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Impact of Financial Regulation and Innovation on Bubbles and Crashes due to Limited Arbitrage: Awareness Heterogeneity

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Abstract

We examine the impact of financial regulation and innovation on bubbles and crashes due to limited arbitrage by modeling timing games among strategic arbitrageurs whose rationality is not commonly known. An unproductive company raises funds by issuing shares, and for purchasing shares, arbitrageurs borrow money from positive feedback traders. The key concept is awareness heterogeneity: positive feedback traders are unaware of euphoria, but arbitrageurs are aware of it. We show the impact of high leverage ratio depends on whether naked CDS is available, and the impact of naked CDS depends on growth balance between positive feedback traders' capital and loan.

Keywords: Bubbles and Crashes, Limited Arbitrage, Awareness Heterogeneity, Leverage, Naked Credit Default Swap, Timing Games with Behavioral Types

JEL Classification Numbers: C720, C730, D820, G140.

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1. Introduction

We investigate a game-theoretic model for a company's stock market with a bounded time horizon. The purpose of this paper is to clarify the impact of financial regulation such as leverage ratio cap and financial innovation such as naked credit default swap (naked CDS) on the bubbles and crashes caused by limited arbitrage.

Several risk-neutral arbitrageurs interact with many positive feedback traders who are influenced by psychological biases or euphoria; these positive feedback traders misperceive the company's fundamental value as greater than the true value, and they reinforce misperception over time, causing the bubble. On the other hand, the arbitrageurs commonly know the correct fundamental value in advance and strategically compete with one another to time the market the earliest, thereby bursting the bubble caused by the positive feedback traders' euphoria.

In order to describe the phenomenon of bubbles and crashes in a manner compatible with the arbitrageurs' incentives, we assume that the arbitrageurs' rationality is not commonly known. With this assumption, in spite of their timing competition, the rational arbitrageurs might be willing to ride the bubble for a while instead of correcting their mispricing immediately.

For the purpose of this paper, we explicitly take the view of *awareness heterogeneity*: the positive feedback traders are naïve enough to be unaware of their own euphoria, i.e., they are unaware of their unconscious reinforcement pattern and the risk of the bubble's crash, while the arbitrageurs are well aware of both aspects. Thanks to this awareness heterogeneity, the arbitrageurs gain an advantageous position in borrowing money from the positive feedback traders for purchasing the company's shares. Hence, thanks to the arbitrageurs' large debt capacity, the company, which itself has only a limited debt capacity, can raise huge funds by issuing shares without having to fear that the resultant selling pressure would dampen the euphoria.

The company's fund-raising efforts in this manner will lead to a big social harm, because the company has no other profitable business opportunity than its fundamental value. Hence, it is important to investigate what kind of regulatory policies on leveraged finance and financial innovation on hedging default risks would deter such harmful bubbles. This is what we attempt in this paper. This paper will emphasize that the availability of non-standard financial instrument such as naked credit default swap (naked CDS) could be an effective policy method in eliminating the socially harmful aspect of a bubble without discouraging its socially beneficial aspect; such favorable viewpoints on naked CDS are in sharp contrast with more standard insurance instruments such as covered credit default swap (covered CDS).

We examine whether weak financial regulation on the arbitrageurs' borrowing activity, e.g., a high leverage ratio, will foster the emergence and long persistence of a bubble. In particular, we show that the impact of a high leverage ratio on the bubbles and crashes will crucially depend on whether a financial innovation such as naked CDS is available. Naked CDS is defined as secured insurance contract against third-party default risks: the purchaser of a naked CDS receives a promised payment from its seller whenever the bubble crashes. Importantly, in contrast with other more standard financial instruments such as covered CDS, which are defined as insurance against own default risks, the purchaser of naked CDS is not required to hold any underlying shares against the default of which he needs to hedge. With awareness heterogeneity, the arbitrageurs are positive purchasers of not only shares but also naked CDSs; by utilizing awareness heterogeneity in a strategic manner, the arbitrageurs can purchase naked CDSs from the positive feedback traders with no premium.

In a timing competition, each arbitrageur is confronted with the trade-off between future benefit, i.e., an increase in winner payoff, and instantaneous gain, i.e., the difference between the current winner and loser payoffs. In this case, an increase in relative future benefit to instantaneous gain encourages the arbitrageurs to postpone their timing, fostering the bubble. Hence, it is crucial for this paper to clarify how the financial regulation and financial innovation influence the arbitrageurs' relative future benefit. In this case, naked CDS is particularly important compared to other more standard insurances, because its availability may effectively decrease the arbitrageurs' relative future benefit in some cases, deterring the harmful bubble from emerging and persisting.

To be more precise, we firstly show that when no naked CDS is available, a high leverage ratio enhances the arbitrageurs' relative future benefit, fostering the bubble to emerge and persist in a socially harmful manner. The company can then raise huge wasting funds even if the positive feedback traders are not very enthusiastic.

In contrast, when naked CDSs are available, a high leverage ratio will deter the bubble from emerging and persisting. As the regulation on the leverage ratio cap gets weakened, the resultant increase of the positive feedback traders' future loan to the arbitrageurs will crowd out the future reserve that the sellers of naked CDSs might have to hold as collaterals. This crowding out of the future reserve reduces the arbitrageurs' relative future benefit, deterring the bubble from emerging and persisting.

These results indicate the following policy implications. Whenever a company is truly productive, the regulator should fix a high leverage ratio in order to support the company's growth, irrespective of whether naked CDS is available or not. Even if the company is unproductive, when naked CDS is available, the regulator should fix a high leverage ratio, because a high leverage ratio will deter the bubble. Hence, when naked CDS is available, the regulator does not have to know whether the company is productive or not.

However, when naked CDS is not available, the regulator should frame different regulatory policies for productive and unproductive companies; the regulator should keep the leverage ratio sufficiently low if the company is unproductive, because a high leverage ratio in this case will foster harmful bubbles. This is a dilemma, because the regulator generally does not know in advance whether a company is productive or not.

We argue a further important point that the impact of the availability of naked CDSs on bubbles and crashes will crucially depend on the growth balance between the positive feedback traders' personal capital and their loan to arbitrageurs. The availability of naked CDS increases both the future benefit and the instantaneous gain. Hence, whether the availability of naked CDS increases the relative future benefit crucially depends on finer details of the model specifications such as the growth rates of the positive feedback traders' personal capital and loan.

We show that the availability of naked CDSs will deter the bubble from emerging and persisting if the arbitrageurs' personal capital does not grow sufficiently compared to their loan to the arbitrageurs; the availability of naked CDSs will decrease the relative future benefit in this case. If the positive feedback traders' personal capital does not grow adequately, their loan to the arbitrageurs will grow more rapidly than their payment of naked CDSs, deterring the bubble. On the other hand, this could foster the bubble if the positive feedback traders' personal capital grows sufficiently compared to their loan to the arbitrageurs; the availability of naked CDSs enhances the relative future benefit.

From these arguments, we can further show that if the leverage ratio is sufficient and the positive feedback traders are enthusiastic, the growth rate of the positive feedback traders' loan to the arbitrageurs can be sufficient compared to that of their personal capital; the availability of naked CDSs will deter the bubble in this case. However, if the leverage ratio is insufficient and the positive feedback traders are not very enthusiastic, the growth rate of the positive feedback traders' loan will be insufficient compared to that of their personal capital; the availability of naked CDSs will foster the bubble.

Thus, ensuring the availability of naked CDSs might be an effective method to deter the bubble from emerging and persisting when there is concern about the huge social harm in terms of wasting funds. This result is consistent with the empirical observations in the United States that the introduction of naked CDSs in 2005 and 2006 brought about the housing bubbles to crash.

However, this paper also indicates that if the arbitrageurs expect the future growth of naked CDSs to be sufficient, the availability of naked CDSs even fosters the bubble. This could occur if the positive feedback traders are not enthusiastic and the leverage ratio is not very high. The emergence and long persistence of bubble are harmful in terms of intrinsic social cost caused by the company's wasteful fund raising, while they have a socially beneficial aspect in supplementing the financial market imperfection; the usage of bubble assets as collaterals for other finances makes up for the fault of financial markets such as financial friction across generations. Hence, we can conclude in this paper that the availability of naked CDSs is expected to be an effective policy method not only in eliminating the socially harmful aspect of a bubble in terms of intrinsic social cost but also in cultivating its socially beneficial aspect in making up for the fault of financial market imperfection.

2. Background and Related Literatures

2.1. Limited Arbitrage

The findings of this paper can be considered our theoretical contributions to the limited arbitrage literature; awareness heterogeneity is a newly introduced concept, for example. The bubble due to limited arbitrage has been investigated by several works such as De Long et al. (1990), Shleifer and Vishny (1992), and Abreu and Brunnermeier (2003). The interactions between the rational arbitrageurs and irrational noise traders (positive feedback traders) have been intensively investigated under assumptions of financial market imperfection such as limitation of short-selling.

Among these, the work by Abreu and Brunnermeier (2003) is especially relevant to this paper. Abreu and Brunnermeier assumed that the arbitrageurs have sufficient ability to correct mispricing back to the fundamental value, and then formulated their strategic aspects as a timing game with a bounded time horizon. In such a timing game, each arbitrageur selects a time to sell his (or her) stockholding; an arbitrageur who sells earlier than the others wins the game. The winner payoff is assumed to be greater than the loser payoff, and increases as time goes on.

However, with a bounded time interval, we need to assume some aspects of information asymmetry in order to describe the phenomenon of bubbles and crashes; otherwise, the backward induction method would logically prove that it is the unique Nash equilibrium that makes all arbitrageurs immediately sell their shareholdings at the initial time. In order to overcome such theoretical difficulties in describing bubbles and crashes, Abreu and Brunnermeier (2003) assumed a particular aspect of informational asymmetry, namely, sequential awareness.

Matsushima (2012) demonstrated an alternative concept of informational asymmetry, which is more tractable than sequential awareness, by defining the timing game with behavioral types. Matsushima eliminated the common knowledge assumption of arbitrageurs' rationality in such a simple manner that each arbitrageur is behavioral with a small but positive probability; a behavioral arbitrageur is committed not to time the market on his (or her) own accord.

The timing games with behavioral types have a strong predictive power in terms of uniqueness. If the probability to be behavioral is sufficient, there will be a unique Nash equilibrium, and this equilibrium would encourage the rational arbitrageurs to ride the bubble for a long time. Importantly, whenever the bubbles and crashes can be sustained as a Nash equilibrium, then almost surely it is a unique Nash equilibrium. In this case, the minimal probability required for them to ride the bubble could be an appropriate and tractable parameter for the likelihood of bubbles. This uniqueness property is in sharp contrast with other formulations such as overlapping generations' models (Samuelson (1958), Tirole (1985), and Martin and Ventura (2012)), which inevitably faced serious multiplicity problems.

Following such tractability, this paper generalizes Matsushima (2012) and then investigates the strategic aspects of arbitrageurs as its applications to the stock market. As an important development in the limited arbitrage literature, this paper permits a company to raise funds by issuing shares and the arbitrageurs to utilize their leverage for purchasing the shares. In order to ensure that these permissions are non-trivial and compatible with the models of limited arbitrage, we make an explicit assumption of awareness heterogeneity between arbitrageurs and positive feedback traders in terms of both positive feedback traders' reinforcement pattern and the risk of bubble's crash.

Note that the assumption that the positive feedback traders are unaware of their reinforcement pattern could be generally necessary for any model of limited arbitrage that assume that the positive feedback traders are unaware of the risk of the bubble's crash. Without this assumption, it would be inevitable for the positive feedback traders to immediately push the share prices up beyond what the model assumes, which would be a contradiction. The previous studies in the limited arbitrage literature, however, have not dealt with the positive feedback traders' unawareness so explicitly. This is the reason why the previous literature has only insufficiently investigated the impact of financial regulations and innovation on harmful bubbles.

2.2. Heterogeneity

Several authors have investigated the impact of financial regulation and innovation on bubbles and crashes in the general equilibrium literature (Geanakoplos (2010), Che and Sethi (2010), and Fostel and Geanakoplos (2012)). These studies assumed that the traders have heterogeneous priors, and the traders have no option to solve this heterogeneity; they have different beliefs about future price movement even if they share the same information. Based on this assumption, Fostel and Geanakoplos argued the possibility that unexpected introduction of naked CDS increases the default risks.

This paper does not assume such prior heterogeneity, but instead assumes awareness heterogeneity. In contrast with prior heterogeneity, the arbitrageurs in the present paper have the option to solve awareness heterogeneity. In this respect, the present paper is relevant to the literature of asymmetric information bubbles such as delegated fund management by Allen and Gorton (1993).

Prior and awareness heterogeneities have another point of difference: with prior heterogeneity, the optimists and pessimists disagree only in terms of default risk, while with awareness heterogeneity, the arbitrageurs and positive feedback traders disagree in terms of both default risk and share price growth. This difference substantially influences the traders' behavioral mode.

For instance, with prior heterogeneity, the optimists are willing to purchase shares and sell naked CDSs, while the pessimists are willing to sell naked CDSs but are not positive to purchase shares. With awareness heterogeneity, on the other hand, the positive feedback traders sell naked CDSs and the arbitrageurs are willing to purchase them. In this respect, the arbitrageurs and positive feedback traders correspond to the pessimists and optimists, respectively.

However, the arbitrageurs, unlike the pessimists, purchase shares in order to prevent a bubble's crash, while the positive feedback traders, unlike the optimists, do not expect any capital gain from shares. In this respect, this paper is related to the approach by Simsek (2012), who emphasized that the impact of belief disagreements on asset prices crucially depends on the specification of what kind of belief disagreement among traders the model assumes.

2.3. Outline of This paper

The remaining part of this paper is organized as follows. Section 3 defines the timing game with behavioral types by generalizing Matsushima (2012). In this definition, we assume that each player (arbitrageur) is not necessarily rational, i.e., they are behavioral with some positive probability. We then specify two symmetric strategy profiles, one for describing the bubbles and crashes, and the other for describing no emergence of bubbles. We show a necessary and sufficient condition for each strategy profile to be a Nash equilibrium. We also show a necessary and sufficient condition for the strategy profile specified for describing bubbles and crashes to be a unique Nash equilibrium. In order to clarify these characterizations, we introduce two measures for the likelihood of bubbles. Since we define the concept of timing game with behavioral types in a rather general manner, we can apply the basic arguments in Section 3 to more general economic problems—other than bubbles and crashes—such as R&D oligopolistic competitions, although this paper does not investigate them further.

Section 4 formulates the stock market and discusses awareness heterogeneity, leveraged finance by arbitrageurs, and share issuance. Section 5 then incorporates this formulation with the timing game with behavioral types as the basic model, which assumes that no CDS is available. We show that a high leverage ratio fosters the bubble.

In Section 6, we incorporate the basic model with the availability of covered CDS and use it as the covered CDS model. Covered CDS is defined as a standard insurance contract against the purchaser’s own default risks. We show that the availability of covered CDS dramatically fosters the bubble; this is the unique Nash equilibrium in which the bubble never crashes until termination.

In Section 7, we incorporate the basic model with the availability of naked CDS and use it as the naked CDS model. The section gives the main results of this paper: a high leverage ratio deters the emergence and persistence of bubbles, and whether the availability of naked CDS fosters the bubble crucially depends on the growth balance between the positive feedback traders’ personal capital and loan. Section 8 shows remarks. Section 9 concludes the paper.

3. Timing Games with Behavioral Types

Let us fix an arbitrary finite set of players $N = \{1, 2, ..., n\}$, where $n \geq 2$. This section defines a *timing game with behavioral types* in a more general manner than Matsushima (2012) in terms of specifications of winner and loser payoffs. From the next section, we will analyze the strategic aspects of arbitrageurs in a company's stock market as its applications.

Let $A_i = [0,1]$ denote the set of all pure strategies for each player $i \in N$. By selecting $a_i \in A_i$, player i plans to time the market at time a_i during a bounded time interval $[0,1]$. A mixed strategy, in short a strategy, for player i is denoted by a cumulative distribution $q_i : A_i \to R_+ \bigcup \{0\}$, where $q_i(t)$ implies the probability that the player times the market at or before time t, which is non-decreasing and right-continuous in t and satisfies $q_i(1) = 1$. Let us denote by Q_i the set of all strategies for player i. Let $Q = \underset{i \in N}{\times} Q_i$ and $q = (q_i)_{i \in N} \in Q$. We denote $q_i = a_i$ if player i selects pure strategy a_i with certainty.

Let us further consider any arbitrary real number $\varepsilon \in (0,1)$. We assume that each player is rational with a probability of $1 - \varepsilon > 0$ and *behavioral* with a probability of $\varepsilon > 0$. If the player is rational, he (or she) will conform to his selected strategy q_i. However, if the player is behavioral, he will not conform to q_i, and will never time the market on his own accord. Whether each player is rational or behavioral is determined independently, and is unknown to the other players.

We now consider an arbitrary pure strategy profile $a = (a_i)_{i \in N} \in A \equiv \underset{i \in N}{\times} A_i$ and an arbitrary non-empty subset of players $H \subset N$. We suppose that any player $i \in H$ is rational, while any player $i \in N \setminus H$ is behavioral. We denote by $\tau = \min_{j \in H} a_j$ the earliest time at which a rational player selects—i.e., the time at which the timing game ends. We denote by $l = \left| \{ j \in H \mid a_j = \tau \} \right|$ the number of rational players who select this ending time τ.

With a probability of $\frac{1}{l}$, each rational player $i \in H$ who selects τ becomes the winner of the timing game, earning the *winner payoff* given by $\bar{v}_i(\tau)$. We assume $\bar{v}_i(\tau)$ is differentiable in τ.[1] With regard to the remaining probability $\frac{l-1}{l}$, the player loses the timing game, earning the *loser payoff* given by $\underline{v}_i(\tau)$. Any player who does not select τ loses the timing game, earning $\underline{v}_i(\tau)$. Hence, the expected payoff for any rational player $i \in H$ can be given by

$$v_i(H,a) = \frac{1}{l}\bar{v}_i(\tau) + \frac{l-1}{l}\underline{v}_i(\tau) \qquad \text{if } a_i = \tau ,$$

and

$$v_i(H,a) \equiv \underline{v}_i(\tau) \qquad \text{if } a_i > \tau .$$

We assume that the winner payoff is not less than the loser payoff, and the winner payoff is non-decreasing:

$$\bar{v}_i(t) \geq \underline{v}_i(t) \quad \text{and} \quad \bar{v}_i'(t) \equiv \frac{\partial \bar{v}_i(t)}{\partial t} \geq 0 .$$[2]

With this assumption, each player is confronted with the trade-off between future benefit and instantaneous gain. Future benefit is expressed by either the derivative of the winner payoff at each time $t \in [0,1]$, $\bar{v}_i'(t)$, or the increase in winner payoff from initial time 0 to termination time 1, $\bar{v}_i(1) - \bar{v}_i(0)$. On the other hand, instantaneous gain is expressed by the difference between the winner and loser payoffs at each time $t \in [0,1]$, $\bar{v}_i(t) - \underline{v}_i(t)$, or simply that at the initial time 0, $\bar{v}_i(0) - \underline{v}_i(0)$. As will be explained later, relative future benefit to instantaneous gain, defined as either $\frac{\bar{v}_i'(t)}{\bar{v}_i(1) - \bar{v}_i(0)}$ or $\frac{\bar{v}_i(1) - \bar{v}_i(0)}{\bar{v}_i(0) - \underline{v}_i(0)}$, will be an appropriate parameter with regard to the player's incentives.

[1] We can easily extend this timing game to the case in which multiple players are winners. See Matsushima (2012).

[2] Abreu and Brunnermeier (2003) and Matsushima (2012) assumed that the winner payoff grows exponentially. This paper does not need to make such a specific assumption.

We define the payoff function $u_i(\cdot,\varepsilon):Q\to R$ for each player $i\in N$ as the expected value of $v_i(H,a)$, when the player is rational, expressed as

(1) $$u_i(q,\varepsilon)\equiv E[\sum_{H\subset N:i\in H} v_i(H,a)\,|\,q,\varepsilon].^3$$

A strategy profile $q\in Q$ is said to be a *Nash equilibrium* in the timing game with behavioral types associated with ε if

$$u_i(q,\varepsilon)\geq u_i(q_i',q_{-i},\varepsilon) \text{ for all } i\in N \text{ and all } q_i'\in Q_i .$$

We further define the probability that the timing game ends at or before $t\in[0,1]$ as

$$D(t;q,\varepsilon)\equiv 1-\prod_{i\in N}\{1-(1-\varepsilon)q_i(t)\}.$$

In case $D(t;q,\varepsilon)$ is differentiable in t, we define the *hazard rate* of the timing game's end at $t\in[0,1]$ as

$$\theta(t)\equiv\frac{D'(t;q)}{1-D(t;q)},$$

where $D'(t:q)\equiv\dfrac{\partial D(t:q)}{\partial t}$. In case the timing game does not end at or before time t, the game will end between time t and time $t+\Delta$, with a probability approximated by $\theta(t)\Delta$, where Δ is an arbitrary positive real number close to zero.

For each $i\in N$, we define the probability that the timing game will end at or before time t, provided player i has never timed the market before, by

$$D_i(t;q_{-i},\varepsilon)=1-\prod_{j\in N\setminus\{i\}}\{1-(1-\varepsilon)q_j(t)\}.$$

In case $D_i(t;q_{-i},\varepsilon)$ is continuous in t, we can rewrite (1) for $q_i=t$ as

$$u_i(t,q_{-i},\varepsilon)=\int_{\tau=0}^{t}\underline{v}_i(\tau)dD_i(\tau:q_{-i},\varepsilon)+\overline{v}_i(t)\{1-D_i(t:q_{-i},\varepsilon)\}.$$

Hence, the first-order condition for Nash equilibrium is given by

(2) $$\{\overline{v}_i(t)-\underline{v}_i(t)\}D_i'(t:q_{-i},\varepsilon)=\overline{v}_i'(t)\{1-D_i(t:q_{-i},\varepsilon)\}.$$

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[3] $E[\cdot\,|\,q,\varepsilon]$ denotes the expectation operator conditional on (q,ε).

The left-hand side of (2) implies the marginal loss induced by decreases in winning probability, while the right-hand side of (2) implies the marginal gain induced by increases in winner payoff.

This paper assumes that the timing game is *symmetric*:

$$\bar{v}_1(t) = \bar{v}_i(t) \text{ and } \underline{v}_1(t) = \underline{v}_i(t) \text{ for all } i \in N \text{ and all } t \in [0,1].$$

A strategy profile $q \in Q$ is said to be *symmetric* if

$$q_i = q_1 \text{ for all } i \in N .$$

For any symmetric strategy profile q, we derive the following equations, both of which express first-order condition (2) in different forms:

(3) $$\theta(t) = \frac{n}{n-1} \frac{\bar{v}_1'(t)}{\bar{v}_1(t) - \underline{v}_1(t)},$$

and

(4) $$\frac{(1-\varepsilon)q_1'(t)}{1-(1-\varepsilon)q_1(t)} = \frac{\bar{v}_1'(t)}{(n-1)\{\bar{v}_1(t) - \underline{v}_1(t)\}}.$$

Note that from (3), the hazard rate $\theta(t)$ is proportional to relative future benefit given by $\dfrac{\bar{v}_1'(t)}{\bar{v}_1(t) - \underline{v}_1(t)}$.

From (4), we can specify a unique symmetric and continuous strategy profile, denoted by $q = \tilde{q} = \tilde{q}(\varepsilon) \in Q$, as follows:

(5) $$\tilde{q}_1(t) = \frac{1 - \{1-(1-\varepsilon)\tilde{q}_1(\tilde{\tau})\}\exp[-\frac{1}{n}\int_{\tau=\tilde{\tau}}^{t} \theta(\tau)d\tau]}{1-\varepsilon} \text{ for all } t \in [\tilde{\tau},1],$$

and

$$\tilde{q}_1(t) = 0 \text{ for all } t \in [0,\tilde{\tau}) .$$

We will name $\tilde{\tau} = \tilde{\tau}(\varepsilon) \in [0,1)$ the *critical time*, which can be uniquely defined as either

(6) $$\tilde{\tau} \geq 0 \text{ and } \varepsilon = \exp[-\frac{1}{n}\int_{\tau=\tilde{\tau}}^{1} \theta(\tau)d\tau],$$

or

$$\tilde{\tau} = 0 \text{ and } \varepsilon = \{1-(1-\varepsilon)\tilde{q}_1(0)\}\exp[-\frac{1}{n}\int_{\tau=0}^{1} \theta(\tau)d\tau].$$

According to $\tilde{q}$, no player times the market before critical time $\tilde{\tau}$. At every time t after $\tilde{\tau}$, the timing game randomly ends according to the hazard rate $\theta(t)$ given by (3). Note that the greater the hazard rate $\theta(\tau)$, the greater is the critical time $\tilde{\tau}$. This paper regards the specified profile $\tilde{q}$ as describing bubbles and crashes.

Let us now define

(7) $$I_1 \equiv \exp[-\frac{1}{n}\int_{\tau=0}^{1}\theta(\tau)d\tau].$$

Note that the value of $-n\ln I_1$ is equivalent to the integral of the hazard rate, i.e., $\int_{\tau=0}^{1}\theta(\tau)d\tau$. We define I_1 as the *first measure* for the likelihood of the timing game's early end. Note that the smaller I_1 is, the greater the hazard rate $\theta(\tau)$, i.e., the greater the critical time $\tilde{\tau}$.

By utilizing this measure of I_1, we show a necessary and sufficient condition for $\tilde{q}$ to be a Nash equilibrium. We also show a necessary and sufficient condition for $\tilde{q}$ to be a unique Nash equilibrium.

Theorem 1: *The strategy profile* $\tilde{q}$ *is a Nash equilibrium if and only if* $I_1 \leq \varepsilon$. *It is a unique Nash equilibrium if strict inequality holds, i.e.,* $I_1 < \varepsilon$.

Proof: See Appendix A.

Theorem 1 implies that the greater I_1 is, the greater the probability of ε to be behavioral, which is necessary for the timing game to persist for a long time. Theorem 1 implies that if $\tilde{q}$ is a Nash equilibrium, it will satisfy (6); according to $\tilde{q}$, no player times the market at the initial time 0; that is,

$$\tilde{q}_i(\tilde{\tau}) = 0 \quad \text{for all} \quad i \in N.$$

Theorem 1 suggests that the timing games with behavioral types have a strong predictive power in terms of uniqueness. If $\tilde{q}$ is a Nash equilibrium, then almost surely it is a unique Nash equilibrium. Moreover, if there is a Nash equilibrium in which no player times the market at the initial time, then almost surely it is a unique Nash equilibrium. In other word, if the bubbles and crashes can be sustained as a Nash equilibrium, almost surely it is a unique Nash equilibrium outcome.

We further specify another symmetric strategy profile $q^* = (q_i^*)_{i \in N}$ as

$$q_i^*(0) = 1 \text{ for all } i \in N.$$

According to q^*, any rational player times the market at the initial time 0 with certainty. This paper regards q^* as describing no emergence of bubbles. Let us define

$$(8) \qquad I_2 \equiv \frac{\overline{v}_1(0) - \underline{v}_1(0)}{\overline{v}_1(1) - \overline{v}_1(0)}.$$

Note that $1/I_2$ is another expression of relative future benefit to instantaneous gain. We will define I_2 as the *second measure* for the likelihood of the timing game's early end. From this measure of I_2, we can show a necessary and sufficient condition for q^* to be a Nash equilibrium.

Theorem 2: *The strategy profile q^* is a Nash equilibrium if and only if*

$$I_2 \geq [\sum_{1 \leq l \leq n-1} \frac{(n-1)!}{l!(n-1-l)!} (\frac{1-\varepsilon}{\varepsilon})^l \frac{1}{l+1}]^{-1}.$$

Proof: See Appendix B.

Since the right-hand side of the inequality in Theorem 2 is increasing in ε, it follows that the greater I_2 is, the greater the probability of ε to be behavioral, which is required for eliminating the possibility that the timing game will end at the initial time 0.

4. Stock Market

This section models the market for a company's stock during the bounded time interval $[0,1]$. The next sections will analyze the strategic aspects in the stock market by incorporating this section's model with the timing game with behavioral types defined in the previous section.

This section extends Matsushima (2012) and is related to Abreu and Brunnermeier (2003). In contrast to these works, this paper explicitly assumes that the positive feedback traders are unaware of their own euphoria, the company can issues shares, and the arbitrageurs can utilize their leveraged finance. We assume financial market imperfection as being necessary for describing limited arbitrage, such that short selling is severely limited. Moreover, for simplicity of arguments, we assume that the market interest rate is set at zero, no dividends are paid, and the arbitrageurs are risk-neutral.

The company has no profitable business opportunity, and therefore, its fundamental value is set at zero. The company has no debt capacity, and it raises funds only by issuing shares during the bubble.[4] Let us denote by $S(t) > 0$ the total share that the company has issued up to time $t \in [0,1]$, where $S(t)$ is non-decreasing in t. During a short time interval $[t, t + \Delta]$, the company issues approximately $S'(t)\Delta$ number of shares.

There are multiple *arbitrageurs,* $n \geq 2$, each of whom decides the time to sell his (or her) shareholding. In case the bubble does not crash at or before $t \in [0,1]$, we denote by $S_i(t) > 0$ the number of shares that arbitrageur i possesses at time t, where $S_i(t)$ is non-decreasing in t. During a short time interval $[t, t + \Delta]$, the arbitrageur purchases approximately $S_i'(t)\Delta$ number of shares.

[4] Empirical studies such as Stein (1996), Baker, Stein, and Wurgler (2003), and Polk and Sapienza (2008) examine whether a deviation from the fundamental value affects real investment. Baker, Stein, and Wurgler (2003) pointed out that the investment of a company with limited debt capacity is sensitive to stock price fluctuations. Empirical studies such as Adrian and Shin (2010) point out that financial intermediaries are sensitive to fluctuations in asset prices. These empirical evidences are consistent with this paper's models.

The share price grows following a continuous and increasing function $P:[0,1]\to(0,\infty)$, and the bubble persists as long as the arbitrageurs continue to hold $n\phi\times 100\,\%$ of the company's stock or more in totality, where $0<\phi<1/n$. Once the arbitrageurs' total shareholdings fall to less than $n\phi\times 100\,\%$, the bubble crashes immediately and the share price declines to zero, i.e., the fundamental value.[5] Even if no arbitrageur sells, the bubble automatically crashes just after termination time 1 for exogenous reasons.[6]

It is reasonable to assume that there are many *positive feedback traders* who are slaves to euphoria; at any time $t\in[0,1]$ during the bubble, they misperceive the current share price $P(t)$ as reflecting the fundamental value, and further reinforce their misperception according to P. We regard them as being *enthusiastic* if $P'(t)\equiv\frac{dP(t)}{dt}$ is sufficiently large for each $t\in[0,1]$. However, once the arbitrageurs' total shareholdings fall to less than $n\phi\times 100\,\%$, the resultant selling pressure would force the positive feedback traders out of euphoria, and they would become aware of the correct fundamental value, immediately bursting the bubble.

For example, we suppose that the number of positive feedback traders is fixed across times, and each positive feedback trader attempts to purchase shares by keeping the proposition of his (or her) shareholding to the total share constant across times. In this case, if some arbitrageurs sell their shareholdings at a time t, the excessive supply exist at the share price $P(t)$, which decrease the market price, dampening the positive feedback trader's euphoria.

The substantial assumption of this paper is *awareness heterogeneity*: the positive feedback traders are unaware of their own euphoria, i.e., their own reinforcement pattern and the risk of the bubble's crash, while the arbitrageurs are well aware of both

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[5] This paper assumes that a single arbitrageur can correct mispricing back to fundamental value. Abreu and Brunnermeier (2003) considered a synchronization problem such that the arbitrageurs need to coordinate to correct mispricing; this need along with some lack of higher-order mutual knowledge makes the bubble persist longer. This paper does not consider such a synchronization issue; however, we can incorporate it with this paper's models without major changes.

[6] For example, if the arbitrageurs expect the share price to stop growing after the termination time 1, the bubble automatically crashes just after termination time 1.

aspects of the positive feedback traders' euphoria. The positive feedback traders incorrectly expect the current price to never change over time, even if they actually change their minds and continue reinforcing their misperception in an unconscious manner.

In contrast, the arbitrageurs well recognize the positive feedback traders' unconscious reinforcement pattern. As will be explained later, by utilizing this awareness heterogeneity, the arbitrageurs gain an advantageous position in borrowing money and purchasing CDSs from the positive feedback traders.

The company raises funds by issuing shares during the bubble, but within a limit. Note that if the company issues too many shares for each arbitrageur to keep his shareholding not less than $\phi \times 100\,\%$, the resultant selling pressure will burst the bubble. Hence, for the company to raise funds without causing the bubble to crash, every arbitrageur has to purchase the sufficient number of shares to keep his shareholding not less than $\phi \times 100\,\%$.

In this case, an effective method would be for the company to encourage each arbitrageur to borrow money from the positive feedback traders. Because of their lack of knowledge and unawareness, the positive feedback traders do not perceive the risk of the share prices slumping. Hence, provided the positive feedback traders have sufficient personal capital, any arbitrageur can enter into short-term debt contracts with the positive feedback traders with no premium in non-recourse manners, where his shareholdings will be utilized as collaterals.

We set an exogenous cap for the *leverage ratio*, i.e., the upper limit for the total asset divided by personal capital, equal to $L \geq 1$. Since any arbitrageur is in a better position when he lets his leverage ratio be equal to this upper limit, he will have a debt obligation of $\frac{L-1}{L} P(t) S_i(t)$ to his debt holders (positive feedback traders) at any time t during the bubble. Because of the non-recourse nature of the contract, any arbitrageur will let his debt holders bear $\frac{L-1}{L} \times 100\%$ of the loss caused by the crash.

From these observations, the *personal capital* of each arbitrageur i, denoted by $W_i(t)$, is expressed as the market value of his shareholding minus his debt obligation:

(9) $$W_i(t) = P(t)S_i(t) - \frac{L-1}{L}P(t)S_i(t) = \frac{P(t)S_i(t)}{L}.$$

Since arbitrageur i earns a capital gain $\{P(t+\Delta) - P(t)\}S_i(t)$ from time t to time $t+\Delta$, his personal capital increases approximately by this amount:

$$W_i(t+\Delta) = W_i(t) + \{P(t+\Delta) - P(t)\}S_i(t),$$

which implies

$$W_i'(t) = P'(t)S_i(t).$$

From (9),

$$W_i'(t) = \frac{P(t)S_i'(t) + P'(t)S_i(t)}{L}.$$

From these two equations, $(P(0), S(0), L)$ uniquely determines arbitrageur $i's$ shareholding, where for every $t \in [0,1]$,

$$P'(t)S_i(t) = \frac{P(t)S_i'(t) + P'(t)S_i(t)}{L},$$

that is,

$$S_i(t) = S_i(0)(\frac{P(t)}{P(0)})^{L-1}.$$

For simplicity, we assume that the arbitrageurs possess the same number of shares at initial time 0:

$$S_i(0) = S_1(0) \text{ for all } i \in \{1,...,n\}.$$

With this assumption, it is clear that the arbitrageurs possess the same number of shares across all time during the bubble:

$$S_i(t) = S_1(t) \text{ for all } i \in \{1,...,n\} \text{ and all } t \in [0,1].$$

In order to maintain the persistence of the bubble, the company has to better keep the number of shares that each arbitrageur possesses equal to $\phi \times 100$%:

$$S_i(t) = S_1(t) = \phi S(t).$$

From the above observations, the total share can be expressed as

(10) $$S(t) = S(0)(\frac{P(t)}{P(0)})^{L-1},$$

and the personal capital of each arbitrageur i can be expressed as

(11) $$W_i(t) = \frac{\phi}{L} P(0)S(0)(\frac{P(t)}{P(0)})^L .$$

Since the company has no profitability, we can define the *intrinsic social cost of the bubble* as the total funds that the company raised through share issuance from the initial time 0 to time t at which the bubble bursts, which can be expressed by

$$C(t) \equiv \int_{\tau=0}^{t} P(\tau)S'(\tau)d\tau .$$

From (10), we derive

$$C(t) = P(0)S(0)\frac{L-1}{L}\{(\frac{P(t)}{P(0)})^L - 1\} = \frac{L-1}{L}\{P(t)S(t) - P(0)S(0)\}.$$

The relative intrinsic social cost to increase in the company's market value is equal to

$$\frac{C(t)}{P(t)S(t) - P(0)S(0)} = \frac{L-1}{L} .$$

Hence, the higher the leverage ratio L, the greater is the relative intrinsic social cost.

5. Bubbles and Crashes: Basic Model

We regard the arbitrageurs as players and model their strategic aspects in the stock market as a timing game with behavioral types, incorporated with the formulation of the stock market in Section 4. This section demonstrates the *basic model*, where CDSs are not available. The next sections will demonstrate two further models, in which some forms of CDSs are available.

By timing the market at time $t \in [0,1]$, arbitrageur (player) $i \in N$ obtains a monetary amount $P(t)S_i(t)$ and repays his debt obligation $\frac{L-1}{L}P(t)S_i(t)$. Hence, the winner payoff could be specified as equivalent to his personal capital:

$$\bar{v}_i(t) = P(t)S_i(t) - \frac{L-1}{L}P(t)S_i(t) = W_i(t)$$
$$= \frac{\phi}{L}P(0)S(0)(\frac{P(t)}{P(0)})^L.$$

If the player fails to sell his shares before the bubble crash, the market value of his shareholding would decline to zero. However, from the non-recourse nature of the contract, the player is exempted from repayment once his defaulted shareholdings are seized by his lenders. Hence, the loser payoff can be specified as zero:

$$\underline{v}_i(t) = 0 .$$

From the above specifications, it follows that

$$\bar{v}_i(t) - \underline{v}_i(t) = \frac{\phi}{L}P(0)S(0)(\frac{P(t)}{P(0)})^L ,$$

and

$$\bar{v}_i'(t) = W_i'(t) = \phi P'(t)S(0)(\frac{P(t)}{P(0)})^{L-1} .$$

Note that

$$\bar{v}_i(0) - \underline{v}_i(0) = \frac{\phi}{L}P(0)S(0) ,$$

and

$$\bar{v}_i(1) - \bar{v}_i(0) = \frac{\phi}{L}P(0)S(0)\{(\frac{P(1)}{P(0)})^L - 1\} .$$

Using these equations, we can replace equations (3), (5), (6), (7), and (8) with

(12) $$\theta(t) = L\frac{n}{n-1}\frac{P'(t)}{P(t)},$$

$$\tilde{q}_1(t) = \frac{1-\{1-(1-\varepsilon)\tilde{q}_1(\tilde{\tau})\}(\frac{P(\tilde{\tau})}{P(t)})^{\frac{L}{n-1}}}{1-\varepsilon},$$

(13) $$\tilde{\tau} \geq 0 \text{ and } \varepsilon = (\frac{P(\tilde{\tau})}{P(1)})^{\frac{L}{n-1}},$$

(14) $$I_1 = (\frac{P(0)}{P(1)})^{\frac{L}{n-1}},$$

and

(15) $$I_2 = \frac{1}{(\frac{P(1)}{P(0)})^L - 1},$$

respectively.

In case the positive feedback traders are more enthusiastic, i.e., $P'(t)$ is greater at any time $t \in [0,1]$, then it follows from (12), (13), (14), and (15) that the hazard rate θ and critical time $\tilde{\tau}$ are greater, and the measures I_1 and I_2 are smaller. Hence, the bubble is more likely to emerge and persist for a long time if the positive feedback traders are more enthusiastic.

Since $\theta(t)$, $\tilde{\tau}$, I_1, and I_2 are dependent on leverage ratio L, we denote

$$\theta(t) = \theta^*(t,L),\ \tilde{\tau} = \tilde{\tau}^*(L),\ I_1 = I_1^*(L), \text{ and } I_2 = I_2^*(L).$$

From (12) and (13), we find that the hazard rate $\theta^*(t,L)$ and critical time $\tilde{\tau}^*(L)$ are increasing in L. Further, from (14) and (15), we find that the first measure $I_1^*(L)$ and second measure $I_2^*(L)$ are decreasing in L. Hence, *in the basic model, the greater the leverage ratio L, the more likely it is for the bubble to emerge and persist for a long time.* Note that even if the positive feedback traders are not very enthusiastic, i.e., even if $P'(t)$ is not sufficient across all time, the bubble is likely to emerge and persist for a long time whenever the leverage ratio L is sufficient.

6. Covered Credit Default Swap

In this section and the next, we investigate the case in which at any time $t \in [0,1]$ before the crash, the arbitrageurs purchase *credit default swaps* (CDSs) from the positive feedback traders. According to the purchased CDSs, each arbitrageur i receives monetary payment $Z_i(t)$ from the sellers at any time t when the bubble crashes.

Since the positive feedback traders misperceive the bubble to never crash, they regard the CDSs *worthless*. On the other hand, the positive feedback traders are sensitive to the price of the CDSs; the observation of positive CDS prices immediately brings them out of their euphoria, because such positivity reveals that the share prices do not reflect the fundamental value. In order to eliminate the possibility of such a positivity causing the crash of the bubble, we assume that the CDS prices are kept at zero across all time frames.

We will now classify the CDSs into two different contractual forms, the *covered CDS* and *naked CDS*. The covered CDS is defined as the insurance against the default risk of the purchaser's own shareholdings in a standard manner, while the naked CDS is defined as insurance against third-party default risks with regard to the company's market value.

This section demonstrates the *covered CDS model*, where only covered CDSs are available. We show that the availability of covered CDSs dramatically fosters a *non-crash* bubble; in this case, the bubble will never crash before its termination time 1.

In order to purchase covered CDSs from the positive feedback traders, any arbitrageur i has to hold some underlying shares against the default of which to hedge. In this case, at any time t before the crash, any arbitrageur i would prefer to cancel his covered CDSs that he purchased, and repurchase the covered CDSs that would provide him with the right to receive payments from the seller amounting to the current share prices multiplied by his current shareholdings. Thus, any arbitrageur i possesses $S_i(t) = \dfrac{Z_i(t)}{P(t)}$ number of shares at any time t; the payment from the sellers of the covered CDS can be expressed as

$$Z_i(t) = P(t)S_i(t) = \phi P(0)S(0)(\frac{P(t)}{P(0)})^L .$$

In this section, we assume that the positive feedback traders have enough reserve money to pay for the covered CDSs.

Since the covered CDSs would hedge only the purchaser's own default risks, it follows that if the arbitrageur i sells before the bubble crashes, he will inevitably lose his right to receive $Z_i(t)$. On the other hand, if arbitrageur i fails to sell before the crash, he can surely receive $Z_i(t)$. This implies that arbitrageur i has debt obligations $\frac{L-1}{L}P(t)S_i(t)$ irrespective of whether he is the winner of the timing game. Hence, in the covered CDS model, the winner payoff could be specified as

$$\overline{v}_i(t) = W_i(t) ,$$

which is the same as the winner payoff in the basic model. On the other hand, the loser payoff can be specified as

$$\underline{v}_i(t) = Z_i(t) - (L-1)W_i(t) = W_i(t) ,$$

which is greater than the winner payoff in the basic model. Clearly, the winner and loser payoffs are equivalent:

$$\overline{v}_i(t) = \underline{v}_i(t) .$$

Thus, in the covered CDS model, any purchaser of a covered CDS will lose the merit of non-recourse nature; the purchaser has to assign the payment from the seller of covered CDSs, $Z_i(t)$, to his debt obligation, $\frac{L-1}{L}P(t)S_i(t)$, even if he is a loser. This is the reason why the winner and loser payoffs are the same, i.e., the default risks are fully insured. This equivalence indicates that no arbitrageur will have an incentive to sell before the crash; *with the positivity of ε, it is the unique Nash equilibrium in the covered CDS model by which the bubble will never crash and will persist up to termination time 1 with certainty.*[7]

This section assumed that the positive feedback traders have enough reserve money to pay for the covered CDSs. Without this assumption, the arbitrageurs are willing to strategically keep their demand for covered CDS not greater than the positive feedback

[7] We need the positivity of ε in order to solve the multiplicity of Nash equilibria.

traders' supply limit. By doing this manner, the arbitrageurs can prevent the positive feedback traders from waking up out of enthusiasm. We can argue on further details in the next section. In this case, the winner payoff is no longer the same as the loser payoff; it is a positive probability with which the bubble crashes before termination 1.

7. Naked Credit Default Swap

This section demonstrates the *naked CDS model*, in which naked CDSs are available. The naked CDS is defined as the insurance against third-party default risks; in contrast with covered CDSs, the purchaser of naked CDSs is not required to hold the underlying shares against the default of which to hedge. Hence, he can receive the payment from the sellers of naked CDS even if he sells out his shareholdings.[8]

The payment from the sellers of naked CDSs (positive feedback traders) in totality would depend on the personal capital of the positive feedback traders in totality, which is denoted by $B(t) > 0$. We assume the personal capital of positive feedback traders $B(t)$ grows across all time frames; $B(t)$ is differentiable and non-decreasing in t; i.e., $B'(t) \geq 0$. We assume that CDSs are secured credits in that its seller is required to have full reserve to pay for its purchaser. Therefore, $B(t)$ must be equal to the market value of the positive feedback traders' shareholdings $(1-n\phi)P(t)S(t)$, their loan to arbitrageurs $\frac{L-1}{L}n\phi P(t)S(t)$, and their reserve for the payment of naked CDSs $nZ_1(t)$:

$$\begin{aligned} B(t) &= (1-n\phi)P(t)S(t) + \frac{L-1}{L}n\phi P(t)S(t) + nZ_1(t) \\ &= (1-\frac{n\phi}{L})P(0)S(0)(\frac{P(t)}{P(0)})^L + nZ_1(t). \end{aligned}$$

Each arbitrageur i, who purchases naked CDSs, has the right to receive payment from the sellers of naked CDSs, which is given by

$$Z_i(t) = \frac{1}{n}B(t) - (\frac{1}{n} - \frac{\phi}{L})P(0)S(0)(\frac{P(t)}{P(0)})^L. \tag{16}$$

It is important to assume that the arbitrageurs' demand for naked CDSs is not greater than the positive feedback traders' supply. Thus, the arbitrageurs can keep the prices of naked CDSs equal to zero; they strategically hide information about mispricing from the positive feedback traders who are sensitive to naked CDS prices.

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[8] Note that the arbitrageurs always prefer to purchase naked CDS rather than covered CDS, even if both naked CDSs and covered CDSs are available.

For simplicity of arguments, we assume that the positive feedback traders have sufficient personal capital, so that $B(t)$ is large enough to satisfy the condition that the payment from the seller of naked CDSs to each arbitrageur i is greater than his debt obligation:

$$Z_1(t) > \frac{L-1}{L}\phi P(0)S(0)(\frac{P(t)}{P(0)})^L \quad \text{for all} \quad t \in [0,1].$$

This assumption guarantees that any arbitrageur can repay his debt obligation in full even if he is a loser; he cannot enjoy the merit of nonrecourse nature of debt finance with positive feedback traders.

From these observations, the winner payoff can be specified as the sum of the personal capital $W_i(t)$ and payment of naked CDSs $Z_i(t)$:

$$\begin{aligned}\overline{v}_i(t) &= W_i(t) + Z_i(t) \\ &= \frac{1}{n}B(t) - (\frac{1}{n} - \frac{2\phi}{L})P(0)S(0)(\frac{P(t)}{P(0)})^L,\end{aligned}$$

which is greater than the winner payoff in the basic model. The loser payoff can be specified as the payment of the sellers of naked CDSs minus their debt obligation:

$$\begin{aligned}\underline{v}_i(t) &= -(L-1)W_i(t) + Z_i(t) \\ &= \frac{1}{n}B(t) - \{\frac{1}{n} + \frac{(L-2)\phi}{L}\}P(0)S(0)(\frac{P(t)}{P(0)})^L,\end{aligned}$$

which is greater than the loser payoff in the basic model too. The difference between the winner payoff and loser payoff, i.e., the instantaneous gain, is equivalent to the market value of his shareholding:

$$\overline{v}_i(t) - \underline{v}_i(t) = W_i(t) = \phi P(0)S(0)(\frac{P(t)}{P(0)})^L, \tag{17}$$

which is greater than the difference in the basic model. The derivative of the winner payoff, i.e., the future benefit, can be given by

$$\overline{v}_i'(t) = W_i'(t) + Z_i'(t) = \frac{1}{n}B'(t) - L(\frac{1}{n} - \frac{2\phi}{L})P'(t)S(0)(\frac{P(t)}{P(0)})^{L-1}, \tag{18}$$

which is greater than the derivative in the basic model too, provided the payment of naked CDSs is increasing; i.e., $Z_i'(t) > 0$. Since both the future benefit and instantaneous gain could be greater in the naked CDS model than in the basic model, whether the relative future benefit is greater in the naked CDS model than in the basic model depends on finer details of the model specifications.

From (17) and (18), we can replace (3) and (8) with

(19) $$\theta(t) = \frac{1}{\phi(n-1)}\{\frac{B'(t)}{P(0)S(0)(\frac{P(t)}{P(0)})^L} - (L-2n\phi)\frac{P'(t)}{P(t)}\},$$

and

(20) $$I_2 = \frac{n\phi P(0)S(0)}{B(1)-B(0)-(1-\frac{2n\phi}{L})P(0)S(0)\{(\frac{P(1)}{P(0)})^L - 1\}},$$

respectively. Note that the more enthusiastic the positive feedback traders are, the greater the hazard rate θ and critical time $\tilde{\tau}$, and the smaller the measures I_1 and I_2. Hence, similarly to the basic model, the bubble is more likely to emerge and persist for a long time in the naked CDS model if the positive feedback traders are more enthusiastic.

7.1. Impact of Availability

From (6), (7), (19), and (20), it follows that $\theta(t)$, $\tilde{\tau}$, I_1, and I_2 are dependent on leverage ratio L; let us now denote

$$\theta(t) = \theta^{**}(t,L),\ \ \tilde{\tau} = \tilde{\tau}^{**}(L),\ \ I_1 = I_1^{**}(L), \text{ and } \ I_2 = I_2^{**}(L).$$

The following theorem clarifies whether the availability of naked CDS will deter the bubble and when.

Theorem 3: *If*

(21) $$Z_1'(t) < (L-1)W_1'(t) \quad \textit{for all} \quad t \in [0,1],$$

then,

$$\theta^{**}(t,L) > \theta^{*}(t,L) \quad \textit{for all} \quad t \in [\tilde{\tau}^{*}(L),1],$$

$$\tilde{\tau}^{**}(L) \le \tilde{\tau}^{*}(L),\ \ I_1^{**}(L) > I_1^{*}(L), \textit{ and } \ I_2^{**}(L) > I_2^{*}(L).$$

If

(22) $$Z_1'(t) > (L-1)W_1'(t) \quad \textit{for all} \quad t \in [0,1],$$

then,

$$\theta^{**}(t,L) < \theta^{*}(t,L) \quad \textit{for all} \quad t \in [\tilde{\tau}^{**}(L),1],$$

$$\tilde{\tau}^{**}(L) \geq \tilde{\tau}^{*}(L), \quad I_1^{**}(L) < I_1^{*}(L), \textit{ and } \quad I_2^{**}(L) < I_2^{*}(L).$$

Proof: From (16),

$$Z_i'(t) = \frac{1}{n}B'(t) - L(\frac{1}{n} - \frac{\phi}{L})P'(t)S(0)(\frac{P(t)}{P(0)})^{L-1}.$$

From (9) and (10),

$$(L-1)W_1'(t) = (L-1)\phi P'(0)S(0)(\frac{P(t)}{P(0)})^{L-1}.$$

Hence, from (21),

(23) $$L\frac{n}{n-1}\frac{P'(t)}{P(t)} > \frac{1}{\phi(n-1)}\{\frac{B'(t)}{P(0)S(0)(\frac{P(t)}{P(0)})^{L}} - (L-2n\phi)\frac{P'(t)}{P(t)}\},$$

which along with (12) and (19) implies

$$\theta^{*}(t) > \theta^{**}(t) \quad \text{for all} \quad t \in [\max[\tilde{\tau}^{*}(L), \tilde{\tau}^{**}(L)],1].$$

Hence, it is clear from (6) and (7) that

$$\tilde{\tau}^{**}(L) \geq \tilde{\tau}^{*}(L) \quad \text{and} \quad I_1^{**}(L) > I_1^{*}(L).$$

Equation (23) is equivalent to

$$B'(t) < L(1-\frac{n\phi}{L})P'(t)S(0)(\frac{P(t)}{P(0)})^{L-1} + (L-1)n\phi P'(0)S(0)(\frac{P(t)}{P(0)})^{L-1}.$$

By integrating both the sides of this inequality from $t=0$ to $t=1$, we obtain

$$B(1) - B(0) < (1+\frac{(L-2)n\phi}{L})P(0)S(0)\{(\frac{P(1)}{P(0)})^{L} - 1\},$$

which is equivalent to

$$\frac{1}{(\frac{P(1)}{P(0)})^{L} - 1} < \frac{n\phi P(0)S(0)}{B(1) - B(0) - (1-\frac{2n\phi}{L})P(0)S(0)\{(\frac{P(1)}{P(0)})^{L} - 1\}}.$$

This inequality, along with (15) and (20), implies that $I_2^{**}(L) > I_2^{*}(L)$.

The latter part of this theorem can be proved in the same manner as above.

Q.E.D.

The first part of Theorem 3 states that *whenever the loan to arbitrageurs* $n(L-1)W_1(t)$ *grows more rapidly than the payment of naked CDS* $nZ_1(t)$*, the bubble is less likely to emerge and persist for a long time in the naked CDS model than in the basic model*; the availability of naked CDSs can deter the bubble in this case.

As time goes on, the greater part of the personal capital of the positive feedback traders must be ploughed back into the loan to arbitrageurs with precedence of the reserve for naked CDSs; otherwise, the resultant selling pressure would burst the bubble. If the personal capital of the positive feedback traders does not grow adequately, the loan to the arbitrageurs will grow more rapidly than the payment of naked CDSs. In this case, the relative future benefit will be lesser in the naked CDS model than in the basic model. This implies that any arbitrageur will have more incentives to sell earlier in the naked CDS model than in the basic model. Even if the availability of naked CDSs inevitably enhances the winner payoffs over time, it would reduce the relative future benefits. This reduction in relative terms is the driving force for the arbitrageurs to sell early.

In contrast, the latter part of Theorem 3 states that *whenever the loan to arbitrageurs grows less rapidly than the payment of naked CDSs, the bubble is more likely to emerge and persist for a long time in the naked CDS model than in the basic model*; the availability of naked CDSs increases the relative future benefit, fostering the bubble, but not as much as in covered CDSs.

We further divide the increase in personal capital of the positive feedback traders $B'(t)\Delta$ into two parts, i.e., the capital gain $(1-n\phi)P'(t)S(t)\Delta$, and the exogenous increase $h(t)\Delta$; we can write

$$B'(t) = (1-n\phi)P'(t)S(t) + h(t) = (1-n\phi)P'(t)S(0)\left(\frac{P(t)}{P(0)}\right)^{L-1} + h(t),$$

which is increasing in L. From (11) and (16), we can replace (21) with

$$h(t) < (L-1+n\phi)P'(t)S(0)\left(\frac{P(t)}{P(0)}\right)^{L-1} \text{ for all } t \in [0,1].$$

It is clear that if the leverage ratio L is sufficient and the positive feedback traders are enthusiastic, i.e., $P'(t)$ and $\frac{P(t)}{P(0)}$ are sufficient, then the above inequalities are likely to hold. It is important to note that the availability of naked CDS is likely to deter the bubble in this case. On the other hand, if the leverage ratio L is insufficient and the positive feedback traders are not very enthusiastic, i.e., $P'(t)$ and $\frac{P(t)}{P(0)}$ are insufficient, then the above inequalities are unlikely to hold; from (22), the availability of naked CDS is likely to foster the bubble.

Hence, we can conclude that ensuring the availability of naked CDSs might be an effective policy method to deter the bubble when there is concern about the huge social harm in terms of wasting funds. On the other hand, the availability of naked CDS can even foster the bubble when there is little concern about such a social harm.

7.2. Impact of High Leverage Ratio

We examine about the impact of the increase in leverage ratio on bubbles and crashes as follows.

Theorem 4: *Suppose* $L > 2n\phi$. *Then, it follows that*

$$\frac{\partial}{\partial L}\theta^{**}(t,L) < 0 \quad \textit{for all} \quad t \in (\tilde{\tau}^*(L),1],$$

$$\frac{\partial}{\partial L}\tilde{\tau}^{**}(L) < 0, \quad \frac{\partial}{\partial L}I_1^{**}(L) > 0, \textit{ and } \quad \frac{\partial}{\partial L}I_2^{**}(L) > 0.$$

Proof: Note that

$$\frac{\partial}{\partial L}\{\frac{B'(t)}{P(0)S(0)(\frac{P(t)}{P(0)})^L}\} = \frac{\partial}{\partial L}\{\frac{(1-n\phi)P'(t)P(0)}{P(t)} + \frac{h(t)}{P(0)S(0)(\frac{P(t)}{P(0)})^L}\} < 0,$$

which along with (19) implies that

$$\frac{\partial}{\partial L}\theta^{**}(t,L) < 0 \quad \text{for all} \quad t \in (\tilde{\tau}^*(L),1].$$

Since $B'(t)$ is increasing in L, it follows that $B(1)$ is increasing in L, which along with (20) implies that

$$\frac{\partial}{\partial L} I_2^{**}(L) > 0.$$

These inequalities along with (6) and (7) imply that

$$\frac{\partial}{\partial L} \tilde{\tau}^{**}(L) < 0 \quad \text{and} \quad \frac{\partial}{\partial L} I_1^{**}(L) > 0.$$

Q.E.D.

Theorem 4 states that as the leverage ratio L increases, the hazard rates $\theta^{**}(t,L)$ decrease, the critical time $\tilde{\tau}^{**}(L)$ decreases, and the first and second measures $I_1^{**}(L)$ and $I_2^{**}(L)$ increase. Hence, *as the leverage ratio L increases, the bubble is less likely to emerge and persist for a long time in the naked CDS model.* This is in contrast with the basic model; a high leverage ratio fosters the bubble when naked CDSs are not available, while it deters the bubble when naked CDSs are available.

The increase in leverage ratio L enhance the future loan to the arbitrageurs, which crowds out the future reserve of naked CDS, decreasing the relative future benefit. This is the driving force for a high leverage ratio to deter the bubble in the naked CDS model.

It is important to note that $I_2^{**}(L)$ increases in L; the greater the leverage ratio L is, the more likely for q^* to act as a Nash equilibrium. Hence, the greater the leverage ratio L, the more likely it is for the resultant intrinsic social cost to be at the minimum level, i.e., zero.

8. Remarks

8.1. Role of Bubbles as Supplementing Financial Friction

Several previous works regarded bubble assets as supplementing the financial market imperfection such as financial friction across generations, where bubble assets are utilized as collaterals. It could be the regulator's policy target to avoid high default risks; Fostel and Geanakoplos (2012), for instance, cautioned regulators that unexpected introduction of naked CDS might dramatically increase the default risks.

The present paper did not take into account such supplementary roles explicitly, because we concentrated on the socially harmful aspect of a bubble in terms of intrinsic social cost. The arguments in Subsection 7.1, however, implied that the availability of naked CDSs can deter the bubble if the positive feedback traders are enthusiastic and the leverage ration is sufficient, while it can foster the bubble otherwise. This indicates that the availability of naked CDSs can foster the bubble if its socially beneficial aspect dominates its socially harmful aspect. Hence, we can conclude that *the availability of naked CDSs is expected to be an effective policy method not only in eliminating the socially harmful aspect of a bubble but also in cultivating its socially beneficial aspect.*

8.2. Multimarket Bubbles

This paper assumed that the bubble can emerge in a single stock market. It would be an important future research to extend this paper's models to the case that the bubble can emerge in multiple financial markets. In such multimarket bubbles, we anticipate slack enforcement power to function across these markets; in order to purchase a company's stock through debt finance, arbitrageurs can utilize other financial assets as collaterals.

The concept of slack enforcement power was firstly studied in the context of implicit collusion with multimarket contact in oligopolistic industries. See Bernheim and Whinston (1990) for the perfect monitoring case, and Matsushima (2001) for the imperfect monitoring case. Ii might be expected that a tiny enthusiasm in a stock market gives a non-negligible impact on promoting bubbles in other markets and making the resultant after-crash crisis severer.

8.3. Informational Role of Naked CDS Price

Hart and Zingales (2011) proposed a capital regulation design for large financial institutions by utilizing the informativeness of the price of naked CDS. The present paper does not assume the price of naked CDSs to always convey sufficient information about default risks; the present paper instead permits the arbitrageurs to have an option to hide such information by keeping their demand for naked CDSs not greater than the positive feedback traders' supply limit. In this manner, this paper indicates that whether the price of naked CDSs can convey such information depends on how the availability of naked CDSs influences the arbitrageurs' incentives to exercise such options. The availability of naked CDS may even blunt the information function of market prices, in contrast with Hart and Zingales (2011).

8.4. Further Extensions

Since this paper formulated models as simple as possible, we did not take into account various features in bubbles and crashes in real situations. For instance, bubble prices are sometimes highly volatile, and huge volumes are traded for speculative purposes. Some positive feedback traders are becoming aware of the bubble and its crash risk. The company, which initially has limited debt capacity, is gradually considering various alternative methods for fund raising. In the middle stage of a bubble, some arbitrageurs earn profits by selling out without dampening the euphoria.

Incorporation of these features into the models of this paper may need to revise this paper's predictions to a certain degree. However, they would not cause substantial changes about this paper's comparative analysis in Section 7.

Some of these features could be partially explained by alternative models for bubbles and crashes such as general equilibrium with prior heterogeneity (Harrison and Kreps (1978), Scheinkman and Xiong (2003), and Fostel and Geanakoplos (2012)), sequential awareness (Abreu and Brunnermeier (2003)), and synchronization (Morris and Shin (2001) and Abreu and Brunnermeier (2003)). We should regard these models as complements to the present paper. We can incorporate them with the present paper, although we did not investigate further.

9. Conclusion

This paper examined the bubbles and crashes due to limited arbitrage, formulated as a timing game with behavioral types, where we assumed that the arbitrageurs' rationality is not commonly known. We investigated the impacts of financial regulation concerning leveraged finance and financial innovation concerning the availability of naked CDSs on deterring socially harmful bubbles.

We explicitly took the view of awareness heterogeneity in which the positive feedback traders are unaware of, but the arbitrageurs are well aware of, the positive feedback traders' unconscious reinforcement pattern and the risk of the bubble's crash. Thanks to such awareness heterogeneity, the arbitrageurs could borrow money from the positive feedback traders to purchase the company's shares, and purchase naked CDSs from positive feedback traders, in advantageous manners.

By analyzing the strategic aspects of arbitrageurs, we characterized the possibility that the bubbles emerged and persisted for a long time as a unique Nash equilibrium. We also characterized the possibility that the bubble quickly crashed at the initial time. We showed that without the availability of naked CDSs, a high leverage ratio could foster the bubble, while with the availability of naked CDSs, a high leverage ratio could deter the bubble. Hence, with naked CDSs, the regulator could consider a high leverage ratio as the best policy, irrespective of whether the company is productive. Without naked CDSs, however, the regulator would have a dilemma in that a high leverage ratio will bring about a social harm if the company is unproductive.

We further showed that the availability of naked CDSs could deter the bubble if the leverage ratio was sufficient and the positive feedback traders are enthusiastic, while it could foster the bubble if the leverage ratio was insufficient and the positive feedback traders are not very enthusiastic. In this respect, the growth balance between the positive feedback traders' loan and personal capital was crucial for determining whether the availability of naked CDSs is effective in deterring the bubble. We concluded that the availability of naked CDSs was expected to be an effective policy method not only in eliminating the socially harmful aspect of bubble in terms of intrinsic social cost but also in encouraging its socially beneficial aspect as making up for the fault of financial market imperfection.

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Appendix A: Proof of Theorem 1

From $\tilde{q}$, for every $\hat{\tau} \in [\tilde{\tau}, 1]$, we specify a symmetric strategy profile $q^{\hat{\tau}} = (q_i^{\hat{\tau}})_{i \in N} \in Q$ as follows:

$$q_1^{\hat{\tau}}(t) = \tilde{q}_1(t) \quad \text{for all} \quad t \in [\hat{\tau}, 1],$$

and

$$q_1^{\hat{\tau}}(t) = \tilde{q}_1(\hat{\tau}) \quad \text{for all} \quad t \in [0, \hat{\tau}).$$

According to $q^{\hat{\tau}}$, any rational player times the market at the initial time 0 with a probability of $\tilde{q}_1(\hat{\tau})$. After the initial time 0, he never times the market until time $\hat{\tau}$. After time $\hat{\tau}$, he conforms to $\tilde{q}$.

Proposition A-1: *A symmetric strategy profile $q \in Q$ is a Nash equilibrium if and only if there exists $\hat{\tau} \in [\tilde{\tau}, 1]$ such that*

$$q = q^{\hat{\tau}},$$

and

(A-1) $$u_1(0, q_{-1}, \varepsilon) = u_1(\hat{\tau}, q_{-1}, \varepsilon) \quad \textit{whenever} \quad \hat{\tau} < 1 \quad \textit{and} \quad q_1(0) > 0,$$

and

(A-2) $$u_1(0, q_{-1}, \varepsilon) \geq u_1(\hat{\tau}, q_{-1}, \varepsilon) \quad \textit{whenever} \quad \hat{\tau} = 1.$$

Proof: We set any symmetric Nash equilibrium $q \in Q$ arbitrarily. It is clear that the inequality (A-2) is necessary and sufficient for the Nash equilibrium property if $\hat{\tau} = 1$, i.e., $q = q^1$. We assume that $q \neq q^1$, i.e., $q_1(0) < 1$.

We show that $q_1(\tau)$ is continuous. Let us suppose that $q_1(\tau)$ is not continuous. Then, there will exist $\tau' > 0$ such that $\lim_{\tau \uparrow \tau'} q_1(\tau) < q_1(\tau')$. From symmetry, it follows that by selecting any time that is slightly earlier than τ', any player can dramatically increase his winning probability. This implies that no player selects τ', which is a contradiction.

Let us specify

$$\hat{\tau} = \max\{\tau \in (0,1] : q_1(\tau) = q_1(0)\} .$$

We show that $q_1(\tau)$ is increasing in $[\hat{\tau},1]$. Suppose that $q_1(\tau)$ is not increasing in $[\hat{\tau},1]$. From the continuity of q_1 and the specification of $\hat{\tau}$, we select $\tau', \tau'' \in [\hat{\tau},1]$ such that $\tau' < \tau''$, $q_1(\tau') = q_1(\tau'')$, and the selection of τ' is a best response. Since no player selects any time τ in (τ', τ''), it follows from the continuity of q that by selecting time τ'' instead of τ', any player can increase his winner payoff without decreasing his winning probability, which is a contradiction.

Any selection $\tau \in [\hat{\tau},1]$ must be a best response, because $q_1(\tau)$ is increasing in $[\hat{\tau},1]$. This implies that the first-order condition holds for all $\tau \in [\hat{\tau},1]$, i.e., $q = q^{\hat{\tau}}$. Given that $\hat{\tau} < 1$, it is clear from the fact that the winner payoff $\bar{v}_i(t)$ is increasing that $q^{\hat{\tau}}$ is a Nash equilibrium if and only if

$$u_1(0, q_{-1}, \varepsilon) = u_1(\hat{\tau}, q_{-1}, \varepsilon) \quad \text{whenever} \quad q_1(0) > 0 .$$

This implies that (A-1) is necessary and sufficient.

Q.E.D.

The first part of Theorem 1 is proved as follows. From Proposition A-1, $\tilde{\tau} < 1$, and $\tilde{q} = q^{\tilde{\tau}}$, it follows that $\tilde{q}$ is a Nash equilibrium if and only if

$$\text{either} \quad \tilde{q}_1(0) = 0 \quad \text{or} \quad u_1(0, \tilde{q}_{-1}, \varepsilon) = u_1(\tilde{\tau}, \tilde{q}_{-1}, \varepsilon) .$$

The inequality in Theorem 1, along with (3), (7), and $\bar{v}_1'(t) \geq 0$, implies (6). Hence, $\tilde{q}_1(0) = 0$, i.e., $\tilde{q}$ is a Nash equilibrium.

Suppose that the inequality in Theorem 1 does not hold. Then, it must hold that $\tilde{\tau} = 0$ and $\tilde{q}_1(0) > 0$. This, however, contradicts the Nash equilibrium property that any selection of $t \in [0,1]$ is a best response: any player prefers time 0 to any time slightly later than time 0, because he can dramatically increase his winning probability without any substantial decrease in winner payoff.

The latter part of Theorem 1 is proved as follows. It follows from (3), (7), $\bar{v}_1'(t) \geq 0$, and the strict inequality in Theorem 1 that the property of (6) holds and $\tilde{\tau} > 0$.

This, along with Proposition A-1, implies that any symmetric Nash equilibrium q must satisfy $q = \tilde{q}$.

Next, we show that $\tilde{q}$ is a unique Nash equilibrium even if all asymmetric Nash equilibria are taken into account. We set any Nash equilibrium $q \in Q$ arbitrarily.

First, we show that $q_i(\tau)$ must be continuous in $[0,1]$ for all $i \in N$. Suppose that $q_i(\tau)$ is not continuous in $[0,1]$. Then, there exists $\tau' > 0$ such that $\lim_{\tau \uparrow \tau'} q_i(\tau) < q_i(\tau')$ for some $i \in N$; any other player can drastically increase his winning probability by selecting any time slightly earlier than time τ'. Hence, no other player selects any time that is either the same as or slightly later than τ'. Hence, player i can postpone timing the market without decreasing his winning probability. This is a contradiction.

Second, we show that $D(\tau;q)$ must be increasing in $[\tau^1,1]$, where we denote

$$\tau^1 = \max\{\tau \in (0,1] : q_i(\tau) = q_i(0) \ \textit{for all } i \in N\}.$$

Now, suppose that $D(\tau;q)$ is not increasing in $[\tau^1,1]$. In this case, from the continuity of q, we can select $\tau', \tau'' \in (\tau^1,1]$ such that $\tau' < \tau''$, $D(\tau';q) = D(\tau'';q)$, and the selection of τ' is a best response for some player. Since no player selects any time τ in (τ', τ''), it follows from the continuity of q that by selecting τ'' instead of τ', any player can postpone the timing from τ' to τ'' without decreasing his winning probability. This is also a contradiction.

Third, we show that q must be symmetric. Now let us suppose that q is asymmetric. The strict inequality in Theorem 1 implies that the selection of 0 is a dominated strategy. Hence, it follows that $\tau^1 > 0$, and

$$q_i(\tau) = 0 \quad \text{for all} \quad i \in N \quad \text{and all} \quad \tau \in [0,\tau^1].$$

Since q is continuous and $D(\tau;q)$ is increasing in $[\tau^1,1]$, it follows that there exist $\tau' > 0$, $\tau'' > \tau'$, and $i \in N$ such that

$$q_1(t) = q_j(t) \quad \text{for all} \quad j \in N \quad \text{and all} \quad t \in [0,\tau'],$$

(A-3) $$\frac{\frac{\partial D_i(\tau;q)}{\partial t}}{1 - D_i(\tau;q)} > \min_{h \neq i} \frac{\frac{\partial D_h(\tau;q)}{\partial t}}{1 - D_h(\tau;q)} \quad \text{for all} \quad t \in (\tau', \tau''),$$

and

$$\text{(A-4)} \qquad \frac{\dfrac{\partial D_i(\tau'';q)}{\partial t}}{1-D_i(\tau'';q)} = \min_{h\neq i} \frac{\dfrac{\partial D_h(\tau'';q)}{\partial t}}{1-D_h(\tau'';q)} > 0 .$$

Since $D(\tau;q)$ is increasing in $[\tau^1,1]$, any selection of t in (τ',τ'') must be a best response for any player $j \in N$ such that

$$\frac{\dfrac{\partial D_j(t;q)}{\partial t}}{1-D_j(t;q)} = \min_{h\neq i} \frac{\dfrac{\partial D_h(t;q)}{\partial t}}{1-D_h(t;q)} .$$

This equality implies that $\dfrac{\partial q_j(t)}{\partial t} > 0$. Hence, it follows from the continuity of q that the first-order condition holds for player j; i.e., for every $t \in (\tau',\tau'')$,

$$\frac{(1-\varepsilon)q_j{}'(t)}{1-(1-\varepsilon)q_j(t)} = \frac{\overline{v}_1'(t)}{(n-1)\{\overline{v}_1(t) - \underline{v}_1(t)\}} .$$

Hence, from (A-3),

$$\frac{(1-\varepsilon)q_i{}'(t)}{1-(1-\varepsilon)q_i(t)} > \frac{\overline{v}_1'(t)}{(n-1)\{\overline{v}_1(t) - \underline{v}_1(t)\}} ,$$

implying that the first-order condition does not hold for player i for every $t \in (\tau',\tau'')$, where the inequality $\dfrac{\partial u_i(\tau, q_{-i}, \varepsilon)}{\partial \tau} < 0$ holds in this case. This inequality implies that player i prefers time τ' to any time in $(\tau', \tau'' + \varepsilon)$, and therefore,

$$\frac{\partial D_i(\tau;q)}{\partial t} = 0 \quad \text{for all} \quad \tau \in (\tau', \tau'' + \eta) ,$$

where η is positive but close to zero. This is a contradiction, because the inequality in (A-4) implies that $\dfrac{\partial D_i(\tau'';q)}{\partial t} > 0$. Hence, we have proved that any Nash equilibrium q must be symmetric.

From the above observations, we have completed the proof of Theorem 1.

Appendix B: Proof of Theorem 2

For every $t \in (0,1]$,

$$u_1(t, q^*_{-1}) = \varepsilon^{n-1}\overline{v}_1(t) + (1-\varepsilon^{n-1})\underline{v}_1(0) \le \varepsilon^{n-1}\overline{v}_1(1) + (1-\varepsilon^{n-1})\underline{v}_1(0) = u_1(1, q^*_{-1}),$$

whereas

$$u_1(0, q^*_{-1}) = \{\sum_{1\le l\le n-1} \frac{(n-1)!}{l!(n-1-l)!}(1-\varepsilon)^l \varepsilon^{n-1-l}\frac{1}{l+1}\}\overline{v}_1(t)$$
$$+\{1 - \sum_{1\le l\le n-1} \frac{(n-1)!}{l!(n-1-l)!}(1-\varepsilon)^l \varepsilon^{n-1-l}\frac{1}{l+1}\}\underline{v}_1(0).$$

Hence, the necessary and sufficient condition for q^* to be a Nash equilibrium is given by

$$u_1(0, q^*_{-1}) \ge u_1(1, q^*_{-1}),$$

that is,

$$\{\sum_{1\le l\le n-1} \frac{(n-1)!}{l!(n-1-l)!}(1-\varepsilon)^l \varepsilon^{n-1-l}\frac{1}{l+1}\}\overline{v}_1(t)$$
$$+\{1 - \sum_{1\le l\le n-1} \frac{(n-1)!}{l!(n-1-l)!}(1-\varepsilon)^l \varepsilon^{n-1-l}\frac{1}{l+1}\}\underline{v}_1(0) \ge \varepsilon^{n-1}\overline{v}_1(1) + (1-\varepsilon^{n-1})\underline{v}_1(0).$$

This inequality is equivalent to

$$\{\sum_{1\le l\le n-1} \frac{(n-1)!}{l!(n-1-l)!}(1-\varepsilon)^l \varepsilon^{n-1-l}\frac{1}{l+1}\}\{\overline{v}_1(t) - \underline{v}_1(0)\}$$
$$\ge \varepsilon^{n-1}\{\overline{v}_1(1) - \underline{v}_1(0)\},$$

which is the same as the inequality in Theorem 2, because of (8).