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Asymptotic Expansion Approach in Finance

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Abstract

This paper provides a survey on an asymptotic expansion approach to valuation and hedging problems in finance. The asymptotic expansion is a widely applicable methodology for analytical approximations of expectations of certain Wiener functionals. Hence not only academic researchers but also practitioners have been applying the scheme to a variety of problems in finance such as pricing and hedging derivatives under high-dimensional stochastic environments. The present note gives an overview of the approach.

Keywords: Asymptotic Expansion, Derivatives, Option Pricing, hedge, Greeks, Stochastic Volatility, Interest Rate, Term Structure Model, Malliavin Calculus, Watanabe Theory

1 Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ denote a probability space with filtration, on which a r-dimensional standard Wiener process W is defined, where P is an appropriate pricing measure (a risk neutral measure) in finance, and T denotes some positive constant. Now, let $F(\omega)$ be a Wiener functional and then $\mathbf{V}$, the security or portfolio value can be expressed as $\mathbf{V} = \mathbf{E}[F(\omega)]$ under certain conditions. Evaluating this expectation is one of the main issues in finance. Moreover, if F depends on the parameter θ, computation of $\frac{\partial \mathbf{V}}{\partial \theta} = \frac{\partial}{\partial \theta}\mathbf{E}[F(\omega;\theta)]$, the sensitivity of the security value with respect to the change in this parameter (so called *Greeks*) is also an important task in practice.

As an example, let us consider a d-dimensional diffusion process $X^{(\epsilon)}$ which is obtained as a strong solution to the stochastic differential equation;

$$dX_t^{(\epsilon)} = V_0(X_t^{(\epsilon)}, \epsilon)dt + V(X_t^{(\epsilon)}, \epsilon)dW_t,\ t \in [0,T];\ X_0^{(\epsilon)} = x_0,$$

where $\epsilon \in [0,1]$ is a known parameter. Here, the coefficients are assumed to satisfy some regularity conditions. In finance, many problems of pricing derivatives and evaluating the portfolios in investment theories are reduced to the problems of computing $\mathbf{E}[f(X_T^{(\epsilon)})]$, the expectation of $f(X_T^{(\epsilon)})$, that is a function of $X_T^{(\epsilon)}$.

In finance applications, it is important to deal with not only a smooth function $f(x)$ but also non-smooth one. For example, when various options are evaluated, f is expressed as $f = T \circ g$, where $T(x) = \max\{x, 0\}$ and g stands for a smooth function of $\mathbf{R}^d \mapsto \mathbf{R}$. In general, it is difficult to represent this expectation explicitly except for special cases. Hence, numerical methods such as Monte Carlo simulations or numerical solutions of partial differential equations (PDEs) are employed and various speeding up techniques are developed, since fast and precise computation is required in practice.

As a different approach, an approximation of the expectation by an asymptotic expansion of the stochastic differential equation around $\epsilon = 0$ may be considered. Furthermore, because $\frac{\partial}{\partial x_0}\mathbf{E}[f(X_T^{(\epsilon)})]$

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and $\frac{\partial}{\partial \epsilon}\mathbf{E}[f(X_T^{(\epsilon)})]$, the sensitivities of the security value with respect to the changes in the initial value x_0 and in the parameter ϵ are important indicators for practical purposes, the approximations with high accuracies are so valuable. Moreover, some schemes that combine Monte Carlo simulations with asymptotic expansions with low orders are developed, since the asymptotic expansion up to the first or second order can be easily evaluated. Those schemes are able to improve the efficiencies of Monte Carlo simulations and the accuracies of approximations obtained by the asymptotic expansions.

An asymptotic expansion approach in finance has been developed for the past two decades, which is mathematically justified by Watanabe theory (Watanabe [111]) in Malliavin calculus (e.g. Malliavin [64], Chapter V-8 in Ikeda. and Watanabe [39], Nualart [73]). To the best of our knowledge, the asymptotic expansion technique is firstly applied to finance for evaluation of average options that are popular derivatives in commodity markets. Kunitomo and Takahashi [48] and [85] derive approximation formulas for average options by an asymptotic expansion method based on log-normal approximations for average prices distributions, when the underlying asset prices follow geometric Brownian motions. Yoshida [119] derives an asymptotic expansion of an average option price around a normal distribution for a general diffusion model, which is a byproduct of his result in statistics [118] based on the Watanabe theory.

Thereafter, the asymptotic expansion approach have been applied to a broad class of valuation problems in finance, which includes pricing options with stochastic volatility models, pricing options under Heath-Jarrow-Morton (HJM) models ([37]) or Libor market models (LMM) (Brace, Gatarek and Musiela [7], Jamshidian [43]) of interest rates, and pricing so called exotic-type options such as basket and barrier options in addition to average options.

For instance, please see Kawai [44], Kobayashi, Takahashi and Tokioka [45], Kunitomo and Takahashi [49], [50], [51], Li [59] Matsuoka, Takahshi and Uchida [66], Muroi [67], Nishiba [71], Osajima [75], Shiraya and Takahashi [78], [79], [80], Shiraya, Takahashi and Toda [81], Shiraya, Takahashi and Yamada [83], Shiraya, Takahashi and Yamazaki [82], Takahashi and Matsushima [88], Takahashi and Saito [89], Takahashi and Takehara [90], [91], [92], [93], [94], Takahashi, Takehara and Toda [90], [91], Takahashi and Tsuzuki[98], Takahashi and Uchida [99], Takahashi and Yamada [100], [101], [102], [103], [104], Takahashi and Yoshida [106], [107], Takehara, Takahashi and Toda[92], [93], Violante[110], Xu and Zheng [112], [113], and [86], [87].

We briefly introduce some of above works in Section 3.6. Moreover, we remark that the asymptotic expansion approach is employed by Yamanobe [116], [117] in physics for analyses of the impulse-driven stochastic biological oscillator and global dynamics of a stochastic neuronal oscillator.

We also note that there exist many other types of the expansion/perturbation methods which have turned out to be so useful for applications in finance. For example, see Bayer and Laurence [2], Ben Arous and Laurence [3], Benaim, Friz and Lee [4], Col, Gnoatto and Grasselli [9], Davydov and Linetsky [11], Deuschel, Friz, Jacquier and Violante [12], [13], Forde and Jacquier [18], Forde, Jacquier and Lee [17], Foschi, Pagliarani, Pascucci [19], Fouque, Papanicolaou and Sircar [20], [21], Fujii [24], Fujii and Takahashi [25], [26], [27], [29], Gatheral, Hsu, Laurence, Ouyang, and Wang [30], Gnoatto and Grasselli [31], Gulisashvili [32], Hagan, Kumar, Lesniewski and Woodward [33], Henry-Labordere [38], Kato Takahashi and Yamada [46], [47], Kusuoka and Osajima [57], Lee [58], Lipton [60], Linetsky[61], Osajima [76], Pagliarani and Pascucci [77], Siopacha and Teichmann [84], Yamamoto, Sato and Takahashi [114], Yamamoto and Takahashi [115], and references therein.

The organization of the paper is as follows. The next section describes the outline of the asymptotic expansion approach in a general diffusion setting. Then, Section 3 explains a computational scheme for the expansion method. Section 4 provides an extension of the general computational scheme in the previous section, and Section 5 briefly introduces two improvement scheme for the expansion method. Section 6 extends the approach to non-diffusion Wiener functionals by using an instantaneous forward rates model as an example. Section 7 and Section 8 introduce an asymptotic expansion in jump-diffusion models and a perturbation scheme in forward backward stochastic differential equations (FBSDEs). Section 9 concludes.

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