tau_1 < T\}} \frac{1}{2} (\partial_{z^a z^b} \hat{f}_{t,\tau_1}) \left(\mathbf{1}_{\{\tau_1 < \tau_2^p < T\}} (Y_{\tau_1,\tau_2^p} \gamma_{\tau_1})_a^i \nabla_{i,\tau_2^p} \hat{f}_{\tau_1,\tau_2^p} \right)^{p=1} \\ & \left. \times \left(\mathbf{1}_{\{\tau_1 < \tau_2^p < T\}} (Y_{\tau_1,\tau_2^p} \gamma_{\tau_1})_b^j \nabla_{j,\tau_2^p} \hat{f}_{\tau_1,\tau_2^p} \right)^{p=2} \middle| \mathcal{F}_t \right] \quad (4.42) \end{aligned}$$

where the contents within each bracket of $p \in \{1, 2\}$ must be calculated according to the diffusion processes $(X^{x_{\tau_1}}, Y_{\tau_1, \cdot})_{p=\{1,2\}}$ that follow the identical diffusion laws with the same initial values, but are independent with each other. $\{\tau_i\}_{i \geq 1}$ are sequential random times of interactions drawn with intensity λ . $\{\tau_2^p\}_{p=1,2}$ should be drawn independently.

Note that, we have introduced two sets of particles labeled by $p \in \{1, 2\}$ that follow the same SDEs of (X, Y) but driven by two independent sets of Brownian motions to eliminate τ_1 -conditional expectations. In this way, one can avoid the use of nested Monte Carlo simulation. In Figures 2 and 3, we have provided two Feynman diagrams, one for the first half, and the other for the second half of the expression of $V_t^{(3)}$. In simulations,

one has to store the interaction times and all the relevant particles values at those points to evaluate the expectations.

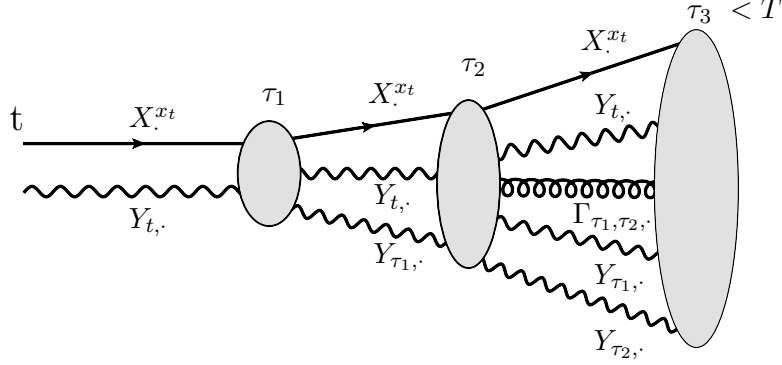


Figure 2: A particle interpretation for the first half of $V_t^{(3)}$.

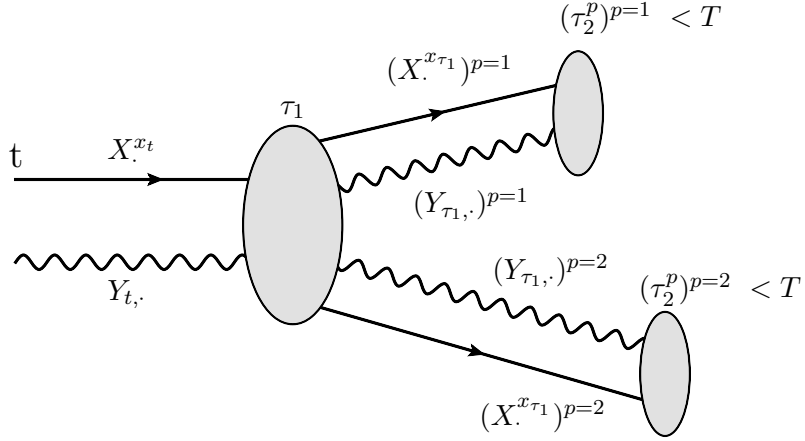


Figure 3: A particle interpretation for the second half of $V_t^{(3)}$.

4.5 $Z^{(3)}$ and ϵ -higher order terms

The valuation procedures for $Z^{(3)}$ are almost the same as that of $Z^{(2)}$, but we need to introduce a new type of particle corresponding to $(\frac{\partial}{\partial x_t} \Gamma_{s,u,v,\cdot})$. As easily guessed from the previous examples, we need to add one new particle corresponding to a higher order stochastic flow to complete the particle picture at every time when we proceed a higher order approximation of the control variable Z . A remarkable fact is that all the initial conditions of the new particles created at random times are known beforehand thanks to the characteristics of the Malliavin derivatives. This feature makes one can perform numerical simulations that describe full history of the evolution of particles.

5 Extension to Fully-Coupled Cases

We now consider the situation where the underlying state process X also gets the feedbacks from the backward components. By making use of the perturbative technique in PDE framework [12], we shall show that the same strategy in the previous sections works well also in this seemingly much more complicated situation.

The dynamics of whole system is assumed to be given by

$$\begin{cases} dV_t = -f(t, X_t, V_t, Z_t)dt + Z_t \cdot dW_t \\ V_T = \Psi(X_T) \\ dX_t = \gamma_0(t, X_t, V_t, Z_t)dt + \gamma(t, X_t, V_t, Z_t) \cdot dW_t \\ X_0 = x \end{cases} \quad (5.1)$$

where we have distinguished time arguments from X to make PDE generator have a familiar form. As before, we assume that V, Z, X take value in \mathbb{R}, \mathbb{R}^r and \mathbb{R}^d respectively, and W denotes a r -dimensional Brownian motion, $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^r \rightarrow \mathbb{R}$, $\gamma_0 : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^r \rightarrow \mathbb{R}^d$ and $\gamma : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^r \rightarrow \mathbb{R}^{d \times r}$ are smooth deterministic functions.

Following the idea of four-step scheme [26], we postulate that V_t is given as $V_t = v(t, X_t)$ by using some appropriate function $v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$. Then it needs to satisfy the relevant PDE:

$$\begin{cases} \partial_t v(t, x) + \left\{ \partial_i v(t, x) \gamma_0^i(t, x, v(t, x), z(t, x)) + \frac{1}{2} \partial_{ij} v(t, x) (\gamma^i \cdot \gamma^j)(t, x, v(t, x), z(t, x)) \right\} \\ \quad + f(t, x, v(t, x), z(t, x)) = 0 \\ z(t, x) = \partial_i v(t, x) \gamma^i(t, x, v(t, x), z(t, x)) \\ v(T, x) = \Psi(T, x) \end{cases} \quad (5.2)$$

The above non-linear PDE cannot be solved in general. Therefore, let us introduce perturbation parameter ϵ as before,

$$\begin{cases} dV_t^{(\epsilon)} = -\epsilon f(t, X_t^{(\epsilon)}, V_t^{(\epsilon)}, Z_t^{(\epsilon)})dt + Z_t^{(\epsilon)} \cdot dW_t \\ V_T^{(\epsilon)} = \Psi(X_T^{(\epsilon)}) \\ dX_t^{(\epsilon)} = \left(r(t, X_t^{(\epsilon)}) + \epsilon \mu(t, X_t^{(\epsilon)}, V_t^{(\epsilon)}, Z_t^{(\epsilon)}) \right) dt \\ \quad + \left(\sigma(t, X_t^{(\epsilon)}) + \epsilon \eta(t, X_t^{(\epsilon)}, V_t^{(\epsilon)}, Z_t^{(\epsilon)}) \right) \cdot dW_t \\ X_0^{(\epsilon)} = x \end{cases}$$

and its corresponding PDE

$$\begin{cases} \partial_t v^{(\epsilon)}(t, x) + \left\{ \partial_i v^{(\epsilon)}(t, x) \gamma_0^i(t, x, v^{(\epsilon)}, z^{(\epsilon)}) + \frac{1}{2} \partial_{ij} v^{(\epsilon)}(t, x) (\gamma^i \cdot \gamma^j)(t, x, v^{(\epsilon)}, z^{(\epsilon)}) \right\} \\ \quad + \epsilon f(t, x, v^{(\epsilon)}, z^{(\epsilon)}) = 0 \\ z^{(\epsilon)}(t, x) = \partial_i v^{(\epsilon)}(t, x) \gamma^i(t, x, v^{(\epsilon)}, z^{(\epsilon)}) \\ v^{(\epsilon)}(T, x) = \Psi(x) \end{cases} \quad (5.3)$$

Here, we have extracted the terms (r, σ) free from feedback effects from X 's dynamics ⁴:

$$\begin{cases} \gamma_0(t, x, v^{(\epsilon)}, z^{(\epsilon)}) = r(t, x) + \epsilon\mu(t, x, v^{(\epsilon)}(t, x), z^{(\epsilon)}(t, x)) \\ \gamma(t, x, v^{(\epsilon)}, z^{(\epsilon)}) = \sigma(t, x) + \epsilon\eta(t, x, v^{(\epsilon)}(t, x), z^{(\epsilon)}(t, x)) \end{cases} \quad (5.4)$$

We suppose that the solution of the above PDE can be expanded perturbatively in terms of ϵ as

$$v^{(\epsilon)}(t, x) = v^{(0)}(t, x) + \epsilon v^{(1)}(t, x) + \epsilon^2 v^{(2)}(t, x) + \dots \quad (5.5)$$

$$z^{(\epsilon)}(t, x) = z^{(0)}(t, x) + \epsilon z^{(1)}(t, x) + \epsilon^2 z^{(2)}(t, x) + \dots \quad (5.6)$$

As in the previous sections, putting $\epsilon = 1$ is expected to give the approximation of the original system as long as the non-linear effects are perturbative.

5.1 Expansion of non-linear PDE

Straightforward calculation allows us to expand the original PDE into a series of linear parabolic PDEs. See [12] for details. Firstly, let us define the differential operator \mathcal{L} :

$$\mathcal{L}(t, x) := r^i(t, x)\partial_i + \frac{1}{2}(\sigma^i \cdot \sigma^j)(t, x)\partial_{ij} \quad (5.7)$$

which corresponds to the infinitesimal generator of $X^{(0)}$, ie., the free forward component

$$dX_t^{(0)} = r(t, X_t)dt + \sigma(t, X_t) \cdot dW_t \quad (5.8)$$

$$X_0^{(0)} = x \quad (5.9)$$

Using this generator, we can show that the backward components in each order satisfy:
Zero-th order

$$\begin{cases} \left(\partial_t + \mathcal{L}(t, x)\right)v^{(0)}(t, x) = 0 \\ v^{(0)}(T, x) = \Psi(x) \end{cases} \quad (5.10)$$

and

$$z^{(0)}(t, x) = \partial_i v^{(0)}(t, x)\sigma^i(t, x) \quad (5.11)$$

Higher expansion order ($n \geq 1$)

$$\begin{cases} \left(\partial_t + \mathcal{L}(t, x)\right)v^{(n)}(t, x) + G^{(n)}(t, x) = 0 \\ v^{(n)}(T, x) = 0 \end{cases} \quad (5.12)$$

where the expression of $G^{(n)}$ and $z^{(n)}$ can be obtained straightforwardly by extracting $\mathcal{O}(\epsilon^n)$ terms from (5.3). See the following discussion for explicit expressions of $G^{(1)}$ and $G^{(2)}$.

⁴Although this can be done somewhat arbitrarily, it may be natural to set $r(t, x)$ and $\sigma(t, x)$ as the expected dynamics of X when all the feedback effects are switched off.

5.2 Particle Interpretation

The crucial point in the previous subsection is, because of the perturbation structure in (5.4), the relevant infinitesimal generator is always given by $\mathcal{L}(t, x)$ for all the expansion orders. In addition, since we put a ϵ -factor in front of the non-linear terms, $G^{(n)}$ contains the backward components with ϵ -order only up to $(n - 1)$. Furthermore, it is clear to see that $z^{(n)}$ can only contain the backward components of $\{v^{(m)}\}_{\{m \leq n\}}$ and $\{z^{(m)}\}_{\{m \leq (n-1)\}}$. Therefore, using Feynman-Kac theorem, we see that the PDE in (5.12) is equivalently expressed by, with some abuse of notation,

$$\begin{cases} dV_t^{(n)} = -G^{(n)}(t, X_t^{(0)}, V_t^{(n-1)}, Z_t^{(n-1)}, \dots)dt + \tilde{Z}_t^{(n)} \cdot dW_t \\ V_T^{(n)} = 0 \end{cases} \quad (5.13)$$

where the dynamics of the forward component $X^{(0)}$ is kept unchanged and decoupled from the BSDE. Note that every variable in $G^{(n)}$ can be written eventually as a function of $X^{(0)}$. Because of the very nature of the perturbative expansion, all the $(V^{(m)}, Z^{(m)})_{\{m \geq 1\}}$ appear as a power series and not contained within the non-linear functions. Thus, $V_t^{(n)}$ can be solved by the same procedures studied in the previous sections, and also the nice properties of explicitly capped number of branches and interaction points still hold.

Note that $\tilde{Z}_t^{(n)}$ is not equal to $Z_t^{(n)}$ that contains additional terms through the feedbacks to X . However, it is not difficult to calculate these terms. For example, one can observe:
1st order ($n = 1$)

$$G^{(1)}(t, x) = f^{(0)}(t, x) + \partial_i v^{(0)}(t, x) \mu^{i(0)}(t, x) + \partial_{ij} v^{(0)}(t, x) (\sigma^i \cdot \eta^{j(0)})(t, x) \quad (5.14)$$

$$z^{(1)}(t, x) = \partial_i v^{(1)}(t, x) \sigma^i(t, x) + \partial_i v^{(0)}(t, x) \eta^{i(0)}(t, x) \quad (5.15)$$

2nd order ($n = 2$)

$$\begin{aligned} G^{(2)}(t, x) &= (v^{(1)}(t, x) \partial_v + z^{a(1)}(t, x) \partial_{z^a}) f^{(0)}(t, x) \\ &\quad + \partial_i v^{(1)}(t, x) \mu^{i(0)}(t, x) + \partial_i v^{(0)}(t, x) (v^{(1)}(t, x) \partial_v + z^{a(1)}(t, x) \partial_{z^a}) \mu^{i(0)}(t, x) \\ &\quad + \partial_{ij} v^{(1)}(t, x) (\sigma^i \cdot \eta^{j(0)})(t, x) + \frac{1}{2} \partial_{ij} v^{(0)}(t, x) (\eta^{i(0)} \cdot \eta^{j(0)})(t, x) \\ &\quad + \partial_{ij} v^{(0)}(t, x) \sigma^i(t, x) \cdot (v^{(1)}(t, x) \partial_v + z^{a(1)}(t, x) \partial_{z^a}) \eta^{j(0)}(t, x) \end{aligned} \quad (5.16)$$

$$\begin{aligned} z^{(2)}(t, x) &= \partial_i v^{(2)}(t, x) \sigma^i(t, x) + \partial_i v^{(1)}(t, x) \eta^{i(0)}(t, x) \\ &\quad + \partial_i v^{(0)}(t, x) (v^{(1)}(t, x) \partial_v + z^{a(1)}(t, x) \partial_{z^a}) \eta^{i(0)}(t, x) \end{aligned} \quad (5.17)$$

Higher order cases can be obtained similarly.

Let us now consider the particle method to evaluate the relevant terms. Let us fix the initial time as t as before: For the zero-th order, the problem is exactly the same as the decoupled case and we can derive easily $v^{(0)}(t, x)$ and $z^{(0)}(t, x)$ as a function of x by asymptotic expansion⁵. For notational simplicity, we write $X_s^{(0)}$ as X_s since the underlying process is the same for every expansion order.

⁵As before, this is only to use higher order expansion. For the valuation of the zero-th order itself, one can use the standard Monte Carlo simulation

1st order

As for the first order, observe that $G^{(1)}(t, x)$ is given as an explicit function of x after the completion of the zero-th order calculation. Then, $V_s^{(1)}$ follows

$$\begin{cases} dV_s^{(1)} = -G^{(1)}(t, X_s)ds + \tilde{Z}_s^{(1)} \cdot dW_s \\ V_T^{(1)} = 0 \end{cases} \quad (5.18)$$

and hence, by the same arguments, for $(s > t)$, we have a particle representation as

$$V_t^{(1)} = \mathbf{1}_{\{\tau > t\}} \mathbb{E} \left[\mathbf{1}_{\{\tau < T\}} \hat{G}_t^{(1)}(\tau, X_\tau) \middle| \mathcal{F}_t \right] \quad (5.19)$$

where $\hat{G}_t^{(1)}$ is defined as

$$\hat{G}_t^{(1)}(s, X_s) := \frac{1}{\lambda_s} e^{\int_t^s \lambda_u du} G^{(1)}(s, X_s) \quad (5.20)$$

with some appropriate positive deterministic (or independent) intensity λ . For martingale component, it is easy to see

$$Z_t^{(1)} = \tilde{Z}_t^{(1)} + \partial_i v^{(0)}(t, x_t) \eta^{i(0)}(t, x_t) \quad (5.21)$$

from (5.15). Here, the particle representation of $\tilde{Z}^{(1)}$ can be derived in the same way as in the decoupled case:

$$\tilde{Z}_t^{a(1)} = \mathbf{1}_{\{\tau > t\}} \mathbb{E} \left[\mathbf{1}_{\{\tau < T\}} (Y_{t,\tau} \sigma_t)_a^i \partial_i \hat{G}_t^{(1)}(\tau, X_\tau) \middle| \mathcal{F}_t \right] \quad (5.22)$$

where $Y_{t,s}$ ($s > t$) is the stochastic flow of X and is given by

$$(Y_{t,u})_j^i = \delta_j^i + \int_t^u (Y_{t,s})_j^k \left\{ \partial_k r^i(s, X_s) ds + \partial_k \sigma^i(s, X_s) \cdot dW_s \right\} \quad (5.23)$$

The second term of $Z^{(1)}$ is already given as an explicit function of x_t .

2nd order

We can proceed to higher orders in similar fashion. For the second order, the contribution to $V^{(2)}$ from the first line of $G^{(2)}$ can be calculated in the same way as the decoupled case. Let us consider non-trivial remaining terms. The contribution from $\partial_i v^{(1)}(t, x) \mu^{i(0)}(t, x)$, for example, can be calculated as

$$\begin{aligned} & \mathbf{1}_{\{\tau_1 > t\}} \mathbb{E} \left[\mathbf{1}_{\{\tau_1 < T\}} \hat{\mu}_t^{i(0)}(\tau_1, X_{\tau_1}) \frac{\partial}{\partial x_{\tau_1}^i} \left(\mathbf{1}_{\{\tau_2 > \tau_1\}} \mathbb{E} \left[\mathbf{1}_{\{\tau_2 < T\}} \hat{G}_{\tau_1}^{(1)}(\tau_2, X_{\tau_2}) \middle| \mathcal{F}_{\tau_1} \right] \right) \middle| \mathcal{F}_t \right] \\ &= \mathbf{1}_{\{\tau_1 > t\}} \mathbb{E} \left[\mathbf{1}_{\{\tau_1 < \tau_2 < T\}} \hat{\mu}_t^{i(0)}(\tau_1, X_{\tau_1}) (Y_{\tau_1, \tau_2})_j^i \partial_j \hat{G}_{\tau_1}^{(1)}(\tau_2, X_{\tau_2}) \middle| \mathcal{F}_t \right] \end{aligned} \quad (5.24)$$

where

$$\hat{\mu}_t^{i(0)}(s, X_s) = \frac{1}{\lambda_s} e^{\int_t^s \lambda_u du} \mu^{i(0)}(s, X_s) \quad (5.25)$$

Note that the partial derivative of x in $\partial_i v^{(1)}(\tau_1, X_{\tau_1})$ should be recognized as the shift of X at the time of τ_1 , which leads to the first order stochastic flow Y in the above expression.

Next, let us consider the contribution from $\partial_{ij} v^{(1)}(t, x)(\sigma^i \cdot \eta^{j(0)})(t, x)$. As is the previous example, it is calculated as

$$\begin{aligned} & \mathbf{1}_{\{\tau_1 > t\}} \mathbb{E} \left[\mathbf{1}_{\{\tau_1 < T\}} (\widehat{\sigma^i \cdot \eta^{j(0)}})(\tau_1, X_{\tau_1}) \frac{\partial^2}{\partial x_{\tau_1}^i \partial x_{\tau_1}^j} \left(\mathbf{1}_{\{\tau_2 > \tau_1\}} \mathbb{E} \left[\mathbf{1}_{\{\tau_2 < T\}} \hat{G}_{\tau_1}^{(1)}(\tau_2, X_{\tau_2}) \middle| \mathcal{F}_{\tau_1} \right] \right) \middle| \mathcal{F}_t \right] \\ &= \mathbf{1}_{\{\tau_1 > t\}} \mathbb{E} \left[\mathbf{1}_{\{\tau_1 < \tau_2 < T\}} (\widehat{\sigma^i \cdot \eta^{j(0)}})(\tau_1, X_{\tau_1}) \left\{ (\Gamma_{\tau_1, \tau_2})_{ij}^k \partial_k \hat{G}_{\tau_1}^{(1)}(\tau_2, X_{\tau_2}) \right. \right. \\ & \quad \left. \left. + (Y_{\tau_1, \tau_2})_i^k (Y_{\tau_1, \tau_2})_j^l \partial_{kl} \hat{G}_{\tau_1}^{(1)}(\tau_2, X_{\tau_2}) \right\} \middle| \mathcal{F}_t \right] \end{aligned} \quad (5.26)$$

where $\widehat{\sigma^i \cdot \eta^{j(0)}}(s, X_s) := e^{\int_t^s \lambda_u du} (\sigma^i \cdot \eta^{j(0)})(s, X_s) / \lambda_s$. Note that the second order stochastic flow $(\Gamma_{t,s})_{i,j}^k$ is defined, for $(u > t)$, as

$$(\Gamma_{t,u})_{i,j}^k := \frac{\partial}{\partial x_t^i \partial x_t^j} (X_u^{x_t})^k \quad (5.27)$$

and is given by

$$\begin{aligned} (\Gamma_{t,u})_{i,j}^k &= \int_t^u (\Gamma_{t,s})_{ij}^l \left\{ \partial_l r^k(s, X_s) ds + \partial_l \sigma^k(s, X_s) \cdot dW_s \right\} \\ & \quad + \int_t^u (Y_{t,s})_i^l (Y_{t,s})_j^m \left\{ \partial_{lm} r^k(s, X_s) ds + \partial_{lm} \sigma^k(s, X_s) \cdot dW_s \right\} \end{aligned} \quad (5.28)$$

The remaining contributions to $V^{(2)}$ as well as $Z^{(2)}$ can be calculated by the same technique.

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Although tedious calculation is needed, we can proceed to an arbitrary higher order in the same fashion. Note that, also in fully-coupled cases, new particles required in simulation are all derived as stochastic flows of X and hence the initial values at their creations are known beforehand.

6 Conclusions and Discussions

In this paper, we have developed an efficient Monte Carlo scheme with an interacting particle representation. It allows straightforward numerical implementation to solve fully non-linear decoupled as well as coupled FBSDEs at each order of perturbative expansion. The appearance of unknown backward components in the expressions of higher order approximations is solved by introducing an appropriate particle interpretation. Although a couple of new particles are created at random interaction times, their initial values are known beforehand. This is due to their properties as the stochastic flows of the underlying state, which is the crucial point to make straightforward Monte Carlo simulation possible. The proposed method can be applied to various problems, such as American option pricing, Credit and Funding Value Adjustment (CVA & FVA), as well as the optimal portfolio

problems in incomplete and/or constrained markets. In fact, there already appeared several papers adopting our proposed method for the analysis. It looks also interesting to use the current method to study higher order FBSDEs, where the higher order Malliavin derivatives exist in the non-linear driver, such as $f(t, X_t, V_t, \mathcal{D}_t V, \mathcal{D}_t^2 V)$. It can be done straightforwardly by introducing higher order stochastic flows.

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