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Abstract

This paper studies impacts of imperfect collateralization on derivatives values. Particularly, we investigate option prices in no collateral posting and time-lagged collateral posting cases with stochastic volatility, interest rate and default intensity models, where a stochastic collateral asset value may depend on the values of the assets different from the underlying contract. We also derive an approximation of the credit value adjustment (CVA)'s density function in pricing forward contract with bilateral counter party risk, which seems useful in evaluation of the CVA's Value-at-Risk(VaR).

Keywords: credit risk; collateral; CVA; pre-default value; derivatives pricing

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1. Introduction

After several financial crises, most financial institutions have been forced to take stringent credit risk management in derivatives trading. For example, collateralized contracts are widely used in interbank markets, which substantially decreases the impact of the counter parties' credit deterioration.

On the other hand, collateralized contracts are not so common in non-financial institutions, as their contracts are still uncollateralized or imperfectly collateralized. In that case, the evaluation of counter parties' credit risks should be taken into account. For instance, credit valuation adjustments (CVAs) are charged for those contracts. As a result, there may exist significant difference in values between collateralized and the corresponding uncollateralized contracts.

Following the recent trend, many researchers in academia and industries have been considering derivatives pricing with taking counter party risks into account and CVAs. The traditional CVA in practice is roughly estimated by a default-free value of the derivatives contract multiplied by the counter party's default probability. In order to obtain more accurate values of the derivatives contracts with default risks, a significant number of researches have investigated so-called the *wrong way risk*. For instance, see Redon [2006], Lipton-Sepp [2009], Brigo et al. [2011], Hull-White[2012] and Fujii-Takahashi [2012c, 2013].

However, a lot of problems still remain unsolved for the derivatives pricing. For example, a collateral is posted with time lag and/or with currencies different from the payment currency or assets such as treasuries suffering from their own price fluctuations. Although the traditional CVA in practice is estimated based on a pre-default value of the corresponding contract with default-free parties, in order to obtain more accurate estimations, the credit risks of the contract parties should be taken into consideration in the pre-default value. Therefore, the traditional CVA is not always enough in the value adjustment of derivatives with the contract parties' default risks.

With regard to formulation of derivatives values with default risks, one typical method is to employ backward stochastic differential equations (BSDEs). Applications of BSDEs in finance are discussed, for instance by El Karoui-Peng-Quenez [1997], Ma-Yong [2000], Carmona[2000] and Crépey [2011]. In this paper, we use Markovian BSDEs, that is forward BSDEs (FBSDEs), where the underlying variables (factors) follow diffusion processes, which are characterized by the solution of forward stochastic differential equations (FSDEs).

More concretely, we investigate the derivatives values of over-the-counter (OTC) forward contracts and European options. In particular, we suppose the underlying variables to be the following random factors: the (forwards' and options') underlying asset prices, their volatilities, (contract parties') default probabilities, the risk-free interest rate, a collateral asset price and those volatilities. Under the setting, we are interested in analyzing derivatives prices with counter party risk and imperfect collateralization. Particularly, we would like to derive the density function of the CVA(credit value adjustment) for a forward contract, which seems useful in evaluation of the CVA's VaR (Value at Risk). Also, we often need to investigate the impacts on the option values of the changes in the parameters of the underlying factors and the times to maturities of the options.

However, it is a very tough task to numerically evaluate the solutions to high dimensional FBSDEs as encountered in this paper. To overcome this problem, we apply *a perturbative expansion method* and *a perturbative expansion technique with interacting particle method*, a new computational scheme for FBSDEs recently developed by Fujii and Takahashi [2012a,b] and Takahashi and Yamada [2013]. Then, it turns out to be true that in certain situations, the traditional CVAs are not enough for the price adjustment of the derivatives with the counter parties' default risks, but more precise evaluation is necessary. Particularly, we concretely show that overestimated amounts of traditional CVAs are not to be neglected under some circumstances.

The organization of the paper is as follows: the next section briefly explains a general result for pre-default values of financial derivatives. Section 3 explains the framework of the approximation methods of the solutions to the FBSDEs. As an application, Section 4 provides an approximation for the density function of the CVA (Credit Value Adjustment) in the valuation of forward contract with bilateral counter party risk. Applying a Monte Carlo scheme to calculate approximated values of FBSDEs, Section 5 analyzes the impacts on the option values of the changes in the parameters of the underlying factors and the times to maturities of the options. Moreover, we examine the shapes of implied volatility curves of the options. Section 6 concludes. Appendix provides the derivation of an expression for the pre-default value of a derivatives contract, and explains the Monte Carlo method to compute approximated values of FBSDEs.

2. Pricing Derivatives with Default Risks under Imperfect Collateralization

This section briefly explains a general method for pricing derivatives with default risks under imperfect collateralization.

Let us define a base probability space as $(\Omega, \mathcal{F}, \mathbb{F}, Q^p)$, where $\mathbb{F}$ is a filtration which satisfies the usual conditions, and Q^p is a risk-neutral measure under a currency p. In these settings, we consider two firms, an investor $(i = 1)$ and a counter party $(i = 2)$, whose default times are defined as $\tau^i \in [0, \infty]$, $(i = 1, 2)$. Also, we define $\tau = \tau^1 \wedge \tau^2 := \min\{\tau_1, \tau_2\}$. Here, τ^i (and hence τ) is assumed to be a totally-inaccessible $\mathbb{F}$-stopping time. Then, indicator functions H_t^i and H_t are defined as $H_t^i = \mathbf{1}_{\{\tau^i \leq t\}}$ and $H_t = \mathbf{1}_{\{\tau \leq t\}}$, respectively. Moreover, we suppose the existence of absolutely continuous compensator for each H^i and h^i is a hazard rate of H^i. We also assume that there are no simultaneous defaults and hence, the hazard rate of H is given by

$$h_t = h_t^1 + h_t^2 \ . \tag{1}$$

Other than the default times $\tau^i (i = 1, 2)$, we introduce a $\mathbb{R}^d$- valued stochastic process, $X = \{X_t : t \geq 0\}$ as a vector of state variables, which affects market values of assets considered in this paper. Specifically, X is the solution to the following $\mathbb{R}^d$-valued stochastic differential equation defined on $(\Omega, \mathcal{F}, \mathbb{F}, Q^p)$:

$$\begin{aligned} dX_t &= \gamma_0(X_t)dt + \gamma(X_t) \cdot dW_t, \\ X_0 &= x_0, \end{aligned} \tag{2}$$

where W is a n dimensional Brownian motion. (for $x, y \in \mathbb{R}^n$, we use notations $x \cdot y = \sum_{i=1}^n x_i y_i$.) $\gamma_0(x) : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\gamma(x) : \mathbb{R}^d \mapsto \mathbb{R}^{d \times n}$ satisfy conditions so that X has the unique strong solution. Moreover, we define $\mathbb{G} = (\mathcal{G}_t)_{\{t \geq 0\}} (\subset \mathbb{F})$ as the augmented filtration generated by X.

Under a bilateral collateral contract, which is a recent market convention, each contract party $i = 1$ or 2 needs to post a collateral to $j (\neq i)$ when the value of derivatives becomes a negative present value for i. Then, when it posts a cash collateral, in practice an overnight interest rate of a collateral currency is paid by a party receiving the collateral. Future values of the overnight interest rates are fixed by overnight index swaps (OISs). We note that an OIS itself is a collateralized contract.

Hereafter, we assume that p is a payment currency, q is a collateral currency, Ψ represents a maturity payoff, Γ_t represents the value process of the collateral and r_t^p (or r_t^q) represents the risk-free interest rate process under currency p (or currency q). Also, c_t^p (or c_t^q) represents the collateral rate process under currency p (or currency q) and $l_t^i (\geq 0)$, $i \in \{1, 2\}$ represents the party i's loss rate process.

In general, c^p, c^q, r^p, r^q, $l^{[i]}$ and $h^{[i]}$ $(i = 1, 2)$ are assumed to be $\mathbb{F}$ adapted processes. In this article, we also suppose that c^p, c^q, r^p, r^q and Ψ are functions of X, and that under non-default states of both parties, that is conditioned on $\{\tau > t\}$, $h^{[i]}$ and $l^{[i]}$ $(i = 1, 2)$ are some functions of X. Moreover, when either an investor or counter party is default, the derivatives contract is terminated after the settlement of the present values of the collateral and derivatives.

Under these assumptions, conditioned on $\{\tau > t\}$, we consider the pre-default value V_t of a derivatives with maturity $T(> 0)$ and the payoff Ψ from the viewpoint of the investor, party 1. Under the no-jump condition which means V does not jump when one party defaults ($\Delta V_\tau := V_\tau - V_{\tau-} = 0$) [1], V_t is expressed as follows. The derivation is given in Appendix.

$$\begin{aligned} V_t &= E^{Q^p}\Bigg[e^{-\int_t^T r_u^p du}\Psi + \int_t^T e^{-\int_t^u r_s^p ds}\Big\{(r_u^q - c_u^q)\Gamma_u \\ &\quad + h_u^{[1]} l_u^{[1]}(\Gamma_u - V_u)^+ - h_u^{[2]} l_u^{[2]}(V_u - \Gamma_u)^+\Big\}du\Big|\mathcal{F}_t\Bigg], \end{aligned} \tag{3}$$

where the first term in the right hand side of (3) is the default-free price of the original derivatives and the second term is the return from posting or posted collaterals. The third and fourth terms express the effects of over (or under) collateral when the investor ($i = 1$) or the counter party ($i = 2$) defaults.

At each time t, when a collateral is posted perfectly as the value of derivatives V_t by the settlement currency of derivatives, the collateral value Γ_t coincides with the present value V_t of the derivatives.

In this article, as examples of imperfect collateralization, we take a no collateral case and a time-lag collateral case. Moreover, in Section 5 we consider a case that collateral values are dependent on not only the original derivatives values, but also other factors. (e.g. the collateral is posted by a currency different from the payment currency of the original derivatives.) Hence, we suppose that the collateral value Γ is a function of X as well as of $V_{t-\Delta}$. Then, because the other variables, c^p, c^q, r^p, r^q, Ψ, $h^{[i]}$, $l^{[i]}$ $(i = 1, 2)$, which determine a value V in (3) on $\{\tau > t\}$, are also some functions of X, V does not jump at a time of default.

From (3), we also remark that

$$\begin{aligned} &e^{-\int_0^t r_u^p du}V_t + \int_0^t e^{-\int_0^u r_s^p ds} \\ &\quad \times\Big\{(r_u^q - c_u^q)\Gamma_u + h_u^{[1]} l_u^{[1]}(\Gamma_u - V_u)^+ - h_u^{[2]} l_u^{[2]}(V_u - \Gamma_u)^+\Big\}du \end{aligned} \tag{4}$$

is a martingale, and the drift term of the stochastic differential equation which above the equation satisfies is zero. Then, a pair of (V, Z) is the solution to the following backward stochastic equation (BSDE). Here, Z stands for the volatility of the derivatives value (or volatility multiplied by V)

$$dV_t = c_t^p V_t dt - f(t, X, V, \Gamma)dt + Z_t \cdot dW_t, \tag{5}$$

$$V_T = \Psi(X_T), \tag{6}$$

$$\begin{aligned} f(t, X, V, \Gamma) &= (r_t^q - c_t^q)\Gamma_t - (r_t^p - c_t^p)V_t \\ &\quad + h_t^{[1]} l_t^{[1]}(\Gamma_t - V_t)^+ - h_t^{[2]} l_t^{[2]}(V_t - \Gamma_t)^+ \end{aligned} \tag{7}$$

We summarize the discussion above as the following proposition:

Proposition 2.1. *Suppose that p is a payment currency, q is a collateral currency, Ψ represents a maturity T's payoff of a derivative, Γ_t represents the value process of the collateral and r_t^p (or r_t^q) represents the risk-free interest rate process under currency p (or currency q). Also, c_t^p (or*

[1] In our subsequent numerical analyses, since Γ_t is a function of $V_{t-\Delta}$ and a diffusion process X_t, with assumptions that c^p, c^q, r^p, r^q, Ψ $h^{[i]}$ and $l^{[i]}$ $(i = 1, 2)$ on $\{\tau > t\}$ are some functions of X, the no-jump condition is fulfilled.

c_t^q) represents the collateral rate process under currency p (or currency q) and $l_t^i(\geq 0)$, $i \in \{1,2\}$ represents the party i's loss rate process.

We also assume that c^p, c^q, r^p, r^q and Ψ are functions of X, the solution to the SDE (2), and that under non-default states of both parties (i.e. on $\{\tau > t\}$), the default hazard rates $h^{[i]}$ and $l^{[i]}$ $(i = 1,2)$ are some functions of X. Moreover, when either an investor or counter party is default, the derivatives contract is terminated after the settlement of the present values of the collateral and derivatives.

Then, when Γ_t is a function of X_t and $V_{t-\Delta}$, a past pre-default value of the derivative, the current pre-default value V_t from the viewpoint of the party 1 is expressed as the equation (3). Moreover, V and its volatility Z are given by the solution to the FBSDE (2), (5)-(7).

In the BSDE, (6) shows that a payoff of the derivatives contracts with maturity T is expressed as $\Psi(X_T)$, which is a function of state variables X_T at time T.

The first term of (5), $c_t^p V_t$ means the discount by a collateral rate of the currency p, which is the same as the payment currency of the derivatives. That is, the term corresponds to the discount in the perfect collateralized contract with payment and collateral currency p.

Moreover, the first term in (7), $(r_t^q - c_t^q)\Gamma_t$, represents a collateral cost caused from a collateral asset whose value is Γ. The second term, $(r_t^p - c_t^p)V_t$, stands for a cost of the cash collateral of currency p. These first and second terms, $(r_t^q - c_t^q)\Gamma_t - (r_t^p - c_t^p)V_t$, express a funding spread between the cash of currency p and a collateral asset valued as Γ.

Finally, the third term in (7), $h_t^{[1]} l_t^{[1]} (\Gamma_t - V_t)^+$, shows an (instantaneous) expected gain of the investor $i = 1$ caused by imperfect collateralization when the investor $i = 1$ defaults. On the other hand, the fourth term in (7), $h_t^{[2]} l_t^{[2]} (V_t - \Gamma_t)^+$, means an (instantaneous) expected loss of the investor $i = 1$ caused by an imperfect collateralization when the counterparty $i = 2$ defaults.

In general, it is difficult to solve this BSDE. Then, in this article, we approximate the solution by a perturbation method introduced in Section 3. Especially, we consider a perturbed BSDE (10), where a perturbation parameter ϵ is introduced in the driver f, and the solution of the BSDE (5)-(7) is expanded around the solution of the following equation, that is in BSDE (10) we set $\epsilon = 0$:

$$dV_t = c_t^p V_t dt + Z_t \cdot dW_t, \tag{8}$$

$$V_T = \Psi(X_T) \tag{9}$$

In other words, we propose to expand the solution around the value of the derivatives which is perfectly collateralized with currency p, the same currency as the settlement currency of the derivatives.

Moreover, in order to implement computation of this approximation, we apply Monte Carlo simulations based on an interacting particle method, which is explained in Appendix.

3. Perturbative Expansion Method

This section summarizes an approximation method for the solution to the BSDEs (5)-(7) by following Fujii-Takahahi [2012a]. For a mathematical validity of the method, see Takahashi and Yamada [2013].

First, we approximate the FBSDE with a perturbative expansion technique. Let us introduce the perturbation parameter ϵ as follows:

$$\begin{cases} dV_t^{(\epsilon)} = c_t^p V_t^{(\epsilon)} dt - \epsilon f\left(t, X, V_t^{(\epsilon)}, \Gamma_t^{(\epsilon)}\right) dt + Z_t^{(\epsilon)} \cdot dW_t \\ V_T^{(\epsilon)} = \Psi(X_T), \end{cases} \tag{10}$$

Here, we make Γ depend explicitly on ϵ as $\Gamma_t^{(\epsilon)}$, since Γ is dependent on V such as $\Gamma_t = V_{t-\Delta}(\Delta > 0)$ in the analyses below.

Next, let us expand a solution of FBSDE (10) with respect to ϵ around $\epsilon = 0$. That is, when f is small enough, we suppose that $V_t^{(\epsilon)}$ and $Z_t^{(\epsilon)}$ are expanded as follows:

$$V_t^{(\epsilon)} = V_t^{(0)} + \epsilon V_t^{(1)} + \epsilon^2 V_t^{(2)} + \cdots \tag{11}$$

$$Z_t^{(\epsilon)} = Z_t^{(0)} + \epsilon Z_t^{(1)} + \epsilon^2 Z_t^{(2)} + \cdots . \tag{12}$$

For instance, by calculating the expansions of $V_t^{(\epsilon)}$ and $Z_t^{(\epsilon)}$ up to the k-th order with putting $\epsilon = 1$, we obtain the k-th order approximation of V_t, Z_t as follows:

$$\tilde{V}_t = \sum_{i=0}^{k} V_t^{(i)}, \qquad \tilde{Z}_t = \sum_{i=0}^{k} Z_t^{(i)}, \tag{13}$$

where $V_t^{(i)}$ and $Z_t^{(i)}$ are calculated recursively using the results of the lower order approximations and X.

Next, we explain how to derive $V_t^{(i)}$ concretely. For the zero-th order of ϵ, one can easily derive $V_t^{(0)}$ by substituting 0 for ϵ in the equation (10), and it is expressed as follows:

$$dV_t^{(0)} = c_t^p V_t^{(0)} dt + Z_t^{(0)} \cdot dW_t \tag{14}$$

$$V_T^{(0)} = \Psi(X_T) . \tag{15}$$

It can be integrated as

$$V_t^{(0)} = E\left[e^{-\int_t^T c_s^p ds} \Psi(X_T) \middle| \mathcal{F}_t \right] . \tag{16}$$

We remark that $V_t^{(0)}$ is equivalent to the price of a standard European contingent claim without default risks, and $V_t^{(0)}$ is a function of X_t. Then, applying Itô's formula, we obtain $Z_t^{(0)}$ as a function of X_t.

It is clear that they can be evaluated by standard Monte Carlo simulation. However, in order to obtain the higher order approximations, it is crucial to derive an explicit or closed form approximation of $V_t^{(0)}$. For instance, the SABR formula for plain vanilla options derived by Hagan et al. [2002] is useful for an approximation of $V_t^{(0)}$, which is applied in the numerical experiments of this paper. To approximate derivatives values in general models and prices of multi-asset exotic options such as basket and average options, which are mainly traded in the energy market, it is useful to employ an asymptotic expansion method. (See Shiraya-Takahashi [2011], [2014] for the details.)

Next, $V_t^{(1)}$ is derived by differentiating (10) with respect to ϵ, and substituting 0 for ϵ.

$$dV_t^{(1)} = c_t^p V_t^{(1)} dt - f(t, X_t, V_t^{(0)}, \Gamma_t^{(0)}) dt + Z_t^{(1)} \cdot dW_t, \tag{17}$$

$$V_T^{(1)} = 0 , \tag{18}$$

Then, we have the following by solving the above equations.

$$V_t^{(1)} = E\left[\int_t^T e^{-\int_t^u c_s^p ds} f(u, X_u, V_u^{(0)}, \Gamma_u^{(0)}) du \middle| \mathcal{F}_t \right] . \tag{19}$$

Because $V_u^{(0)}$ and $Z_u^{(0)}$ are functions of X_u, we obtain $V_t^{(1)}$ as a function of X_t, and again by Itô's formula, we have $Z_t^{(1)}$ as a function of X_t, too. We note that this first order approximation term $V_t^{(1)}$ can be regarded as a traditional CVA (Credit Value Adjustment) which is often used in practice. [2].

[2]Our convention of CVA is different from the one, which is used in practice by sign, where it is defined as the "charge" to the clients. Thus, our CVA = -CVA.

In the similar manner, an arbitrarily higher order correction term can be derived. For example, the second order correction term is expressed as follows:

$$\begin{aligned}
dV_t^{(2)} &= c_t^p V_t^{(2)} dt - \frac{\partial}{\partial v} f(t, X_t, V_t^{(0)}, \Gamma_t^{(0)}) V_t^{(1)} dt - \frac{\partial}{\partial \gamma} f(t, X_t, V_t^{(0)}, \Gamma_t^{(0)}) \Gamma_t^{(1)} dt \\
&\quad + Z_t^{(2)} \cdot dW_t, \qquad (20) \\
V_T^{(2)} &= 0, \qquad (21) \\
V_t^{(2)} &= E\left[\int_t^T e^{-\int_t^u c_s^p ds} \left[\frac{\partial}{\partial v} f(u, X_u, V_u^{(0)}, \Gamma_u^{(0)}) V_u^{(1)} + \frac{\partial}{\partial \gamma} f(t, X_u, V_u^{(0)}, \Gamma_u^{(0)}) \Gamma_u^{(1)} \right] du \middle| \mathcal{F}_t \right]. \\
&= E\Bigg[\int_t^T e^{-\int_t^u c_s^p ds} \Big[\Big(-(r_u^p - c_u^p) - h_u^{[1]} l_u^{[1]} \mathbf{1}_{\{\Gamma_u^{(0)} > V_u^{(0)}\}} - h_u^{[2]} l_u^{[2]} \mathbf{1}_{\{\Gamma_u^{(0)} < V_u^{(0)}\}} \Big) V_u^{(1)} \\
&\qquad + \Big((r_u^q - c_u^q) + h_u^{[1]} l_u^{[1]} \mathbf{1}_{\{\Gamma_u^{(0)} > V_u^{(0)}\}} + h_u^{[2]} l_u^{[2]} \mathbf{1}_{\{\Gamma_u^{(0)} < V_u^{(0)}\}} \Big) \Gamma_u^{(1)} \Big] du \Big| \mathcal{F}_t \Bigg]. \qquad (22)
\end{aligned}$$

4. Approximation of Density Function of CVA in a Multi-factor Model

The first concrete example applies a perturbative expansion method in the relevant FBSDE to an approximation for the density function of the CVA (Credit Value Adjustment) in the valuation of a pre-default contract with bilateral counter party risk. We note that the first order expansion term in the driver of the pricing BSDE is regarded as CVA which is typically used in practice, and that an approximate density function of the CVA seems useful in evaluation of its VaR(Value at Risk).

In particular, we take a forward contract of a foreign exchange (forex) rate with bilateral counter party risk, where both parties post their collateral perfectly with the constant time-lag (Δ) by the same currency as the payment currency. For simplicity we also assume the constant risk-free interest rate r is equal to the collateral rate.

We consider a forward contract on the forex rate $S^{\hat\epsilon}$ with strike K and maturity T. The relevant FBSDE for the pre-default contract value is given with perturbation parameters $\epsilon, \hat\epsilon \in (0,1]$. In particular, the state vector consisting of the FSDE is specified as $X^{\hat\epsilon} = (h^{[1],\hat\epsilon}, h^{[2],\hat\epsilon}, S^{\hat\epsilon}, \nu^{\hat\epsilon})$, where $S^{\hat\epsilon}$ is the forex rate, $\nu^{\hat\epsilon}$ is its volatility and $h^{j,\hat\epsilon}$, $j = 1, 2$ stands for each counter party's hazard rate process.

$$dV_t^{(\epsilon),\hat\epsilon} = rV_t^{(\epsilon),\hat\epsilon} dt - \epsilon f(h_t^{[1],\hat\epsilon}, h_t^{[2],\hat\epsilon}, V_t^{(\epsilon),\hat\epsilon}, V_{t-\Delta}^{(\epsilon),\hat\epsilon}) dt + Z_t^{(\epsilon),\hat\epsilon} \cdot dW_t, \qquad (23)$$

$$V_T^{(\epsilon),\hat\epsilon} = S_T^{\hat\epsilon} - K, \qquad (24)$$

$$f(h_t^{[1],\hat\epsilon}, h_t^{[2],\hat\epsilon}, V_t^{(\epsilon),\hat\epsilon}, V_{t-\Delta}^{(\epsilon),\hat\epsilon}) = h_t^{[1],\hat\epsilon} (V_{t-\Delta}^{(\epsilon),\hat\epsilon} - V_t^{(\epsilon),\hat\epsilon})^+ - h_t^{[2],\hat\epsilon} (V_t^{(\epsilon),\hat\epsilon} - V_{t-\Delta}^{(\epsilon),\hat\epsilon})^+, \qquad (25)$$

$$dh_t^{[j],\hat\epsilon} = \mu^{[j]} h_t^{[j],\hat\epsilon} dt + \hat\epsilon \sigma_{h^{[j]}} h_t^{[j],\hat\epsilon} \left(\sum_{\eta=1}^{j} c_{j,\eta} dW_t^\eta \right), \; h_0^{[j],\hat\epsilon} = h_0^{[j]}, (j = 1, 2), \qquad (26)$$

$$dS_t^{\hat\epsilon} = (r - r_f) S_t^{\hat\epsilon} dt + \hat\epsilon \nu_t^{\hat\epsilon} \left(S_t^{\hat\epsilon} \right)^\beta \left(\sum_{\eta=1}^{3} c_{3,\eta} dW_t^\eta \right), \; S_0^{\hat\epsilon} = s_0, \beta \in (0, 1], \qquad (27)$$

$$d\nu_t^{\hat\epsilon} = \kappa(\theta - \nu_t^{\hat\epsilon}) dt + \hat\epsilon \xi \nu_t^{\hat\epsilon} \left(\sum_{\eta=1}^{4} c_{4,\eta} dW_t^\eta \right), \; \nu_0^{\hat\epsilon} = \nu_0. \qquad (28)$$

Here, $W = (W^1, W^2, W^3, W^4)$ is a four dimensional Brownian motion, and $c_{j,\eta}$ ($j = 1, 2, 3, 4$, $\eta = 1, \cdots, j$), r, r_f, κ θ, ξ, $\mu_{h^{[j]}}$, $\sigma_{h^{[j]}}$, h_0^j ($j = 1, 2$), s_0 and ν_0 are some constants.

Then, the derivative price with the bilateral counter party risk is given by

$$V_t^{(\epsilon),\hat{\epsilon}} = E\left[e^{-r(T-t)}(S_T^{\hat{\epsilon}} - K)\right] + \epsilon E\left[\int_t^T e^{-r(u-t)} f(h_u^{[1],\hat{\epsilon}}, h_u^{[2],\hat{\epsilon}}, V_u^{(\epsilon),\hat{\epsilon}}, V_{u-\Delta}^{(\epsilon),\hat{\epsilon}}) du\right]. \tag{29}$$

Hereafter, we pursue an approximation of the equation above. Firstly, the equation with regard to the first order of the ϵ-expansion is expressed as follows:

$$dV_t^{(1),\hat{\epsilon}} = rV_t^{(1),\hat{\epsilon}} dt - f(h_t^{[1],\hat{\epsilon}}, h_t^{[2],\hat{\epsilon}}, V_t^{(0),\hat{\epsilon}}, V_{t-\Delta}^{(0),\hat{\epsilon}})dt + Z_t^{(1),\hat{\epsilon}} \cdot dW_t, \tag{30}$$

$$V_T^{(1),\hat{\epsilon}} = 0. \tag{31}$$

Then, the derivative price with the bilateral counter party risk is approximated by

$$V_t^{(\epsilon),\hat{\epsilon}} \simeq V_t^{(0),\hat{\epsilon}} + \epsilon V_t^{(1),\hat{\epsilon}}, \tag{32}$$

and the term $\epsilon V_t^{(1),\hat{\epsilon}}$ is regarded as the CVA at time t represented by the following equation:

$$V_t^{(1),\hat{\epsilon}} = E\left[\int_t^T e^{-r(u-t)} f(h_u^{[1],\hat{\epsilon}}, h_u^{[2],\hat{\epsilon}}, V_u^{(0),\hat{\epsilon}}, V_{u-\Delta}^{(0),\hat{\epsilon}}) du\right], \tag{33}$$

where

$$f(h_u^{[1],\hat{\epsilon}}, h_u^{[2],\hat{\epsilon}}, V_s^{(0),\hat{\epsilon}}, V_{u-\Delta}^{(0),\hat{\epsilon}}) = h_u^{[1],\hat{\epsilon}} \cdot (V_{u-\Delta}^{(0),\hat{\epsilon}} - V_u^{(0),\hat{\epsilon}})^+ - h_u^{[2],\hat{\epsilon}} \cdot (V_u^{(0),\hat{\epsilon}} - V_{u-\Delta}^{(0),\hat{\epsilon}})^+. \tag{34}$$

Here, $V_{u-\Delta}^{(0),\hat{\epsilon}} = 0$ when $u < t + \Delta$.

Then, $V_u^{(0),\hat{\epsilon}}$ and $V_u^{(0),\hat{\epsilon}} - V_{u-\Delta}^{(0),\hat{\epsilon}}$ are explicitly calculated as

$$V_u^{(0),\hat{\epsilon}} = e^{-r_f(T-u)} S_u^{\hat{\epsilon}} - e^{-r(T-u)} K, \tag{35}$$

$$V_u^{(0),\hat{\epsilon}} - V_{u-\Delta}^{(0),\hat{\epsilon}} = e^{-r_f(T-u)} S_u^{\hat{\epsilon}} - e^{-r_f(T-u+\Delta)} S_{u-\Delta}^{\hat{\epsilon}} - k(u; \Delta, r), \tag{36}$$

where

$$k(u; \Delta, r) := e^{-r(T-u)}(1 - e^{-r\Delta})K. \tag{37}$$

Next, we apply the asymptotic expansion method to evaluation of $C(u; t, x) = e^{-r(u-t)} E\left[f(h_u^{[1],\hat{\epsilon}}, h_u^{[2],\hat{\epsilon}}, V_u^{(0),\hat{\epsilon}}, V_{u-\Delta}^{(0),\hat{\epsilon}})\right]$ up to $\hat{\epsilon}^3$ where (t, x) represents the values of the state variables $x = (h^{[1]}, h^{[2]}, s, \nu)$ at time t, that is the third order of $\hat{\epsilon}$. Then, we obtain the following result.

Proposition 4.1. *The value of CVA at time t, $\epsilon V_t^{(1),\hat{\epsilon}}$ is approximated by*

$$\epsilon V_t^{(1),\hat{\epsilon}} = \epsilon \int_t^T C_{AE}(u; t, x) du + O(\hat{\epsilon}^4), \tag{38}$$

where, $C_{AE}(u; t, x)$ stands for the approximation of $C(u; t, x)$ based on the asymptotic expansion up to the third order, which is expressed as follows:

$$C_{AE}(u; t, x) = e^{-r(u-t)} \sum_{j=1}^{2} \Bigg\{ \hat{\epsilon}\left(y^j N\left(\frac{y^j}{\sqrt{\Sigma^j}}\right) + \Sigma^j n[y^j; 0, \Sigma^j]\right)$$
$$+\hat{\epsilon}^2 \left(-\frac{C_1^j}{\Sigma^j} y^j n[y^j; 0, \Sigma^j] + C_0^j N\left(\frac{y^j}{\sqrt{\Sigma^j}}\right)\right)$$

$$+\hat{\epsilon}^3\left(C_2^j\left(\frac{-1}{\Sigma^j}+\frac{(y^j)^2}{(\Sigma^j)^2}\right)n[y^j;0,\Sigma^j]+C_3^j n[y^j;0,\Sigma^j]\right.$$
$$+\left(C_4^j\left(\frac{(y^j)^4}{(\Sigma^j)^4}-\frac{6(y^j)^2}{(\Sigma^j)^3}+\frac{3}{(\Sigma^j)^2}\right)+C_5^j\left(\frac{(y^j)^2}{(\Sigma^j)^2}-\frac{1}{\Sigma^j}\right)+C_6^j\right)n[y^j;0,\Sigma^j]$$
$$\left.\left.+\frac{(C_0^j)^2}{2}N\left(\frac{y^j}{\sqrt{\Sigma^j}}\right)-\frac{C_0^jC_1^j}{\Sigma^j}y^j n[y^j;0,\Sigma^j]\right)\right\}. \qquad (39)$$

Here, $N(\cdot)$ and $n[\cdot,\mu,\Sigma]$ stand for the standard normal distribution function and the normal density function with mean μ and variance Σ, respectively. Also, C_i^j and y^j $i=0,1,\cdots,6$, $j=1,2$ are some constants. [3]

We emphasize that due to the analytical approximation of each $C_{AE}(u;t,x)$, we have no problem in computation of the integral in (38), which is very fast.

The parameters in the factors are set as follows with $\epsilon=\hat{\epsilon}=1$:

- parameters of $h^{[1]}$:
  $h_0^{[1]}=0.02$, $\mu^{[1]}=-0.02$, $\sigma_{h^{[1]}}=0.2$.
- parameters of $h^{[2]}$:
  $h_0^{[2]}=0.01$, $\mu^{[2]}=0.02$, $\sigma_{h^{[2]}}=0.3$.
- parameters of S:
  $S_0=10,000$, $r=r_f=0.01$, $\beta=1$.
- parameters of ν:
  $\nu_0=0.1$, $\kappa=1$, $\theta=0.2$, $\xi=0.3$.
- correlations are in Table 1.

Table 1: Correlation Matrix

	$h^{[1]}$	$h^{[2]}$	S	ν
$h^{[1]}$	1	0.5	-0.3	0.2
$h^{[2]}$	0.5	1	0.1	0.1
S	-0.3	0.1	1	-0.8
ν	0.2	0.1	-0.8	1

In this setting, we show the density function of the approximate CVA above by the asymptotic expansion method with Monte Carlo simulations. T stands for the maturity of the forward contract, t denotes the future time when CVA is evaluated, and Δ is the lag of the collateral posting. In addition to the the parameters above, the setup and the procedure of the calculation are summarized as follows:

- maturity (T): 5 years, evaluation date (t): 2.5 years.
- strike price (K): 10,000.
- time step size in Monte Carlo: $\frac{1}{400}$ year.
- the number of trials in Monte Carlo: 325,000 with antithetic variates.

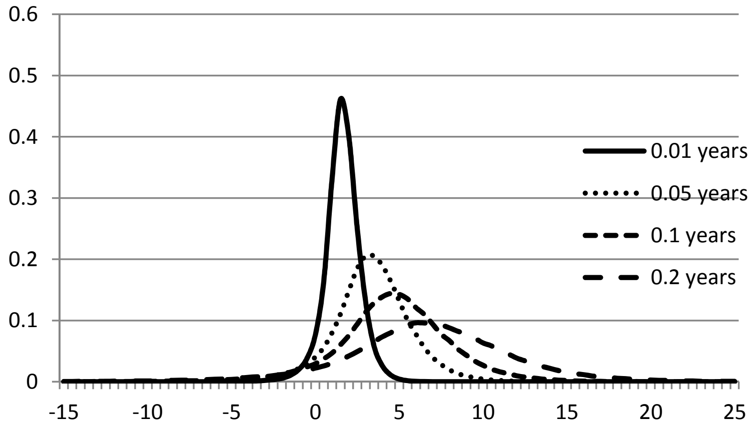
Procedure:

[3] Those are given upon request.

1. implement Monte Carlo simulations of the state variables ($h^{[1]}$, $h^{[2]}$, S, ν) until time t (from time 0).
2. given each realization of the state variables, compute $C_{AE}(u; t, x)$.
3. integrate $C_{AE}(u; t, x)$ numerically with respect to the time parameter u from t to T, and plot the values and their frequencies after normalization.

Figure 1 shows the density functions of CVA with different time-lags.

Figure 1: Density Functions of CVA with Different Time-Lags



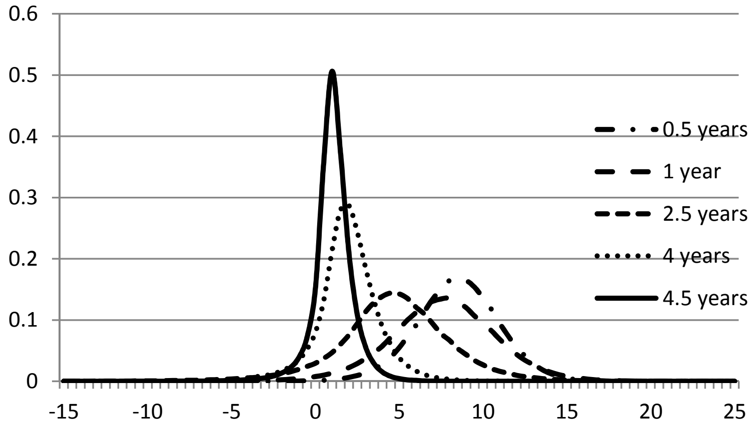
It is observed that the longer the time lag is, the wider the density is, and that the mode (or average) moves to the right when the time-lag becomes longer. Here, we recall that in the CVA equation (33), we have

$$f(h_u^{[1],\hat{\epsilon}}, h_u^{[2],\hat{\epsilon}}, V_u^{(0),\hat{\epsilon}}, V_{u-\Delta}^{(0),\hat{\epsilon}})$$
$$= h_u^{[1],\hat{\epsilon}} \cdot (V_{u-\Delta}^{(0),\hat{\epsilon}} - V_u^{(0),\hat{\epsilon}})^+ - h_u^{[2],\hat{\epsilon}} \cdot (V_u^{(0),\hat{\epsilon}} - V_{u-\Delta}^{(0),\hat{\epsilon}})^+. \tag{40}$$

Then, we are able to see that when the first term on the right hand side increases, the CVA also increases. This is because in our parameterization the hazard rate $h^{[1],\hat{\epsilon}}$ in the first term tends to be larger than $h^{[2],\hat{\epsilon}}$ in the second term mainly due to $h_0^{[1]} > h_0^{[2]}$.

Figure 2 shows the density functions of CVA with different evaluation dates.

Figure 2: Density Functions of CVA with Different Evaluation Dates



Because the CVA depends on the time to maturity $T-t$, we can see that when the evaluation date t is in the more future $(0.5, 1, \cdots, 4.5)$, that is, the shorter the time to maturity $(T-t)$ becomes, the CVA becomes smaller.

5. Option Pricing with Counter Party Risk and Imperfect Collateralization

This section analyzes option prices with counter party risk and imperfect collateralization by applying the so-called interacting particle method, which is explained in Appendix. Particularly, we investigate the impacts on the option values of the changes in the parameters of the underlying factors and the times to maturities of the options. We also examine the shapes of implied volatility curves.

5.1. Models

First, we explain the models which are used in simulations. Hereafter, W stands for a eight dimensional standard Brownian motion. For $x, y \in \mathbb{R}^n$, we use notations $x \cdot y = \sum_{i=1}^{n} x_i y_i$.

- The underlying asset price S is described by a SABR model:

$$dS_t = (r_t^p - \delta_t) S_t dt + \nu_t (S_t)^{\beta} \Sigma_S \cdot dW_t;\ S_0 = s_0, \quad (41)$$

$$d\nu_t = \nu_t \Sigma_\nu \cdot dW_t;\ \nu_0 = \hat{\nu}_0. \quad (42)$$

- Both of the hazard rates ($h^{[i]}$, $i = 1, 2$) and the risk-free rates (r^p, r^q) of currencies p and q follow CIR models:

$$dh_t^{[i]} = \kappa_i \left(\theta_i - h_t^{[i]}\right) dt + \sqrt{h_t^{[i]}} \Sigma_{h^{[i]}} \cdot dW_t;\ h_0^{[i]} = \hat{h}_0^{[i]}, \quad (43)$$

$$dr_t^p = \kappa_{r^p} (\theta_{r^p} - r_t^p)\, dt + \sqrt{r_t^p} \Sigma_{r^p} \cdot dW_t;\ r_0^p = \hat{r}_0^p, \quad (44)$$

$$dr_t^q = \kappa_{r^q} (\theta_{r^q} - r_t^q)\, dt + \sqrt{r_t^q} \Sigma_{r^q} \cdot dW_t;\ r_0^q = \hat{r}_0^q. \quad (45)$$

- The collateral asset price (A) follows a SABR model:

$$dA_t = \mu_A A_t dt + A_t^{\beta_A} \nu_t^A \Sigma_A \cdot dW_t;\ A_0 = a_0, \quad (46)$$

$$d\nu_t^A = \nu_t^A \Sigma_{\nu^A} \cdot dW_t;\ \nu_0^A = \hat{\nu}_0^A. \quad (47)$$

Here, Σ_S, Σ_ν, $\Sigma_{h^{[i]}}(i = 1, 2)$, Σ_{r^p}, Σ_{r^q}, Σ_A and Σ_{ν^A} are eight dimensional vectors, which are determined by an instantaneous correlation matrix among S, ν, $h^{[i]}(i = 1, 2)$, r^p, r^q and A.

For simplicity, we assume that collateral rates (c^p, c^q) and loss rates ($l^{[1]}$,$l^{[2]}$) are constants.

The payoff Ψ at maturity T of a derivatives is expressed by a function of the underlying asset price S. Particularly, for the case of a European call option with strike K, Ψ is given as

$$\Psi(S_T) = (S_T - K)^+ := \max\{S_T - K, 0\}. \quad (48)$$

Moreover, we allow a collateral asset to be different from the cash of the settlement currency of the derivatives, such as the cash of a different currency. In addition, we consider the cases that both parties post no collaterals or post collaterals by the asset with its value A and a constant time-lag Δ.

Under the setting, the state variable vector $\{X_t : t \geq 0\}$ is specified as $X_t = (S_t, \nu_t, h_t^{[1]}, h_t^{[2]}, r_t^p, r_t^q, A_t, \hat{A}_t, \nu_t^A, \nu_t^{\hat{A}})$, where $\hat{A}_t := A_{t-\Delta}$. Then, the stochastic differential equation (2) is formulated by the above stochastic differential equations (41)-(47).

Then, the driver f for the no collateral case is expressed as follows:

$$f(t,X,V,\Gamma) \;=\; f(t,X,V) = -y^p_t V_t + h^{[1]}_t l^{[1]}(-V_t)^+ - h^{[2]}_t l^{[2]}(V_t)^+, \tag{49}$$

where $y^p_t = r^p_t - c^p$.

On the other hand, the driver f for the case that the collateral asset with its value A is posted with a constant time-lag Δ is expressed as follows:

$$\begin{aligned} f(t,X,V,\Gamma) \;=\;& y^q_t V_{t-\Delta}\frac{A_t}{A_{t-\Delta}} - y^p_t V_t + h^{[1]}_t l^{[1]}\left(V_{t-\Delta}\frac{A_t}{A_{t-\Delta}} - V_t\right)^+ \\ & -h^{[2]}_t l^{[2]}\left(V_t - V_{t-\Delta}\frac{A_t}{A_{t-\Delta}}\right)^+, \end{aligned} \tag{50}$$

where $\Gamma_t = V_{t-\Delta}\frac{A_t}{A_{t-\Delta}}$, $y^q_t = r^q_t - c^q_t$ and $y^p_t = r^p_t - c^p_t$.

5.2. Concrete Setup in Numerical Experiments

We calculate the values of OTC (over the counter) European call options on an asset price S.

Hereafter, to avoid complexity, we assume that the investor is default-free ($h^{[1]} \equiv 0$, $h = h^{[2]}$), and the loss rate of the counter party is 1 ($l^{[2]} = 1$), that is there is no recovery at default.

Under this setup, we have the following specifications:

- The driver of the BSDE with the no collateral case is expressed as follows:

$$f(t,X,V,\Gamma) \;=\; -y^p_t V_t - h^{[2]}_t V_t. \tag{51}$$

  The first order approximation $V^{(1)}_t$ in (19) (the general expression of the first order approximation in Section 3) is given as follows:

$$V^{(1)}_t = E\left[\int_t^T e^{-\int_t^u c^p_s ds}\left(-y^p_u V^{(0)}_u - h^{[2]}_u V^{(0)}_u\right) du \middle| \mathcal{F}_t\right]. \tag{52}$$

- The driver f of the BSDE for the case that the collateral asset with its value A is posted with a constant time-lag Δ is expressed as follows:

$$f(t,X,V,\Gamma) \;=\; y^q_t V_{t-\Delta}\frac{A_t}{A_{t-\Delta}} - y^p_t V_t - h^{[2]}_t\left(V_t - V_{t-\Delta}\frac{A_t}{A_{t-\Delta}}\right)^+. \tag{53}$$

  In this case, the first order approximation $V^{(1)}_t$ in (19), is obtained as follows:

$$V^{(1)}_t = E\left[\int_t^T e^{-\int_t^u c^p_s ds}\left[y^q_u V^{(0)}_{u-\Delta}\frac{A_u}{A_{u-\Delta}} - y^p_u V^{(0)}_u - h^{[2]}_u\left(V^{(0)}_u - V^{(0)}_{u-\Delta}\frac{A_u}{A_{u-\Delta}}\right)^+\right] du \middle| \mathcal{F}_t\right], \tag{54}$$

  where we use the relation that $\Gamma^{(0)}_u = V^{(0)}_{u-\Delta}\frac{A_u}{A_{u-\Delta}}$.

We remark that in the following tables, $h^{[2]}$ is denoted as h.

In numerical examples, we investigate the following points:

1. effects of parameters
   - no collateral
     - effects of parameters of the hazard rate

- effects of parameters of the interest rate r^p

- asset collateral
  - effects of parameters of the hazard rate
    - time-lags : 0.25 or 0.02
  - effects of parameters of the interest rate r^p
    - time-lags : 0.25 or 0.02
  - effects of parameters of the interest rate r^q
  - effects of parameters of the collateral asset A

2. effects of the maturities
   - no collateral
   - asset collateral

The Monte Carlo simulations with the interacting particle method are implemented with time-step size 1/200 year and 5 million sample paths, and $V^{(0)}$ is evaluated by the formula of Hagan et al. [2002].

Hereafter, we make the following assumptions otherwise mentioned.

- We set the correlations which are not under consideration for the effects in the changes as 0.

- We set the risk free rates and the collateral rates as $r^p = c^p = 1\%$ or $r^q = c^q = 1\%$ when r^p or r^q is a constant.

  Under these settings with no collateral posting, the driver f of the BSDE (51) is expressed as

$$f(t, X, V, \Gamma) = -h_t^{[2]} V_t, (r^p : \text{constant}) \tag{55}$$
$$f(t, X, V, \Gamma) = -y_t^p V_t - h_t^{[2]} V_t, (r^p : \text{stochastic}) \tag{56}$$

  On the other hand, when the collateral is posted with a constant time-lag Δ by the asset whose value is A, the driver (53) is expressed as

$$f(t, X, V, \Gamma) = -h_t^{[2]} \left(V_t - V_{t-\Delta} \frac{A_t}{A_{t-\Delta}} \right)^+, (r^p \text{and } r^q : \text{constant}) \tag{57}$$
$$f(t, X, V, \Gamma) = y_t^q V_{t-\Delta} \frac{A_t}{A_{t-\Delta}} - h_t^{[2]} \left(V_t - V_{t-\Delta} \frac{A_t}{A_{t-\Delta}} \right)^+, (r^p : \text{constant}) \tag{58}$$
$$f(t, X, V, \Gamma) = -y_t^p V_t - h_t^{[2]} \left(V_t - V_{t-\Delta} \frac{A_t}{A_{t-\Delta}} \right)^+, (r^q : \text{constant}) \tag{59}$$
$$f(t, X, V, \Gamma) = y_t^q V_{t-\Delta} \frac{A_t}{A_{t-\Delta}} - y_t^p V_t - h_t^{[2]} \left(V_t - V_{t-\Delta} \frac{A_t}{A_{t-\Delta}} \right)^+, (r^p \text{and } r^q : \text{stochastic}) \tag{60}$$

- The time-lag Δ is equal to 0.25.

- The OTC European option is ATM ($K = S_0$) call option with 6 years maturity, where the underlying asset yields a dividend, which is equal to the risk free rate (i.e. the drift term of the risk-neutral asset price process is 0).

  The underlying asset price follows the SABR model and its parameters are listed in Table 2.

Table 2: Parameters of the underlying asset price (SABR model)

	s_0	β	ν_0	σ_ν	ρ
underlying asset and its volatility	100	0.5	2	0.4	0

Here, $\sigma_X := |\Sigma_X|$ where $|x| = \sqrt{\sum_{i=1}^n x_i^2}$ for $x \in \mathbb{R}^n$. ρ is the instantaneous correlation between S and ν. We note that the corresponding Black-Scholes volatility is about 20%, where the Black-Scholes volatility is defined as σ such that $\nu_0 S_0^\beta = \sigma S_0$.

- The parameters of the stochastic differential equation of the collateral asset value (46) are generally assumed to be a SABR model. However, otherwise mentioned, we use the following parameters in Table 3, that is A follows a log-normal model.

Table 3: Parameters of the collateral asset price

	A_0	μ_A	β	ν_0^A	σ_A	ρ
collateral asset and its volatility	1	0	1	50%	0	0

ρ is the instantaneous correlation between A and ν^A.

- The parameters of the stochastic differential equation (SDE) of the hazard rate are listed in Table 4.

Table 4: Parameters of Hazard Rate

	h_0	κ	θ	σ_h
hazard rate	4%	1	4%	40%

Here, the initial value of the hazard rate $h_0 = 4\%$ is taken from the results of Hull - White [2005], which is regarded as the default probability about between Ba and Baa ratings.

- When the risk free rate r^p or r^q is stochastic, the parameters of the SDE are listed in Table 5.

Table 5: Parameters of risk free interest rate ($r^x = r^p$ or r^q)

	$\hat{r}_0^x$	κ_{r^x}	θ_{r^x}	σ_{r^x}	OIS c^x
risk free rate (r^p)	4%	1	4%	40%	1%
risk free rate (r^q)	4%	1	4%	40%	1%

- "0th", "1st" and "2nd" in the tables stand for the values of $V^{(0)}$, $V^{(1)}$ and $V^{(2)}$, respectively. "total" means the sum of 0th, 1st and 2nd values, that is the sum of $V^{(0)}$, $V^{(1)}$ and $V^{(2)}$ (Total $= V^{(0)} + V^{(1)} + V^{(2)}$).

5.3. Effects of Parameters

We investigate the effects of the changes in the parameters of the underlying factors with or without collateral.

5.3.1. No Collateral

Here, we change the parameters of the hazard rate h and the interest rate r^p for uncollateralized contracts.

First, we investigate the effects of the changes in the parameters of the hazard rate. The cases of the parameters of the hazard rate are listed in Table 6.

Table 6: Parameters of Hazard Rate

	h_0	κ	θ	σ_h
i	2%	1	2%	20%
ii	2%	1	2%	40%
iii	4%	1	4%	20%
iv	4%	1	4%	40%
v	6%	1	6%	20%
vi	6%	1	6%	40%

From the results in Hull - White [2005], the default probability 2% is the one for a rating between A and Baa, and 4% is for a rating between Baa and Ba, and 6% is for a rating between Ba and B. Here, in order to concentrate on the effects of the parameters of h, we assume that r^p is a constant as $r^p = c^p$. The results are listed in Table 7.

Table 7: Effects of Hazard Rate - no collateral -

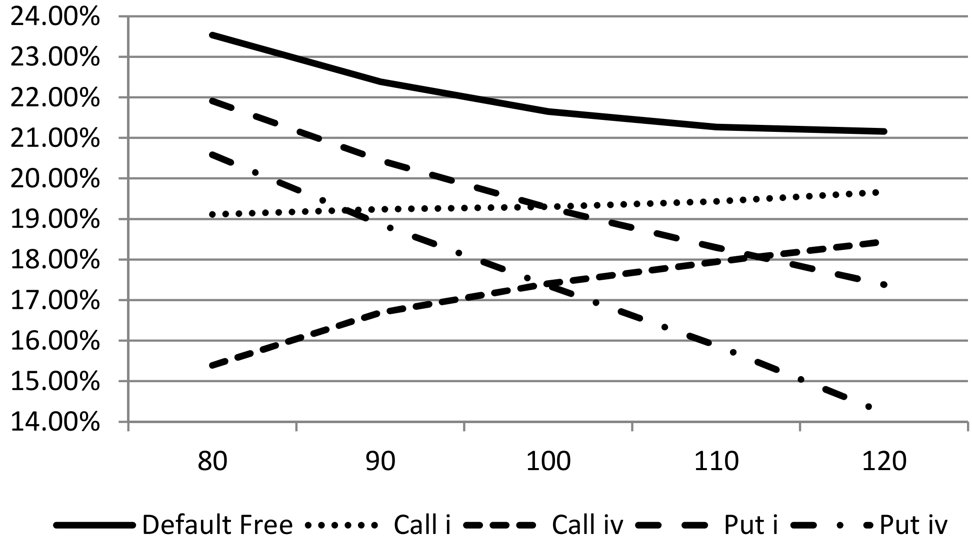
	i	ii	iii	iv	v	vi
0th	19.69	19.69	19.69	19.69	19.69	19.69
1st	-2.26	-2.54	-4.52	-4.62	-6.79	-6.81
2nd	0.17	0.31	0.60	0.82	1.31	1.61
Total	17.60	17.46	15.77	15.90	14.21	14.50

This result shows that the level of hazard rate (h_0 and θ) has a large impact on the 1st order value $V^{(1)}$. However, the effect of volatility of the hazard rate is small (see case i and ii, or iii and iv). In the cases of iii and iv, the volatility of the hazard rate has a larger impact on the 2nd order value $V^{(2)}$ than on the 1st order value $V^{(1)}$.

We also note that the sign of the 2nd order value is different from that of the 1st order value. This is because while $V^{(1)}$ is evaluated based on the default-free price, it should be based on the value including the default risk, and $V^{(2)}$ makes its adjustment.

In terms of call and put option's implied volatilities, these results are in Figure 3.

Figure 3: Implied volatilities of call and put option prices with no collateral posting



The low rating (high value of hazard rate) causes decrease in implied volatilities. Especially, this is the case for the deeper In-The-Money (ITM) options. In fact, for the call options the shapes of the skew curves have upward slopes, as opposed to that for the default-free case. Moreover, this results means that the put-call parity does not hold and that implied volatilities of call and put options with the same strike do not coincide.

We can understand it from the following observation: the losses become larger in the deeper ITMs when the counter party defaults, and the default probability is higher for the worse rating of the counter party. Consequently, the values of correction terms become larger. That is, the decreases in the implied volatilities become larger.

Next, we study the effects of interest rate r^p. The cases of parameters are listed in Table 8.

Table 8: Parameters of Interest Rate r^p

	r_0^p	κ	θ	σ_{r^p}	OIS
i	1%	1	1%	0%	1%
ii	1%	1	1%	20%	1%
iii	1%	1	1%	40%	1%
iv	4%	1	4%	20%	1%
v	4%	1	4%	40%	1%
vi	4%	1	4%	20%	4%
vii	4%	1	4%	40%	4%

The results are listed in Table 9.

Table 9: Effects of Interest Rate r^p - no collateral -

	i	ii	iii	iv	v	vi	vii
0th	19.69	19.69	19.69	19.69	19.69	16.45	16.45
1st	-4.63	-4.65	-5.05	-7.92	-8.03	-3.54	-3.61
2nd	0.82	0.85	1.01	2.00	2.23	0.65	0.80
Total	15.89	15.89	15.65	13.77	13.90	13.56	13.64

The effect of the collateral cost $y^p = r^p - c^p$ is the same as that of h because of the functional form f. Hence, the results are similar to that of h in no collateral case. Although the cases ii, iii and vi, vii have the same initial and long term collateral spreads ($r_0^p = \theta_{r^p} = c^p$), the difference of the interest rate level affects the level of the pre-default value. Moreover, when the initial value of the risk free rate is different from the collateral rate (see iv and v), the second order value $V^{(2)}$ affects more than 10% of the default free derivatives price. Thus, one needs to consider about the effects of interest rate, especially when the collateral cost y^p is not equal to 0.

5.3.2. Asset Collateral

Next, we investigate the cases of an asset collateralized contract.

First, we study the effects of parameters of the hazard rate for different volatilities of the asset collateral price. The parameters of the hazard rate are the same as in Table 6. We consider the cases that the time-lag of collateral posting is 0.25 years or 0.02 years. As in the no collateral case, to concentrate on the effects of the parameter changes of h, we assume that r^p and r^q are constants as $r^p = c^p$ and $r^q = c^q$.

The results are in Table 10 and 11.

Table 10: Effects of Hazard Rate - time-lag : 0.25 years -

	i	ii	iii	iv	v	vi
0th	19.69	19.69	19.69	19.69	19.69	19.69
1st	-0.44	-0.48	-0.87	-0.89	-1.31	-1.32
2nd	0.002	0.003	0.006	0.008	0.012	0.016
Total	19.26	19.21	18.82	18.81	18.39	18.39

Table 11: Effects of Hazard Rate - time-lag : 0.02 years -

	i	ii	iii	iv	v	vi
0th	19.69	19.69	19.69	19.69	19.69	19.69
1st	-0.11	-0.12	-0.21	-0.22	-0.32	-0.32
2nd	0.000	0.000	0.000	0.001	0.001	0.001
Total	19.59	19.57	19.48	19.48	19.38	19.38

The absolute values of $V^{(1)}$ and $V^{(2)}$ are smaller than those in the no collateral cases. The first order value $V^{(1)}$ is mainly changed by the initial value (h_0) and its long term value (θ) of the hazard rate, and the effect of the volatility on the hazard rate is small as in the no collateral case. The 2nd order value $V^{(2)}$ is very small, even if the volatility of collateral asset price is 50%. Especially, in the case of short time-lag ($\Delta = 0.02$), the effect of the 2nd order value is almost 0.

Next, we change parameters of the interest rate r^p or r^q for the asset collateralized contracts.

First, we study the effects of the interest rate r^p (r^q is a constant as $r^q = c^q$). The parameters are the same as in Table 8. The results are in Table 12.

Table 12: Effects of Interest Rate r^p - asset collateral, time-lag : 0.25 years -

	i	ii	iii	iv	v	vi	vii
0th	19.69	19.69	19.69	19.69	19.69	16.45	16.45
1st	-0.89	-0.92	-1.32	-4.29	-4.40	-0.68	-0.76
2nd	0.01	0.03	0.10	0.46	0.67	0.05	0.19
Total	18.81	18.80	18.47	15.86	15.97	15.82	15.88

The absolute values of $V^{(1)}$ and $V^{(2)}$ are smaller than those in the no collateral cases, as in the results of the effect of the hazard rate case. As opposed to the no collateral case, the impact of r^p is much larger than that of hazard rate h. Under the current assumption, h affects only $\left(V_t - V_{t-\Delta}\frac{A_t}{A_{t-\Delta}}\right)^+$, which is seen from the equation (53). On the other hand, r_t^p affects V_t itself. Thus, in the collateral posted case, the change in the y^p has the larger effect on the derivatives value.

As in the Table 8, when the level of r^p is different from c^p, the second order value $V^{(2)}$ has a large impact, even if the collateral is posted (see iv and v). Thus, we cannot ignore the second order value, especially when the risk free rate is different from the collateral rate.

The Table 13 shows the results of 0.02 years time-lag:

Table 13: Effects of Interest Rate r^p - asset collateral, time-lag : 0.02 years -

	i	ii	iii	iv	v	vi	vii
0th	19.69	19.69	19.69	19.69	19.69	16.45	16.45
1st	-0.22	-0.24	-0.65	-3.61	-3.72	-0.17	-0.25
2nd	0.00	0.02	0.09	0.39	0.61	0.05	0.19
Total	19.48	19.47	19.13	16.47	16.58	16.33	16.39

The short time lag makes the first order values $V^{(1)}$ decrease. However, the second order values $V^{(2)}$ are not so small. From these results, we need to consider the second order values in the cases of wide collateral spreads and the stochastic interest rate, even if the time lag of collateralization is short.

Next, we check the effects of another interest rate r^q based on the parameters listed in Table 14.

Table 14: Parameters of Interest Rate r^q

	r_0^q	κ	θ	σ_{r^q}	OIS
i	1%	1	1%	0%	1%
ii	1%	1	1%	20%	1%
iii	1%	1	1%	20%	1%
iv	4%	1	4%	20%	1%
v	4%	1	4%	40%	1%
vi	4%	1	4%	20%	4%
vii	4%	1	4%	40%	4%

The results are listed in Table 15.

Table 15: Effects of Interest Rate r^q - asset collateral -

	i	ii	iii	iv	v	vi	vii
0th	19.69	19.69	19.69	19.69	19.69	19.69	19.69
1st	-0.89	-0.86	-0.45	2.37	2.46	-0.89	-0.78
2nd	0.01	0.06	0.28	0.44	1.12	0.23	0.91
Total	18.81	18.89	19.53	22.50	23.28	19.04	19.83

It is observed that when the volatility becomes high, the first order value contributes to the plus side in the total value, as opposed to the effects of the changes in the other factors' volatilities, where the first order values contribute to the minus sides in the total values. The reason is that the sign of the term concerning with r^q is plus that is different from the sign of another term concerning with $h^{[2]}$ (see (58)). Thus, the first order value contributes to the plus side in the total value, when the r^q moves widely or $y^q > 0$. (This phenomenon is observed in the case of $y^p < 0$, because of the sign of y^p in (59).) However, in the cases of $y^q < 0$, the first order value moves conversely.

Moreover, as in the case of r^p in Table 12, the cases that risk free rate r^q is different from collateral rate c^q have a large impact, and we need to treat carefully these cases.

Finally, we study the effects of the parameters of the collateral asset price. The cases that we consider are listed in Table 16.

Table 16: Parameters of the Collateral Asset Price

Collateral Asset Price	A_0	μ_A	β	ν_0^A	σ_A
i	1	0	1	20%	0%
ii	1	0	1	20%	30%
iii	1	0	1	50%	0%
iv	1	0	1	50%	30%
iv	1	0	0	0%	0%

The risk free rates are set as a constant ($r^p = c^p$ and $r^q = c^q$). The results of collateral asset price are in Table 17.

Table 17: Effects of the collateral asset price

	i	ii	iii	iv	v
0th	19.69	19.69	19.69	19.69	19.69
1st	-0.75	-0.75	-0.89	-0.91	-0.71
2nd	0.001	0.002	0.008	0.011	0.000
Total	18.95	18.94	18.81	18.80	18.98

This result shows that the volatility on volatility of the collateral asset price does not have a large impact on the pre-default value. The effects of the volatility on the collateral asset price are also not so large.

5.4. Maturity Effect

In this subsection we investigate the effects of the option values of the differences on the contract maturity.

First, we assume that the collateral cost y^q is set as 0, while y^p follows a stochastic process, whose parameters are in Table 5. (The driver of the BSDE is given by (56) or (59).) Maturities of options are 2, 4, 6, 8 and 10 years.

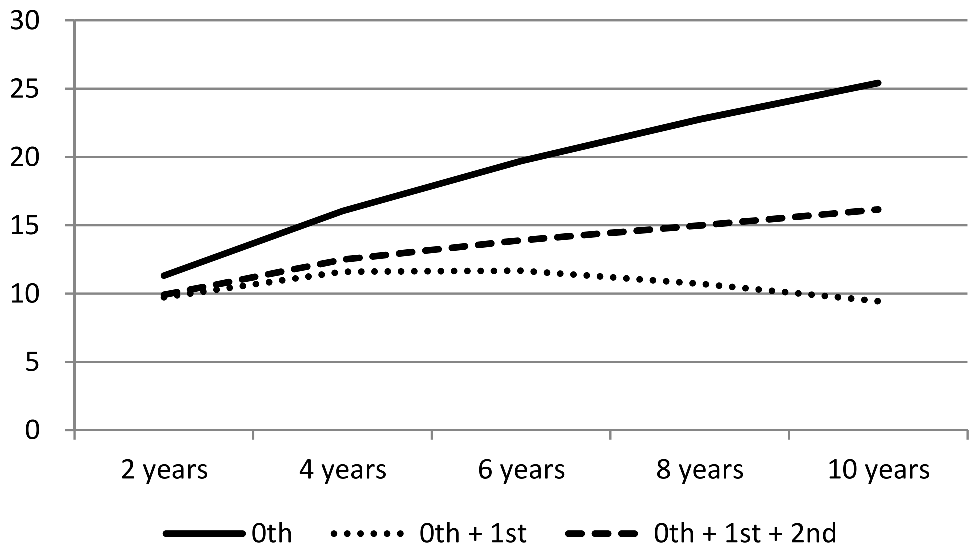
5.4.1. No Collateral

Table 18 and Figure 4 show the options prices without default risks: $V^{(0)}$ denoted by 0th, the traditional CVAs in practice $V^{(1)}$ denoted by 1st and the second order correction term $V^{(2)}$ denoted by 2nd.

Table 18: Call option prices with no collateral posting

	2 years	4 years	6 years	8 years	10 years
0th	11.32	16.06	19.69	22.75	25.42
0th + 1st	9.74	11.61	11.67	10.71	9.45
0th + 1st + 2nd	9.90	12.49	13.90	14.98	16.16

Figure 4: Call option prices with no collateral posting



This result shows that the second order value ($V^{(2)}$) largely affects the pre-default values in the long maturity such as 10 years. The effect of the second order value is increased when the maturity is longer: in the case of 2 years maturity, the second order effect ($V^{(2)}$) is less than 1% of the corresponding default-free price. However, in the case of the 10 years maturity, the second order value ($V^{(2)}$) affects more than 10%. This result also reveals that if the rating of the counter party is not good, the traditional CVA in practice could overestimate the adjustment for an option price.

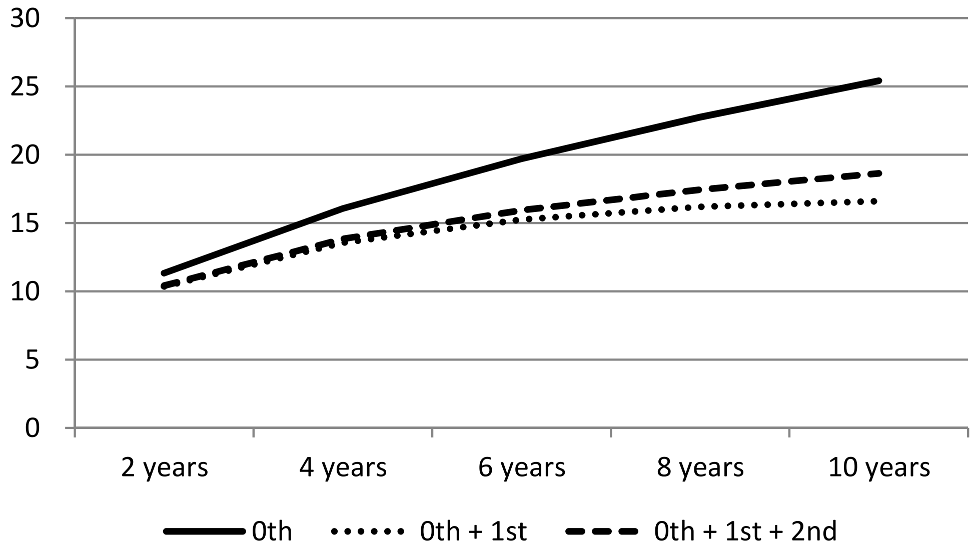
5.4.2. Asset Collateral

Next, we study the case of the asset collateral posting. Table 19 and Figure 5 show the results for the collateral posting case.

Table 19: Call option prices with collateral posting

	2 years	4 years	6 years	8 years	10 years
0th	11.32	16.06	19.69	22.75	25.42
0th + 1st	10.35	13.55	15.25	16.19	16.60
0th + 1st + 2nd	10.41	13.83	15.92	17.45	18.62

Figure 5: Call option prices with collateral posting



Even if the collateral is posted, the effect of the second order value ($V^{(2)}$) becomes important when there exists a non negligible collateral cost y^p.

6. Conclusion

We have studied the impacts of imperfect collateralization on forward and option values. In particular, we examine the cases of no collateral posting and collateral posting with time-lag. We also derive an approximation for the density function of the traditional CVA (Credit Value Adjustment) in the valuation of forward contract with bilateral counter party risk, which seems useful in evaluation of the CVA ' s VaR(Value t Risk). Moreover, we have considered the case that the collateral values depend not only on the underlying contract prices, but also on other asset values: for instance, currencies different from the payment currency or assets such as treasuries suffering from their own price fluctuations.

In the numerical experiments we have shown that in the uncollateralized cases, we should include higher order correction terms in our approximation method for the solutions to the pricing FBSDEs (forward backward stochastic differential equations). Particularly, for contracts with long maturities and low rating counter parties, the second order approximation term ($V^{(2)}$) should not be ignored.

In addition, under the existence of default risks we have shown that the put-call parity does not hold, and that implied volatilities of call and put options with the same strike do not coincide.

In the collateralized contract cases, when the time-lag of collateral posting is long, the collateral asset value is volatile or the interest rate is stochastic, our analyses have revealed that one needs to appropriately estimate those effects through the higher order correction terms in the approximate solution to the pricing FBSDE.

More realistically, it is necessary to analyze large-scale portfolios in financial institutions that consist of various types of financial assets and derivatives, which will be one of our next research topics.

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A. Derivation of (3)

For a derivatives contract whose payoff is Ψ at maturity $T(>0)$, we derive the pre-default value V_t $(\tau > t)$ from the investor($i = 1$)'s viewpoint, where $\tau > t$ means that both the investor and the counter party do not default until time t.

We assume that the regularity conditions necessary for the discussion are satisfied, and that V does not jump at default of the contract parties.

In addition, the discussion below can be applied to not only options, but also general derivatives including forward contracts.

We set the derivatives value as S_t, and define $S_t = 0$ on $\{\tau \leq t\}$. From the viewpoint of the investor $(i = 1)$, the payoffs at default of the investor and the counter party are expressed respectively as follows:

$$\eta_u^{[1]} := \left(S_u - l_u^{[1]}(S_u - \Gamma_u)^+\right)\mathbf{1}_{\{S_u<0\}} + \left(S_u + l_u^{[1]}(\Gamma_u - S_u)^+\right)\mathbf{1}_{\{S_u\geq 0\}}, \tag{61}$$

$$\eta_u^{[2]} := \left(S_u - l_u^{[2]}(S_u - \Gamma_u)^+\right)\mathbf{1}_{\{S_u\geq 0\}} + \left(S_u + l_u^{[2]}(\Gamma_u - S_u)^+\right)\mathbf{1}_{\{S_u<0\}}. \tag{62}$$

Thus, the value of derivatives at time $t(<\tau)$ is given as follows:

$$\begin{aligned} S_t &= e^{\int_0^t r_s^p ds} E^{Q^p}\Bigg[e^{-\int_0^T r_s^p ds}\Psi\mathbf{1}_{\{\tau>T\}} + \int_t^T e^{-\int_0^u r_s^p ds}(r_u^q - c_u^q)\Gamma_u\mathbf{1}_{\{\tau>u\}}du \\ &\quad + \int_t^T e^{-\int_0^u r_s^p ds}\eta_{u-}^{[1]}\mathbf{1}_{\{\tau>u-\}}dH_u^1 + \int_t^T e^{-\int_0^u r_s^p ds}\eta_{u-}^{[2]}\mathbf{1}_{\{\tau>u-\}}dH_u^2\Big|\mathcal{F}_t\Bigg], \end{aligned} \tag{63}$$

where the first term in the right hand side stands for the value of the derivatives payoff, and the second term represents the gain (in the case of a positive sign) or the loss (in the case of a negative sign), which is generated from the collateral posting. The third and fourth terms express the payoffs at default of the investor $(i = 1)$ and the counter party $(i = 2)$, respectively.

Hereafter, we consider the pre-default value V which satisfies the relation that $S_t = V_t\mathbf{1}_{\{\tau>t\}}$ at $\{\tau > t\}$.

First, the next equation holds by (63):

$$\begin{aligned} & S_t e^{-\int_0^t r_s^p ds} + \int_0^t e^{-\int_0^u r_s^p ds}(r_u^q - c_u^q)\Gamma_u\mathbf{1}_{\{\tau>u\}}du \\ & + \int_0^t e^{-\int_0^u r_s^p ds}\eta_{u-}^{[1]}\mathbf{1}_{\{\tau>u-\}}dH_u^1 + \int_0^t e^{-\int_0^u r_s^p ds}\eta_{u-}^{[2]}\mathbf{1}_{\{\tau>u-\}}dH_u^2 \\ = & E^{Q^p}\Bigg[e^{-\int_0^T r_s^p ds}\Psi\mathbf{1}_{\{\tau>T\}} + \int_0^T e^{-\int_0^u r_s^p ds}(r_u^q - c_u^q)\Gamma_u\mathbf{1}_{\{\tau>u\}}du \\ & + \int_0^T e^{-\int_0^u r_s^p ds}\eta_{u-}^{[1]}\mathbf{1}_{\{\tau>u-\}}dH_u^1 + \int_0^T e^{-\int_0^u r_s^p ds}\eta_{u-}^{[2]}\mathbf{1}_{\{\tau>u-\}}dH_u^2\Big|\mathcal{F}_t\Bigg] \\ = & M_t, \end{aligned} \tag{64}$$

where M_t is a Q^p-martingale. If we set $H_t^i = M_t^{[i]} + \int_0^t h_s^{[i]}\mathbf{1}_{\{\tau^i>s\}}ds$, then

$$\begin{aligned} dM_t &= e^{-\int_0^t r_s^p ds}dS_t - r_t^p e^{-\int_0^t r_s^p ds}S_t dt + e^{-\int_0^t r_s^p ds}(r_t^q - c_t^q)\Gamma_t\mathbf{1}_{\{\tau>t\}}dt \\ &\quad + e^{-\int_0^t r_s^p ds}\eta_{t-}^{[1]}\mathbf{1}_{\{\tau>t-\}}dH_t^1 + e^{-\int_0^t r_s^p ds}\eta_{t-}^{[2]}\mathbf{1}_{\{\tau>t-\}}dH_t^2 \\ &= e^{-\int_0^t r_s^p ds}dS_t - r_t^p e^{-\int_0^t r_s^p ds}S_t dt + e^{-\int_0^t r_s^p ds}(r_t^q - c_t^q)\Gamma_t\mathbf{1}_{\{\tau>t\}}dt \end{aligned}$$

$$
\begin{aligned}
&+e^{-\int_0^t r_s^p ds}\eta_{t-}^{[1]}\mathbf{1}_{\{\tau>t-\}}dM_t^{[1]} + e^{-\int_0^t r_s^p ds}\eta_{t-}^{[2]}\mathbf{1}_{\{\tau>t-\}}dM_t^{[2]} \\
&+e^{-\int_0^t r_s^p ds}\eta_t^{[1]}\mathbf{1}_{\{\tau>t\}}h_t^{[1]}dt + e^{-\int_0^t r_s^p ds}\eta_t^{[2]}\mathbf{1}_{\{\tau>t\}}h_t^{[2]}dt. \qquad (65)
\end{aligned}
$$

When we set $dm_t = e^{-\int_0^t r_s^p ds}dM_t + \eta_{t-}^{[1]}\mathbf{1}_{\{\tau>t-\}}dM_t^{[1]} + \eta_{t-}^{[2]}\mathbf{1}_{\{\tau>t-\}}dM_t^{[2]}$, m_t is also a Q^p-martingale, and we have

$$
dm_t = dS_t - r_t^p S_t dt + (r_t^q - c_t^q)\Gamma_t \mathbf{1}_{\{\tau>t\}}dt + \eta_t^{[1]}\mathbf{1}_{\{\tau>t\}}h_t^{[1]}dt + \eta_t^{[2]}\mathbf{1}_{\{\tau>t\}}h_t^{[2]}dt. \qquad (66)
$$

Since $S_t = 0$ on $\{\tau \le t\}$, it holds that

$$
\begin{aligned}
dS_t &= \mathbf{1}_{\{\tau>t\}}\Big(r_t^p S_t dt - (r_t^q - c_t^q)\Gamma_t dt - \eta_t^{[1]}h_t^{[1]}dt - \eta_t^{[2]}h_t^{[2]}dt\Big) + dm_t \\
&= \mathbf{1}_{\{\tau>t\}}\Big(r_t^p S_t dt - (r_t^q - c_t^q)\Gamma_t dt - S_t h_t dt \\
&\quad + \Big(l_t^{[1]}(S_t - \Gamma_t)^+ \mathbf{1}_{\{S_t<0\}} - l_t^{[1]}(\Gamma_t - S_t)^+\mathbf{1}_{\{S_t\ge 0\}}\Big)h_t^{[1]}dt \\
&\quad + \Big(l_t^{[2]}(S_t - \Gamma_t)^+ \mathbf{1}_{\{S_t\ge 0\}} - l_t^{[2]}(\Gamma_t - S_t)^+\mathbf{1}_{\{S_t<0\}}\Big)h_t^{[2]}dt\Big) + dm_t, \qquad (67)
\end{aligned}
$$

On the other hand, if we define

$$
\begin{aligned}
V_t &:= E^{Q^p}\left[e^{-\int_t^T r_u^p du}\Psi + \int_t^T e^{-\int_t^u r_s^p ds}\varpi_u du \Big| \mathcal{F}_t\right], \qquad (68) \\
\varpi_u &:= (r_u^q - c_u^q)\Gamma_u + \vartheta_u^{[1]}h_u^{[1]} + \vartheta_u^{[2]}h_u^{[2]}, \qquad (69) \\
\vartheta_u^{[1]} &:= -\Big(l_u^{[1]}(V_u - \Gamma_u)^+\Big)\mathbf{1}_{\{V_u<0\}} + \Big(l_u^{[1]}(\Gamma_u - V_u)^+\Big)\mathbf{1}_{\{V_u\ge 0\}}, \qquad (70) \\
\vartheta_u^{[2]} &:= -\Big(l_u^{[2]}(V_u - \Gamma_u)^+\Big)\mathbf{1}_{\{V_u\ge 0\}} + \Big(l_u^{[2]}(\Gamma_u - V_u)^+\Big)\mathbf{1}_{\{V_u<0\}}, \qquad (71)
\end{aligned}
$$

and then, the similar argument derives

$$
\begin{aligned}
e^{-\int_0^t r_u^p du}V_t + \int_0^t e^{-\int_0^u r_s^p ds}\varpi_u du &= E^{Q^p}\left[e^{-\int_0^T r_u^p du}\Psi + \int_0^T e^{-\int_0^u r_s^p ds}\varpi_u du \Big| \mathcal{F}_t\right] \\
&= \tilde{M}_t, \qquad (72)
\end{aligned}
$$

where $\tilde{M}_t$ is a Q^p-martingale. Since $\tilde{m}_t$ is also a Q-martingale where $d\tilde{m}_t = e^{\int_0^t r_u^p du}d\tilde{M}_t$, it holds that

$$
dV_t = (r_t^p V_t - \varpi_t)\,dt + d\tilde{m}_t. \qquad (73)
$$

Since the assumption that V does not jump at default ($\Delta V_\tau := V_\tau - V_{\tau-} = 0$), it holds that

$$
\begin{aligned}
d(V_t\mathbf{1}_{\{\tau>t\}}) &= \mathbf{1}_{\{\tau>t\}}dV_t - V_{t-}dH_t - \Delta V_\tau \Delta H_\tau \\
&= \mathbf{1}_{\{\tau>t\}}(r_t^p V_t - \varpi_t)\,dt - V_t h_t \mathbf{1}_{\{\tau>t\}}dt - \Delta V_\tau \Delta H_\tau + d\tilde{n}_t \\
&= \mathbf{1}_{\{\tau>t\}}\Big(r_t^p V_t dt - (r_t^q - c_t^q)\Gamma_t dt - V_t h_t dt \\
&\quad + \Big(l_t^{[1]}(V_t - \Gamma_t)^+\mathbf{1}_{\{V_t<0\}} - l_t^{[1]}(\Gamma_t - V_t)^+\mathbf{1}_{\{V_t\ge 0\}}\Big)h_t^{[1]}dt \\
&\quad + \Big(l_t^{[2]}(V_t - \Gamma_t)^+\mathbf{1}_{\{V_t\ge 0\}} - l_t^{[2]}(\Gamma_t - V_t)^+\mathbf{1}_{\{V_t<0\}}\Big)h_t^{[2]}dt\Big) + d\tilde{n}_t, \qquad (74)
\end{aligned}
$$

where $d\tilde{n}_t = \mathbf{1}_{\{\tau>t\}}\Big(d\tilde{m}_t + V_{t-}(dM_t^{[1]} + dM_t^{[2]})\Big)$. Both of the drivers of the BSDEs in (67) and (74) are the same, and the boundary conditions are also the same because of $S_T = \mathbf{1}_{\{\tau>T\}}V_T = \mathbf{1}_{\{\tau>T\}}\Psi$. Thus, we can regard the solution of the BSDE S_t as that of $\mathbf{1}_{\{\tau>t\}}V_t$.

Finally, we note that when the derivatives is an option contract, (3) is obtained since $V_t \ge 0$.

B. Interacting Particle Method

As it is still a tough task to approximate FBSDEs more than the first order values analytically, we make use of a Monte Carlo method based on a so-called *interacting particle technique*.

Hereafter, we assume that both parties post their collaterals perfectly with the constant time-lag Δ (i.e. $\Gamma_t = V_{t-\Delta}$), and the notations are the same as those in Section 3.

To calculate the values of $V^{(1)}$ and $V^{(2)}$, we apply interacting particle method. $V^{(1)}$ is expressed as follows:

$$V_t^{(1)} = \mathbf{1}_{\{\tau>t\}}\mathbf{E}\left[\mathbf{1}_{\{\tau<T\}}\hat{f}_t\left(X_\tau, V_\tau^{(0)}, V_{\tau-\Delta}^{(0)}\right)|\mathcal{F}_t\right], \tag{75}$$

where τ represents an interaction time, which is drawn independently from a Poisson distribution with an arbitrary deterministic positive intensity λ, and $\hat{f}$ is defined as follows:

$$\hat{f}_t(X_s, V_s^{(0)}, V_{s-\Delta}^{(0)}) = \frac{1}{\lambda}e^{\lambda(s-t)}e^{-c^p(s-t)}f(s, X_s, V_s^{(0)}, V_{s-\Delta}^{(0)}). \tag{76}$$

The above equations can be understood by intuition. That is, firstly using (17) with $\Gamma_t^{(0)} = V_{t-\Delta}^{(0)}$, we derive a stochastic differential equation which is satisfied by $\hat{V}_{t,s}^{(1)} := e^{\lambda(s-t)}V_s^{(1)}$. Then, noting $V_t^{(1)} = \hat{V}_{t,t}^{(1)}$, we obtain the following equation:

$$V_t^{(1)} = \mathbf{E}\left[\int_t^T e^{-\lambda(u-t)}\lambda\hat{f}_t\left(X_u, V_u^{(0)}, V_{u-\Delta}^{(0)}\right)du\middle|\mathcal{F}_t\right]. \tag{77}$$

We remark that by standard results of credit risk models (e.g. Bielecki-Rutkowski [2000]), the right hand side in this equation is known as the present value of a contract whose maturity T, hazard rate λ and payment $\hat{f}_t\left(X_u, V_u^{(0)}, V_{u-\Delta}^{(0)}\right)$ at a default time $u(>t)$. Consequently, we express $V_t^{(1)}$ as (75).

Similarly, the expression of $V^{(2)}$ is given as

$$\begin{aligned} V_t^{(2)} = {} & \mathbf{1}_{\{\tau_1>t\}}\mathbf{E}\Big[\mathbf{1}_{\{\tau_1<T\}}V_{\tau_1}^{(1)}\partial_{v_{\tau_1}}\hat{f}_t\left(X_{\tau_1}, V_{\tau_1}^{(0)}, V_{\tau_1-\Delta}^{(0)}\right) \\ & +\mathbf{1}_{\{\tau_1<T\}}V_{\tau_1-\Delta}^{(1)}\partial_{v_{\tau_1-\Delta}}\hat{f}_t\left(X_{\tau_1}, V_{\tau_1}^{(0)}, V_{\tau_1-\Delta}^{(0)}\right)|\mathcal{F}_t\Big]. \end{aligned} \tag{78}$$

Moreover, using tower property for conditional expectations, $V^{(2)}$ is expressed as follows:

$$\begin{aligned} V_t^{(2)} = {} & \mathbf{1}_{\{\tau_1>t\}}\mathbf{E}\Big[\mathbf{1}_{\{\tau_1<\tau_2<T\}}\hat{f}_{\tau_1}\left(X_{\tau_2}, V_{\tau_2}^{(0)}, V_{\tau_2-\Delta}^{(0)}\right)\partial_{v_{\tau_1}}\hat{f}_t\left(X_{\tau_1}, V_{\tau_1}^{(0)}, V_{\tau_1-\Delta}^{(0)}\right) \\ & +\mathbf{1}_{\{\tau_1<\tau_2<T\}}\hat{f}_{\tau_1}\left(X_{\tau_2}, V_{\tau_2}^{(0)}, V_{\tau_2-\Delta}^{(0)}\right)\partial_{v_{\tau_1-\Delta}}\hat{f}_t\left(X_{\tau_1}, V_{\tau_1}^{(0)}, V_{\tau_1-\Delta}^{(0)}\right)\Big|\mathcal{F}_t\Big] \\ = {} & \mathbf{1}_{\{\tau_1>t\}}\mathbf{E}\Big[\mathbf{1}_{\{\tau_1<\tau_2<T\}}\hat{f}_{\tau_1}\left(X_{\tau_2}, V_{\tau_2}^{(0)}, V_{\tau_2-\Delta}^{(0)}\right) \\ & \times\frac{1}{\lambda}e^{\lambda(\tau_1-t)}e^{-c^p(\tau_1-t)}\left(-(r_{\tau_1}^p - c_{\tau_1}^p) - h_{\tau_1}^{[1]}l_{\tau_1}^{[1]}\mathbf{1}_{\{V_{\tau_1-\Delta}^{(0)}>V_{\tau_1}^{(0)}\}} - h_{\tau_1}^{[2]}l_{\tau_1}^{[2]}\mathbf{1}_{\{V_{\tau_1-\Delta}^{(0)}<V_{\tau_1}^{(0)}\}}\right) \\ & +\mathbf{1}_{\{\tau_1<\tau_2<T\}}\hat{f}_{\tau_1}\left(X_{\tau_2}, V_{\tau_2}^{(0)}, V_{\tau_2-\Delta}^{(0)}\right) \\ & \times\frac{1}{\lambda}e^{\lambda(\tau_1-t)}e^{-c^p(\tau_1-t)}\left((r_{\tau_1}^q - c_{\tau_1}^q) + h_{\tau_1}^{[1]}l_{\tau_1}^{[1]}\mathbf{1}_{\{V_{\tau_1-\Delta}^{(0)}>V_{\tau_1}^{(0)}\}} + h_{\tau_1}^{[2]}l_{\tau_1}^{[2]}\mathbf{1}_{\{V_{\tau_1-\Delta}^{(0)}<V_{\tau_1}^{(0)}\}}\right)\Big|\mathcal{F}_t\Big]. \end{aligned} \tag{79}$$

In the cases that $\tau < \Delta$, we define $V_{\tau-\Delta}$ and the second term of right hand side of (79) as 0.

Based on above preparations, we summarize a procedure to calculate $V^{(i)}$, $i = 1, 2$ by a Monte Carlo method. We set the number of discretization of $[0, T]$ as N, this is, reference times are $\{0, \frac{T}{N}, \frac{2T}{N}, \cdots, T\}$, and set the number of simulation as M. Then, the procedure of calculation is follows.

1. In order to get the stopping time τ, we generate uniform random numbers $(u_i,\ i = 1, \cdots, N)$ corresponding to the reference times $\{0, \frac{T}{N}, \frac{2T}{N}, \cdots, T\}$.

   Then, using the first i which satisfies $1 - e^{-\lambda \frac{T}{N}} > u_i$, we set $\tau_1 = \frac{Ti}{N}$.

   Next, using the first j which is larger than i $(i < j)$ satisfying $1 - e^{-\lambda \frac{T}{N}} > u_j$, we set $\tau_2 = \frac{Tj}{N}$.

   If $0 < \tau_1 < T$ for $V^{(1)}$ $(\tau_1 < \tau_2 < T$ for $V^{(2)})$, we proceed to Step 2 below.

2. We compute the realized values of diffusion processes of the underlying asset prices, hazard rates and a collateral asset value until τ_1 (τ_2) by Monte Carlo simulations[4].

3. Using values $X_{\tau_1}, V^{(0)}_{\tau_1}, V^{(0)}_{\tau_1-\Delta}$ at τ_1 (τ_2) and $X_{\tau_2}, V^{(0)}_{\tau_2}, V^{(0)}_{\tau_2-\Delta}$ at τ_2, we calculate

$$\hat{f}_0 \left(X_{\tau_1}, V^{(0)}_{\tau_1}, V^{(0)}_{\tau_1-\Delta} \right), \tag{80}$$

$$\begin{aligned}
&\hat{f}_{\tau_1} \left(X_{\tau_2}, V^{(0)}_{\tau_2}, V^{(0)}_{\tau_2-\Delta} \right) \\
&\times \frac{1}{\lambda} e^{\lambda(\tau_1 - t)} e^{-c^p(\tau_1 - t)} \left(-(r^p_{\tau_1} - c^p_{\tau_1}) - h^{[1]}_{\tau_1} l^{[1]}_{\tau_1} \mathbf{1}_{\{V^{(0)}_{\tau_1-\Delta} > V^{(0)}_{\tau_1}\}} - h^{[2]}_{\tau_1} l^{[2]}_{\tau_1} \mathbf{1}_{\{V^{(0)}_{\tau_1-\Delta} < V^{(0)}_{\tau_1}\}} \right) \\
&+ \hat{f}_{\tau_1} \left(X_{\tau_2}, V^{(0)}_{\tau_2}, V^{(0)}_{\tau_2-\Delta} \right) \\
&\times \frac{1}{\lambda} e^{\lambda(\tau_1 - t)} e^{-c^p(\tau_1 - t)} \left((r^q_{\tau_1} - c^q_{\tau_1}) + h^{[1]}_{\tau_1} l^{[1]}_{\tau_1} \mathbf{1}_{\{V^{(0)}_{\tau_1-\Delta} > V^{(0)}_{\tau_1}\}} + h^{[2]}_{\tau_1} l^{[2]}_{\tau_1} \mathbf{1}_{\{V^{(0)}_{\tau_1-\Delta} < V^{(0)}_{\tau_1}\}} \right),
\end{aligned} \tag{81}$$

   and store these values.

4. Reiterate M times from Step 1 to Step 3, and take the average.

Finally, we note that in our models, as the driver f of the BSDE (5)-(7) does not depend on the volatility Z of the BSDE, we do not need to calculate Z.

[4]When the value of diffusion process is smaller than 0, we set a value as 0.