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Collateralization and its Implications**

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# Modeling of Interest Rate Term Structures under Collateralization and its Implications\*

Masaaki Fujii<sup>†</sup>, Akihiko Takahashi<sup>‡</sup>

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## Abstract

In recent years, we have observed dramatic increase of collateralization as an important credit risk mitigation tool in over the counter (OTC) market [6]. Combined with the significant and persistent widening of various basis spreads, such as Libor-OIS and cross currency basis, the practitioners have started to notice the importance of difference between the funding cost of contracts and Libors of the relevant currencies. In this article, we integrate the series of our recent works [1, 2, 4] and explain the consistent construction of term structures of interest rates in the presence of collateralization and all the relevant basis spreads, their no-arbitrage dynamics as well as their implications for derivative pricing and risk management. Particularly, we have shown the importance of the choice of collateral currency and embedded "cheapest-to-deliver" (CTD) option in a collateral agreement.

**Keywords :** swap, collateral, Libor, OIS, EONIA, Fed-Fund, cross currency, basis, HJM, CSA, CVA

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# 1 Introduction

The recent financial crisis and the following liquidity and credit squeeze have caused significant and persistent widening of various basis spreads<sup>1</sup>. In particular, we have witnessed drastic movement of cross currency swap (CCS), Libor-OIS, and tenor swap<sup>2</sup> (TS) basis spreads. In some occasions, the size of spreads has exceeded several tens of basis points, which is far wider than the general size of bid/offer spreads. Furthermore, there has been a dramatic increase of collateralization in financial contracts recent years, and it has become almost a market standard among the major financial institutions, at least [6]. The importance of the collateralization is mainly twofold: 1) reduction of the counterparty credit risk, and 2) change of the funding cost of the trade. The first one is well recognized, and there exist a large number of studies in the context of credit value adjustment (CVA). Although it is not as obvious as the first one, the second effect is also important and it has started to attract strong attentions among practitioners.

As we will see later, the details of collateralization specified in CSA (credit support annex), the existence of various basis spreads change the effective discounting rate appropriate for the specific contract. Because of the characteristic of the products, the effects are most relevant in interest rate and long-dated foreign exchange (FX) markets. These findings cast a strong warning to the financial institutions using the standard Libor Market Model (LMM), which treats the Libor as a risk-free interest rate and hence cannot reflect the existence of various basis spreads and collateralization. These drawbacks make the LMM incapable of calibrating to the relevant swap markets nor their dynamics, which is likely to cause the financial firms overlooking critical risk exposures.

In this article, we provide a systematic solution for the above market developments by integrating our series works, Fujii, Shimada & Takahashi (2009, 2009, 2010) [1, 2, 4]. We first present the pricing formula of derivatives under the collateralization, including the case where the payment and collateral currencies are different. Based on this result, we propose the procedures for the consistent term structure construction of collateralized swaps in multi-currency environment. Secondly, we provide the no-arbitrage dynamics under Heath-Jarrow-Morton (HJM) framework which is capable of calibrating to all the relevant swaps and allow full stochastic modeling of basis spreads. Thirdly, we explain the importance of the choice of collateral currency and the effects of embedded "cheapest-to-deliver" option in collateral agreement when it allows the replacement of collateral currency. And then, we conclude.

As related works, we refer to Johannes & Sundareshan (2007) [7] (and its working paper) as the first work focusing on the cost of collateral posting and studying its effect on the swap par rates based on empirical analysis of the US treasury and swap markets. More recently, Piterbarg (2010) [8] has used the similar formula given in [7] to take the collateral cost into account for the option pricing in general, but he has not discussed the case where the payment and collateral currencies are different, nor the term structure modeling of interest rate and FX under the collateralization.

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<sup>1</sup>A basis spread generally means the interest rate differentials between two different floating rates.

<sup>2</sup>It is a floating-vs-floating swap that exchanges Libors with two different tenors with a fixed spread in one side.

## 2 Pricing under the collateralization

This section reviews [1], our results on pricing derivatives under the collateralization. Let us make the following simplifying assumptions about the collateral contract.

1. *Full collateralization (zero threshold) by cash.*
2. *The collateral is adjusted continuously with zero minimum transfer amount.*

In fact, the daily margin call is now quite popular in the market and there is also an increasing tendency to include a provision for "add-hoc call", which allows the participants to call margin arbitrary time in addition to the periodical regular calls. Furthermore, there is a wide movement to reduce the threshold and minimum transfer amount since the various studies indicate that the existence of meaningful size of these variables significantly reduces the effectiveness of credit risk mitigation of collateralization, which results in higher CVA charge in turn. These developments make our assumptions a good proxy for the market reality at least for the standard fixed income products. Since the assumptions allow us to neglect the loss given default of the counterparty, we can treat each trade/payment separately without worrying about the non-linearity arising from the netting effects and the asymmetric handling of exposure.

We consider a derivative whose payoff at time  $T$  is given by  $h^{(i)}(T)$  in terms of currency "i". We suppose that the currency "j" is used as the collateral for the contract. Note that the instantaneous return (or cost when it is negative ) by holding the cash collateral at time  $t$  is given by

$$y^{(j)}(t) = r^{(j)}(t) - c^{(j)}(t) , \quad (2.1)$$

where  $r^{(j)}$  and  $c^{(j)}$  denote the risk-free interest rate and collateral rate of the currency  $j$ , respectively. In the case of cash collateral, the collateral rate  $c^{(i)}$  is usually given by the overnight rate of the corresponding currency, such as Fed-Fund rate for USD. If we denote the present value of the derivative at time  $t$  by  $h^{(i)}(t)$  (in terms of currency  $i$ ), the collateral amount posted from the counterparty is given by  $\left( h^{(i)}(t)/f_x^{(i,j)}(t) \right)$ , where  $f_x^{(i,j)}(t)$  is the foreign exchange rate at time  $t$  representing the price of the unit amount of currency  $j$  in terms of currency  $i$ . These considerations lead to the following calculation for the collateralized derivative price,

$$\begin{aligned} h^{(i)}(t) &= E_t^{Q_i} \left[ e^{-\int_t^T r^{(i)}(s)ds} h^{(i)}(T) \right] \\ &+ f_x^{(i,j)}(t) E_t^{Q_j} \left[ \int_t^T e^{-\int_t^s r^{(j)}(u)du} y^{(j)}(s) \left( \frac{h^{(i)}(s)}{f_x^{(i,j)}(s)} \right) ds \right] , \end{aligned} \quad (2.2)$$

where  $E_t^{Q_i}[\cdot]$  is the time  $t$  conditional expectation under the risk-neutral measure of currency  $i$ , where the money-market account of currency  $i$  is used as the numeraire. Here, the second term takes into account the return/cost from holding the collateral. By aligning the measure to  $Q_i$ , we obtain

$$h^{(i)}(t) = E_t^{Q_i} \left[ e^{-\int_t^T r^{(i)}(s)ds} h^{(i)}(T) + \int_t^T e^{-\int_t^s r^{(i)}(u)du} y^{(j)}(s) h^{(i)}(s) ds \right] . \quad (2.3)$$

Now, it is easy to see that

$$X(t) := e^{-\int_0^t r^{(i)}(s)ds} h^{(i)}(t) + \int_0^t e^{-\int_0^s r^{(i)}(u)du} y^{(j)}(s) h^{(i)}(s) ds \quad (2.4)$$

is a  $Q_i$ -martingale under appropriate integrability conditions. This tells us that the process of the option value can be written as

$$dh^{(i)}(t) = \left( r^{(i)}(t) - y^{(j)}(t) \right) h^{(i)}(t) dt + dM(t) \quad (2.5)$$

with some  $Q_i$ -martingale  $M$ .

As a result, we have the following theorem:

**Theorem 1**<sup>3</sup> *Suppose that  $h^{(i)}(T)$  is a derivative's payoff at time  $T$  in terms of currency "i" and that the currency "j" is used as the collateral for the contract. Then, the value of the derivative at time  $t$ ,  $h^{(i)}(t)$  is given by*

$$h^{(i)}(t) = E_t^{Q_i} \left[ e^{-\int_t^T r^{(i)}(s)ds} \left( e^{\int_t^T y^{(j)}(s)ds} \right) h^{(i)}(T) \right] \quad (2.6)$$

$$= D^{(i)}(t, T) E_t^{\mathcal{T}_i^c} \left[ e^{-\int_t^T y^{(i,j)}(s)ds} h^{(i)}(T) \right], \quad (2.7)$$

where

$$y^{(i,j)}(s) = y^{(i)}(s) - y^{(j)}(s) \quad (2.8)$$

with  $y^{(i)}(s) = r^{(i)}(s) - c^{(i)}(s)$  and  $y^{(j)}(s) = r^{(j)}(s) - c^{(j)}(s)$ . Here, we have also defined the collateralized zero-coupon bond of currency  $i$  as

$$D^{(i)}(t, T) = E_t^{Q_i} \left[ e^{-\int_t^T c^{(i)}(s)ds} \right] \quad (2.9)$$

and the collateralized forward measure  $\mathcal{T}_i^c$ , where the collateralized zero-coupon bond is used as the numeraire; thus,  $E_t^{\mathcal{T}_i^c}[\cdot]$  denotes the time  $t$  conditional expectation under the measure  $\mathcal{T}_i^c$ .

As a corollary of the theorem, we have

$$\begin{aligned} h(t) &= E_t^Q \left[ e^{-\int_t^T c(s)ds} h(T) \right] \\ &= D(t, T) E_t^{\mathcal{T}^c} [h(T)] \end{aligned} \quad (2.10)$$

when the payment and collateral currencies are the same.

Finally, let us also define the forward collateral rate of currency  $i$ ,  $c^{(i)}(t, T)$ , for the convenience of later discussion:

$$c^{(i)}(t, T) = -\frac{\partial}{\partial T} \ln D^{(i)}(t, T), \quad (2.11)$$

or equivalently

$$D^{(i)}(t, T) = \exp \left( -\int_t^T c^{(i)}(t, s) ds \right). \quad (2.12)$$

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<sup>3</sup>Although we are dealing with continuous processes here, we obtain the same result as long as there is no simultaneous jump of underlying assets when the counterparty defaults even in more general setting.

### 3 Calibration to Single Currency Swaps

In this section, we construct the relevant yield curves in a single currency market, where there exist three different types of interest rate swap, which are overnight index swap (OIS), standard interest rate swap (IRS), and tenor swap (TS). In the following, we explain how we can extract relevant forwards based on the result of previous section following the works [1, 3, 4]. Throughout this section, we assume that the relevant swap is collateralized by the domestic (or payment) currency.

#### 3.1 Overnight Index Swap

As we have seen in the previous section, it is critical to determine the forward curve of overnight rate for the pricing of collateralized swaps. Fortunately, there is a product called "overnight index swap" (OIS), which exchanges the fixed coupon and the daily-compounded overnight rate. Here, let us assume that the OIS itself is continuously collateralized by the domestic currency. In this case, using the Eq.(2.10), we get the consistency condition of  $T_0$ -start  $T_N$ -maturing OIS rate as

$$\text{OIS}_N \sum_{n=1}^N \Delta_n E^Q \left[ e^{-\int_0^{T_n} c(s) ds} \right] = \sum_{n=1}^N E_t^Q \left[ e^{-\int_0^{T_n} c(s) ds} \left( e^{\int_{T_{n-1}}^{T_n} c(s) ds} - 1 \right) \right]. \quad (3.1)$$

Here,  $\text{OIS}_N$  is the market quote of par rate for the length- $N$  OIS, and  $c(t)$  is the overnight (and hence collateral) rate at time  $t$  of the domestic currency.  $\Delta$  and  $\delta$  denote day-count fractions for fixed and floating legs, respectively. We can simplify the above equation into the form

$$\text{OIS}_N \sum_{n=1}^N \Delta_n D(0, T_n) = D(0, T_0) - D(0, T_N) \quad (3.2)$$

by using the collateralized zero-coupon bond. Now, from Eq.(3.2), we can easily obtain the term structure of  $D(0, \cdot)$  by the standard bootstrap and spline techniques [5].

#### 3.2 Interest rate swap

In the standard interest rate swap (IRS), two parties exchange a fixed coupon and Libor for a certain period with a given frequency. The tenor of Libor " $\tau$ " is determined by the frequency of floating payments, i.e., 6m-tenor for semi-annual payments, for example. For a  $T_0$ -start  $T_M$ -maturing IRS with the Libor of tenor  $\tau$ , we have

$$\text{IRS}_M \sum_{m=1}^M \Delta_m D(0, T_m) = \sum_{m=1}^M \delta_m D(0, T_m) E^{T_m^c} [L(T_{m-1}, T_m; \tau)] \quad (3.3)$$

as a consistency condition. Here,  $\text{IRS}_M$  is the market IRS quote,  $L(T_{m-1}, T_m; \tau)$  is the Libor rate with tenor  $\tau$  for a period of  $(T_{m-1}, T_m)$ . Since we already have the term structure of  $D(0, \cdot)$ , it is straightforward to extract the set of  $E^{T_m^c} [L(T_{m-1}, T_m; \tau)]$  for each  $T_m$  from Eq. (3.3).

### 3.3 Tenor swap

A tenor swap is a floating-vs-floating swap where the parties exchange Libors with different tenors with a fixed spread on one side, which we call TS basis spread in this article. Usually, the spread is added on top of the Libor with shorter tenor. For example, in a 3m/6m tenor swap, quarterly payments with 3m Libor plus spread are exchanged by semi-annual payments of 6m Libor flat. The condition that the tenor spread should satisfy is

$$\sum_{n=1}^N \delta_n D(0, T_n) (E^{T_n^c} [L(T_{n-1}, T_n; \tau_S)] + \text{TS}_N) = \sum_{m=1}^M \delta_m D(0, T_m) E^{T_m^c} [L(T_{m-1}, T_m; \tau_L)] , \quad (3.4)$$

where  $T_N = T_M$ , "m" and "n" distinguish the difference of payment frequency.  $\text{TS}_N$  denotes the market quote of the basis spread for the  $T_0$ -start  $T_N$ -maturing tenor swap. The spread is added on the Libor with the shorter tenor  $\tau_S$  in exchange for the Libor with longer tenor  $\tau_L$ . From the above relation, one can extract the forward Libor with different tenors.

Here, we have explained using slightly simplified terms of contract. In the actual market, it is more common that the coupons of the Leg with the shorter tenor are compounded by Libor flat and being paid with the same frequency of the other Leg. However, the size of correction from the above simplification can be shown negligibly small.

### 3.4 Calibration Example

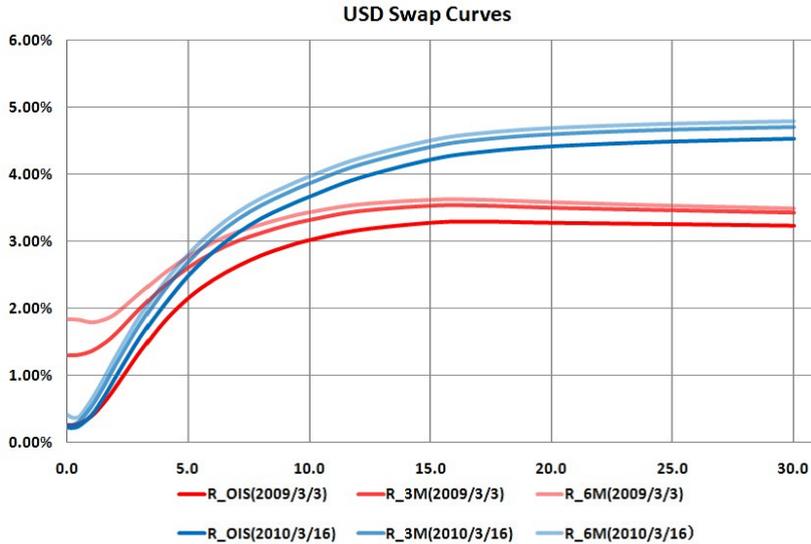


Figure 1: USD zero rate curves of Fed-Fund rate, 3m and 6m Libors.

In Fig. 1, we have given examples of calibrated yield curves for USD market on 2009/3/3 and 2010/3/16, where  $R_{\text{OIS}}$ ,  $R_{3m}$  and  $R_{6m}$  denote the zero rates for OIS (Fed-Fund rate), 3m and 6m forward Libor, respectively.  $R_{\text{OIS}}(\cdot)$  is defined as  $R_{\text{OIS}}(T) =$

$-\ln(D(0, T))/T$ . For the forward Libor, the zero-rate curve  $R_\tau(\cdot)$  is determined recursively through the relation

$$E_m^{T_m^c}[L(T_{m-1}, T_m; \tau)] = \frac{1}{\delta_m} \left( \frac{e^{-R_\tau(T_{m-1})T_{m-1}}}{e^{-R_\tau(T_m)T_m}} - 1 \right). \quad (3.5)$$

In the actual calculation of  $D(0, \cdot)$ , we have used the Fed-Fund vs 3m-Libor basis swap, where the two parties exchange 3m Libor and the compounded Fed-Fund rate with spread, which seems more liquid and a larger number of quotes available than the usual OIS. In Fig. 2, one can see the historical behavior of the spread between 1yr IRS and OIS for USD, JPY and EUR, where the underlying floating rates of IRS are 3m-Libor for USD and EUR and 6m-Libor for JPY.

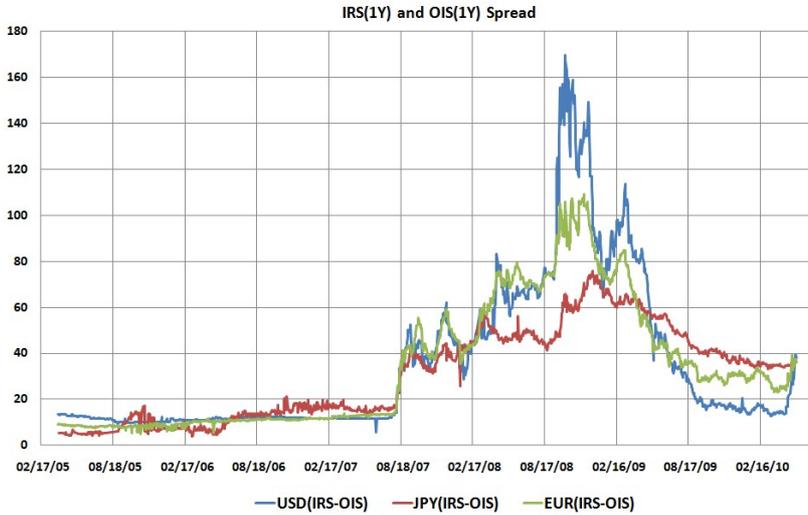


Figure 2: Difference between 1yr IRS and OIS. Underlying floating rates are 3m-Libor for USD and EUR, and 6m-Libor for JPY.

*Remarks:* In the above calculations, we have assumed that the conditions given in the previous section are satisfied, and also that all the instruments are collateralized by the cash of domestic currency which is the same as the payment currency. Cautious readers may worry about the possibility that the market quotes contain significant contributions from market participants who use a foreign currency as collateral. However, the induced changes in IRS/TS quotes are very small and impossible to distinguish from the bid/offer spreads in normal circumstances, because the correction appears both in the fixed and floating legs which keeps the market quotes almost unchanged <sup>4</sup>. However, as we will see in the later sections, the present values of off-the-market swaps will be significantly affected when the collateral currency is different.

<sup>4</sup>As for cross currency swaps, the change can be a few basis point, and hence comparable to the market bid/offer spreads.

## 4 Calibration to Cross Currency Swaps

After completing the calibration to the single currency swaps, we should have obtained the terms structures of  $D^{(i)}(0, T)$  and  $E^{\mathcal{T}^c(i)} [L^{(i)}(T_{m-1}, T_m; \tau)]$  for each currency and tenor. The remaining freedom of the model is the term structure of  $y^{(i,j)}(\cdot)$  for each relevant currency pair. As we will see, this is the most important ingredient to determine the value of CCS.

### 4.1 Mark-to-Market Cross Currency Swap

In this section, we will discuss how to make the term structure consistent with CCS (cross currency swap) markets [1, 3]. The current market is dominated by USD crosses where 3m USD Libor flat is exchanged with 3m Libor of a different currency with additional basis spread. There are two types of CCS, one is CNCCS (Constant Notional CCS), and the other is MtMCCS (Mark-to-Market CCS). In a CNCCS, the notionals of both legs are fixed at the inception of the trade and kept constant until its maturity. On the other hand, in a MtMCCS, the notional of USD leg is reset at the start of every calculation period of the Libor while the notional of the other leg is kept constant throughout the contract period. Although the required calculation becomes a bit more complicated, we will use MtMCCS for calibration due to its better liquidity<sup>5</sup>.

First, let us define  $y^{(i,j)}(t, s)$ , the forward rate of  $y^{(i,j)}(s)$  at time  $t$  as

$$e^{-\int_t^T y^{(j,i)}(t,s)ds} = E_t^{Q_j} \left[ e^{-\int_t^T y^{(j,i)}(s)ds} \right]. \quad (4.1)$$

We consider a MtMCCS of  $(i, j)$  currency pair, where the leg of currency  $i$  (intended to be USD) needs notional refreshments. We assume that the collateral is posted in currency  $i$ , which seems common in the market. The value of  $j$ -leg of a  $T_0$ -start  $T_N$ -maturing MtMCCS is calculated as

$$\begin{aligned} PV_j &= -D^{(j)}(0, T_0)E^{\mathcal{T}^c(j)} \left[ e^{-\int_0^{T_0} y^{(j,i)}(s)ds} \right] + D^{(j)}(0, T_N)E^{\mathcal{T}^c(j)} \left[ e^{-\int_0^{T_N} y^{(j,i)}(s)ds} \right] \\ &+ \sum_{n=1}^N \delta_n^{(j)} D^{(j)}(0, T_n)E^{\mathcal{T}^c(j)} \left[ e^{-\int_0^{T_n} y^{(j,i)}(s)ds} \left( L^{(j)}(T_{n-1}, T_n; \tau) + B_N \right) \right], \end{aligned} \quad (4.2)$$

where the basis spread  $B_N$  is available as a market quote. In [2], we have assumed that all of the  $\{y^{(k)}(\cdot)\}$  and hence  $\{y^{(i,j)}(\cdot)\}$  are deterministic functions of time to make the curve construction more tractable. Here, we slightly relax the assumption allowing randomness of  $\{y^{(i,j)}(\cdot)\}$ . As long as we assume that  $\{y^{(i,j)}(\cdot)\}$  is independent from the dynamics of Libors and collateral rates, the procedures of bootstrapping given in [2] can be applied in the same way<sup>6</sup>. Under the assumption of independence, we obtain

$$\begin{aligned} PV_j &= -D^{(j)}(0, T_0)e^{-\int_0^{T_0} y^{(j,i)}(0,s)ds} + D^{(j)}(0, T_N)e^{-\int_0^{T_N} y^{(j,i)}(0,s)ds} \\ &+ \sum_{n=1}^N \delta_n^{(j)} D^{(j)}(0, T_n)e^{-\int_0^{T_n} y^{(j,i)}(0,s)ds} \left( E^{\mathcal{T}^c(j)} [L^{(j)}(T_{n-1}, T_n; \tau)] + B_N \right). \end{aligned} \quad (4.3)$$

<sup>5</sup>As for the details of MtMCCS and CNCCS, see [1, 3].

<sup>6</sup>In practice, it would not be a problem even if there is a non-zero correlation as long as it does not meaningfully change the model implied quotes compared to the market bid/offer spreads.

On the other hand, the present value of  $i$ -leg in terms of currency  $j$  is given by

$$\begin{aligned}
PV_i &= - \sum_{n=1}^N E^{Q_i} \left[ e^{-\int_0^{T_{n-1}} c^{(i)}(s) ds} f_x^{(i,j)}(T_{n-1}) \right] / f_x^{(i,j)}(0) \\
&\quad + \sum_{n=1}^N E^{Q_i} \left[ e^{-\int_0^{T_n} c^{(i)}(s) ds} f_x^{(i,j)}(T_{n-1}) \left( 1 + \delta_n^{(i)} L^{(i)}(T_{n-1}, T_n; \tau) \right) \right] / f_x^{(i,j)}(0) \\
&= \sum_{n=1}^N \delta_n^{(i)} D^{(i)}(0, T_n) E^{\mathcal{T}_{n,(i)}^c} \left[ \frac{f_x^{(i,j)}(T_{n-1})}{f_x^{(i,j)}(0)} B^{(i)}(T_{n-1}, T_n; \tau) \right], \tag{4.4}
\end{aligned}$$

where

$$B^{(i)}(t, T_k; \tau) = E_t^{\mathcal{T}_{k,(i)}^c} \left[ L^{(i)}(T_{k-1}, T_k; \tau) \right] - \frac{1}{\delta_k^{(i)}} \left( \frac{D^{(i)}(t, T_{k-1})}{D^{(i)}(t, T_k)} - 1 \right), \tag{4.5}$$

which represents a Libor-OIS spread. Since we found no persistent correlation between FX and Libor-OIS spread in historical data, we have treated them as independent variables. Even if a non-zero correlation exists in a certain period, the expected correction seems not numerically important due to the typical size of bid/offer spreads for MtMCCS (about a few bps at the time of writing). Since 3-month timing adjustment of FX is safely negligible, an approximate value of  $i$ -leg is obtained as

$$PV_i \simeq \sum_{n=1}^N \delta_n^{(i)} D^{(i)}(0, T_n) \frac{D^{(j)}(0, T_{n-1})}{D^{(i)}(0, T_{n-1})} e^{-\int_0^{T_{n-1}} y^{(j,i)}(0,s) ds} B^{(i)}(0, T_n; \tau). \tag{4.6}$$

Here, we have used the following expression of the forward FX collateralized with currency  $i$ :

$$f_x^{(i,j)}(t, T) = f_x^{(i,j)}(t) \frac{D^{(j)}(t, T)}{D^{(i)}(t, T)} e^{-\int_t^T y^{(j,i)}(t,s) ds}. \tag{4.7}$$

Notice that, after calibrating to the single currency swaps for each currency, the only unknown in Eqs. (4.3) and (4.6) is  $y^{(j,i)}(0, \cdot)$ . Therefore, one can easily see that the consistency condition  $PV_i = PV_j$  with given market spread  $B_N$  for each maturity allows us to bootstrap the term structure of  $\{y^{(i,j)}(0, \cdot)\}$ . Finally, let us mention the fact that the  $(i, j)$ -MtMCCS par spread is expressed as

$$\begin{aligned}
B_N &= \left\{ \left[ \sum_{n=1}^N \delta_n^{(i)} D_{T_n}^{(i)} \left( \frac{D_{T_{n-1}}^{(j)}}{D_{T_{n-1}}^{(i)}} \right) e^{-\int_0^{T_{n-1}} y^{(j,i)}(0,s) ds} B_{T_n}^{(i)} - \sum_{n=1}^N \delta_n^{(j)} D_{T_n}^{(j)} e^{-\int_0^{T_n} y^{(j,i)}(0,s) ds} B_{T_n}^{(j)} \right] \right. \\
&\quad \left. - \sum_{n=1}^N D_{T_{n-1}}^{(j)} e^{-\int_0^{T_{n-1}} y^{(j,i)}(0,s) ds} \left( e^{-\int_{T_{n-1}}^{T_n} y^{(j,i)}(0,s) ds} - 1 \right) \right] / \sum_{n=1}^N \delta_n^{(j)} D_{T_n}^{(j)} e^{-\int_0^{T_n} y^{(j,i)}(0,s) ds}, \tag{4.8}
\end{aligned}$$

in the above mentioned approximation, where we have shortened the notations as  $D^{(k)}(0, T) = D_T^{(k)}$  and  $B^{(k)}(0, T; \tau) = B_T^{(k)}$ .

## 4.2 Calibration Example and Historical Behavior

In Fig. 3, we have given examples of calibration for EUR/USD and USD/JPY MtMCCS as of 2010/3/16. We have plotted the zero rates of  $y^{(j,i)}$  defined as

$$R_{y^{(j,i)}}(T) = -\frac{\ln\left(E^{Q_j}\left[e^{-\int_0^T y^{(j,i)}(s)ds}\right]\right)}{T} = \frac{1}{T}\int_0^T y^{(j,i)}(0,s)ds \quad (4.9)$$

together with the term structure of MtMCCS basis spreads. It is easy to expect that there are significant contributions from the second line of Eq. (4.8) to the CCS basis spreads from the similarities between  $R_{y^{(X,USD)}}$  and CCS quotes. The implied forward FXs derived from Eq. (4.7) were well within the bid/offer spreads <sup>7</sup>.

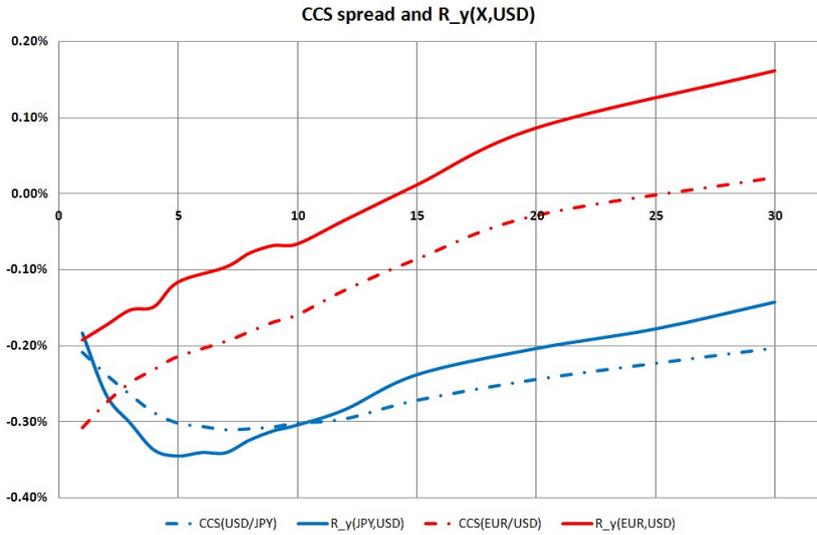


Figure 3: MtMCCS par spreads,  $R_{y^{(JPY,USD)}}$  and  $R_{y^{(EUR,USD)}}$  as of 2010/3/16.

Let us also check the historical behavior of  $R_{y^{(EUR,USD)}}$  and  $R_{y^{(JPY,USD)}}$  given in Fig. 4 to 8 <sup>8</sup>. For both cases, the term structures of  $R_y$  have quite similar shapes and levels to those of the corresponding CCS basis spreads. In Fig. 4, historical behavior of  $R_{y^{(X,USD)}}(T = 5y)$  ( $X = \text{EUR}, \text{JPY}$ ) and corresponding 5y-MtMCCS spreads are given. One can see that a significant portion of CCS spreads movement stems from the change of  $y^{(i,j)}$ , rather than the difference of Libor-OIS spread between two currencies. The level (difference)-correlation between  $R_y$  and CCS spread is quite high, which is about 93% (69%) for EUR or about 70% (48%) for JPY for the historical series used in the figure, for example.

The 3m-roll historical volatilities of  $y^{(EUR,USD)}$  instantaneous forwards, which are annualized in absolute terms, are given in Fig. 9. In a calm market, they tend to be 50 bps or so, but they were more than a percentage point just after the market crisis,

<sup>7</sup>In any case, it is quite wide for long maturities.

<sup>8</sup>Due to the lack of OIS data for JPY market, we have only a limited data for (JPY,USD) pair. We have used Cubic Monotone Spline for calibration although the figures are given in linear plots for ease. For spline technique, see [5], for example.

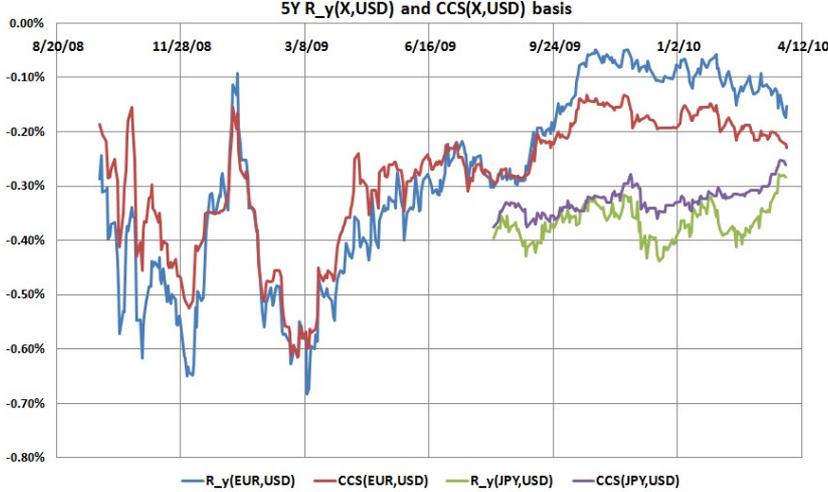


Figure 4:  $R_y(EUR,USD)(5y)$ ,  $R_y(JPY,USD)(5y)$  and corresponding quotes of 5y-MtMCCS.

which is reflecting a significant widening of the CCS basis spread to seek USD cash in the low liquidity market. Except the CCS basis spread,  $y$  does not seem to have persistent correlations with other variables such as OIS, IRS and FX forwards. At least, within our limited data, the 3m-roll historical correlations with these variables fluctuate mainly around  $\pm 20\%$  or so.

## 5 No-arbitrage dynamics in Heath-Jarrow-Morton Framework

In this section, we give the set of SDEs for the whole system <sup>9</sup>. From the previous discussion, we have seen that the relevant building blocks of term structures are given by

$$\{D^{(i)}(t, T)\}, \quad \{E_t^{T_n, (i)} [L^{(i)}(T_{n-1}, T_n; \tau)]\}, \quad \text{and} \quad \{y^{(i,j)}(t, T)\}, \quad (5.1)$$

or equivalently

$$\{c^{(i)}(t, T)\}, \quad \{B^{(i)}(t, T; \tau)\}, \quad \text{and} \quad \{y^{(i,j)}(t, T)\}, \quad (5.2)$$

for each maturity  $T$ , tenor  $\tau$ , currency  $i$ , and currency pair  $(i, j)$ . As seen in Sec.2, the collateral rate plays a critical role as the effective discounting rate. Thus, let us fix the base currency  $i$ , and consider the dynamics of  $c^{(i)}(t, s)$ . Suppose that the dynamics of the forward collateral rate under the  $Q_i$  is given by

$$dc^{(i)}(t, s) = \alpha^{(i)}(t, s)dt + \sigma_c^{(i)}(t, s) \cdot dW^{Q_i}(t), \quad (5.3)$$

where  $\alpha^{(i)}(t, s)$  is a scalar function for its drift, and  $W^{Q_i}(t)$  is a d-dimensional Brownian motion under the  $Q_i$ -measure.  $\sigma_c^{(i)}(t, s)$  is a d-dimensional vector and the following

<sup>9</sup>See [2, 3, 4] for details.

abbreviation have been used:

$$\sigma_c^{(i)}(t, s) \cdot dW^{Q_i}(t) = \sum_{j=1}^d \left[ \sigma_c^{(i)}(t, s) \right]_j dW_j^{Q_i}(t). \quad (5.4)$$

Here, we will not specify the details of the volatility process : It can depend on the collateral rate itself, or any other state variables.

Applying Itô's formula to Eq.(2.12), we have

$$\frac{dD^{(i)}(t, T)}{D^{(i)}(t, T)} = \left\{ c^{(i)}(t) - \int_t^T \alpha^{(i)}(t, s) ds + \frac{1}{2} \left\| \int_t^T \sigma_c^{(i)}(t, s) ds \right\|^2 \right\} dt - \left( \int_t^T \sigma_c^{(i)}(t, s) ds \right) \cdot dW_t^{Q_i}. \quad (5.5)$$

On the other hand, by definition, the drift rate of  $D^{(i)}(t, T)$  should be  $c^{(i)}(t) = c^{(i)}(t, t)$ . Therefore, it is necessary that

$$\alpha^{(i)}(t, s) = \sum_{j=1}^d [\sigma_c^{(i)}(t, s)]_j \left( \int_t^s \sigma_c^{(i)}(t, u) du \right)_j \quad (5.6)$$

$$= \sigma_c^{(i)}(t, s) \cdot \left( \int_t^s \sigma_c^{(i)}(t, u) du \right), \quad (5.7)$$

and as a result, the process of  $c^{(i)}(t, s)$  under the  $Q_i$ -measure is given by

$$dc^{(i)}(t, s) = \sigma_c^{(i)}(t, s) \cdot \left( \int_t^s \sigma_c^{(i)}(t, u) du \right) dt + \sigma_c^{(i)}(t, s) \cdot dW^{Q_i}(t). \quad (5.8)$$

In exactly the same way, we obtain

$$dy^{(i,j)}(t, s) = \sigma_y^{(i,j)}(t, s) \cdot \left( \int_t^s \sigma_y^{(i,j)}(t, u) du \right) dt + \sigma_y^{(i,k)}(t, s) \cdot dW^{Q_i}(t). \quad (5.9)$$

Next, let us consider the dynamics of Libor-OIS spread,  $B^{(i)}(t, T; \tau)$ . From the definition in Eq. (4.5), it is clear that  $B^{(i)}(\cdot, T; \tau)$  is a martingale under the collateralized forward measure  $\mathcal{T}_{(i)}^c$ , where the numeraire is given by  $D^{(i)}(\cdot, T)$ . Using the Maruyama-Girsanov theorem, one can see that Brownian motion under the forward measure  $\mathcal{T}_{(i)}^c$ , or  $W^{\mathcal{T}_{(i)}^c}$ , is related to the  $W^{Q_i}$  as

$$dW^{\mathcal{T}_{(i)}^c}(t) = \left( \int_t^T \sigma_c^{(i)}(t, s) ds \right) dt + dW^{Q_i}(t), \quad (5.10)$$

and hence, one easily obtains

$$\frac{dB^{(i)}(t, T; \tau)}{B^{(i)}(t, T; \tau)} = \sigma_B^{(i)}(t, T; \tau) \cdot \left( \int_t^T \sigma_c^{(i)}(t, s) ds \right) dt + \sigma_B^{(i)}(t, T; \tau) \cdot dW^{Q_i}(t). \quad (5.11)$$

Since we have the relation

$$r^{(i)}(t) - r^{(j)}(t) = c^{(i)}(t) - c^{(j)}(t) + y^{(i,j)}(t) \quad (5.12)$$

the SDE for the spot FX process is given by

$$df_x^{(i,j)}(t)/f_x^{(i,j)}(t) = \left( c^{(i)}(t) - c^{(j)}(t) + y^{(i,j)}(t) \right) dt + \sigma_X^{(i,j)}(t) \cdot dW^{Q_i}(t) . \quad (5.13)$$

Maruyama-Girsanov theorem tells us that Brownian motions in two different currencies are related by the formula

$$dW^{Q_j}(t) = -\sigma_X^{(i,j)}(t)dt + dW^{Q_i}(t) , \quad (5.14)$$

which allows us to derive the SDEs of these building blocks under a different base currency, too. For example, the SDE of collateral rate of the foreign currency  $j$  is given by

$$dc^{(j)}(t, s) = \sigma_c^{(j)}(t, s) \cdot \left[ \left( \int_t^s \sigma_c^{(j)}(t, u) du \right) - \sigma_X^{(i,j)}(t) \right] dt + \sigma_c^{(j)}(t, s) \cdot dW^{Q_i}(t) . \quad (5.15)$$

Pricing formulas for some of the vanilla options are available in [2].

## 6 Implications for Derivative Pricing

Although it is worth exploring various implications of collateralization by using the dynamics given in the previous section, the leading order effects are expected to arise from the change of the effective discounting rate. In this section, we discuss some of these important implications by using the calibrated yield curves.

### 6.1 Choice of Collateral Currency

When the payment and collateral currencies are the same, the discounting factor is given by the collateral rate which is under control of the relevant central bank as indicated in Eq. (2.10). Traditionally, among financial firms, the Libor curve has been widely used to discount the future cash flows. However, this method would easily underestimate their values by several percentage points for long maturities, even with the current level of Libor-OIS spread, or 10 ~ 20 bps. Considering the mechanism of collateralization, financial firms need to hedge the change of OIS in addition to the standard hedge against the movement of Libors. Especially, the risk of floating-rate payments needs to be checked carefully, since the overnight rate can move in the opposite direction to the Libor as was observed in this financial crisis. In Fig. 10, the present values of Libor floating legs with final principal (= 1) payment

$$PV = \sum_{n=1}^N \delta_n D(0, T_n) E^{\mathcal{T}_n^c} [L(T_{n-1}, T_n; \tau)] + D(0, T_N) \quad (6.1)$$

are given for various maturities. If traditional Libor discounting is being used, the stream of Libor payments has the constant present value "1", which is obviously wrong from our results. This point is very important in risk-management, since financial firms may overlook the quite significant interest-rate risk exposure when they adopt the traditional interest rate model in their system.

If a trade with payment currency  $j$  is collateralized by foreign currency  $i$ , an additional modification to the discounting factor appears ( See, Eq. (4.1).) <sup>10</sup>:

$$e^{-\int_t^T y^{(j,i)}(t,s)ds} = E_t^{Q_j} \left[ e^{-\int_t^T y^{(j,i)}(s)ds} \right] . \quad (6.2)$$

From Figs. 6 and 8, one can see that posting USD as collateral tends to be expensive from the view point of collateral payers, which is particularly the case when the market is illiquid. For example, from Fig. 8, one can see that the value of JPY payment in 10 years time is more expensive by around 3% when it is collateralized by USD instead of JPY. The effects should be more profound for emerging currencies where the implied CCS basis spread can easily be 100 bps or more.

## 6.2 Embedded Cheapest-to-Deliver Option

We now discuss the embedded CTD option in a collateral agreement. In some cases, financial firms make contracts with CSA allowing several currencies as eligible collateral. Suppose that the payer of collateral has a right to replace a collateral currency whenever he wants. If this is the case, the collateral payer should choose the cheapest collateral currency to post, which leads to the modification of the discounting factor as

$$E_t^{Q_j} \left[ e^{-\int_t^T \max_{i \in C} \{y^{(j,i)}(s)\} ds} \right] , \quad (6.3)$$

where  $C$  is the set of eligible currencies. Note that, by the definition of collateral payers, they want to make  $(-PV)$  ( $> 0$ ) as small as possible. Although there is a tendency toward a CSA allowing only one collateral currency to reduce the operational burden, it does not seem uncommon to accept the domestic currency and USD as eligible collateral, for example. In this case, the above formula turns out to be

$$E_t^{Q_j} \left[ e^{-\int_t^T \max\{y^{(j,USD)}(s), 0\} ds} \right] . \quad (6.4)$$

In Figs. 11 and 12, we have plotted the modifications of discounting factors given in Eq. (6.4), for  $j = \text{EUR}$  and  $\text{JPY}$  as of 2010/3/16. We have used the Hull-White model for the dynamics of  $y^{(\text{EUR},\text{USD})}(\cdot)$  and  $y^{(\text{JPY},\text{USD})}(\cdot)$ , with a mean reversion parameter 1.5% per annum and the set of volatilities,  $\sigma = 0, 25, 50$  and  $75$  bps <sup>11</sup>, respectively. As can be seen from the historical volatilities given in Fig. 9,  $\sigma$  can be much higher under volatile environment. The curve labeled by USD (EUR, JPY) denotes the modification of the discount factor when only USD (EUR, JPY) is eligible collateral for the ease of comparison. One can easily see that there is significant impact when the collateral currency chosen optimally. For example, from Fig. 12, one can see if the parties choose the collateral currency from JPY and USD optimally, it roughly increases the effective discounting rate by around 50 bps annually even when the annualized volatility of spread  $y^{(\text{JPY},\text{USD})}$  is 50 bps.

In the calculation, we have used daily-step Monte Carlo simulation. Although we can expect that there are various obstacles to implement the optimal strategy in practice, the development of common infrastructure for collateral management, such as the electronic automation of the margin call and collateral delivery, will make the optimal choice of collateral currency be an important issue in coming years.

<sup>10</sup>Here, we are assuming independence of  $y$  from reference assets.

<sup>11</sup>These are annualized volatilities in absolute terms.

## 7 Conclusions

In this article, which integrates the series of our recent works [1, 2, 4], we explain the consistent construction of multiple swap curves in the presence of collateralization and cross currency basis spreads, their no-arbitrage dynamics, and implications for derivative pricing. Especially, we have shown the importance of the choice of collateral currency and embedded "cheapest-to-deliver" (CTD) option in collateral agreements. We have also emphasized dangers to use the standard LMM in actual financial business since it allows the financial firms to overlook potentially critical risk exposures.

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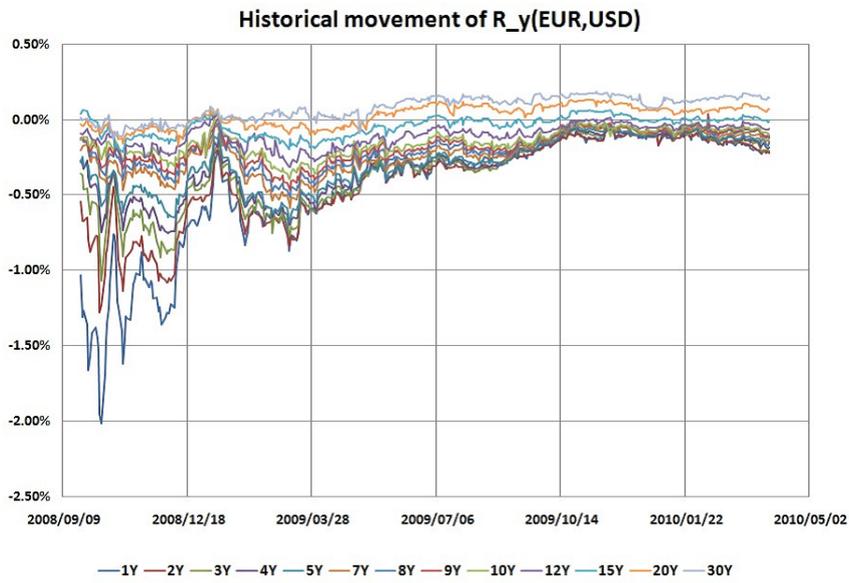


Figure 5: Historical movement of calibrated  $R_y(EUR,USD)$ .

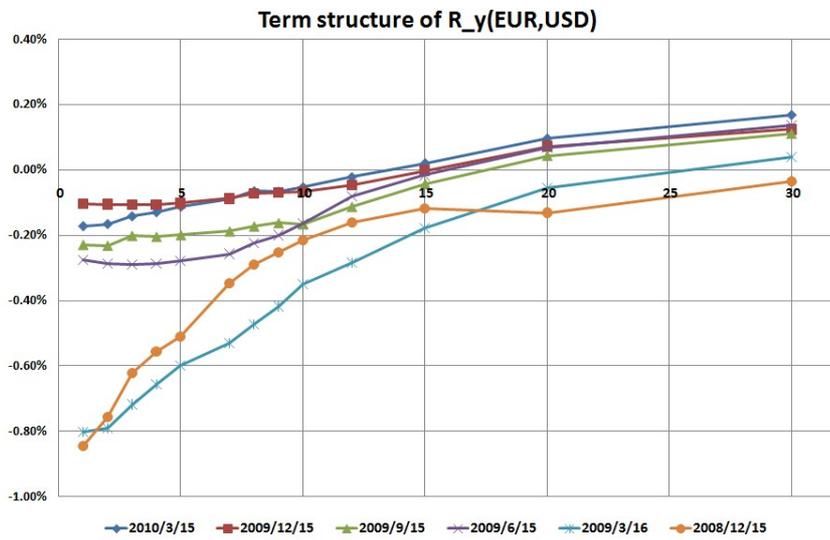


Figure 6: Examples of  $R_y(EUR,USD)$  term structure.

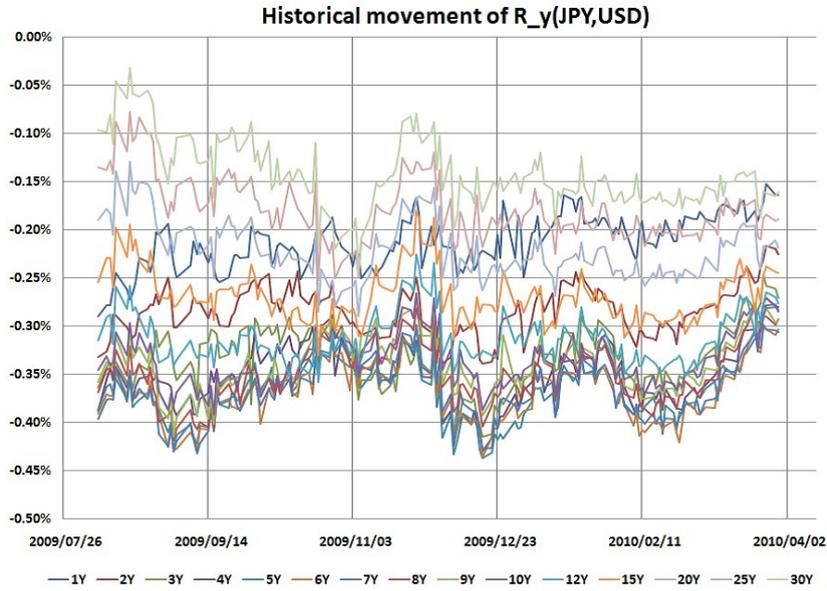


Figure 7: Historical movement of calibrated  $R_y(JPY,USD)$ .

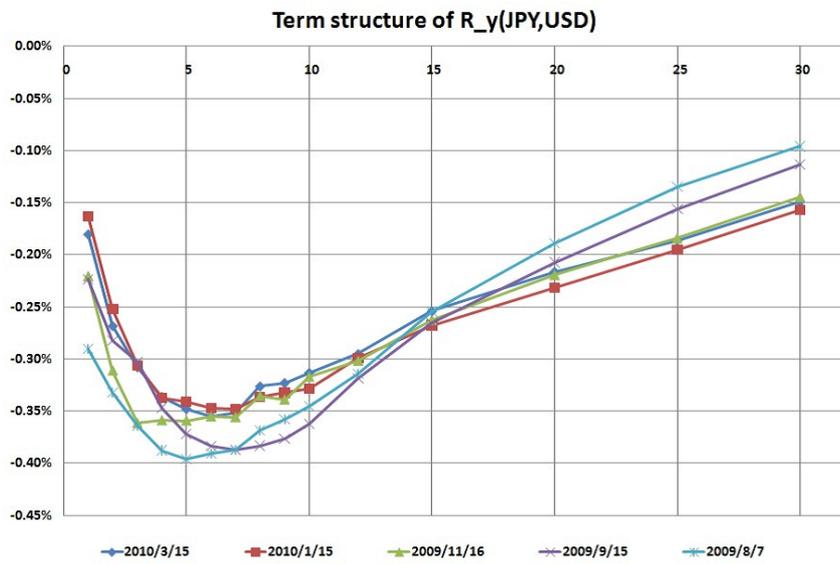


Figure 8: Examples of  $R_y(JPY,USD)$  term structure.

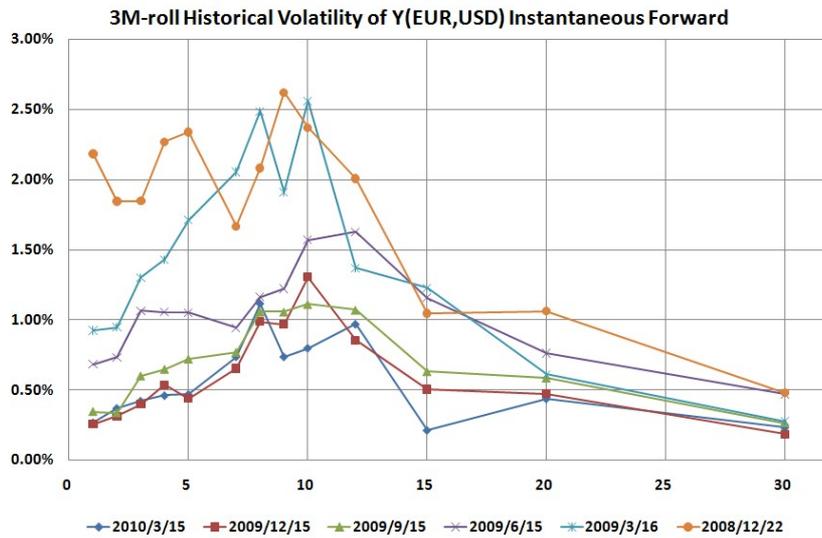


Figure 9: 3M-Roll historical volatility of  $y^{(EUR,USD)}$  instantaneous forward. Annualized in absolute terms.

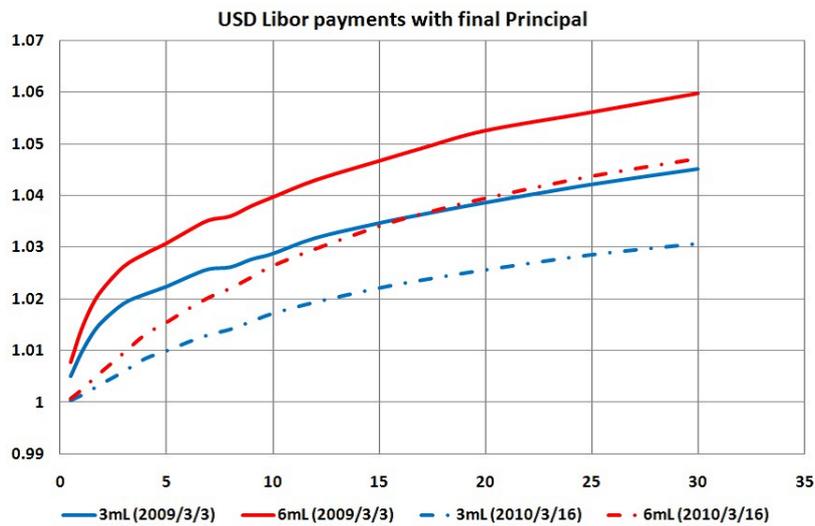


Figure 10: Present value of USD Libor stream with final principal (= 1) payment.

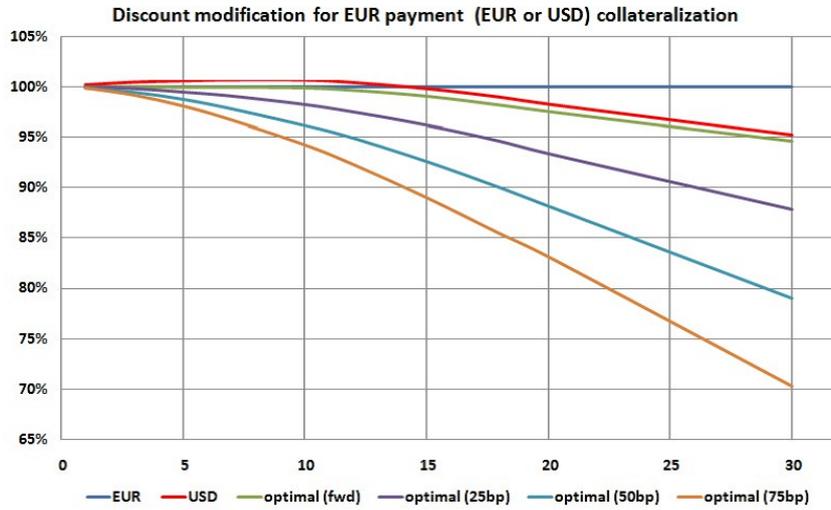


Figure 11: Modification of EUR discounting factors based on HW model for  $y^{(EUR,USD)}$  as of 2010/3/16. The mean-reversion parameter is 1.5%, and the volatility is given at each label.

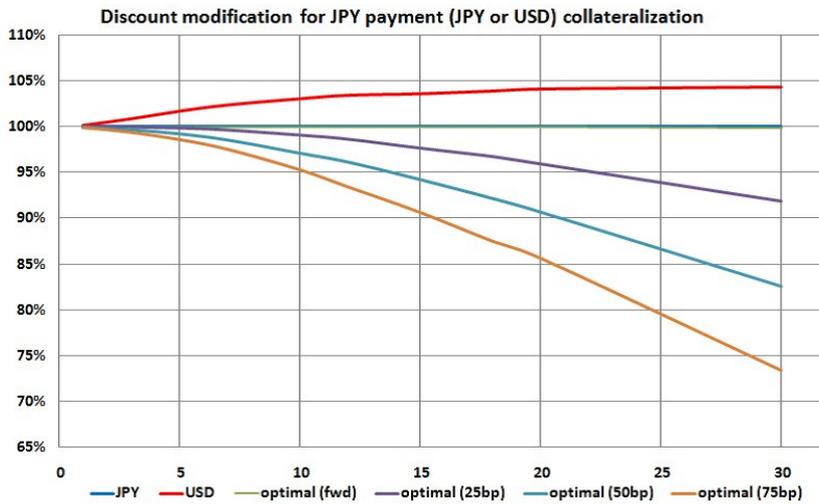


Figure 12: Modification of JPY discounting factors based on HW model for  $y^{(JPY,USD)}$  as of 2010/3/16. The mean-reversion parameter is 1.5%, and the volatility is given at each label.