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SHARP BOUNDS IN THE BINARY ROY MODEL

MARC HENRY AND ISMAEL MOURIFIÉ

Abstract. We derive the empirical content of an instrumental variables model of sectorial choice with binary outcomes. Assumptions on selection include the simple, extended and generalized Roy models. The derived bounds are nonparametric intersection bounds and are simple enough to lend themselves to existing inference methods. Identification implications of exclusion restrictions are also derived.

Keywords: treatment effect, discrete outcomes, sectorial choice, partial identification, intersection bounds.

JEL subject classification: C21, C25, C26

Introduction

A large literature has developed since Heckman and Honoré (1990) on the empirical content of the Roy model of sectorial choice with sector specific unobserved heterogeneity. Most of this literature, however, concerns the case of continuous outcomes and many applications, where outcomes are discrete, fall outside its scope. They include analysis of the effects of different training programs on the probability of renewed employment, of competing medical treatments or surgical procedures on the probability of survival, of higher education on the probability of migration and of competing

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policies on schooling decisions in developing countries among numerous others. The Roy model is still highly relevant to those applications, but very little is known of its empirical content in such cases. The case of discrete outcomes is considered in Chesher (2010) but the analysis doesn't apply to binary outcomes. Sharp bounds are derived in binary outcome models with a binary endogenous regressor in Shaikh and Vytlacil (2011), Chiburis (2010), Jun, Pinkse, and Xu (2010) and Mourifié (2011) under a variety of assumptions, which all rule out sector specific unobserved heterogeneity. Finally, Heckman and Vytlacil (1999) derive identification conditions in a parametric version of the binary Roy model.

We consider three distinct versions of the binary Roy model: the original model, where selection is based solely on the probability of success; the extended Roy model, where selection depends on the probability of success and a function of observable variables (sometimes called "nonpecuniary component"); and the generalized Roy model, with selection specific unobservable heterogeneity. When considering the generalized Roy model, we further distinguish restrictions on the selection equation and restrictions on the joint distribution of sector specific unobserved heterogeneity. We specifically consider the case, where selection variables are independent of sector specific unobserved heterogeneity and the case, where sector specific unobserved heterogeneity follows a factor structure proposed in Aakvik, Heckman, and Vytlacil (2005). Following Heckman, Smith, and Clements (1997), we apply results from optimal transportation theory to derive sharp bounds on the structural parameters, from which a range of treatment parameters can be derived. More specifically, we apply Theorem 1 of (Galichon and Henry 2011) (equivalently Theorem 3.2 of Beresteanu, Molchanov, and Molinari (2011)) to derive bounds for the generalized binary Roy model. The latter Theorem was recently applied in a similar context by Chesher, Rosen, and Smolinski (2011) to derive sharp bounds for instrumental variable models of discrete choice. We spell out the point identification implications of the bounds under certain exclusion restrictions. The bounds are simple enough to lend themselves to existing inferential methods, specifically Chernozhukov, Lee, and Rosen (2009) in the instrumental variables case.

The remainder of the paper is organized as follows. Section 1 clarifies the analytical framework. In Section 2, sharp bounds are derived for the binary Roy model, when selection depends only on the probability of success and possibly on observable variables. Identification implications are spelled out under exclusion restrictions. Section 3 considers the generalized binary Roy model and the last section concludes.

1. Analytical framework

We adopt the framework of the potential outcomes model $Y = Y_1 D + Y_0(1-D)$, where Y is an observed outcome, D is an observed selection indicator and Y_1, Y_0 are unobserved potential outcomes. Heckman and Vytlacil (1999) trace the genealogy of this model and we refer to them for terminology and attribution. Potential outcomes are as follows:

$$Y_d = 1\{Y_d^* > 0\} = 1\{F(d, X_d, u_d) > 0\},\ d = 1, 0, \tag{1.1}$$

where $1\{.\}$ denotes the indicator function and F is an unknown function of the vector of observable random variables X_d and unobserved random variable u_d. We make the following assumptions throughout the paper.

Assumption 1 (Weak separability)**.** *The functions $F(d, X_d, u_d)$, d=1,0, both have weakly separable errors. As shown in Vytlacil (2002), potential outcomes can then be written $Y_d = 1\{f_d(X_d) > u_d\}$ without loss of generality.*

Assumption 2 (Regularity)**.** *The sector specific unobserved variables u_d, $d = 1, 0$, are uniformly continuous with respect to Lebesgue measure, so that they may be assumed without loss of generality to be distributed uniformly on $[0, 1]$.*

The normalization of Assumption 2 is very convenient, since it implies $f_d(x_d) = \mathbb{E}(Y_d|x_d, z)$ and bounds on treatment effects parameters can be derived from bounds on the structural parameters f_1 and f_0.

Assumption 3 (Instruments)**.** *Observable variables X_d, $d = 1, 0$, and instruments Z are independent of (u_1, u_0). Common components of X_1 and X_0 will be dropped from the notation in the remainder of the paper and by slight abuse of notation, X_d will refer only to the variables that are excluded from the equation for Y_{1-d} and Z to variables that are excluded from both outcome equations (when the case arises).*

As was the case in Aakvik, Heckman, and Vytlacil (2005), many of the results apply to the Tobit version of the model, where $Y_d = Y_d^* 1\{Y_d^* > 0\}$, but for clarity of exposition, we only report results pertaining to the binary case.

2. Sharp bounds for the binary Roy and extended Roy models

2.1. **Simple binary Roy model.** In the original model proposed by Roy (1951), the sector yielding the highest outcome is selected, i.e., $D = 1\{Y_1^* > Y_0^*\}$. In the binary case, this is equivalent to selecting the sector with the highest probability of success. The empirical content of the model under this selection rule is characterized in Figures 1 and 2.

For each value of the exogenous observable variables and each value of the pair (u_1, u_0), the outcome is uniquely determined. If the joint distribution were known, the likelihood of each of the potential outcomes $(Y = 1, D = 1)$, $(Y = 1, D = 0)$, $(Y = 0, D = 1)$ and $(Y = 0, D = 0)$ would be determined. However, only the marginal distributions of u_1 and u_0 are fixed, not the copula, so that only the probability of vertical and horizontal bands in Figures 1 and 2 are uniquely determined. Thus we see for instance that $f_1 = \mathbb{P}(Y = 1, D = 1)$ is identified when $f_0 = 0$ (as in Figure 2)

FIGURE 1. Characterization of the empirical content of the simple binary Roy model in the unit square of the (u_1, u_0) space.

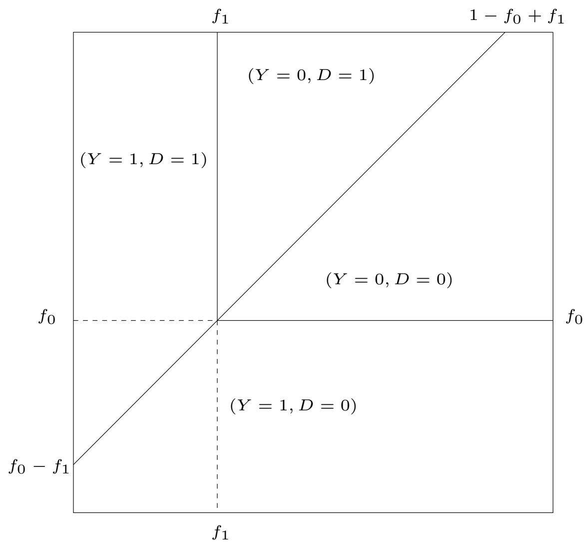
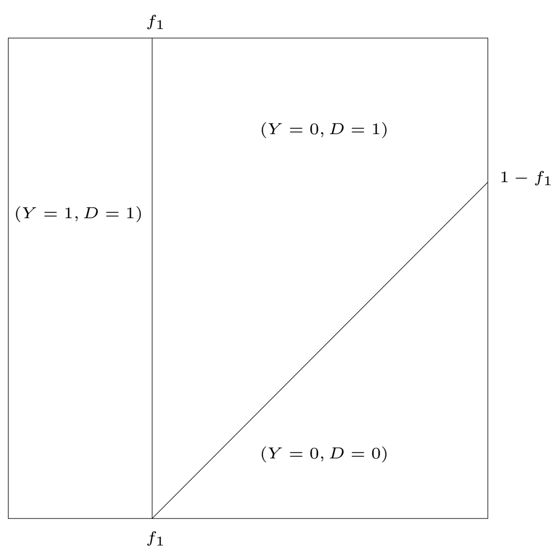


FIGURE 2. Characterization of the empirical content of the simple binary Roy model in the unit square of the (u_1, u_0) space in case $f_0 = 0$.



and $f_0 = \mathbb{P}(Y = 1, D = 0)$ is identified when $f_1 = 0$. But in other cases (as in Figure 1), we only know $\mathbb{P}(Y = 1, D = 1) \le f_1 \le \mathbb{P}(Y = 1)$ and $\mathbb{P}(Y = 1, D = 0) \le f_0 \le \mathbb{P}(Y = 1)$. The following proposition, proved in the Appendix, shows that these bounds are sharp. In all that follows, we shall use the notation $\mathbb{P}(i, j|X)$ for $\mathbb{P}(Y = i, D = j|X)$ and $W = (Z, X_1, X_0)$, $\omega = (z, x_1, x_0)$.

Proposition 1 (Roy model). *Under Assumptions 1-3, the following inequalities characterize the empirical content of the model.*

$$\sup_{x_0,z} \mathbb{P}(1,1|x_1,x_0,z) \le \quad f_1(x_1) \quad \le \inf_{x_0,z} \Big[\mathbb{P}(1,1|\omega) + \mathbb{P}(1,0|\omega)1\{f_0(x_0) > 0\}\Big] \tag{2.1}$$

$$\sup_{x_1,z} \mathbb{P}(1,0|\omega) \le \quad f_0(x_0) \quad \le \inf_{x_1,z} \Big[\mathbb{P}(1,0|\omega) + \mathbb{P}(1,1|\omega)1\{f_1(x_1) > 0\}\Big] \tag{2.2}$$

where the infima and suprema are taken over the domains of the excluded variables Z, X_1 or X_0 as indicated and when they exist.

Since the bounds in Proposition 1 are obtained as intersections over the domains of the excluded variables, they are called "intersection bounds". They are also semiparametric in the non excluded variables. Inference on such bounds can be conducted with existing methods described in Chernozhukov, Lee, and Rosen (2009).

A simple implication of selection equation $D = 1\{Y_1^* > Y_0^*\}$ is that actual success is more likely than counterfactual success.

Assumption 4 (Roy model). $\mathbb{E}(Y_d|D = d, Z, X_1, X_0) \ge \mathbb{E}(Y_{1-d}|D = d, Z, X_1, X_0)$ *for* $d = 1, 0$.

Under Assumption 4, omitting conditioning variables for ease of notation,

$$
\begin{aligned}
f_d &= \mathbb{E}[Y_d] \\
&= \mathbb{E}[Y_d|D=d]\mathbb{P}(D=d) + \mathbb{E}[Y_d|D=1-d]\mathbb{P}(D=1-d) \\
&\leq \mathbb{P}[Y=1, D=d] + \mathbb{E}[Y_{1-d}|D=1-d]\mathbb{P}(D=1-d) \\
&= \mathbb{P}(Y=1, D=d) + \mathbb{P}(Y=1, D=1-d).
\end{aligned}
$$

Moreover, if $f_d > 0$ and $f_{1-d} = 0$, $\mathbb{P}(D = 1-d) = 0$. This implies that

$$
\mathbb{P}(1, d|\omega) \leq \mathbb{E}[Y_d|\omega] \leq \mathbb{P}(1, d|\omega) + \mathbb{P}(1, 1-d|\omega)1\{\mathbb{E}[Y_{1-d}|\omega] > 0\}
$$

(with $\omega = (z, x_1, x_0)$) characterizes the empirical content of the potential outcomes model $Y = Y_1 D + Y_0(1-D)$ in all generality (i.e., without weak separability and without assumptions on the dimension of unobservable heterogeneity). It also shows that the simple binary Roy model has no empirical content beyond Assumption 4. Indeed, bounds (2.1) and (2.1) still hold under Assumptions 1-4. They are also sharp, since $D = 1\{Y_1^* > Y_0^*\}$ implies Assumption 4. Therefore, the empirical content of the model defined by Assumption 4 is the same as the empirical content of the model defined by the selection equation $D = 1\{Y_1^* > Y_0^*\}$.

Corollary 1. *The empirical content of the model defined by Assumptions 1-4 is characterized by inequalities (2.1) and (2.2).*

In case of exclusion restrictions, an immediate corollary to Proposition 1 gives conditions for identification of the outcome equations.

Corollary 2 (Identification)**.** *Under Assumptions 1-4, the following hold (writing $\omega = (z, x_1, x_0)$ as before).*

a. *If there is $x_0 \in Dom(X_0)$ such that $f_0(x_0) = 0$, then f_1 is identified over $Dom(X_1)$.*

b. *If there is* $x_1 \in Dom(X_1)$ *such that* $f_1(x_1) = 0$, *then* f_0 *is identified over* $Dom(X_0)$.

a'. *Take* $x_1 \in Dom(X_1)$. *If there is* $x_0 \in Dom(X_0)$ *or* $z \in Dom(Z)$ *such that* $\mathbb{P}(1, 0|\omega) = 0$, *then* $f_1(x_1)$ *is identified.*

b'. *Take* $x_0 \in Dom(X_0)$. *If there is* $x_1 \in Dom(X_1)$ *or* $z \in Dom(Z)$ *such that* $\mathbb{P}(1, 1|\omega) = 0$, *then* $f_0(x_0)$ *is identified.*

The existence of valid instruments or exclusion restrictions is often problematic in applications of discrete choice models. However, in the Roy model of sectorial choice with sector specific unobserved heterogeneity, it is natural to expect some sector specific observed heterogeneity as well. Such sector specific observed heterogeneity would provide exclusion restrictions in the form of variables affecting outcome equation for Y_d without affecting outcome equation for Y_{1-d}. Such exclusion restrictions would yield intersection bounds in Proposition 1. Of course, even if it exists, sector specific observed heterogeneity may not satisfy a. or b. of Corollary 2. However, the availability of an exclusion restriction as in a. or b. of Corollary 2 is consistent with the spirit of a model of sector specific heterogeneity.

2.2. **Extended binary Roy model.** Assumption 4 is very restrictive and recent research by Haultfoeuille and Maurel (2011) and Bayer, Khan, and Timmins (2011) on the Roy model with continuous outcomes has focused on an extended version, where selection depends on $Y_1^* - Y_0^*$ and a function of observable variables $g(Z, X_1, X_0)$ sometimes called "non pecuniary component". We now investigate the implications of this selection assumption in the binary case.

Assumption 5 (Observable heterogeneity in selection)**.** $D = 1\{Y_1^* - Y_0^* > g(Z, X_1, X_0)\}$ *for some unknown function* g *of the vector of the observable variables* Z, X_1 *and* X_0.

Under Assumptions 1-3 and 5, we may still characterize the empirical content of the model graphically, in Figures 3 and 4. We drop Z, X_1 and X_0 from the notation in the discussion below.

FIGURE 3. Characterization of the empirical content of the extended binary Roy model in the unit square of the (u_1, u_0) space in case $0 \leq g < f_1$.

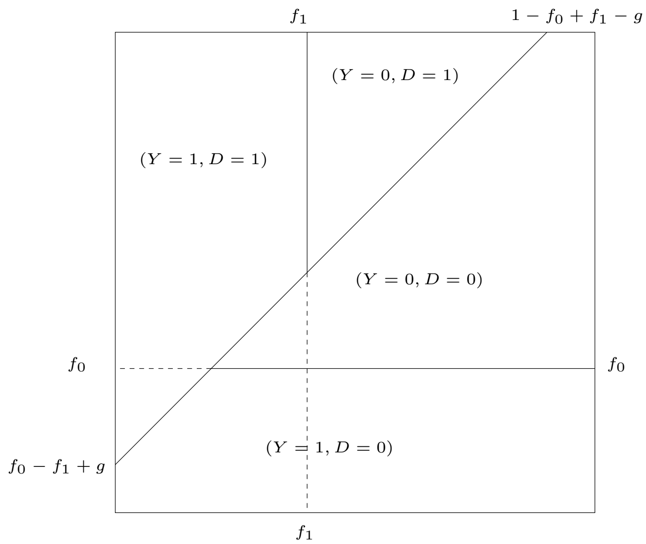
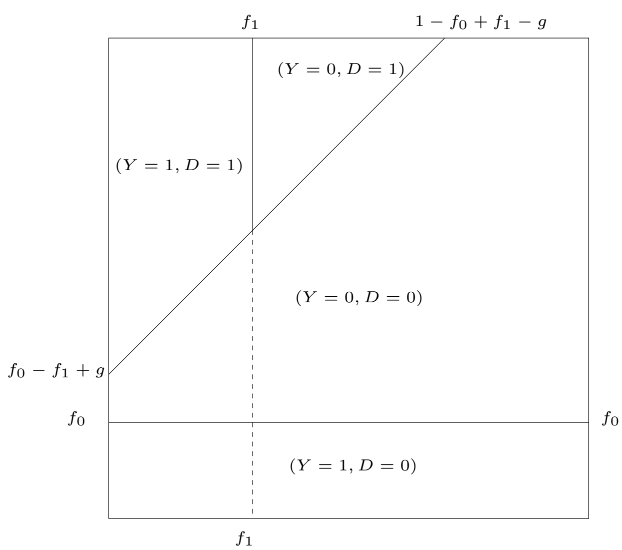


FIGURE 4. Characterization of the empirical content of the extended binary Roy model in the unit square of the (u_1, u_0) space in case $g \geq f_1$.



For each value of (u_1, u_0), the outcome is uniquely determined by f_1, f_0 and g. Again, the missing piece to compute the likelihood of outcomes $\mathbb{P}(i,j)$, $i,j = 1,0$, is the copula for (u_1, u_0). From the knowledge of the probabilities of horizontal and vertical bands in the (u_1, u_0) space, we can derive the sharp bounds on structural parameters f_1, f_0 and g. Four cases are considered below to explain the bounds, which are derived formally and shown to be sharp in Proposition 2.

a. Case where $g \geq f_1$ on Figure 4. The probability of outcome $(Y = 1, D = 0)$ is seen to be exactly equal to the area of the lower horizontal band. Hence $f_0 = \mathbb{P}(1,0)$ is identified. Moreover, the area of the horizontal band $(f_0, f_0 - f_1 + g)$ is smaller than the probability of outcome $(Y = 1, D = 1)$. Hence $g \leq f_1 + \mathbb{P}(1,1)$. Similar reasoning yields $\mathbb{P}(1,1) \leq f_1 \leq \mathbb{P}(Y = 1) + \mathbb{P}(0,0)$.

b. Case where $0 \leq g < f_1$ on Figure 3. The area of the lower horizontal band $(0, f_0 - f_1 + g)$ is smaller than the probability of outcome $(Y = 1, D = 0)$. Hence $g \leq f_1 - f_0 + \mathbb{P}(1,0)$. Moreover, the area of the horizontal band $(0, f_0)$ is larger than the probability of outcome $(Y = 1, D = 0)$ and smaller than the probability of outcome $(Y = 1)$. Hence $\mathbb{P}(1,0) \leq f_0 \leq \mathbb{P}(Y = 1)$. Finally, $\mathbb{P}(1,1) \leq f_1 \leq \mathbb{P}(Y = 1) + \mathbb{P}(0,0)$ still holds.

c. Case where $-f_0 < g \leq 0$. Similarly to Case b., we obtain bounds $g \geq f_1 - f_0 + \mathbb{P}(1,1)$, $\mathbb{P}(1,0) \leq f_0 \leq \mathbb{P}(Y = 1) + \mathbb{P}(0,1)$ and $\mathbb{P}(1,1) \leq f_1 \leq \mathbb{P}(Y = 1)$.

d. Case where $g \leq -f_0$. Similarly to Case a., $f_1 = \mathbb{P}(1,1)$ is identified and $\mathbb{P}(1,0) \leq f_0 \leq \mathbb{P}(Y = 1) + \mathbb{P}(0,1)$ and $g \geq -f_0 - \mathbb{P}(0,1)$.

It is shown formally in Proposition 2 that the bounds discussed above hold and cannot be improved upon. The same arguments can be applied to derive the empirical content of the model where the selection equation generalizes Assumption 5 with $D = 1\{u_0 > h(u_1, W)\}$ and h strictly increasing in u_1, for all W. Assumption 5 is the special case, where $h(u_1, W) = u_1 + f_0(X_0) - f_1(X_1) + g(W)$.

Proposition 2 (Sharp bounds for the extended binary Roy model)**.** *Under Assumptions 1-3 and 5, the empirical content of the model is characterized by the following (writing $\omega = (z, x_1, x_0)$ as before).*

$$\begin{array}{rcccl} \sup_{z,x_0} \mathbb{P}(1,1|\omega) & \leq & f_1(x_1) & \leq & \inf_{z,x_0} \Big[\mathbb{P}(1,1|\omega) + \mathbb{P}(1,0|\omega)1\{g(\omega) > -f_0(x_0)\} \\ & & & & \qquad +\mathbb{P}(0,0|\omega)1\{g(\omega) > 0\}\Big], \\ \sup_{z,x_1} \mathbb{P}(1,0|\omega) & \leq & f_0(x_0) & \leq & \inf_{z,x_1} \Big[\mathbb{P}(1,0|\omega) + \mathbb{P}(1,1|\omega)1\{g(\omega) < f_1(x_1)\} \\ & & & & \qquad +\mathbb{P}(0,1|\omega)1\{g(\omega) < 0\}\Big] \end{array} \tag{2.3}$$

and

$$\begin{array}{rcl} g(\omega) & \in & \Big[-f_0(x_0) - \mathbb{P}(0,1|\omega), -f_0(x_0)\Big] \cup \Big[f_1(x_1) - f_0(x_0) - \mathbb{P}(1,1|\omega), \\ & & \qquad f_1(x_1) - f_0(x_0) + \mathbb{P}(1,0|\omega)\Big] \cup \Big[f_1(x_1), f_1(x_1) + \mathbb{P}(0,0|\omega)\Big], \end{array} \tag{2.4}$$

where the infima and suprema are taken over the domain of Z, X_1 or X_0 as indicated and when they arise.

Simple identification conditions can be derived for f_1 and f_0 from the bounds of Proposition 2 under exclusion restrictions. However, it can be seen immediately that exclusion restrictions cannot identify $g(\,)$.

Corollary 3 (Identification)**.** *Under Assumptions 1-3 and 5, the following hold (writing $\omega = (z, x_1, x_0)$ as before).*

a. *If there is $z \in Dom(Z)$ and $x_0 \in Dom(X_0)$ such that $g(\omega) \leq -f_0(x_0)$, then $f_1(x_1) = \mathbb{P}(1,1|\omega)$ is identified.*

b. *If there is $z \in Dom(Z)$ and $x_1 \in Dom(X_1)$ such that $g(\omega) \geq f_1(x_1)$, then $f_0(x_0) = \mathbb{P}(1,0|\omega)$ is identified.*

a'. *Take $x_1 \in Dom(X_1)$. If there is $x_0 \in Dom(X_0)$ or $z \in Dom(Z)$ such that $\mathbb{P}(1,0|\omega) = \mathbb{P}(0,0|\omega) = 0$, then $f_1(x_1)$ is identified.*

b'. *Take $x_0 \in Dom(X_0)$. If there is $x_1 \in Dom(X_1)$ or $z \in Dom(Z)$ such that $\mathbb{P}(1,1|\omega) = \mathbb{P}(0,1|\omega) = 0$, then $f_0(x_0)$ is identified.*

As in the case of the simple Roy model, the sharp bounds of Proposition 2 take the form of intersection bounds and inference can be conducted with existing methods. When there are no instruments (or exclusion restrictions), however, the bounds are no longer intersection bounds. They become:

$$\begin{aligned} \mathbb{P}(1,1) &\leq f_1 \leq \mathbb{P}(1,1)+\mathbb{P}(1,0)1\{g>-f_0\}+\mathbb{P}(0,0)1\{g>0\}, \\ \mathbb{P}(1,0) &\leq f_0 \leq \mathbb{P}(1,0)+\mathbb{P}(1,1)1\{g<f_1\}+\mathbb{P}(0,1)1\{g<0\} \end{aligned}$$

and

$$g \in \Big[-f_0-\mathbb{P}(0,1),-f_0\Big] \cup \Big[f_1-f_0-\mathbb{P}(1,1),f_1-f_0+\mathbb{P}(1,0)\Big] \cup \Big[f_1,f_1+\mathbb{P}(0,0)\Big].$$

When the object of interest is treatment parameters only, the three dimensional identification region defined by the sharp bounds on (f_1,f_0,g) is projected on the two-dimensional space (f_1,f_0). When there are no exclusion restrictions, this projection yields the Manski (2000) nonparametric bounds. If the object of interest is the non pecuniary component g, the three dimensional identification region is projected on the one-dimensional space for g into the single interval $[-\mathbb{P}(1,1)-\mathbb{P}(1,0)-2\mathbb{P}(0,1),\mathbb{P}(1,1)+\mathbb{P}(1,0)+2\mathbb{P}(0,0)]$. In the presence of instruments (or exclusion restrictions), the projections on (f_1,f_0) and g can be much tighter and the projection on (f_1,f_0) may even be reduced to a point, as in Corollary 3.

Testing for the presence of a non pecuniary component. As we have just seen, in the absence of exclusion restrictions, the projection of the identified region on the g space always contains zero, so that it is impossible to test the hypothesis $g=0$. However, in the presence of exclusion restrictions, the hypothesis $g=0$ may become testable. There is a non zero non pecuniary component in the selection equation if and only if the projection of the sharp bounds does not contain 0 or equivalently, if the hyperplane $g=0$ does not intersect the three dimensional identification region for (f_1,f_0,g) defined by the sharp bounds in Proposition 2. It is also equivalent to the crossing of the intersection bounds in Proposition 1, in the sense that

$$\sup_{x_0,z} \mathbb{P}(1,1|x_1,x_0,z) \quad > \quad \inf_{x_0,z} \Big[\mathbb{P}(1,1|\omega) + \mathbb{P}(1,0|\omega) 1\{f_0(x_0) > 0\} \Big]$$

or

$$\sup_{x_1,z} \mathbb{P}(1,0|\omega) \quad > \quad \inf_{x_1,z} \Big[\mathbb{P}(1,0|\omega) + \mathbb{P}(1,1|\omega) 1\{f_1(x_1) > 0\} \Big]$$

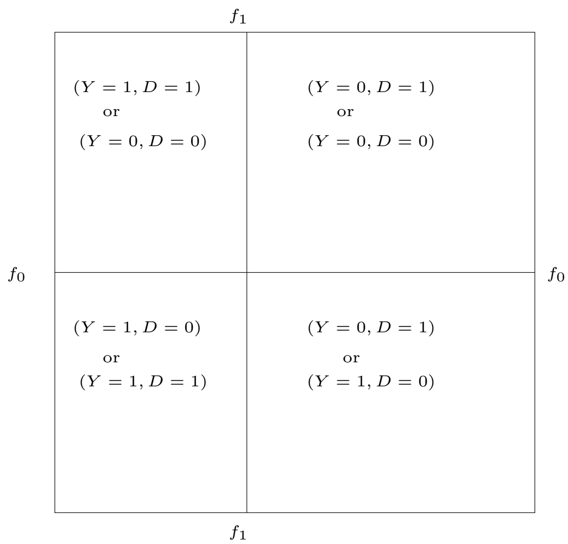
so that by Proposition 1, the simple Roy model is rejected. In practice, the test for the existence of a non pecuniary component would be carried out by constructing a confidence region according to the methods proposed in Chernozhukov, Lee, and Rosen (2009) and checking, whether the hyperplane $g = 0$ intersects the confidence region. If it does, we fail to reject the hypothesis of existence of a non pecuniary component g and if it doesn't, we reject the hypothesis at significance level equal to 1 minus the confidence level chosen for the confidence region. The hypotheses $g \geq 0$ or $g \leq 0$ may be tested in the same way.

3. Sharp bounds for the generalized binary Roy model

So far, we have assumed that selection occurs on the basis of success probability and other observable variables. We now turn to the general case, where unobservable heterogeneity, beyond $u_0 - u_1$, may play a role in sectorial selection. Knowledge of (u_1, u_0) now no longer uniquely determines the outcome $(Y = i, D = j)$ as seen on Figure 5. Multiplicity of equilibria and lack of coherence of the model can be dealt with, however, with the optimal transportation approach of Galichon and Henry (2011), as shown in the proof of Theorem 1 below.

Theorem 1 (Sharp bounds for the generalized Roy model)**.** *Under Assumption 1-3, the empirical content of the model is characterized by inequalities (3.1)-(3.3) below (writing $\omega = (z, x_1, x_0)$ as*

FIGURE 5. Characterization of the empirical content of the generalized binary Roy model in the unit square of the (u_1, u_0) space.



before).

$$\sup_{z,x_0} \mathbb{P}(1,1|\omega) \quad \leq f_1(x_1) \leq \quad 1 - \sup_{z,x_0} \mathbb{P}(0,1|\omega), \tag{3.1}$$

$$\sup_{z,x_1} \mathbb{P}(1,0|\omega) \quad \leq f_0(x_0) \leq \quad 1 - \sup_{z,x_1} \mathbb{P}(0,0|\omega) \tag{3.2}$$

and

$$\begin{aligned} & \sup_z \max\Big(0, f_0(x_0) - \mathbb{P}(1,0|\omega) - \mathbb{P}(0,1|\omega), f_1(x_1) - \mathbb{P}(1,1|\omega) - \mathbb{P}(0,0|\omega)\Big) \\ \leq \quad & \mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1, x_0) \\ \leq \quad & \inf_z \min\Big(\mathbb{P}(Y=1|\omega), f_1(x_1) + f_0(x_0) - \mathbb{P}(Y=1|\omega)\Big). \end{aligned} \tag{3.3}$$

Theorem 1 is not an operational characterization of the empirical content of the model since the sharp bounds involve the unknown quantity $\mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1, x_0)$, which, by the normalization of Assumption 2, is exactly the copula of (u_1, u_0). In the case of total ignorance about the copula of (u_1, u_0), after plugging Fréchet bounds $\max(f_1(x_1) + f_0(x_0) - 1, 0) \leq \mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1, x_0) \leq \min(f_1(x_1), f_0(x_0))$, inequalities (3.3) are shown to be redundant. Hence we have the following.

Corollary 4. *Sharp bounds for the generalized Roy model under Assumption 1-3 are given by inequalities (3.1) and (3.2).*

In order to sharpen those bounds, we may consider restrictions on the copula for (u_1, u_0) or restrictions on the selection equation. We consider both strategies in turn.

3.1. **Restrictions on selection.** Consider the following selection model, where selection depends on $Y_1^* - Y_0^*$ and $g(Z, X_1, X_0)$ and selection specific unobserved heterogeneity v, which is independent of sector specific unobserved heterogeneity (u_1, u_0).

Assumption 6. $D = 1\{Y_1^* - Y_0^* > g(W) + v\}$*, with* $v \perp\!\!\!\perp (u_1, u_0)$ *and* $v \perp\!\!\!\perp W$*,* $\mathbb{E}v = 0$ *(without loss of generality) and* $W = (Z, X_1, X_0)$*.*

With $v \perp\!\!\!\perp (u_1, u_0)$, we have $\mathbb{P}(u_d \leq g(z, x_1, x_0) + v + f_1(x_1) - f_0(x_0)|z, x_1, x_0) = \mathbb{E}_v\mathbb{E}[1\{u_d \leq g(z, x_1, x_0) + v - f_1(x_1) + f_0(x_0)\}|z, x_1, x_0, v] = \max(0, g(z, x_1, x_0) - f_1(x_1) + f_0(x_0))$ and it is shown in Corollary 5 that the bounds on $g(\,)$ derived in Section 2 remain valid.

Corollary 5. *Under assumptions 1-3 and 6, (2.4) holds.*

As for the bounds on (f_1, f_0), they remain valid under specific domain restrictions for v.

3.2. **Restrictions on the joint distribution of sector specific heterogeneity.**

3.2.1. *Parametric restrictions on the copula.* In case the copula for (u_1, u_2) is parameterized with parameter vector θ, sharp bounds are obtained straightforwardly by replacing $\mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1, x_0)$ with the parametric version $F(f_1(x_1), f_0(x_0); \theta)$ in (3.3).

3.2.2. *Perfect correlation.* In the case of perfect correlation between the two sector specific unobserved heterogeneity variables, $\mathbb{P}(u_1 \leq f_1(x_0), u_0 \leq f_0(x_0)) = \min(f_1(x_1), f_0(x_0))$ so that the sharp bounds of Theorem 1 specialize to (3.1), (3.2), $\min(f_1(x_1), f_0(x_0)) \leq \inf_z \mathbb{P}(Y = 1|z, x_1, x_0)$ and $\sup_z \mathbb{P}(Y = 1|z, x_1, x_0) \leq \max(f_1(x_1), f_0(x_0))$, which are the bounds derived in Chiburis (2010).

3.2.3. *Independence.* In the special case, where the two sector specific errors are independent of each other $u_1 \perp\!\!\!\perp u_0$, sharp bounds can be derived from Theorem 1 and $\mathbb{P}(u_1 \leq f_1(x_0), u_0 \leq f_0(x_0)) = \mathbb{P}(u_1 \leq f_1(x_1))\mathbb{P}(u_0 \leq f_0(x_0)) = f_1(x_1)f_0(x_0)$.

3.2.4. *Factor structure.* Theorem 1 also allows us to characterize the empirical content of the factor model for sector specific unobserved heterogeneity proposed in Aakvik, Heckman, and Vytlacil (2005).

Assumption 7 (Factor model)**.** *Sector specific unobserved heterogeneity has factor structure* $u_d = \alpha_d u + \eta_d$, $d = 1, 0$, *with* $\mathbb{E}u = 0$, $\mathbb{E}u^2 = 1$ *(without loss of generality) and* $\eta_1 \perp\!\!\!\perp \eta_0 | u$. η_d *is uniformly distributed on* $[0, 1]$ *for* $d = 1, 0$, *conditionally on* u.

This factor specification for sector specific unobserved heterogeneity is particularly appealing in applications to the effects of employment programs. Success in securing a job depends on common unobservable heterogeneity in talent and motivation and sector specific noise. Under Assumptions 1, 3 and 7, we still have $\mathbb{E}[Y_d|z, x_1, x_0] = f_d(x_d)$ and

$$\begin{aligned}
\mathbb{P}(u_1 \leq f_1(x_1), u_0 \leq f_0(x_0)|x_1, x_0) &= \mathbb{E}_u\mathbb{P}(\eta_1 \leq f_1(x_1) - \alpha_1 u, \eta_0 \leq f_0(x_0) - \alpha_0 u|x_1, x_0, u) \\
&= \mathbb{E}_u\mathbb{P}(\eta_1 \leq f_1(x_1) - \alpha_1 u|x_1, u)\mathbb{P}(\eta_0 \leq f_0(x_0) - \alpha_0 u|x_1, x_0, u) \\
&= f_1(x_1)f_0(x_0) + \alpha_1\alpha_0.
\end{aligned}$$

Hence we can obtain sharp bounds on parameters f_1, f_0, α_1 and α_0 as follows.

Corollary 6 (Sharp bounds for the factor model)**.** *Under Assumptions 1, 3 and 7, the empirical content of the model is characterized by (3.1), (3.2) and (writing $\omega = (z, x_1, x_0)$ as before)*

$$\begin{aligned}
& \sup_z \max\Big(0, f_0(x_0) - \mathbb{P}(1,0|\omega) - \mathbb{P}(0,1|\omega), f_1(x_1) - \mathbb{P}(1,1|\omega) - \mathbb{P}(0,0|\omega)\Big) \\
\leq \;& f_1(x_1) f_0(x_0) + \alpha_1 \alpha_0 \\
\leq \;& \inf_z \min\Big(\mathbb{P}(Y=1|\omega), f_1(x_1) + f_0(x_0) - \mathbb{P}(Y=1|\omega)\Big)
\end{aligned}$$

We recover the case of independent sector specific heterogeneity variables, when $\alpha_1 = \alpha_0 = 0$.

Conclusion

We have derived sharp bounds in the simple, extended and generalized binary Roy models, including a factor specification proposed by Aakvik, Heckman, and Vytlacil (2005). The bounds are simple enough to lend themselves to existing inference methods for intersection bounds as in Chernozhukov, Lee, and Rosen (2009).

Appendix A. Proofs

In all the proofs, we use the notation $\omega = (z, x_1, x_0)$. When there is no ambiguity, we shall write $f_1 = f_1(x_1)$, $f_0 = f_0(x_0)$ and $g = g(\omega)$.

A.1. **Proof of Proposition 1.**

A.1.1. *Validity of the bounds.* See main text.

A.1.2. *Sharpness of the bounds.* To show that these bounds are sharp for $f_1(x_1)$ it is sufficient to construct joint distributions for (u_0^*, u_1^*) such that $f_1(x_1)$ equals $P(Y = 1, D = 1|\omega)$ or $P(Y = 1|\omega)$ (and similarly for $f_0(x_0)$) and which is compatible with the observed data in the following sense:

(1) $P(u_0^* \leq f_0(x_0), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0)|x_1, x_0) = P(Y = 1, D = 0|\omega)$,

(2) $P(u_1^* \leq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0)|x_1, x_0) = P(Y = 1, D = 1|\omega)$,

(3) $P(u_0^* \geq f_0(x_0), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0)|x_1, x_0) = P(Y = 0, D = 0|\omega)$,

(4) $P(u_1^* \geq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0)|x_1, x_0) = P(Y = 0, D = 1|\omega)$,

(5) $P(u_0^* \leq f_0(x_0)|x_0) \in [P(Y = 1, D = 0|\omega), P(Y = 1|\omega)]$.

We assume in the following that $f_0(x_0) \geq f_1(x_1)$ (the opposite case can be treated similarly). Consider the following function $f(u_0^*, u_1^*)$ with values:

$$
\begin{array}{lll}
\frac{2P(Y=1,D=0|\omega)}{2f_1(x_1)f_0(x_0)-f_1(x_1)^2} & \text{if} & u_1^* \leq f_1(x_1), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0), \\
0 & \text{if} & u_1^* \geq f_1(x_1), u_0^* \leq f_0(x_0), \\
\frac{2P(Y=1,D=1|\omega)}{(2-2f_0(x_0)+f_1(x_1))f_1(x_1)} & \text{if} & u_1^* \leq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0), \\
\frac{2P(Y=0,D=1|\omega)}{(1-f_0(x_0))^2} & \text{if} & u_1^* \geq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0), \\
\frac{2P(Y=0,D=0|\omega)}{(1+f_0(x_0)-2f_1(x_1))(1-f_0(x_0))} & \text{if} & u_0^* \geq f_0(x_0), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0).
\end{array}
$$

It is easy to verify that this function is a density of a joint distribution which is compatible with the observed data (i.e respects conditions 1 to 5 above) and such as $f_1(x_1) = P(u_1^* \leq f_1(x_1)|x_1) = P(Y = 1|\omega)$. This fact shows that $P(Y = 1|\omega)$ is the sharp upper bound for $f_1(x_1)$. Now, we will propose another joint distribution compatible with the observed data such that: $f_1(x_1) = P(u_1^* \leq f_1(x_1)|x_1) = P(Y = 1, D = 1|\omega)$. Consider now the function $f(u_0^*, u_1^*) =$ with values:

$$
\begin{array}{lll}
0 & \text{if} & u_1^* \leq f_1(x_1), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0), \\
\frac{P(Y=1,D=0|\omega)}{f_0(x_0)(1-f_1(x_1))} & \text{if} & u_1^* \geq f_1(x_1), u_0^* \leq f_0(x_0), \\
\frac{2P(Y=1,D=1|\omega)}{(2-2f_0(x_0)+f_1(x_1))f_1(x_1)} & \text{if} & u_1^* \leq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0), \\
\frac{2P(Y=0,D=1|\omega)}{(1-f_0(x_0))^2} & \text{if} & u_1^* \geq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0), \\
\frac{2P(Y=0,D=0|\omega)}{(1+f_0(x_0)-2f_1(x_1))(1-f_0(x_0))} & \text{if} & u_0^* \geq f_0(x_0), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0).
\end{array}
$$

It is also easy to verify that this function is a density of a joint distribution which is compatible with the observed data (i.e respects conditions 1 to 5) and such as $f_1(x_1) = P(u_1^* \leq f_1(x_1)|x_1) = P(Y =$

$1, D = 1|\omega)$. This fact shows that $P(Y = 1, D = 1|\omega)$ is the sharp lower bound for $f_1(x_1)$. With the same strategy we can show that the bounds $P(Y = 1, D = 0|\omega) \leq f_0(x_0) \leq P(Y = 1|\omega)$ are sharp. This fact completes our Proof.

A.2. **Proof of Proposition 2.**

A.2.1. *Validity of the bounds.* To show validity of the bounds, we drop all the conditioning variables $\omega = (z, x_1, x_0)$ from the notation. We have $D = 1 \Rightarrow Y_0^* + g \leq Y_1^* \Rightarrow 1\{Y_0^* + g \geq 0\} \leq 1\{Y_1^* \geq 0\} \Rightarrow 1\{Y_0^* + g \geq 0\}1\{D = 1\} \leq 1\{Y_1^* \geq 0\}1\{D = 1\} \Rightarrow \mathbb{E}[1\{Y_0^* + g \geq 0\}|D = 1] \leq \mathbb{E}[1\{Y_1^* \geq 0\}|D = 1] \Rightarrow \mathbb{E}[1\{Y_0^* + g \geq 0\}|D = 1] \leq \mathbb{E}[Y_1|D = 1]$. We can easily derive equivalent inequalities when $D = 0$. Hence, if $D = 1\{Y_1^* > Y_0^* + g\}$ then $\mathbb{E}[1\{Y_0^* + g \geq 0\}|D = 1] \leq \mathbb{E}[Y_1|D = 1]$ and $\mathbb{E}[Y_1|D = 0] \leq \mathbb{E}[1\{Y_0^* + g \geq 0\}|D = 0]$. Hence, when $g \geq 0$, $\mathbb{E}[Y_0|D = 1] \leq \mathbb{E}[Y_1|D = 1]$ and when $g \leq 0$, $\mathbb{E}[Y_1|D = 0] \leq \mathbb{E}[Y_0|D = 0]$. Finally, if $g = 0$ we have $\mathbb{E}[Y_d|D = d] \geq \mathbb{E}[Y_d|D = 1 - d]$ where $d \in \{0, 1\}$. Those inequalities allow us to construct the sharp bounds for f_1 and f_0 in the case where $D = 1\{Y_1^* > Y_0^* + g\}$.Indeed, $f_1 = \mathbb{E}[Y_1] = \mathbb{E}[Y_1, D = 1] + \mathbb{E}[Y_1|D = 0]P(D = 0)$ and $f_0 = \mathbb{E}[Y_0] = \mathbb{E}[Y_0, D = 0] + \mathbb{E}[Y_0|D = 1]P(D = 1)$. Now, if $g \geq 0$, then $P(Y = 1, D = 1) \leq f_1 \leq P(Y = 1, D = 1) + P(D = 0)$ and $P(Y = 1, D = 0) \leq f_0 \leq P(Y = 1)$. On the other hand, if $g \leq 0$, $P(Y = 1, D = 1) \leq f_1 \leq P(Y = 1)$ and $P(Y = 1, D = 0) \leq f_0 \leq P(Y = 1, D = 0) + P(D = 1)$. Finally, $f_0 = \mathbb{E}[1\{u_0 \leq f_0\}1\{u_1 \geq u_0 + f_1 - f_0 - g\}] + \mathbb{E}[1\{u_0 \leq f_0\}1\{u_1 \leq u_0 + f_1 - f_0 - g\}]$. Hence, if $g \geq f_1$, then $\{u_1 \leq u_0 + f_1 - f_0 - g\} \Rightarrow \{u_0 \geq f_0\}$ and $f_0(X_0) \leq \mathbb{E}[1\{u_0 \leq f_0\}1\{u_1 \geq u_0 + f_1 - f_0 - g\}] + \mathbb{E}[1\{u_0 \leq f_0\}1\{u_0 \geq f_0\}] \leq \mathbb{E}[1\{u_0 \leq f_0\}1\{u_1 \geq u_0 + f_1 - f_0 - g\}] = P(Y = 1, D = 0)$.

Now the bounds for g can be obtained as follows.

- If $g + f_0 - f_1 \geq 0$ and $g \leq f_1$, then $\{u_0 \leq g + f_0 - f_1\} \Rightarrow \{u_0 \leq u_1 + g + f_0 - f_1\}$ and $\{u_0 \leq g + f_0 - f_1\} \Rightarrow \{u_0 \leq f_0\}$. So $\{u_0 \leq g + f_0 - f_1\} \Rightarrow \{u_0 \leq u_1 + g + f_0 - f_1\} \cap \{u_0 \leq f_0\}$.

Hence $g+f_0-f_1 = P(u_0 \leq g+f_0-f_1) \leq P(\{u_0 \leq u_1+g+f_0-f_1\} \cap \{u_0 \leq f_0\}) = P(Y = 1, D = 0)$.

- If $g + f_0 - f_1 \geq 0$ and $g \geq f_1$, then $\{u_0 \leq g + f_0 - f_1\} \Rightarrow \{u_0 \leq u_1 + g + f_0 - f_1\}$, hence $g + f_0 - f_1 = P(u_0 \leq g + f_0 - f_1) \leq P(\{u_0 \leq u_1 + g + f_0 - f_1\}) = P(D = 0)$. As $f_0 = P(Y = 1, D = 0)$ we have $g - f_1 \leq P(Y = 0, D = 0)$.
- If $g+f_0-f_1 \leq 0$ and $g \geq -f_0$, then by similar arguments, we have $g+f_0-f_1 \geq -P(Y = 1, D = 1)$.
- If $g + f_0 - f_1 \leq 0$ and $g \leq -f_0$, then $g + f_0 \geq -P(Y = 0, D = 1)$.

A.2.2. *Sharpness of the bounds.* As previously our method consist in constructing joint distributions compatible with the observed data such that:

- if $g(\omega) > 0$, $f_1(x_1)$ equals $P(Y = 1, D = 1|\omega)$ or $P(Y = 1, D = 1|\omega) + P(D = 0|\omega)$,
- if $g(\omega) < 0$, $f_1(x_1)$ equals $P(Y = 1, D = 1|\omega)$ or $P(Y = 1|\omega)$,

and similarly for $f_0(x_0)$. The compatibility between the joint distribution and the observed data can be expressed as follows:

(1) $P(u_0^* \leq f_0(x_0), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega)|\omega) = P(Y = 1, D = 0|\omega)$,

(2) $P(u_1^* \leq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega)|\omega) = P(Y = 1, D = 1|\omega)$,

(3) $P(u_0^* \geq f_0(x_0), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega)|\omega) = P(Y = 0, D = 0|\omega)$,

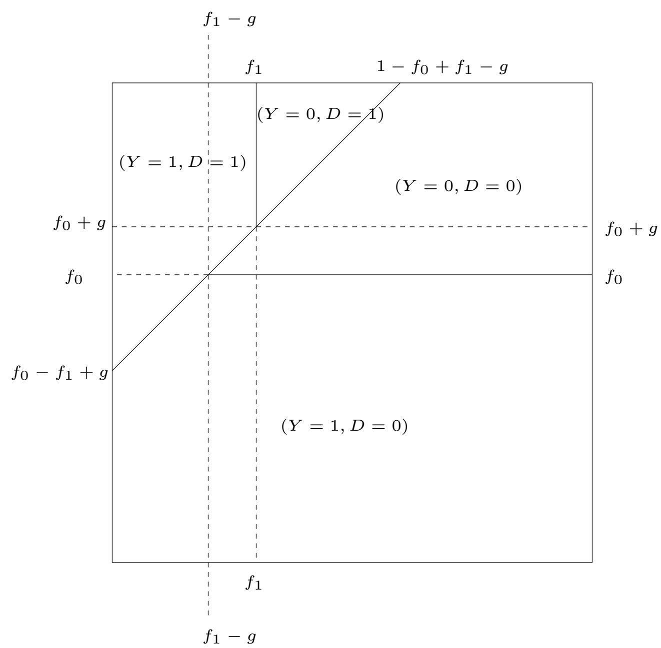
(4) $P(u_1^* \geq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega)|\omega) = P(Y = 0, D = 1|\omega)$,

(5) (a) if $g > 0$, $P(u_0^* \leq f_0(x_0)|x_0) \in [P(Y = 1, D = 0|\omega), P(Y = 1|\omega)]$,

(b) if $g < 0$, $P(u_0^* \leq f_0(x_0)|x_0) \in [P(Y = 1, D = 0|\omega), P(Y = 1, D = 0|\omega) + P(D = 1|\omega)]$.

Assume that $f_0(x_0) > f_1(x_1)$, $0 < g(\omega) < f_1(x_1)$ and $f_0(x_0) + g(\omega) < 1$ as in Figure 6. Other cases

FIGURE 6. Characterization of the empirical content of the extended binary Roy model in the unit square of the (u_1, u_0) space in case $f_0(x_0) > f_1(x_1)$, $0 < g(\omega) < f_1(x_1)$ and $f_0(x_0) + g(\omega) < 1$.



can be treated similarly. Consider the function $f(u_0^*, u_1^*)$ with values:

$$
\begin{array}{lll}
0 & \text{if} & u_0^* \leq g(\omega) + f_0(x_0) - f_1(x_1), u_1^* \geq f_1(x_1), \\
\frac{P(Y=1,D=0|\omega)}{f_1(x_1)(g(\omega)+f_0(x_0)-f_1(x_1))} & \text{if} & u_0^* \leq g(\omega) + f_0(x_0) - f_1(x_1), u_1^* \leq f_1(x_1), \\
0 & \text{if} & u_0^* \geq g(\omega) + f_0(x_0) - f_1(x_1), u_0^* \leq f_0(x_0) \\
 & & \qquad \text{and } u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega), \\
\frac{2P(Y=0,D=0|\omega)}{g(\omega)g(\omega)} & \text{if} & u_0^* \geq f_0(x_0), u_1^* \leq f_1(x_1), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega), \\
0 & \text{if} & u_0^* \geq f_0(x_0), u_1^* \geq f_1(x_1), u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega), \\
\frac{2P(Y=1,D=1|\omega)}{(2-2(f_0(x_0)+g(\omega))+f_1(x_1))f_1(x_1)} & \text{if} & u_1^* \leq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega), \\
\frac{2P(Y=0,D=1|\omega)}{(1-f_0(x_0)-g(\omega))^2} & \text{if} & u_1^* \leq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega).
\end{array}
$$

It is easy to verify that this function is a density of a joint distribution which is compatible with the observed data (i.e respects conditions 1 to 5a) and such that $f_1(x_1) = P(u_1^* \leq f_1(x_1)|x_1) = P(Y = 1, D = 1|\omega) + P(D = 0|\omega)$ and $g(\omega) + f_0(x_0) - f_1(x_1) = P(u_0^* \leq g(\omega) + f_0(x_0) - f_1(x_1)|\omega) = P(Y = 1, D = 0|\omega)$. In the previous section, we showed that $P(Y = 1, D = 0|\omega)$ is an upper bound for $g(\omega) + f_0(x) - f_1(x)$ in case $g(\omega) < f_1(x_1)$. Here we construct a joint distribution which hits this upper bound. This fact shows that $P(Y = 1, D = 1|\omega) + P(D = 0|\omega)$ is the sharp upper bound for $f_1(x_1)$ and that $P(Y = 1, D = 0|\omega)$ is the sharp upper bound for $g(\omega) + f_0(x) - f_1(x)$.

We now propose a joint distribution such that $f_1(x_1) = P(u_1^* \leq f_1(x_1)|x_1) = P(Y = 1, D = 1|\omega)$ and $g(\omega) + f_0(x_0) - f_1(x_1) = P(u_0^* \leq g(\omega) + f_0(x_0) - f_1(x_1)|\omega) = P(Y = 1, D = 0|\omega)$. Consider the function $f(u_0^*, u_1^*)$ with values:

$$
\begin{array}{ll}
\frac{P(Y=1,D=0|\omega)}{(1-f_1(x_1))(g(\omega)+f_0(x_0)-f_1(x_1))} & \text{if} \quad u_0^* \leq g(\omega) + f_0(x_0) - f_1(x_1), u_1^* \geq f_1(x_1), \\
0 & \text{if} \quad u_0^* \leq g(\omega) + f_0(x_0) - f_1(x_1), u_1^* \leq f_1(x_1), \\
0 & \text{if} \quad u_0^* \geq g(\omega) + f_0(x_0) - f_1(x_1), u_0^* \leq f_0(x_0) \\
 & \qquad \text{and} \quad u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega), \\
0 & \text{if} \quad u_0^* \geq f_0(x_0), u_1^* \leq f_1(x_1) \\
 & \qquad \text{and} \quad u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega), \\
\frac{P(Y=0,D=0|\omega)}{(1-f_1(x_1))(1-f_0(x_0))-\frac{1}{2}(1-f_0(x_0)-g(\omega))^2} & \text{if} \quad u_0^* \geq f_0(x_0), u_1^* \geq f_1(x_1) \\
 & \qquad \text{and} \quad u_1^* \geq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega), \\
\frac{2P(Y=1,D=1|\omega)}{(2-2(f_0(x_0)+g(\omega))+f_1(x_1))f_1(x_1)} & \text{if} \quad u_1^* \leq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega), \\
\frac{2P(Y=0,D=1|\omega)}{(1-f_0(x_0)-g(\omega))^2} & \text{if} \quad u_1^* \geq f_1(x_1), u_1^* \leq u_0^* + f_1(x_1) - f_0(x_0) - g(\omega).
\end{array}
$$

It is also easy to verify that this function is a density of a joint distribution which is compatible with the observed data (i.e respects conditions 1 to 5a) and such that $f_1(x_1) = P(u_1^* \leq f_1(x_1)|x_1) = P(Y = 1, D = 1|\omega)$ and $g(\omega) + f_0(x_0) - f_1(x_1) = P(u_0^* \leq g(\omega) + f_0(x_0) - f_1(x_1)|\omega) = P(Y = 1, D = 0|\omega)$. This fact shows that $P(Y = 1, D = 1|\omega)$ is the sharp lower bound for $f_1(x_1)$. With the same strategy we can show that $P(Y=1, D=0|\omega) \leq f_0(x_0) \leq P(Y=1|\omega)]$ is sharp. This fact completes our Proof.

A.3. **Proof of Theorem 1.** Under Assumptions 1-3, the model can be equivalently written $(Y, D) \in G((u_1, u_0)|W)$ almost surely conditionally on $W = (Z, X_1, X_0)$, where G is a multi-valued mapping, which to (u_1, u_0) associates $(y, d) = G((u_1, u_0)|W) = \{(1,1), (1,0)\}$ if $u_1 \leq f_1(x_1)$ and $u_0 \leq f_0(x_0)$, $\{(0,1), (1,0)\}$ if $u_1 > f_1(x_1)$ and $u_0 \leq f_0(x_0)$, $\{(1,1), (0,0)\}$ if $u_1 \leq f_1(x_1)$ and $u_0 > f_0(x_0)$ and $\{(0,1), (0,0)\}$ if $u_1 > f_1(x_1)$ and $u_0 > f_0(x_0)$. Hence Theorem 1 of Galichon and Henry (2011) applies and the empirical content of the model is characterized by the collection of inequalities $P(A|W) \leq P((u_1, u_0) : \; G((u_1, u_0)|W)$ hits $A|W)$ for each subset A of $\{(0,0), (0,1), (1,0), (1,1)\}$ (i.e., 16 inequalities). The only non redundant inequalities are $P(1,1|W) \leq f_1(X_1)$, $P(1,0|W) \leq f_0(X_0)$, $P(0,1|W) \leq 1 - f_1(X_1)$, $P(0,0|W) \leq 1 - f_0(X_0)$, $P(Y=0|W) \leq 1 - P(u_1 \leq f_1(X_1), u_0 \leq f_0(X_0)|X_1, X_0)$, $P(Y=1|W) \leq 1 - P(u_1 > f_1(X_1), u_0 > f_0(X_0)|X_1, X_0)$, $P(0,0|W) + P(1,1|W) \leq P(u_1 \leq f_1(X_1), u_0 \leq f_0(X_0)|X_1, X_0) + P(u_0 > f_0(X_0)|X_0)$ and $P(0,1|W) + P(1,0|W) \leq P(u_1 \leq f_1(X_1), u_0 \leq f_0(X_0)|X_1, X_0) + P(u_1 > f_1(X_1)|X_1)$. After some manipulation, the result follows.

A.4. **Proof of Corollary 5.** We only need to show that the bounds (2.4) for g remain valid. We drop conditioning variables from the notation throughout this section.

- If $g+v+f_0-f_1 \geq 0$ and $g+v \leq f_1$, then $\{u_0 \leq g+v+f_0-f_1\} \Rightarrow \{u_0 \leq u_1+g+v+f_0-f_1\}$ and $\{u_0 \leq g+v+f_0-f_1\} \Rightarrow \{u_0 \leq f_0\}$. So $\{u_0 \leq g+v+f_0-f_1\} \Rightarrow \{u_0 \leq u_1+g+v+f_0-f_1\} \cap \{u_0 \leq f_0\}$. Therefore $P(u_0 - v \leq g + f_0 - f_1) \leq P(\{u_0 \leq u_1 + g + v + f_0 - f_1\} \cap \{u_0 \leq f_0\}) = P(Y=1, D=0)$.
- If $g+v+f_0-f_1 \geq 0$ and $g+v \geq f_1$, then $\{u_0 \leq g+v+f_0-f_1\} \Rightarrow \{u_0 \leq u_1+g+v+f_0-f_1\}$. Therefore $P(u_0 - v \leq g + f_0 - f_1) \leq P(\{u_0 \leq u_1 + g + v + f_0 - f_1\}) = P(D=0)$.

- If $g+v+f_0-f_1 \leq 0$ and $g+v \geq -f_0$, then $\{u_1 \leq f_1-f_0-g-v\} \Rightarrow \{u_1 \leq u_0+f_1-f_0-g-v\}$ and $\{u_1 \leq f_1-f_0-g-v\} \Rightarrow \{u_1 \leq f_1\}$. So $\{u_1 \leq f_1-f_0-g-v\} \Rightarrow \{u_1 \leq u_0+f_1-f_0-g-v\} \cap \{u_1 \leq f_1\}$. Therefore $P(u_1+v \leq f_1-f_0-g) \leq P(\{u_1 \leq u_0+f_1-f_0-g-v\} \cap \{u_1 \leq f_1\}) = P(Y=1, D=1)$.
- If $g+v+f_0-f_1 \leq 0$ and $g+v \leq -f_0$, then $\{u_1 \leq f_1-f_0-g-v\} \Rightarrow \{u_1 \leq u_0+f_1-f_0-g-v\}$. Hence $P(u_1+v \leq f_1-f_0-g) \leq P(u_1 \leq u_0+f_1-f_0-g-v) = P(D=1)$.

Now, since $v \perp\!\!\!\perp (u_0,u_1)$, we have: $P(u_0 \leq g+v+f_0-f_1) = \mathbb{E}_v[\mathbb{E}[1\{u_0 \leq g+v+f_0-f_1\}|v]] = \mathbb{E}_v[g+v+f_0-f_1] = g+f_0-f_1$. Then, we get the following:

- If $g+v+f_0-f_1 \geq 0$ and $g+v \leq f_1$, then $g+f_0-f_1 \leq P(Y=1, D=0)$.
- If $g+v+f_0-f_1 \geq 0$ and $g+v \geq f_1$, then $g-f_1 \leq P(Y=0, D=0)$.
- If $g+v+f_0-f_1 \leq 0$ and $g+v \geq -f_0$, then $g+f_0-f_1 \geq -P(Y=1, D=1)$.
- If $g+v+f_0-f_1 \leq 0$ and $g+v \leq -f_0$, then $g+f_0 \geq -P(Y=0, D=1)$.

which completes the proof.

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