CIRJE-F-810

A Note on the Decomposition Technique of Economic Indices

Kohta Mori
Yale University

Saki Sugano
Graduate School of Economics, University of Tokyo

Joe Chen
National Chengchi University

Yun Jeong Choi
Kyung Hee University

Yasuyuki Sawada
University of Tokyo

July 2011

A Note on the Decomposition Technique of Economic Indices

Kohta Mori* Saki Sugano† Joe Chen‡ Yun Jeong Choi§ Yasuyuki Sawada¶

July 28, 2011

1 Introduction

Economic indices we are interested in are often compounds of multiple components. For instance, national product is the summation of the values of outputs from various sectors in the economy, and unemployment rate is the ratio of the number of the unemployed to the number of people in the labor force. In many applications, it is necessary to look into changes from each constituent components, rather than the overall growth of an index. Decomposition technique seeks for ways to disintegrate the overall growth rate of an index to gain insight on the roles of each individual component on the overall change.

In demography, Kitagawa (1955) developed a method named “components of a difference between two rates” to explain the difference between the total rates of two groups in terms of differences in their specific rates and differences in their composition. Its extensions to multiple component decomposition are done by Cho and Retherford (1973), Das Gupta (1978), and Kim and Strobino (1984). In economics, decomposition method dates back to Leontief who analyzed the change in the structure of production (Dietzenbacher and Los , 1998; Canudas-Romo , 2003). Oosterhaven and Van der Linden (1997) suggested polar decomposition, which was extended by Dietzenbacher and Los (1998) and Andreev *et al.* (2002). These methods give very similar decomposition results. For a more detail review on decomposition technique, see Canudas-Romo (2003).

This note introduces a generalized version of various multi-component decomposition, which can be applied to two classes of indices, the *additive-product form* and the *product-additive form*. Section 2 introduces the method and the formula, and Section 3 provides an example of the application.

2 Decomposition Method

Let $y(t)$ be a generic positive economic index. Let $X(t) := (x_{j,k}(t))$, a $J \times K$ matrix, be the components of $y(t)$, so that $y(t) \equiv f(X(t))$, where f is some function. In a typical application, j may index for groups and k may index for variables. Consider the following two examples.

Example 1 (Nominal GDP)
Let y be nominal GDP, $x_{j,1}$ be the price level of the j-th sector, and $x_{j,2}$ be the quantity produced in the j-th sector. Then y is represented as:

$$y = \sum_{j=1}^{J} x_{j,1} x_{j,2}.$$

*Department of Economics, Yale University. kota.mori@yale.edu

†Graduate School of Economics, University of Tokyo; Japan Society for the Promotion of Science. ee097017@mail.ecc.u-tokyo.ac.jp

‡Department of Public Finance, National Chengchi University. joe@nccu.edu.tw

§College of International Studies, Kyung Hee University. yun.choi@khu.ac.kr

¶Faculty of Economics, University of Tokyo. sawada@e.u-tokyo.ac.jp

Example 2 (Unemployment rate)
Let y denote unemployment rate. Suppose the population is devided exhautively into J exclusive groups. Let $x_{j,1}$ be the number of unemployed of group j, and $x_{j,2}$ be the labor force of group j. Then y is represented as:

$$y = \frac{\sum_{j=1}^{J} x_{j,1}}{\sum_{j=1}^{J} x_{j,2}}.$$

Now, let us consider a simple case where y is a summation of a single variable x_j of $J > 1$ groups, *i.e.* $K = 1$:

$$y(t) = \sum_{j=1}^{J} x_j(t).$$

Take the first order difference of $y(y)$ and devide it by $y(t)$,

$$\frac{\Delta y(t)}{y(t)} = \sum_{j=1}^{J} \frac{\Delta x_j(t)}{y(t)} = \sum_{j=1}^{J} \mathrm{cd}_j(t), \tag{1}$$

where $\Delta z(t) := z(t+1) - z(t)$. The growth rate of $y(t)$, shown in the left-hand side, is decomposed into J components, and $\mathrm{cd}_j(t) \equiv \Delta x_j(t)/y(t)$ is the *degree of contribution* of variable x_j to the growth of y at time t. Notice that decomposition formula (1) can be seen as a weighted average of the variable's group-specific growth rates. To see this, suppose $x_j(t) \neq 0$ for all j, then:

$$\frac{\Delta y(t)}{y(t)} = \sum_{j=1}^{J} \frac{x_j(t)}{y(t)} \left[\frac{\Delta x_j(t)}{x_j(t)}\right].$$

In the following, we generalize the decomposition technique to two classes of indices, the *additive-product form* and the *product-additive form.*

2.1 Additive-Product Form

Suppose y is in the additive-product form defined as:

$$y = \sum_{j=1}^{J} \prod_{k=1}^{K} x_{j,k}. \tag{2}$$

Note that the nominal GDP in Example 1 belongs to this class.

Differentiate the logarithm of both sides of (2), and replace the derivatives by differences:

$$\frac{\Delta y(t)}{y(t)} \simeq \sum_{j=1}^{J} \sum_{k=1}^{K} \left[\frac{\prod_{l \neq k} x_{j,l}(t)}{y(t)}\right] \Delta x_{j,k}(t) = \sum_{j=1}^{J} \sum_{k=1}^{K} \mathrm{cd}_{j,k}(t), \tag{3}$$

where $\mathrm{cd}_{j,k}(t) \equiv \left[\prod_{l \neq k} x_{j,l}(t)/y(t)\right] \cdot \Delta x_{j,k}(t)$. Notice that decomposition formula (3) can be seen as a weighted sum of the variables' group-specific growth rates. To see this, suppose $x_{j,k}(t) \neq 0$ for all j and k, then:

$$\frac{\Delta y(t)}{y(t)} \simeq \sum_{j=1}^{J} \sum_{k=1}^{K} \frac{\prod_{l=1}^{K} x_{j,l}(t)}{y(t)} \left[\frac{\Delta x_{j,k}(t)}{x_{j,k}(t)}\right].$$

2.2 Product-Additive Form

Suppose y is in the product-addtive form defined as:

$$y = \prod_{k=1}^{K} \left(\sum_{j=1}^{J} x_{j,k} \right)^{\gamma_k}, \tag{4}$$

where $\gamma \equiv (\gamma_1, \ldots, \gamma_K)'$ is a fixed K-vector. Note that the unemployment rate in Example 2 belongs to this class when $K = 2$ and $\gamma = (1, -1)'$.

Differentiate the logarithm of both sides of (4), and replace the derivatives by differences:

$$\frac{\Delta y(t)}{y(t)} \simeq \sum_{k=1}^{K} \sum_{j=1}^{J} \left[\frac{\gamma_k}{\sum_{j=1}^{J} x_{j,k}(t)} \right] \Delta x_{j,k}(t) = \sum_{k=1}^{K} \sum_{j=1}^{J} \mathrm{cd}_{j,k}(t), \tag{5}$$

where $\mathrm{cd}_{j,k}(t) \equiv \left[\gamma_k / \sum_{j=1}^{J} x_{j,k}(t) \right] \cdot \Delta x_{j,k}(t)$. Notice that decomposition formula (5) can be seen as a weighted sum of the variables' group-specific growth rates. To see this, suppose $x_{j,k}(t) \neq 0$ for all j and k, then:

$$\frac{\Delta y(t)}{y(t)} \simeq \sum_{k=1}^{K} \sum_{j=1}^{J} \left[\frac{\gamma_k x_{j,k}(t)}{\sum_{j=1}^{J} x_{j,k}(t)} \right] \frac{\Delta x_{j,k}(t)}{x_{j,k}(t)}.$$

3 Application

In this section, we provide an application of the decomposition technique in order to measure the impacts of various components of an index of interest.[1]

Figure 1 plots the 10-year Japan mortality rate of cardiac disease from 1999.[2] The graph exhibits a significant increase in the mortality rate, from 120.00 to 142.21 per 100,000 population, or by 18.5%.

It is known that the overall death by cardiac diseases can be characterized by the age group specific rates and the age structure of the population, thereby we decompose the overall growth rate of death by cardiac diseases into changes in the age group specific rates and changes in the age structure of the population. The following equation suggests that this index belongs to the class of the additive-product form:

$$\begin{aligned}&\text{overall cardiac disease mortality rate}\\ &= \sum_j [\text{the } j\text{-th age group mortality rate by cardiac disease} \times \text{the population share of the } j\text{-th age group}]\end{aligned}$$

The decomposition result is shown in Table 1. The numbers in the cells are the cumulated degrees of contribution of the age group specific rates and the age structure of the population. The result implies that the increase in the mortality rate by cardiac diseases is due to the aging problem of the population, as the largest contributions are from the population share of those aged 60–79, and 80 and older. Notice also that the numbers in the 'Mortality Rate' column for age groups 60–79 and 80 and over show large negative contributions, probably due to progresses made in the treatments for cardiac diseases and improvement in the welfare system for the elderly. Rapid aging, however, outweighs such progresses, resulting in steady growth of mortality rate over all.

As a final point, notice that the decomposition is not exact, due to the approximation of the derivatives by differences. In this application, the sum of cumulative degrees of contribution is 18.21%, while the growth rate of original mortality rate is 18.5%.[3] Since approximation becomes more accurate as the time interval shortens, one should use cumulative degrees of contribution for the change of an index in long periods.

[1] See Chen *et al.* (2011) for an applications of the decomposition technique to investigate recent suicide trends in Japan.

[2] Mortality rate is the ratio of the number of death to population. We obtained the mortality data from *Vital Statistics*, the Ministry of Health, Labour, and Welfare, Japan, and the population data from *Demographics based on the Basic Resident Register*, the Ministry of Internal Affairs and Communications, Japan.

[3] The sum of annual growth rates is 17.58%.

Figure 1: Japan Mortality Rate of Cardiac Diseases, 1999–2009

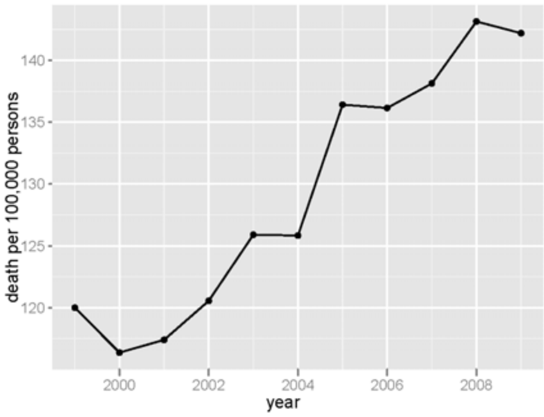

Table 1: Cumulative Degrees of Contribution, 1999–2009

Age Group	Mortality Rate	Population Share	Sum
00–19	-0.06%	-0.03%	-0.10%
20–39	-0.13%	-0.07%	-0.20%
40–59	-0.55%	-0.63%	-1.17%
60–79	-8.75%	7.32%	-1.43%
80 and above	-8.56%	29.67%	21.11%
Sum	-18.05%	36.26%	18.21%

References

Andreev, E.M., V.M. Shkolnikov, A.Z. Begun, 2002. Algorithm for decomposition of differences between aggregate demographic measures and its application to life expectancies, Gini coefficients, health expectancies, parity-progression ratios, and total fertility rates. Demographic Research 7, article 14.

Canudas-Romo, V., 2003. Decomposition methods in demography. Netherlands: Rozenberg publishers.

Chen, J., Y.J. Choi, K. Mori, Y. Sawada, and S. Sugano, 2011, An Analysis of Suicides in Japan, 1997–2007: Changes in Incidence, Persistence, and Age Profiles.

Cho, L.J. and R.D. Retherford, 1973. Comparative analysis of recent fertility trends in East Asia. International Union for the Scientific Study of Population (Ed.), International Population Conference, Liege, 2, 163–181.

Das Gupta, P., 1978. A general method of decomposing a difference between two rates into several components. Demography 15(1), 99–112.

Dietzenbacher, E. and B. Los, 1998. Structural decomposition techniques: sense and sensitivity. Economic System Research 10(4), 307–324.

Kim, Y.J. and D.M. Strobino, 1984. Decomposition of the difference between two rates with hierarchical factors. Demography 21(3), 361–372.

Kitagawa, E.M., 1955. Components of a difference between two rates. Journal of the American Statistical Association 50 (272), 1168–1194.

Oosterhaven, J. and J.A. Van der Linden, 1997. European technology, trade, and income changes for 1975?985: an intercountry input-output decomposition. Economic Systems Research 9(4), 393–412.