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Abstract

We formulate strategic aspects of speculative arbitrageurs in a stock market as a generalization of timing game with behavioral types explored by Matsushima (2013b). A company raises huge funds during the bubble driven by positive feedback traders' euphoria by issuing shares in a socially harmful manner. The arbitrageurs borrow money from positive feedback traders under a regulation on leverage ratio and purchase credit default swaps defined as bubble-contingent claim from them. We demonstrate a theoretical ground for considering the availability of credit default swap associated with a high leverage ratio as a powerful policy method to deter harmful bubbles.

Keywords: Harmful Bubbles, Awareness Heterogeneity, Timing Games with Behavioral Types, Leverage, Credit Default Swap.
JEL Classification Numbers: C720, C730, D820, G140.

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1. Introduction

The **e**mergence and long persistence of the bubble in a market for a company's stock would be socially harmful, because even if the company is unproductive, it can raise huge funds by issuing new shares during the bubble. The policy maker generally cannot identify whether the booming company is unproductive. Hence, it is important to answer the question about how the policy maker deters such harmful bubbles without this identification.

This paper shows that the availability of credit default swap (in short CDS) could be a powerful policy method for deterring harmful bubbles. We define CDS as a financial instrument for bubble-contingent claim such that the seller of a CDS pays a promised monetary amount to its purchaser if and only if the bubble crashes. The purchaser of a CDS can receive this payment irrespective of whether he (or she) has underlying assets that are defaulted because of the bubble's crash.[1]

This paper assumes that a seller of a CDS is required to hold the full reserve for the payment of this CDS to the purchaser, and that the payment of a CDS is utilized for the purchaser's debt obligation. Based on these assumptions, we show that the availability of CDS to trade can deter bubbles if the total payment of CDS grows less rapidly than the market value of the personal capital for the purchasers of CDS, i.e., the arbitrageurs who pursue speculative benefits.

If an arbitrageur purchases no CDS and fails to sell his shareholding before the bubble's crash, he (or she) is exempted from his debt obligation because of the non-recourse nature of debt contracts. On the other hand, if he purchases CDSs and fails to sell before the crash, he is not exempted from his entire debt obligation; he has to utilize the payment of the CDSs for paying off his debt obligation. This makes the instantaneous gain from timing the market greater, i.e., incentivizes any arbitrageur to time the market earlier, when CDS is available to trade than when CDS is not available.

[1] We should distinguish CDS from 'covered' CDS the purchaser of which needs to have underlying default assets for receiving the payment. The definition of CDS in this paper corresponds to so-called 'naked' CDS. See Fostel and Geanakoplos (2012). We should also distinguish CDS from insurance contract against own default risks. See Section 3.

We assume, in an implicit manner, that, besides these arbitrageurs, there are many positive feedback traders (noise traders) who have a plenty of personal capital but are slaves of euphoria; they misperceive the fundamental value of the company and reinforce their misperception through time as long as the market share price matches their misperception. It is important to assume that the positive feedback traders are unaware of their own euphoria; they neither expect the share price to increase nor slump even if they actually, unconsciously, reinforce their misperception through time. In contrast, the arbitrageurs are well aware of the positive feedback traders' euphoria and expect future speculative benefits as well as the bubble's crash risk.

Because of such awareness heterogeneity, the arbitrageurs can borrow money from the positive feedback traders *with no premium* for purchasing the newly issued shares. The arbitrageurs can also purchase CDSs from the positive feedback traders *with no premium*.

In order to deter harmful bubbles, the policy maker must determine whether CDS is available to trade and the upper limit of leverage ratio that regulates the arbitrageurs' borrowing activities. We show that, in a wide class of environments, irrespective of whether the company is unproductive, it is the best policy determination that CDS is made available and the regulation on leverage ratio is weakened.

With a high leverage ratio (a weak regulation), the market value of the positive feedback trader's shareholdings and their loan are sufficient, crowding out the reserve for the total payment of CDS. This decreases the arbitrageurs' relative future benefits from riding the bubble to the instantaneous gain from timing the market, and therefore, can incentivize the arbitrageurs to time the market at early times.

Based on these observations, we show that with a sufficient leverage ratio, the total payment of CDS grows less rapidly than the total personal capital of the positive feedback traders. Hence, the bubble is less likely to emerge and persist for a long time when CDS is available than when CDS is not available. We further show that the higher the leverage ratio is, the less likely the bubble is to emerge and persist for a long time. Moreover, provided the growth rate of the positive feedback traders' personal capital is sufficient, the expected *social cost*, i.e., the expected amount of the raised funds, induced by the bubble decreases as the leverage ratio increases. These are in contrast with the case that CDS is not available. In this case, the higher the leverage ratio is, the more likely the bubble is to emerge and persist for a long time; the greater the expected social cost induced by the bubble is.

In order to describe strategic aspects in the stock market, we formulate a *timing game with behavioral types* as a generalization of Matsushima (2013b). There are multiple arbitrageurs as the players of this game, who decide whether to ride the bubble or to time the market at any time during the bubble in a bounded time interval $[0,1]$. The arbitrageur who times the market at the earliest wins the game and obtains the winner payoff, which is greater than the loser payoff. The winner payoff is increasing through time. The important assumption is that each arbitrageur is almost certainly rational, but, with a small but positive probability, he is behavioral in the sense that he never time the market on his own accord, i.e., he is committed to ride the bubble. Based on this assumption, this paper shows a non-trivial necessary and sufficient condition for the existence of Nash equilibrium, namely the bubble-crash equilibrium, according to which, any arbitrageur never time the market at the initial time, i.e., the bubble emerges as a Nash equilibrium outcome. This paper also shows that almost the same condition as the above guarantees that the bubble-crash strategy profile is the unique Nash equilibrium. We further show a necessary and sufficient condition for the existence of a Nash equilibrium namely the no-bubble equilibrium, according to which, any rational arbitrageur certainly times the market at the initial time.

Based on these characterization results, we argue that the greater the relative future benefits for each arbitrageur are, the more likely the bubble is to emerge and persist for a long time, and that their relative future benefits crucially depend on whether CDS is available and on the stock market environments such as the leverage ratio cap and the degree of the positive feedback traders' enthusiasm.

The findings of this paper can be considered theoretical contributions to the limited arbitrage literature (De Long et al. (1990), Shleifer and Vishny (1992), Abreu and Brunnermeier (2003), Matsushima (2013b), and others), where the interaction between rational arbitrageurs and positive feedback traders were intensively studied.

Abreu and Brunnermeier (2003) formulated a stock market as a timing game among arbitrageurs, where a particular aspect of informational asymmetry namely sequential awareness was assumed. Matsushima (2013b) demonstrated an alternative model of timing game by replacing sequential awareness with behavioral types.

The main departure of this paper from Matsushima (2013b) and the other works is to permit the company to raise funds during the bubble, permit arbitrageurs to make debt contracts with positive feedback traders, and permit arbitrageurs to purchase CDS from positive feedback traders. With these permissions, the awareness heterogeneity needs to be explicitly treated.

Besides the limited arbitrage literature, there are various theoretical approaches for understanding the phenomenon of bubbles and crashes, such as overlapping generation models (Tirole (1985), Martin and Ventura (2012), Hirano and Yanagawa (2010), and others) and prior heterogeneity (Harrison and Kreps (1978), Simsek (2012), Maekawa (2013), and others).[2]

Fostel and Geanakoplos (2012)) studied prior heterogeneity where traders have different beliefs about future price movement even if they share the same information.[3] Fostel and Geanakoplos argued that unexpected introduction of CDS increases default risks. This paper does not assume prior heterogeneity but awareness heterogeneity. In contrast with prior heterogeneity, arbitrageurs have the option to solve awareness heterogeneity. Moreover, with prior heterogeneity, traders disagree only in terms of default risk, while, with awareness heterogeneity, they disagree in terms of not only default risk but also share price growth.

In summary, this paper demonstrates a theoretical ground for considering credit default swap as a powerful policy method to deter harmful bubbles. The organization of this paper is as follows. Section 2 demonstrates a formulation for timing game with behavioral types. Section 3 introduces the basic model, where CDS is not available, and the CDS model, where CDS is available. Section 4 explains the fine details of the stock market formulation. Sections 5 and 6 make further investigations about the basic model and the CDS model. Section 7 considers the social cost induced by the bubble. Section 8 concludes.

[2] For a general survey about bubbles and crashes, see Brunnermeier and Oehmke (2013).
[3] See also Geanakoplos (2010) and Che and Sethi (2010).

2. Timing Games with Behavioral Types

Fix a finite set of arbitrageurs (players) $N=\{1,2,...,n\}$, where $n\geq 2$. We define a *timing game with behavioral types* as follows. Let $A_i=[0,1]$ denote the set of all pure strategies for each arbitrageur $i\in N$. By selecting $a_i\in A_i$, arbitrageur i plans to time the market at time a_i during a bounded time interval $[0,1]$. A mixed strategy, in short a strategy, for arbitrageur i is defined as a cumulative distribution $q_i : A_i \to R_+ \bigcup \{0\}$, where $q_i(t)$ implies the probability that arbitrageur i times the market at or before time t, which is non-decreasing and right-continuous in t and satisfies $q_i(1)=1$. Let us denote by Q_i the set of all strategies for arbitrageur i. Let $Q=\underset{i\in N}{\times} Q_i$ and $q=(q_i)_{i\in N}\in Q$. We write $q_i=a_i$ if arbitrageur i selects pure strategy a_i with certainty.

Let us fix an arbitrary real number $\varepsilon\in(0,1)$. We assume that each arbitrageur is rational with a probability $1-\varepsilon>0$, while he (or she) is *behavioral* with a probability $\varepsilon>0$. If an arbitrageur is rational, he will conform to his selected strategy q_i. If he is behavioral, he will not conform to q_i and instead never time the market on his own accord. Whether each arbitrageur is rational or behavioral is determined independently, and is unknown to the other arbitrageurs.

Consider an arbitrary pure strategy profile $a=(a_i)_{i\in N}\in A\equiv \underset{i\in N}{\times} A_i$, and an arbitrary non-empty subset of arbitrageurs $H\subset N$. Suppose that any arbitrageur in H is rational, while any arbitrageur in $N\setminus H$ is behavioral. We denote by $\tau=\min_{j\in H} a_j$ the earliest time at which a rational arbitrageur selects, i.e., the time at which the timing game ends, i.e., the bubble crashes. We denote by $l=\left|\{j\in H \mid a_j=\tau\}\right|$ the number of rational arbitrageurs who select this ending time τ.

With a probability $\frac{1}{l}$, each rational arbitrageur $i\in H$, who selects τ, becomes the winner of the timing game, and earns the *winner payoff* denoted by $\bar{v}_i(\tau)$. We assume that $\overline{v}_i(\tau)$ is differentiable in τ. With regard to the remaining probability $\frac{l-1}{l}$, he loses the timing game, and earns the *loser payoff* denoted by $\underline{v}_i(\tau)$. Any arbitrageur who does not select τ loses the timing game. Hence, the expected payoff for any rational arbitrageur $i \in H$ can be given by

$$v_i(H,a) = \frac{1}{l}\overline{v}_i(\tau) + \frac{l-1}{l}\underline{v}_i(\tau) \qquad \text{if } a_i = \tau,$$

and

$$v_i(H,a) \equiv \underline{v}_i(\tau) \qquad \text{if } a_i > \tau.$$

We assume that the winner payoff is greater than the loser payoff, and the winner payoff is non-decreasing:

$$\overline{v}_i(t) > \underline{v}_i(t) \quad \text{and} \quad \overline{v}_i'(t) \equiv \frac{\partial \overline{v}_i(t)}{\partial t} \geq 0.$$

We define the payoff function $u_i(\cdot,\varepsilon): Q \to R$ for each arbitrageur $i \in N$ as the expected value of $v_i(H,a)$ when he is rational, which is expressed as

(1) $$u_i(q,\varepsilon) \equiv E[\sum_{H \subset N: i \in H} v_i(H,a) \mid q,\varepsilon].$$ [4]

A strategy profile $q \in Q$ is said to be a *Nash equilibrium* associated with ε if

$$u_i(q,\varepsilon) \geq u_i(q_i', q_{-i}, \varepsilon) \quad \text{for all } i \in N \text{ and all } q_i' \in Q_i.$$

We define the probability that the timing game ends, i.e., the bubble crashes, at or before $t \in [0,1]$ as

$$D(t;q,\varepsilon) \equiv 1 - \prod_{i \in N} \{1 - (1-\varepsilon) q_i(t)\}.$$

In case $D(t;q,\varepsilon)$ is differentiable in t, we can define the *hazard rate* of the timing game's end at $t \in [0,1]$ as

$$\theta(t) \equiv \frac{D'(t;q,\varepsilon)}{1 - D(t;q,\varepsilon)},$$

[4] $E[\cdot \mid q,\varepsilon]$ denotes the expectation operator conditional on (q,ε).

where $D'(t:q,\varepsilon) \equiv \dfrac{\partial D(t:q,\varepsilon)}{\partial t}$. For each $i \in N$, we define the probability that the timing game ends at or before time t, provided arbitrageur i has never timed the market before, by

$$D_i(t;q_{-i},\varepsilon) = 1 - \prod_{j \in N \backslash \{i\}} \{1-(1-\varepsilon)q_j(t)\}.$$

In case $D_i(t;q_{-i},\varepsilon)$ is continuous in t, we can rewrite (1) for $q_i = t$ as

$$u_i(t,q_{-i},\varepsilon) = \int_{\tau=0}^{t} \underline{v}_i(\tau) dD_i(\tau : q_{-i},\varepsilon) + \overline{v}_i(t)\{1 - D_i(t:q_{-i},\varepsilon)\}.$$

Hence, the first-order condition for Nash equilibrium is given by

(2) $$\{\overline{v}_i(t) - \underline{v}_i(t)\} D_i'(t:q_{-i},\varepsilon) = \overline{v}_i'(t)\{1 - D_i(t:q_{-i},\varepsilon)\},$$

where $D_i'(t:q_{-i},\varepsilon) \equiv \dfrac{\partial D_i(t:q_{-i},\varepsilon)}{\partial t}$.

Throughout this paper, we assume that the timing game is *symmetric*:

$$\overline{v}_1(t) = \overline{v}_i(t) \text{ and } \underline{v}_1(t) = \underline{v}_i(t) \text{ for all } i \in N \text{ and all } t \in [0,1].$$

We define the *relative future benefit* at time t as

$$R(t) \equiv \frac{\overline{v}_1'(t)}{\overline{v}_1(t) - \underline{v}_1(t)}.$$

A strategy profile $q \in Q$ is said to be *symmetric* if

$$q_i = q_1 \text{ for all } i \in N.$$

Note that if a strategy profile $q \in Q$ is symmetric, then

(3) $$\theta(t) = \frac{n(1-\varepsilon)q_1'(t)}{1-(1-\varepsilon)q_1(t)},$$

and, from (2),

(4) $$\theta(t) = \frac{n}{n-1} R(t).$$

Hence, the hazard rate $\theta(t)$ is proportional to the relative future benefit $R_1(t)$.

We specify a symmetric and continuous strategy profile $\tilde{q} = \tilde{q}(\varepsilon) \in Q$, namely the *bubble-crash strategy profile*, as follows:

(5) $$\tilde{q}_1(t) = \frac{1 - \{1-(1-\varepsilon)\tilde{q}_1(\tilde{\tau})\}\exp[-\frac{1}{n-1}\int_{\tau=\tilde{\tau}}^{t} R(\tau)d\tau]}{1-\varepsilon} \text{ for all } t \in [\tilde{\tau},1],$$

and

$$\tilde{q}_1(t) = 0 \quad \text{for all} \quad t \in [0, \tilde{\tau}),$$

where we name $\tilde{\tau} = \tilde{\tau}(\varepsilon) \in [0,1)$ the *critical time*, which is uniquely defined as either

(6) $$\tilde{\tau} \geq 0 \quad \text{and} \quad \varepsilon = \exp[-\frac{1}{n-1}\int_{\tau=\tilde{\tau}}^{1} R(\tau)d\tau],$$

or

$$\tilde{\tau} = 0 \quad \text{and} \quad \varepsilon = \{1-(1-\varepsilon)\tilde{q}_1(0)\}\exp[-\frac{1}{n-1}\int_{\tau=0}^{1} R(\tau)d\tau].$$

According to the bubble-crash strategy profile $\tilde{q}$, no arbitrageur times the market before the critical time $\tilde{\tau}$. After the critical time $\tilde{\tau}$, the timing game randomly ends, i.e., the bubble crashes randomly, according to the hazard rate given by $\theta(t) = \frac{n}{n-1}R(t)$. It must be noted that the greater the relative future benefit $R(\tau)$ is, the greater the critical time $\tilde{\tau}$ and the hazard rate $\theta(t)$ are, i.e., the more likely the bubble is to persist for a long time. Note that $q = \tilde{q}$ satisfies the first order condition for Nash equilibrium (2) for each $t \in (\tilde{\tau}, 1)$.

The following proposition shows a necessary and sufficient condition for the bubble-crash strategy profile $\tilde{q}$ to be a Nash equilibrium, and also shows a necessary and sufficient condition for $\tilde{q}$ to be a *unique* Nash equilibrium. The proposition is regarded as a generalization of Proposition 1, Theorem 2, and Theorem 3 in Matsushima (2013b).

Proposition 1: *The bubble-crash strategy profile* $\tilde{q}$ *is a Nash equilibrium if and only if*

$$\exp[-\frac{1}{n-1}\int_{\tau=0}^{1} R(\tau)d\tau] \geq \varepsilon.$$

It is a unique Nash equilibrium if the strict inequality holds, i.e.,

$$\exp[-\frac{1}{n-1}\int_{\tau=0}^{1} R(\tau)d\tau] > \varepsilon.$$

Proof: See Appendix A.

It follows from Proposition 1 that the greater the relative future benefits $R(t)$ are, the lesser ε is required for the inequalities in Proposition 1, i.e., the more likely the bubble-crash strategy profile $\tilde{q}$ is to be a Nash equilibrium; the more likely the bubble is to persist for a long time.

If the bubble-crash strategy profile $\tilde{q}$ is a Nash equilibrium, then almost surely it is the unique Nash equilibrium. If it is a Nash equilibrium, then the critical time $\tilde{\tau}$ satisfies (6); no arbitrageur times the market at the initial time 0, i.e.,

$$\tilde{q}_i(\tilde{\tau}) = 0 \text{ for all } i \in N .$$

We further specify another symmetric strategy profile $\hat{q} = (\hat{q}_i)_{i \in N}$, namely the *no-bubble strategy profile*, as

$$\hat{q}_i(0) = 1 \text{ for all } i \in N .$$

The following proposition shows a necessary and sufficient condition for the non-bubble strategy profile q^* to be a Nash equilibrium. The proposition is a generalization of Proposition 4 in Matsushima (2013b). Let us define the *overall relative future benefit* by

$$R \equiv \frac{\bar{v}_1(1) - \bar{v}_1(0)}{\bar{v}_1(0) - \underline{v}_1(0)} .$$

Proposition 2: *The no-bubble strategy profile* $\hat{q}$ *is a Nash equilibrium if and only if*

$$R \leq \sum_{1 \leq l \leq n-1} \frac{(n-1)!}{l!(n-1-l)!} \left(\frac{1-\varepsilon}{\varepsilon}\right)^l \frac{1}{l+1} .$$

Proof: See Appendix B.

Note that the right-hand side of the inequality in Proposition 2 is decreasing in ε. It follows from Proposition 2 that the smaller the overall relative future benefit R is, the greater ε is permitted for the inequalities in Proposition 2, i.e., the less likely the no-bubble strategy profile is to be a Nash equilibrium; the more likely the bubble is to emerge.

3. Basic Model and CDS Model

We demonstrate the *basic model*, in which credit default swap (in short CDS) is not available. If arbitrageur i wins the associated timing game with the basic model, he receives the monetary value of his shareholding evaluated according to the bubble share price minus his debt obligation, i.e., he earns his personal capital, denoted by $W_i(t)$, evaluated according to the bubble share price. If arbitrageur i loses the timing game, he receives nothing; after the bubble's crash, the market value of his shareholding declines to zero. Because of the non-recourse nature of debt contracts, he is exempted from his entire debt obligation. Hence, the winner payoff and loser payoff in the basic model are given by

$$\overline{v}_i(t) = W_i(t) \quad \text{and} \quad \underline{v}_i(t) = 0 .$$

On the symmetry assumption in that

$$W_1(t) = W_i(t) \quad \text{for all} \quad i \in N ,$$

the relative future benefit in the basic model, denoted by $R(t) = R^*(t)$, is given by

$$R^*(t) = \frac{W_1'(t)}{W_1(t)},$$

and the overall relative future benefit , denoted by $R = R^*$, is given by

$$R^* = \frac{W_1(1)}{W_1(0)} - 1 .$$

The critical time and the hazard rate associated with the basic model are denoted by

$$\tilde{\tau} = \tilde{\tau}^* \quad \text{and} \quad \theta(t) = \theta^*(t) ,$$

respectively.

We further demonstrate the *CDS model*, in which CDS is available. Any arbitrageur purchases CDS from the positive feedback traders. Each arbitrageur i receives the payment from the seller of CDS, denoted by $Z_i(t)$, when the bubble crashes at time t, irrespective of whether he wins the timing game associated with the CDS model or not. If arbitrageur i wins the timing game, he earns, not only his personal capital $W_i(t)$, but also the payment of CDS $Z_i(t)$. If he loses the timing game, he earns the payment of CDS $Z_i(t)$, but is not exempted from his entire debt obligation.

His debt obligation is given by

$$\{L_i(t)-1\}W_i(t),$$

where $L_i(t)$ denotes the *leverage ratio* defined as the market value of the arbitrageur's stockholding divided by the market value of his personal capital. By utilizing the payment of CDS, any loser has to pay back to his debt holders the monetary amount of

$$\min[Z_i(t),\{L_i(t)-1\}W_i(t)].$$

Hence, the winner payoff and the loser payoff in the CDS model are given by

$$\overline{v}_i(t)=W_i(t)+Z_i(t),$$

and

$$\underline{v}_i(t)=\max[Z_i(t)-\{L_i(t)-1\}W_i(t),0].$$

On the symmetry assumption in that

$$W_1(t)=W_i(t),\quad Z_1(t)=Z_i(t),\text{ and }\quad L_1(t)=L_i(t)\quad\text{for all}\quad i\in N,$$

the relative future benefit in the CDS model, denoted by $R(t)=R^{**}(t)$, is given by

$$R^{**}(t)=\frac{W_1'(t)+Z_1'(t)}{W_1(t)+Z_1(t)}\qquad\text{if}\quad\{L_1(t)-1\}W_1(t)\geq Z_1(t),$$

$$R^{**}(t)=\frac{W_1'(t)+Z_1'(t)}{L_1(t)W_1(t)}\qquad\text{if}\quad\{L_1(t)-1\}W_1(t)<Z_1(t),$$

and the overall relative future benefit , denoted by $R=R^{**}$, is given by

$$R^{**}=\frac{W_1(1)+Z_1(1)}{W_1(0)+Z_1(0)}-1\qquad\text{if}\quad\{L_1(0)-1\}W_1(0)\geq Z_1(0),$$

and

$$R^{**}=\frac{W_1(1)+Z_1(1)-\{W_1(0)+Z_1(0)\}}{L_1(0)W_1(0)}\qquad\text{if}\quad\{L_1(0)-1\}W_1(0)<Z_1(0).$$

The hazard rate and the critical time associated with the CDS model are denoted by

$$\theta(t)=\theta^{**}(t)\quad\text{and}\quad\tilde{\tau}=\tilde{\tau}^{**},$$

respectively.

It is clear from (4) and (6) that

$$[R^*(t)>R^{**}(t)]\Leftrightarrow[\theta^*(t)>\theta^{**}(t)]$$

$$[R^*(t) > R^{**}(t) \text{ for all } t \in [0,1]] \Rightarrow [\tilde{\tau}^* > \tilde{\tau}^{**}].$$

and

$$[R^*(t) < R^{**}(t) \text{ for all } t \in [0,1]] \Rightarrow [\tilde{\tau}^* < \tilde{\tau}^{**}].$$

Hence, if the relative future benefits are greater (smaller) in the basic model than in the CDS model, then the bubble is more (less, respectively) likely to persist for a long time in the basic model than in the CDS model. If the overall relative future benefit is greater (smaller) in the basic model than the CDS model, i.e.,

$$R^* > R^{**} \quad (R^* < R^{**}),$$

then the bubble is more (less, respectively) likely to emerge in the basic model than in the CDS model.

The following theorem shows a characterization result concerning which is greater between the basic model and the CDS model in terms of relative future benefits and overall relative future benefit. This theorem generally says that the relative future benefits and overall relative future benefit are greater, and therefore, the bubble is more likely to emerge and persist for a long time, in the basic model than in the CDS model, provided the payment of CDS $Z_1(t)$ sufficiently grows through time compared with the arbitrageur's personal capital $W_1(t)$.

Theorem 3: *For each* $t \in [0,1]$, *if*

$$\{L_1(t) - 1\}W_1(t) \geq Z_1(t),$$

then

$$[R^*(t) > R^{**}(t)] \Leftrightarrow [\frac{W_1'(t)}{W_1(t)} > \frac{W_1'(t) + Z_1'(t)}{W_1(t) + Z_1(t)}].$$

For each $t \in [0,1]$, *if*

$$\{L_1(0) - 1\}W_1(0) < Z_1(0),$$

then

$$[R^* > R^{**}] \Leftrightarrow [\{L_1(t) - 1\}W_1'(t) > Z_1'(t)].$$

If

$$\{L_1(0) - 1\}W_1(0) \geq Z_1(0),$$

then

$$[R^* > R^{**}] \Leftrightarrow [\frac{W_1(1)}{W_1(0)} > \frac{W_1(1)+Z_1(1)}{W_1(0)+Z_1(0)}].$$

If

$$\{L_1(0)-1\}W_1(0) < Z_1(0),$$

then

$$[R^* > R^{**}] \Leftrightarrow [\{L_1(t)-1\}\{W_1(1)-W_1(0)\} > Z_1(1)-Z_1(0)].$$

Proof: The proof of this theorem is straightforward from the definitions of $R^*(t)$, R^*, $R^{**}(t)$, and R^{**}.

Q.E.D.

We should distinguish the CDS model from the ‘covered’ CDS model, where covered CDS is available to trade, which pays its purchaser the difference between his debt obligation and the monetary value of his defaulted shareholding whenever the bubble crashes. This implies that the winner payoff and the loser payoff in the covered CDS model are the same as the winner payoff and the loser payoff in the basic model, respectively.

We should also distinguish the CDS model from the ‘insurance’ model, where any arbitrageur can make a full insurance contract with positive feedback traders against the crash risk of his own shareholding. In the insurance model, the loser payoff is the same as the winner payoff, and therefore, all arbitrageurs are incentivized to ride the bubble at all times: the bubble never crashes.

4. Stock Market

This section formulates the details of the market for a company's stock. The company has no profitable business opportunity: its fundamental value is set at zero. We denote by $S(t) > 0$ the total share that the company has issued up to time $t \in [0,1]$, where $S(t)$ is non-decreasing in t. The company raises funds by issuing shares during the bubble. During a short time interval $[t, t+\Delta]$, the company issues approximately $S'(t)\Delta$ number of shares.

We denote by $S_i(t) > 0$ the number of shares that arbitrageur i possesses at time t during the bubble, where $S_i(t)$ is non-decreasing in t. During a short time interval $[t, t+\Delta]$, arbitrageur i purchases approximately $S_i'(t)\Delta$ number of shares.

The share price grows during the bubble following a continuous and increasing function $P:[0,1] \to (0,\infty)$. The bubble persists as long as the arbitrageurs continue to hold $n\phi \times 100$ % of the company's stock or more, where $0 < \phi < 1/n$. Once the arbitrageurs' total shareholdings fall to less than $n\phi \times 100$ %, the bubble crashes immediately and the share price declines to zero. Even if no arbitrageur sells, the bubble automatically crashes just after termination time 1.[5]

It is implicit to assume that there are many positive feedback traders who are slaves to euphoria. During the bubble, at any time $t \in [0,1]$, they misperceive the current share price $P(t)$ as reflecting the correct fundamental value, and they further unconsciously reinforce their misperception according to P. However, once the arbitrageurs' total shareholdings fall to less than $n\phi \times 100$ %, the resultant selling pressure would force the positive feedback traders out of euphoria; they would become aware of the correct fundamental value, immediately bursting the bubble.

It is substantial to make the assumption of *awareness heterogeneity* between arbitrageurs and positive feedback traders as follows. The positive feedback traders are unaware of their own euphoria, i.e., their reinforcement pattern and the bubble's crash

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[5] For example, if the arbitrageurs expect the share price to stop growing after the termination time 1, the bubble automatically crashes just after termination time 1.

risk, while the arbitrageurs are well aware of the positive feedback traders' euphoria. The positive feedback traders incorrectly expect the current share price to never change through time, even if they actually change their minds and reinforce their misperception. Hence, the positive feedback traders incorrectly neither expect the current share price to increase nor to stump.

The company raises funds by issuing shares during the bubble, but within a limit. If the company issues too many shares for each arbitrageur to keep his shareholding not less than $\phi \times 100$%, the resultant selling pressure will burst the bubble. Hence, for the company to raise funds without causing the bubble's crash, the arbitrageurs need to purchase the sufficient number of shares to keep their shareholdings not less than $n\phi \times 100$%.

In this case, an effective method would be for the company to encourage each arbitrageur to borrow money from the positive feedback traders. Because of the assumption of awareness heterogeneity, the positive feedback traders neither perceive the bubble's crash risk nor any speculative merit. Hence, each arbitrageur can enter into short-term debt contracts with the positive feedback traders with *no premium*.

We set an exogenous cap for the leverage ratio, denoted by $L \geq 1$. Since any arbitrageur is in a better position when he lets his leverage ratio be equal to this upper limit, he will have a debt obligation of $\frac{L-1}{L} P(t) S_i(t)$ to his debt holders at any time t during the bubble, i.e., $L_i(t) = L$.

The personal capital $W_i(t)$ of each arbitrageur i is expressed as the market value of his shareholding minus his debt obligation:

$$W_i(t) = P(t)S_i(t) - \frac{L-1}{L} P(t) S_i(t) = \frac{P(t)S_i(t)}{L}. \tag{7}$$

Since arbitrageur i earns a capital gain $\{P(t+\Delta) - P(t)\}S_i(t)$ from time t to time $t+\Delta$, his personal capital increases approximately by this amount:

$$W_i(t+\Delta) = W_i(t) + \{P(t+\Delta) - P(t)\}S_i(t),$$

which implies

$$W_i'(t) = P'(t)S_i(t).$$

From (7),

$$W_i'(t) = \frac{P(t)S_i'(t) + P'(t)S_i(t)}{L}.$$

Hence, from these equations, $(P(0), S(0), L)$ uniquely determines arbitrageur $i's$ shareholding; for every $t \in [0,1]$,

$$P'(t)S_i(t) = \frac{P(t)S_i'(t) + P'(t)S_i(t)}{L},$$

that is,

$$S_i(t) = S_i(0)(\frac{P(t)}{P(0)})^{L-1}.$$

We assume symmetry in that the arbitrageurs possess the same number of shares at initial time: $S_i(0) = S_1(0)$ for all $i \in \{1,...,n\}$. In this case, the arbitrageurs possess the same number of shares through time:

$$S_i(t) = S_1(t) \text{ for all } i \in \{1,...,n\} \text{ and all } t \in [0,1].$$

The company better keeps the number of shares that each arbitrageur possesses equal to $\phi \times 100\%$:

$$S_i(t) = S_1(t) = \phi S(t).$$

From the above observations, the total share can be expressed as

(8) $$S(t) = S(0)(\frac{P(t)}{P(0)})^{L-1},$$

and the personal capital of each arbitrageur i can be expressed as

(9) $$W_i(t) = \frac{\phi}{L} P(0)S(0)(\frac{P(t)}{P(0)})^{L}.$$

5. Basic Model: Specification

According to Section 4, we specify the details of the basic model as follows. If arbitrageur i wins the timing game, he obtains a monetary amount $P(t)S_i(t)$ and repays his debt obligation $\frac{L-1}{L}P(t)S_i(t)$. Hence, the winner payoff, which is equivalent to his personal capital, is specified as

$$\bar{v}_i(t) = W_i(t) = P(t)S_i(t) - \frac{L-1}{L}P(t)S_i(t) = \frac{\phi}{L}P(0)S(0)(\frac{P(t)}{P(0)})^L .$$

It follows from Section 3 that

(10) $$R^*(t) = L\frac{P'(t)}{P(t)},$$

(11) $$R^* = (\frac{P(1)}{P(0)})^L - 1 .$$

(12) $$\theta^*(t) = L\frac{n}{n-1}\frac{P'(t)}{P(t)},$$

and

(13) $$\varepsilon = (\frac{P(\tilde{\tau}^*)}{P(1)})^{\frac{L}{n-1}} .$$

From these equalities, it follows that the greater $\frac{P'(t)}{P(t)}$ is at any time $t \in [0,1]$, the greater the relative future benefits $R^*(t)$, the overall relative future benefit R^*, the hazard rate $\theta^*(t)$, and the critical time $\tilde{\tau}^*$ are. Hence, the more enthusiastic the positive feedback traders are, the more likely the bubble is to emerge and persist for a long time.

Let us denote

$$R^*(t) = R^*(t,L), \quad R^* = R^*(L), \quad \theta^*(t) = \theta^*(t,L), \text{ and } \quad \tilde{\tau}^* = \tilde{\tau}^*(L) .$$

From (10), (11), (12), and (13),

$$\frac{\partial R^*(t,L)}{\partial L} > 0, \quad \frac{\partial R^*(L)}{\partial L} > 0, \quad \frac{\partial \theta^*(t,L)}{\partial L} > 0, \text{ and } \quad \frac{\partial \tilde{\tau}^*(L)}{\partial L} > 0 .$$

Hence, the greater the leverage ratio L is, the more likely the bubble is to emerge and persist for a long time.

From (10), (11), (12), and (13),

$$\lim_{L\to\infty} R^*(t,L) = \lim_{L\to\infty} R^*(L) = \lim_{L\to\infty} \theta^*(t,L) = \infty \quad \text{and} \quad \lim_{L\to\infty} \tilde{\tau}^*(L) = 1.$$

Hence, even if the positive feedback traders are not very enthusiastic, the bubble is likely to emerge and persist for a long time whenever the leverage ratio L is sufficient.

6. CDS Model: Specification

According to Section 4, we specify the details of the CDS model as follows. The payment of CDS would depend on the total personal capital of the positive feedback traders (the sellers of CDS), which is denoted by $B(t) > 0$. We assume $B(t)$ is differentiable and non-decreasing in t; i.e., $B'(t) \geq 0$.

The sellers of CDS, i.e., the positive feedback traders, are required to have full reserve to pay for the purchasers of CDS, i.e., the arbitrageurs. Therefore, $B(t)$ must be equal to the sum of the market value of the positive feedback traders' shareholdings $(1-n\phi)P(t)S(t)$, their loan to arbitrageurs $\frac{L-1}{L}n\phi P(t)S(t)$, and their reserve for the payment of CDS $nZ_1(t)$:

$$B(t) = (1-n\phi)P(t)S(t) + \frac{L-1}{L}n\phi P(t)S(t) + nZ_1(t)$$

$$= (1-\frac{n\phi}{L})P(0)S(0)(\frac{P(t)}{P(0)})^L + nZ_1(t),$$

which implies

$$Z_1(t) = \frac{1}{n}B(t) - (\frac{1}{n} - \frac{\phi}{L})P(0)S(0)(\frac{P(t)}{P(0)})^L. \tag{14}$$

Because of awareness heterogeneity, any arbitrageur can purchase CDS from the positive feedback traders with *no premium.* It is important to note that any arbitrageur never demands CDS beyond the right-hand side of (14); otherwise, the CDS demand pressure makes the premium for CDS positive, which makes the positive feedback traders aware of the bubble's crash risk, dampening their euphoria.

For simplicity of arguments, we assume that $B(t)$ is sufficient to keep the payment of CDS to each arbitrageur greater than his debt obligation:

$$Z_1(t) > (L-1)W_1(t).$$

Hence, according to Section 3, the winner payoff and the loser payoff in the CDS model are specified as

$$\bar{v}_1(t) = W_1(t) + Z_1(t) = \frac{1}{n}\{B(t) - (1-\frac{2n\phi}{L})P(0)S(0)(\frac{P(t)}{P(0)})^L\},$$

and

$$\underline{v}_1(t) = Z_1(t) - (L-1)W_1(t) = \frac{1}{n}[B(t) - \{1 + \frac{(L-2)n\phi}{L}\}P(0)S(0)(\frac{P(t)}{P(0)})^L].$$

Note from (14) that

(15) $$Z_1'(t) = \frac{1}{n}\{B'(t) - (L - n\phi)P'(t)S(0)(\frac{P(t)}{P(0)})^{L-1}\}.$$

We assume

$$Z_1'(t) > 0 \quad \text{for all} \quad t \in [0,1].$$

The relative future benefit and the overall relative future benefit are specified as

(16) $$R^{**}(t) = \frac{W_1'(t) + Z_1'(t)}{LW_1(t)}$$

$$= \frac{1}{L\phi n}\{\frac{B'(t)}{P(0)S(0)(\frac{P(t)}{P(0)})^L} - (L - 2n\phi)\frac{P'(t)}{P(t)}\}$$

and

(17) $$R^{**} = \frac{W_1(1) + Z_1(1) - \{W_1(0) + Z_1(0)\}}{LW_1(0)}$$

$$= \frac{B(1) - B(0) - (1 - \frac{2n\phi}{L})P(0)S(0)\{(\frac{P(1)}{P(0)})^L - 1\}}{n\phi P(0)S(0)}.$$

We decompose the increase in positive feedback traders' personal capital $B'(t)\Delta$ into two parts, i.e., the capital gain $(1-n\phi)P'(t)S(t)\Delta$ and the exogenous increase $h(t)\Delta$:

(18) $$B'(t) = (1-n\phi)P'(t)S(t) + h(t) = (1-n\phi)P'(t)S(0)(\frac{P(t)}{P(0)})^{L-1} + h(t).$$

From (9), (15), and (18), it follows that

$$[\,(L-1)W_1'(t) \ge Z_1'(t)\,] \Leftrightarrow [h(t) \le (L-1)(1+n\phi)P'(t)S(0)(\frac{P(t)}{P(0)})^{L-1}].$$

Hence, if the leverage ratio L is sufficient, the positive feedback traders are enthusiastic, i.e., $P'(t)$ and $\frac{P(t)}{P(0)}$ are sufficient, and the rates of exogenous increase in positive feedback traders' personal capital, $h(t)$, are insufficient, then the inequality of

$$h(t) \leq (L-1)(1+n\phi)P'(t)S(0)(\frac{P(t)}{P(0)})^{L-1}$$

is likely to hold. Hence, from Theorem 3, if the leverage ratio is sufficient, the positive feedback traders are enthusiastic, and the rates of exogenous increase in positive feedback traders' personal capital are insufficient, then the relative future benefit in the basic model is greater than in the CDS model, i.e., $R^{*}(t) > R^{**}(t)$; the bubble is more likely to persist for a long time in the basic model than in the CDS model.

In the same manner, if the leverage ratio is insufficient, the positive feedback traders are not very enthusiastic, and the rates of exogenous increase in positive feedback traders' personal capital are sufficient, then the relative future benefit in the basic model is lesser than in the CDS model, i.e., $R^{*}(t) < R^{**}(t)$; the bubble is less likely to persist for a long time in the basic model than in the CDS model.

From (9), (15), and (18), it follows that

$$[(L-1)\{W_1(1) - W_1(0)\} \geq Z_1(1) - Z_1(0)]$$

$$\Leftrightarrow [\int_{t=0}^{1} h(t)dt \leq \frac{L-1}{L}(1+n\phi)P(0)S(0)\{(\frac{P(1)}{P(0)})^{L} - 1\}].$$

Hence, if the leverage ratio L is sufficient, the positive feedback traders are enthusiastic, i.e., $\frac{P(1)}{P(0)}$ is sufficient, and the overall exogenous increase in positive feedback traders' personal capital $\int_{t=0}^{1} h(t)dt$ is insufficient, then the inequality of

$$\int_{t=0}^{1} h(t)dt \leq \frac{L-1}{L}(1+n\phi)P(0)S(0)\{(\frac{P(1)}{P(0)})^{L} - 1\}$$

is likely to hold. Hence, from Theorem 3, if the leverage ratio is sufficient, the positive feedback traders are enthusiastic, and the overall exogenous increase in positive feedback traders' personal capital is insufficient, then the overall relative future benefit in the basic model is greater than in the CDS model, i.e., $R^{*} > R^{**}$; the bubble is more likely to emerge in the basic model than in the CDS model.

In the same manner, if the leverage ratio is insufficient, the positive feedback traders are not very enthusiastic, and the overall exogenous increase in positive feedback traders' personal capital is sufficient, then the overall relative future benefit in the basic model is lesser than in the CDS model, i.e., $R^* < R^{**}$; the bubble is less likely to emerge in the basic model than in the CDS model.

We examine about the impact of the increase in leverage ratio as follows.

Theorem 4: *Suppose* $L > 1 + n\phi$. *Then,*

$$\frac{\partial R^{**}(t,L)}{\partial L} < 0 \quad and \quad \frac{\partial R^{**}(L)}{\partial L} < 0 .$$

Proof: From (16) and (18),

$$R^{**}(t,L) = \frac{1}{L\phi n}\{(1+n\phi - L)\frac{P'(t)}{P(t)} + \frac{h(t)}{P(0)S(0)(\frac{P(t)}{P(0)})^L}\} .$$

Hence, from $L > 1 + n\phi$,

$$\frac{\partial R^{**}(t,L)}{\partial L} = -\frac{1}{L^2\phi n}\{(1+n\phi - L)\frac{P'(t)}{P(t)} + \frac{h(t)}{P(0)S(0)(\frac{P(t)}{P(0)})^L}\}$$

$$-\frac{1}{L\phi n}\{\frac{P'(t)}{P(t)} + \frac{h(t)\log(\frac{P(t)}{P(0)})}{P(0)S(0)(\frac{P(t)}{P(0)})^L}\} < 0 .$$

From (17) and (18),

$$R^{**}(L) = \frac{\frac{1+n\phi - L}{L}P(0)S(0)\{(\frac{P(1)}{P(0)})^L - 1\} + \int_{t=0}^{1} h(t)dt}{n\phi P(0)S(0)} .$$

Hence, from $L > 1 + n\phi$,

$$\frac{\partial R^{**}(L)}{\partial L} = \frac{1}{n\phi P(0)S(0)}[-\frac{1+n\phi}{L^2}P(0)S(0)\{(\frac{P(1)}{P(0)})^L - 1\}$$

$$+\frac{1+n\phi - L}{L}\log(\frac{P(1)}{P(0)})P(0)S(0)(\frac{P(1)}{P(0)})^L] < 0 .$$

Q.E.D.

Theorem 4 states that provided the leverage ratio L is not insufficient, i.e., $L > 1 + n\phi$, as the greater the leverage ratio L is, the less likely the bubble is to emerge and persist for a long time. This is in contrast with the basic model. A high leverage ratio fosters the bubble when CDS is not available, while it does deter the bubble when CDS is available. The increase in leverage ratio L enhances the future loan to the arbitrageurs, which crowds out the future reserve for CDS, decreasing the relative future benefit. This is the driving force for a high leverage ratio to deter the bubble in the CDS model.

7. Social Cost

Since the company has no profitability, we can define the *social cost* of the bubble as the total funds that the company raises during the bubble, i.e., from the initial time 0 to time t at which the bubble crashes, which can be expressed as

$$C(t,L) \equiv \int_{\tau=0}^{t} P(\tau)S'(\tau)d\tau = P(0)S(0)\frac{L-1}{L}\{(\frac{P(t)}{P(0)})^{L}-1\}.$$

We clarify whether the expected social cost induced by the bubble-crash strategy profile $\tilde{q}$, i.e.,

$$\int_{t=0}^{1} C(t,L)dD(t;\tilde{q},\varepsilon),$$

is decreasing in leverage ratio L in the CDS model. Note that $C(t,L)$ is increasing in L, while the relative future benefit $R^{**}(t,L)$ is decreasing in L. Hence, if $\left|\frac{\partial R^{**}(t,L)}{\partial L}\right|$ is sufficient compared with $\frac{\partial C(t,L)}{\partial L}$, then the expected social cost induced by $\tilde{q}$ decreases as L increases.

Importantly, the values of social cost $C(t,L)$ are irrelevant to the rates of exogenous increase in positive feedback traders' personal capital $h(t)$, while $\left|\frac{\partial R^{**}(t,L)}{\partial L}\right|$ is increasing in $h(t)$, which diverges to infinity as $h(t)$ increases. Hence, we can conclude that if the rates of exogenous increase in positive feedback traders' personal capital $h(t)$ are sufficient, then a high leverage ratio not only decreases the critical time and the hazard rate, but also decreases the expected social cost induced by $\tilde{q}$.

8. Conclusion

This paper generalized the timing game with behavioral types explored by Matsushima (2013b), and applied it to the stock market of a company that is unproductive but raises wasteful funds during the bubble that is driven by the positive feedback traders' euphoria. We assumed awareness heterogeneity between arbitrageurs and positive feedback traders in terms of reinforcement and bubble's crash risk. We permitted arbitrageurs to borrow money from positive feedback traders with no premium. We also permitted arbitrageurs to purchase credit default swap (CDS) defined as bubble-contingent claim from positive feedback traders with no premium.

We demonstrated a theoretical ground for considering CDS as a powerful policy method to deter harmful bubbles, even if the policy maker cannot identify whether the company is unproductive. We showed that the bubble is less likely to emerge and persist in the CDS model than in the basic model. A high leverage ratio deters the mergence and long persistence of bubble in the CDS model, while it facilitates the mergence and long persistence of bubble in the basic model. In the CDS model, a high leverage ratio could decrease the expected social cost induced by the company's fund-raising during the bubble.

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Appendix A: Proof of Proposition 1

From $\tilde{q}$, for every $\hat{\tau} \in [\tilde{\tau}, 1]$, we specify a symmetric strategy profile $q^{\hat{\tau}} = (q_i^{\hat{\tau}})_{i \in N} \in Q$ as follows:

$$q_1^{\hat{\tau}}(t) = \tilde{q}_1(t) \quad \text{for all} \quad t \in [\hat{\tau}, 1],$$

and

$$q_1^{\hat{\tau}}(t) = \tilde{q}_1(\hat{\tau}) \quad \text{for all} \quad t \in [0, \hat{\tau}).$$

According to $q^{\hat{\tau}}$, any rational arbitrageur times the market at the initial time 0 with probability $\tilde{q}_1(\hat{\tau})$. After the initial time 0, he never times the market until time $\hat{\tau}$. After time $\hat{\tau}$, he conforms to $\tilde{q}$.

Proposition A-1: *A symmetric strategy profile* $q \in Q$ *is a Nash equilibrium if and only if there exists* $\hat{\tau} \in [\tilde{\tau}, 1]$ *such that*

$$q = q^{\hat{\tau}},$$

and

(A-1) $\quad u_1(0, q_{-1}, \varepsilon) = u_1(\hat{\tau}, q_{-1}, \varepsilon)$ *whenever* $\hat{\tau} < 1$ *and* $q_1(0) > 0$,

and

(A-2) $\quad u_1(0, q_{-1}, \varepsilon) \geq u_1(\hat{\tau}, q_{-1}, \varepsilon)$ *whenever* $\hat{\tau} = 1$.

Proof: We set any symmetric Nash equilibrium $q \in Q$ arbitrarily. It is clear that the inequality (A-2) is necessary and sufficient for the Nash equilibrium property if $\hat{\tau} = 1$, i.e., $q = q^1$. We assume that $q \neq q^1$, i.e., $q_1(0) < 1$.

We show that $q_1(\tau)$ is continuous. Let us suppose that $q_1(\tau)$ is not continuous. Then, there will exist $\tau' > 0$ such that $\lim_{\tau \uparrow \tau'} q_1(\tau) < q_1(\tau')$. From symmetry, it follows that by selecting any time that is slightly earlier than τ', any arbitrageur can dramatically increase his winning probability. This implies that no arbitrageur selects τ', which is a contradiction.

Let us specify

$$\hat{\tau} = \max\{\tau \in (0,1] : q_1(\tau) = q_1(0)\}.$$

We show that $q_1(\tau)$ is increasing in $[\hat{\tau},1]$. Suppose that $q_1(\tau)$ is not increasing in $[\hat{\tau},1]$. From the continuity of q_1 and the specification of $\hat{\tau}$, we select $\tau', \tau'' \in [\hat{\tau},1]$ such that $\tau' < \tau''$, $q_1(\tau') = q_1(\tau'')$, and the selection of τ' is a best response. Since no arbitrageur selects any time τ in (τ', τ''), it follows from the continuity of q that by selecting time τ'' instead of τ', any arbitrageur can increase his winner payoff without decreasing his winning probability, which is a contradiction.

Any selection $\tau \in [\hat{\tau},1]$ must be a best response, because $q_1(\tau)$ is increasing in $[\hat{\tau},1]$. This implies that the first-order condition holds for all $\tau \in [\hat{\tau},1]$, i.e., $q = q^{\hat{\tau}}$. Given that $\hat{\tau} < 1$, it is clear from the fact that the winner payoff $\bar{v}_i(t)$ is increasing that $q^{\hat{\tau}}$ is a Nash equilibrium if and only if

$$u_1(0, q_{-1}, \varepsilon) = u_1(\hat{\tau}, q_{-1}, \varepsilon) \quad \text{whenever} \quad q_1(0) > 0.$$

This implies that (A-1) is necessary and sufficient.

Q.E.D.

The first part of Proposition 1 is proved as follows. From Proposition A-1, $\tilde{\tau} < 1$, and $\tilde{q} = q^{\tilde{\tau}}$, it follows that $\tilde{q}$ is a Nash equilibrium if and only if

$$\text{either} \quad \tilde{q}_1(0) = 0 \quad \text{or} \quad u_1(0, \tilde{q}_{-1}, \varepsilon) = u_1(\tilde{\tau}, \tilde{q}_{-1}, \varepsilon).$$

The weak inequality in Proposition 1 implies (6). Hence, $\tilde{q}_1(0) = 0$, i.e., $\tilde{q}$ is a Nash equilibrium.

Suppose that the strict inequality in Proposition 1 does not hold. Then, it must hold that $\tilde{\tau} = 0$ and $\tilde{q}_1(0) > 0$. This, however, contradicts the Nash equilibrium property that any selection of $t \in [0,1]$ is a best response: any arbitrageur prefers time 0 to any time slightly later than time 0, because he can dramatically increase his winning probability without any substantial decrease in winner payoff.

The latter part of Proposition 1 is proved as follows. It follows from the strict inequality in Proposition 1 that the property of (6) holds and $\tilde{\tau} > 0$. This, along with Proposition A-1, implies that any symmetric Nash equilibrium q must satisfy $q = \tilde{q}$.

Next, we show that $\tilde{q}$ is a unique Nash equilibrium even if all asymmetric Nash equilibria are taken into account. We set any Nash equilibrium $q \in Q$ arbitrarily.

First, we show that $q_i(\tau)$ must be continuous in $[0,1]$ for all $i \in N$. Suppose that $q_i(\tau)$ is not continuous in $[0,1]$. Then, there exists $\tau' > 0$ such that $\lim_{\tau \uparrow \tau'} q_i(\tau) < q_i(\tau')$ for some $i \in N$; any other arbitrageur can drastically increase his winning probability by selecting any time slightly earlier than time τ'. Hence, no other arbitrageur selects any time that is either the same as or slightly later than τ'. Hence, arbitrageur i can postpone timing the market without decreasing his winning probability. This is a contradiction.

Second, we show that $D(\tau;q)$ must be increasing in $[\tau^1,1]$, where we denote

$$\tau^1 = \max\{\tau \in (0,1] : q_i(\tau) = q_i(0) \text{ } \textit{for all } i \in N\}.$$

Now, suppose that $D(\tau;q)$ is not increasing in $[\tau^1,1]$. In this case, from the continuity of q, we can select $\tau', \tau'' \in (\tau^1,1]$ such that $\tau' < \tau''$, $D(\tau';q) = D(\tau'';q)$, and the selection of τ' is a best response for some arbitrageur. Since no arbitrageur selects any time τ in (τ', τ''), it follows from the continuity of q that by selecting τ'' instead of τ', any arbitrageur can postpone the timing from τ' to τ'' without decreasing his winning probability. This is also a contradiction.

Third, we show that q must be symmetric. Now let us suppose that q is asymmetric. The strict inequality in Proposition 1 implies that the selection of 0 is a dominated strategy. Hence, it follows that $\tau^1 > 0$, and

$$q_i(\tau) = 0 \quad \text{for all } \; i \in N \; \text{ and all } \; \tau \in [0,\tau^1].$$

Since q is continuous and $D(\tau;q)$ is increasing in $[\tau^1,1]$, it follows that there exist $\tau' > 0$, $\tau'' > \tau'$, and $i \in N$ such that

$$q_1(t) = q_j(t) \quad \text{for all } \; j \in N \; \text{ and all } \; t \in [0,\tau'],$$

$$\text{(A-3)} \qquad \frac{\dfrac{\partial D_i(\tau;q)}{\partial t}}{1 - D_i(\tau;q)} > \min_{h \neq i} \frac{\dfrac{\partial D_h(\tau;q)}{\partial t}}{1 - D_h(\tau;q)} \quad \text{for all } \; t \in (\tau', \tau''),$$

and

$$\text{(A-4)} \qquad \frac{\dfrac{\partial D_i(\tau'';q)}{\partial t}}{1-D_i(\tau'';q)} = \min_{h\neq i} \frac{\dfrac{\partial D_h(\tau'';q)}{\partial t}}{1-D_h(\tau'';q)} > 0 .$$

Since $D(\tau;q)$ is increasing in $[\tau^1,1]$, any selection of t in (τ',τ'') must be a best response for any arbitrageur $j \in N$ such that

$$\frac{\dfrac{\partial D_j(t;q)}{\partial t}}{1-D_j(t;q)} = \min_{h\neq i} \frac{\dfrac{\partial D_h(t;q)}{\partial t}}{1-D_h(t;q)} .$$

This equality implies that $\dfrac{\partial q_j(t)}{\partial t} > 0$. Hence, it follows from the continuity of q that the first-order condition holds for arbitrageur j; i.e., for every $t \in (\tau',\tau'')$,

$$\frac{(1-\varepsilon)q_j{}'(t)}{1-(1-\varepsilon)q_j(t)} = \frac{\overline{v}_1'(t)}{(n-1)\{\overline{v}_1(t)-\underline{v}_1(t)\}} .$$

Hence, from (A-3),

$$\frac{(1-\varepsilon)q_i{}'(t)}{1-(1-\varepsilon)q_i(t)} > \frac{\overline{v}_1'(t)}{(n-1)\{\overline{v}_1(t)-\underline{v}_1(t)\}} ,$$

implying that the first-order condition does not hold for arbitrageur i for every $t \in (\tau',\tau'')$, where the inequality $\dfrac{\partial u_i(\tau,q_{-i},\varepsilon)}{\partial \tau} < 0$ holds in this case. This inequality implies that arbitrageur i prefers time τ' to any time in $(\tau',\tau''+\varepsilon)$, and therefore,

$$\frac{\partial D_i(\tau;q)}{\partial t} = 0 \quad \text{for all} \quad \tau \in (\tau',\tau''+\eta) ,$$

where η is positive but close to zero. This is a contradiction, because the inequality in (A-4) implies that $\dfrac{\partial D_i(\tau'';q)}{\partial t} > 0$. Hence, we have proved that any Nash equilibrium q must be symmetric.

From the above observations, we have completed the proof of Proposition 1.

Appendix B: Proof of Proposition 2

For every $t \in (0,1]$,

$$u_1(t,\hat{q}_{-1}) = \varepsilon^{n-1}\overline{v}_1(t) + (1-\varepsilon^{n-1})\underline{v}_1(0)$$

$$\leq \varepsilon^{n-1}\overline{v}_1(1) + (1-\varepsilon^{n-1})\underline{v}_1(0) = u_1(1,\hat{q}_{-1}),$$

whereas

$$u_1(0,\hat{q}_{-1}) = \{\sum_{1\leq l\leq n-1}\frac{(n-1)!}{l!(n-1-l)!}(1-\varepsilon)^l\varepsilon^{n-1-l}\frac{1}{l+1}\}\overline{v}_1(t)$$

$$+\{1-\sum_{1\leq l\leq n-1}\frac{(n-1)!}{l!(n-1-l)!}(1-\varepsilon)^l\varepsilon^{n-1-l}\frac{1}{l+1}\}\underline{v}_1(0).$$

Hence, the necessary and sufficient condition for $\hat{q}$ to be a Nash equilibrium is given by

$$u_1(0,\hat{q}_{-1}) \geq u_1(1,\hat{q}_{-1}),$$

that is,

$$\{\sum_{1\leq l\leq n-1}\frac{(n-1)!}{l!(n-1-l)!}(1-\varepsilon)^l\varepsilon^{n-1-l}\frac{1}{l+1}\}\overline{v}_1(t)$$

$$+\{1-\sum_{1\leq l\leq n-1}\frac{(n-1)!}{l!(n-1-l)!}(1-\varepsilon)^l\varepsilon^{n-1-l}\frac{1}{l+1}\}\underline{v}_1(0) \geq \varepsilon^{n-1}\overline{v}_1(1) + (1-\varepsilon^{n-1})\underline{v}_1(0).$$

This inequality is equivalent to

$$\sum_{1\leq l\leq n-1}\frac{(n-1)!}{l!(n-1-l)!}(\frac{1-\varepsilon}{\varepsilon})^l\frac{1}{l+1} \geq \frac{\overline{v}_1(1)-\overline{v}_1(0)}{\overline{v}_1(0)-\underline{v}_1(0)} = R.$$