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Abstract

We investigate the adverse selection problem where a principal delegates multiple tasks to an agent. We characterize the virtually implementable social choice functions by using the linking mechanism proposed by Jackson and Sonnenschein (2007) that restricts the message spaces. The principal does not require any incentive wage schemes and can therefore avoid any information rent and welfare loss. We show the resemblance between the functioning of this message space restriction and that of incentive wage schemes. We also extend the results of the single-agent model to the multi-agent model.

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1. Introduction

This paper investigates the adverse selection problem in which a principal hires a single agent and delegates different tasks to him; these tasks are assumed to be independent of each other and homogeneous. The hired agent observes the private signals relevant to his respective tasks; however, the principal cannot observe these signals. Therefore, the principal will attempt to incentivize the agent to announce his true private signals by designing a well-behaved mechanism or a contract.

The standard approach in the informational economics literature is that the principal is sufficient to design an *individual* wage scheme for each task since each task is independent and identical. The scheme bases a wage payment on the agent's announcement. However, this approach has the following drawback. If the lower bound of the wage payment such as non-negativity exists, each agent can earn a positive information rent and the principal fails to extract the full surplus in a non-negligible manner.

This paper presents an alternative approach to solve the adverse selection problem and suggests a means to overcome the above mentioned drawback. By using a *whole* incentive scheme, which depends on *all* independent tasks, the principal is not required to design an inconstant wage scheme and succeeds in extracting the full surplus without suffering any non-negligible welfare distortion. More precisely, this paper will show that when the number of tasks is sufficiently large, a social choice function is virtually implementable by the *whole* incentive scheme *without* side payments if and only if such a function is exactly implementable by the individual wage scheme *with* unbounded side payment. Thus, the class of implementable social choice functions is almost the *same* in both cases.

In order to prove this, we apply the concept of a *linking mechanism* as a whole incentive scheme, which was proposed by Jackson and Sonnenschein (2007). As in the case of the standard direct mechanism, the principal requires the agent to make an announcement for each task about the observed private signal. The main difference between the linking mechanism and the direct mechanism is that the principal restricts the message space in advance by directing the agent to ensure that the proportion of the tasks for which the agent announces a private signal is approximately equal to the probability of this signal being observed for a single task. Since the total number of the tasks is sufficiently large, it is almost certain, based on the law of large numbers, that the realized proportion of the tasks for which each private signal is observed is almost the same as the probability of this signal being observed for a single task. Therefore, truth-telling, which induces the value of the social choice function for all tasks, is almost compatible with this message space restriction.

The essential finding of this paper is the clear resemblance between the functioning of this message space restriction and that of the incentive wage schemes in the standard approach. This resemblance can be elucidated using the following case in which the principal designs the wage schemes for the agent in order to incentivize him to tell the truth. Let us suppose that the agent adopts a dishonest strategy that causes the frequency of announcing each signal to be different from the probability of this signal being observed. In such a case, a well-designed wage scheme can detect this dishonesty and the agent will be fined a large expected amount. In this sense, the functioning of the incentive wage scheme parallels that of message space restriction. On the other hand, if the agent adopts a dishonest strategy that causes the frequency of announcing each signal to be equal to the probability of this signal being observed, no wage scheme will detect this dishonesty. Therefore, we merely need to examine whether, in the absence of an incentive device, the agent has an incentive to adopt a dishonest strategy that causes the frequency of announcing each signal to be equal to the probability of this signal being observed. This implies that the necessary and sufficient condition for implementability is generally the same for both the individual wage scheme case with wage payment devices and the whole incentive case with no such devices. Therefore, we can conclude that applying a linking mechanism is far more advantageous than designing an incentive wage scheme; this is because a linking mechanism enables us to avoid positive information rents and welfare distortions without narrowing the class of implementable social choice functions.

We can extend our arguments to the case of multiple agents who are in conflict with each other. Jackson and Sonnenschein (2007) showed that the linking mechanism functions effectively with private values, and independent signals across the agents if the social choice function satisfies the *ex ante efficiency*. Contrary to Jackson and Sonnenschein (2007), we consider more general environment, where we do not require the private values assumption. In the general environment, we characterize the class of social choice functions that are virtually implemented by the linking mechanisms if we consider *weak* implementation with ε-*Nash equilibrium*.[1] We also present an alternative sufficient condition, i.e., *supermodularity*, and also investigate the correlated signals case.

In the economics theory literature, we find that some papers have presented concepts related to linking mechanisms before the study by Jackson and Sonnenschein (2007). For instance, bundling goods by a monopolist (Armstrong (1999)), storable votes (Casella (2005) and Casella, Gelman, and Palfrey (2006)), and multimarket contact (Bernheim and Whinston (1990) and Matsushima (2001)). For more recent studies, see Eliaz, Ray, and Razin (2007) and Fang and Norman (2003, 2006). It is important to conduct laboratory experiments to show whether the linking mechanism functions effectively and the extent to what it does so. As Fehr and Falk (2002), Fehr and Gächter (2002), and Fehr, Gächter, and Kirschsteiger (1997) have shown through laboratory experiments, the incentive device of monetary rewards and punishments results in a decline in the reciprocal motives of real individuals. We conjecture that the incentive device of a linking mechanism is more compatible with this reciprocal motive than is that of monetary rewards and punishments.[2]

This paper is organized as follows. Section 2 describes the single agent model. Section 3 presents the necessary condition for the virtual implementation of a social choice function. Section 4 introduces the linking mechanism and characterizes the class of social choice functions that it virtually implements. Section 5 characterizes the class of the social choice functions that are exactly implemented by inconstant wage schemes and shows the resemblance between the functioning of incentive wage schemes and that of the linking mechanism. Section 6 extends our results to the case of multiple agents.

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[1] Note that Jackson and Sonnenschein (2007) consider a sufficient condition for *full* implementation with *Bayesian Nash equilibrium* in the sense that *all* exact equilibria secure the efficiency. For characterization of virtual implementation in more general environment, we relax the notion of equilibrium and focus one equilibrium even if there are multiple equilibria.

[2] Engelmann and Grimm (2008) presents experimental research on linking mechanisms. They reported that linking mechanisms function effectively in laboratories. Moreover, the experiments conducted by Casella, Gelman, and Palfrey (2006) on storable votes closely related to linking mechanisms reported that the storable votes performed very well.

2. The Model

We investigate the following situation in which a principal delegates K distinct tasks to a *single* agent, i.e., the agent is required to choose a profile of K alternatives $(a_1,...,a_K) = (a_k)_{k=1}^K \in \times_{k=1}^K A_k$, where for each $k \in \{1,...,K\}$, A_k denotes the finite set of alternatives for the k-th task, and $a_k \in A_k$. The agent observes a profile of K private signals $(\omega_1,...,\omega_K) = (\omega_k)_{k=1}^K \in \times_{k=1}^K \Omega_k$, where Ω_k denotes the finite set of private signals for the k-th task, and $\omega_k \in \Omega_k$. This paper focuses on *symmetric* models in that $A_k = A$ and $\Omega_k = \Omega$ for all $k \in \{1,...,K\}$. Let $\sharp\Omega = I < \infty$. For each $k \in \{1,...,K\}$, the agent observes a private signal $\omega_k \in \Omega$ for the k-th task with positive probability $p(\omega_k) > 0$, where the private signals are *independently* drawn according to the probability function $p : \Omega \to (0,1]$.

The principal is unaware of the profile of private signals and therefore requires the agent to announce a message on the basis of the mechanism given by $\Gamma = \Gamma(K) \equiv (M,(g,t))$. Here, M is the finite set of messages for the agent, $g : M \to \Delta(A^K)$,[3] and $t : M \to R$. When the agent announces a message $m \in M$, the principal compels him to choose any profile of alternatives $(a_1,...,a_K) \in A^K$ with probability $g(m)(a_1,...,a_K)$ [4]; the principal himself chooses the side payment $t = t(m) \in R$. When the agent observes ω_k and chooses a_k for each task $k \in \{1,...,K\}$, and the principal chooses t, the agent's payoff is given by $\frac{1}{K}\sum_{k=1}^K u(a_k,\omega_k) + t$, where $u : A \times \Omega \to R$, and additive separability is assumed We also assume expected utility.

A *strategy* for the agent is defined as the function $\sigma : \Omega^K \to M$. We denote the set of strategies by Σ. Let

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[3] For every set Φ, the set of simple lotteries over Φ is denoted by $\Delta(\Phi)$.

[4] In order to focus on the adverse selection problem, we assume that the probabilistic alternative choices are verifiable by the court.

$$v(\sigma,\Gamma) = E[\frac{1}{K}\sum_{k=1}^{K} u(a_k,\omega_k) + t \mid \sigma,\Gamma]$$

denote the expected payoff induced by a strategy $\sigma \in \Sigma$ in Γ.[5] A strategy $\sigma \in \Sigma$ is said to be a *best response* in Γ if

$$v(\sigma,\Gamma) \geq v(\sigma',\Gamma) \text{ for all } \sigma' \in \Sigma .$$

A social choice function is defined as $f:\Omega \to A$. Irrespective of $k \in \{1,...,K\}$, $f(\omega_k) \in A$ is regarded as the desirable alternative choice for the k-th task when the agent observes ω_k.[6] A social choice function f is said to be *exactly implementable with respect to* K if there exist a mechanism $\Gamma(K)$ and a best response $\sigma \in \Sigma$ in $\Gamma(K)$ such that

(1) $$g(\sigma(\omega_1,...,\omega_K))(f(\omega_1),..., f(\omega_K)) = 1 \text{ for all } (\omega_k)_{k=1}^{K} \in \Omega^K .$$

An infinite sequence of mechanisms $(\Gamma(K))_{K=1}^{\infty}$ is said to *virtually implement a social choice function* f if for every $\eta > 0$, there exists $\bar{K}$ such that for every $K \geq \bar{K}$, there is a best response $\sigma \in \Sigma$ in $\Gamma(K)$ that satisfies

(2) $$E\left[\frac{\sharp\{k \in \{1,\cdots,K\} \mid a_k = f(\omega_k)\}}{K} \middle| \sigma,\Gamma(K)\right] \geq 1-\eta .$$

3. Necessary Condition for Virtual Implementation

This section shows that the following condition is necessary for virtual implementation. The condition requires that truth-telling is better than any lying in the sense of permutation.

Condition 1: For every $L \in \{2,\cdots,I\}$ and every $(\omega(1),\cdots,\omega(L)) \in \Omega^L$, if

$$\omega(l) \neq \omega(l') \text{ for all } l \in \{1,\ldots,L\} \text{ and all } l' \in \{1,\ldots,L\}\setminus\{l\},$$

then

[5] Here, $E\left[\cdot \middle| \Psi\right]$ denotes the expectation operator given a condition Ψ.

[6] Generally, a social choice function is defined as $f^*:\Omega^K \to A^K$. In this paper, we restrict symmetric separable social choice functions, as in Jackson and Sonnenschein (2007). This restriction enables us to tract the model conveniently when we increase the number of tasks.

(3) $$\sum_{l=1}^{L} u(f(\omega(l)),\omega(l)) \geq \sum_{l=1}^{L} u(f(\omega(l+1)),\omega(l)),$$

where $\omega(l)\in\Omega$ for all $l\in\{1,\ldots,L\}$, and $L+1=1$.[7] [8]

It might be difficult for the principal to detect any deviant if this deviant keeps the relative frequency of announcing each signal unchanged by lying according to any permutation over the private signals with the same probability across the tasks. Condition 1 implies that if an agent lies in this way, his payoff never improves. That is, for every permutation over the tasks $\pi:\{1,\cdots,K\}\to\{1,\cdots,K\}$, the agent's payoff never improves by announcing $\omega_{\pi(k)}$ instead of ω_k for each task k with the same probability across the tasks, i.e.,

$$\sum_{k=1}^{K} u(f(\omega_k),\omega_k) \geq \sum_{k=1}^{K} u(f(\omega_{\pi(k)}),\omega_{\pi(k)}).$$

The following theorem shows that this condition is necessary for virtual implementation.

Theorem 1: *If there exists an infinite sequence of mechanisms* $(\Gamma(K))_{K=1}^{\infty}$ *that virtually implement* f*, then Condition 1 holds.*

Proof: Suppose that Condition 1 does not hold, i.e., there exists $L\in\{2,\cdots,I\}$ and $(\omega(1),\cdots,\omega(L))$ such that $\omega(l)\neq\omega(l')$ for all $l\in\{1,\ldots,L\}$ and all $l'\in\{1,\ldots,L\}\setminus\{l\}$, and $\sum_{l=1}^{L} u(f(\omega(l)),\omega(l)) < \sum_{l=1}^{L} u(f(\omega(l+1)),\omega(l))$. Further, suppose that $(\Gamma(K))_{K=1}^{\infty}$ virtually implements f. By the revelation principle,[9] we can assume without loss of generality that for every K, $\Gamma(K)=(M,g,t)$ is a *direct* mechanism where $M=\Omega^K$;

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[7] Condition 1 is our original condition. However, it is related with Theorem 1 of Fan(1956). We will discuss the detail about that in Section 5.

[8] For example, if $\Omega=\{H,M,L\}$, the Condition 1 implies that

$$u(f(H),H)+u(f(M),M)+u(f(L),L)\geq u(f(M),H)+u(f(L),M)+u(f(H),L),$$
$$u(f(H),H)+u(f(L),L)\geq u(f(L),H)+u(f(H),L),$$

and so on.

[9] See Myerson (1979) and Fudenberg and Tirole (1993, Chapter 7).

the *truthful* strategy $\hat{\sigma} = \hat{\sigma}(K)$, defined by $\hat{\sigma}(\omega_1,...,\omega_K) = (\omega_1,...,\omega_K)$ for all $(\omega_k)_{k=1}^K \in \Omega^K$, is a best response; and

(4) $$\lim_{K\to\infty} E\left[\frac{\sharp\{k \in \{1,\cdots,K\} \mid a_k = f(\omega_k)\}}{K} \middle| \hat{\sigma}(K), \Gamma(K)\right] = 1.$$

Let $M = \times_{k=1}^K M_k$, $M_k = \Omega$, $m = (m_1,\cdots,m_K) \in M$, and $m_k \in M_k$. We denote $\sigma(\omega_1,...,\omega_K) = (\sigma_k(\omega_1,...,\omega_K))_{k=1}^K$, where $\sigma_k(\omega_1,...,\omega_K) \in \Omega$. According to Appendix A, we can assume without loss of generality that for every K, $\Gamma(K) = (M, g, t)$ is *symmetric* in that for every $m \in M$, every $(a_1,...,a_K) \in A^K$, and every permutation $\pi : \{1,\cdots,K\} \to \{1,\cdots,K\}$,

$$g(m)(a_1,...,a_K) = g(m^\pi)(a_1^\pi,...,a_K^\pi),$$

where $m^\pi = (m_1^\pi,...,m_K^\pi) \in M$, $m_{\pi(k)}^\pi = m_k$, and $a_{\pi(k)}^\pi = a_k$.

For any $\lambda > 0$, let

$$\Omega^{K*}(\lambda) \equiv \left\{(\omega_k)_{k=1}^K \in \Omega^K \mid \left|\frac{\sharp\{k \in \{1,\cdots,K\} \mid \omega_k = \omega\}}{K} - p(\omega)\right| < \lambda \ \textit{for all}\ \omega \in \Omega\right\},$$

which is the set of signal profiles such that the proportion of each signal ω is approximated by the probability $p(\omega)$. The law of large numbers implies that $\lim_{K\to\infty} \sum_{(\omega_k)_{k=1}^K \in \Omega^{K*}(\lambda)} \{\prod_{k=1}^K p(\omega_k)\} = 1$ for all $\lambda > 0$. Therefore, there is an infinite sequence of positive real numbers $(\lambda_K)_{K=1}^\infty$ such that $\lim_{K\to\infty} \lambda_K = 0$ and

(5) $$\lim_{K\to\infty} \sum_{(\omega_k)_{k=1}^K \in \Omega^{K*}(\lambda_K)} \{\prod_{k=1}^K p(\omega_k)\} = 1.$$

Assume a sufficiently large K. From (4) and (5), it follows that there exists $(\tilde{\omega}_k)_{k=1}^K \in \Omega^{K*}(\lambda_K)$ such that

(6) $g^K(\tilde{\omega}_1,...,\tilde{\omega}_K)(f(\tilde{\omega}_1),..., f(\tilde{\omega}_K))$ is close to 1.

We specify a strategy $\sigma' \in \Sigma$ as follows.

(i) For every $l \in \{1,\cdots,L\}$, the number of $k \in \{1,\cdots,K\}$ satisfying that $\tilde{\omega}_k = \omega(l)$ and $\sigma'_k(\tilde{\omega}_1,...,\tilde{\omega}_K) = \omega(l+1)$ is set equal to $\min_{l\in\{1,\cdots,L\}} \sharp\{k \in \{1,\cdots,K\} \mid \tilde{\omega}_k = \omega(l)\}$.

(ii) For every $k \in \{1,\cdots,K\}$, either $\sigma'_k(\tilde{\omega}_1,...,\tilde{\omega}_K) = \tilde{\omega}_k$, or

$$\tilde{\omega}_k = \omega(l) \text{ and } \sigma'_k(\tilde{\omega}_1,...,\tilde{\omega}_K) = \omega(l+1) \text{ for some } l \in \{1,\cdots,L\}.$$

(iii) For every $(\omega_1,...,\omega_K) \neq (\tilde{\omega}_1,...,\tilde{\omega}_K)$,

$$\sigma'(\omega_1,...,\omega_K) = \sigma(\omega_1,...,\omega_K).$$

Note that there exists a permutation π on $\{1,...,K\}$ such that

$$\sigma'(\tilde{\omega}_1,...,\tilde{\omega}_K) = (\tilde{\omega}_1^{\pi},...,\tilde{\omega}_K^{\pi}),$$

(7) $$\sum_{k=1}^{K} u(f(\tilde{\omega}_k),\tilde{\omega}_k) < \sum_{k=1}^{K} u(f(\tilde{\omega}_{\pi(k)}),\tilde{\omega}_k), \text{ and}$$

(8) $$g(\sigma'(\tilde{\omega}_1,....,\tilde{\omega}_K))(f(\tilde{\omega}_1^{\pi},....,\tilde{\omega}_K^{\pi})) \text{ is approximated by 1,}$$

where $\tilde{\omega}_{\pi(k)}^{\pi} = \tilde{\omega}_k$. From (6), (7), and (8), it follows that

$$E\left[\sum_{k=1}^{K} u(a_k,\tilde{\omega}_k) \middle| \hat{\sigma},\Gamma(K),(\tilde{\omega}_k)_{k=1}^{K}\right] < E\left[\sum_{k=1}^{K} u(a_k,\tilde{\omega}_k) \middle| \sigma',\Gamma(K),(\tilde{\omega}_k)_{k=1}^{K}\right].$$

Since $\Gamma(K)$ is symmetric, it follows that

$$E\left[\frac{1}{K}\sum_{k=1}^{K} u(a_k,\tilde{\omega}_k) + t \middle| \hat{\sigma},\Gamma(K),(\tilde{\omega}_k)_{k=1}^{K}\right] < E\left[\frac{1}{K}\sum_{k=1}^{K} u(a_k,\tilde{\omega}_k) + t \middle| \sigma',\Gamma(K),(\tilde{\omega}_k)_{k=1}^{K}\right],$$

which contradicts the fact that $\hat{\sigma}$ is a best response in $\Gamma(K)$. **Q.E.D.**

4. Linking Mechanisms

This section shows that Condition 1 is also sufficient for virtual implementation. The proof of this statement is constructive, which shows not only this sufficiency but also that we do not need any side payment devices. Based on Jackson and Sonnenschein (2007), we define the linking mechanism $\Gamma^* = \Gamma^*(K) = (M,(g,t))$ as follows, which uses only constant side payments. We specify $B = B(\cdot,K) : \Omega \to \{0,...,K\}$ such that

$$\sum_{\omega\in\Omega} B(\omega) = K,$$

and for every $b : \Omega \to \{0,...,K\}$,

(9) $$\sum_{\omega\in\Omega}\left|\frac{B(\omega)}{K} - p(\omega)\right| \leq \sum_{\omega\in\Omega}\left|\frac{b(\omega)}{K} - p(\omega)\right| \text{ whenever } \sum_{\omega\in\Omega} b(\omega) = K.$$

Note that $\frac{B(\omega)}{K}$ is approximated by $p(\omega)$ for a sufficiently large K, i.e.,

(10) $$\lim_{K\to\infty}\frac{B(\omega,K)}{K}=p(\omega) \quad \text{for all} \quad \omega\in\Omega .$$

Let $M_k=\Omega$ for all $k\in\{1,...,K\}$. We specify M as a subset of Ω^K defined by

(11) $$M=\left\{m\in\Omega^K \left| \sharp\{k\in\{1,\cdots,K\} \mid m_k=\omega\right\}=B(\omega) \text{ for all } \omega\in\Omega\right\}.$$

For every $m\in M$, let

$$g(m)(f(m_1),...,f(m_K))=1 \quad \text{and} \quad t(m)=z \quad \text{for some} \quad z\in R .$$

According to the linking mechanism, the agent has to announce each $\omega\in\Omega$ exactly $B(\omega)$ times. This along with (10) implies that for a sufficiently large K, the proportion of the tasks for which the agent announces ω is almost the same as the probability $p(\omega)$ of ω occurring. The payment z is adjusted to satisfy the agent's *ex ante* participation constraint. Since we do not incentivize the agent with side payment, the linking mechanism is free from the failure of the principal to extract the full surplus owing to the agent's positive information rent.[10]

The following theorem shows that Condition 1 is sufficient for the linking mechanisms to virtually implement the social choice function.

Theorem 2: *Under Condition 1,* $(\Gamma^*(K))_{K=1}^{\infty}$ *virtually implements* f .

Proof: Suppose that there exists $\eta>0$ such that for every $\bar{K}$, there exists $K\geq\bar{K}$ that satisfies that for every best response $\sigma\in\Sigma$ in $\Gamma^*(K)$,

(12) $$E\left[\frac{\sharp\{k\in\{1,\cdots,K\} \mid a_k\neq f(\omega_k)\}}{K}\middle|\sigma,\Gamma^*(K)\right]>\eta .$$

As in the proof of Theorem 1, we can choose an infinite sequence of positive real numbers $(\lambda_K)_{K=1}^{\infty}$ satisfying (5) and $\lim_{K\to\infty}\lambda_K=0$. From (5) and (10), it follows that for every sufficiently large K, every $(\omega_k)_{k=1}^{K}\in\Omega^{K*}(\lambda_K)$, and every $\omega\in\Omega$,

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[10] If the agent's reservation utility is 0, then the principal can set $z=-E\left[\frac{1}{K}\sum_{k=1}^{K}u(a_k,\omega_k)\middle|\sigma,\Gamma^*\right]$ to extract full surplus, where σ is a best response in Γ^*. Note that our concept of full surplus extraction is in an *ex ante* sense as in Jackson and Sonnenschein (2007). Crèmer and McLean (1988) consider a stronger concept, *interim* full surplus extraction.

(13) $$\frac{\sharp\{k\in\{1,\cdots,K\}\mid\omega_k=\omega\}}{K} \text{ is approximated by } \frac{B(\omega,K)}{K}.$$

Consider a best response $\sigma\in\Sigma$ in $\Gamma^*(K)$ satisfying that for every best response $\tilde{\sigma}\in\Sigma$ in $\Gamma^*(K)$,

(14) $$E\left[\frac{\sharp\{k\in\{1,\cdots,K\}\mid\sigma_k(\omega_1,...,\omega_K)\neq\omega_k\}}{K}\Big|\sigma,\Gamma^*(K)\right]$$

$$\leq E\left[\frac{\sharp\{k\in\{1,\cdots,K\}\mid\sigma_k(\omega_1,...,\omega_K)\neq\omega_k\}}{K}\Big|\tilde{\sigma},\Gamma^*(K)\right]..$$

The left-hand side of (12) is rewritten as

$$\sum_{(\omega_k)_{k=1}^K\in\Omega^K}\prod_{k=1}^K p(\omega_k)\times\frac{\sharp\{k\in\{1,\cdots,K\}\mid\sigma_k(\omega_1,...,\omega_K)\neq\omega_k\}}{K}$$

$$=\sum_{(\omega_k)_{k=1}^K\in\Omega^{K*}(\lambda_K)}\prod_{k=1}^K p(\omega_k)\times\frac{\sharp\{k\in\{1,\cdots,K\}\mid\sigma_k(\omega_1,...,\omega_K)\neq\omega_k\}}{K}$$

$$+\sum_{(\omega_k)_{k=1}^K\notin\Omega^{K*}(\lambda_K)}\prod_{k=1}^K p(\omega_k)\times\frac{\sharp\{k\in\{1,\cdots,K\}\mid\sigma_k(\omega_1,...,\omega_K)\neq\omega_k\}}{K}.$$

For every sufficiently large K, the last term is close to zero; therefore, the left-hand side of (12) is approximated by

$$\sum_{(\omega_k)_{k=1}^K\in\Omega^{K*}(\lambda_K)}\prod_{k=1}^K p(\omega_k)\times\frac{\sharp\{k\in\{1,\cdots,K\}\mid\sigma_k(\omega_1,...,\omega_K)\neq\omega_k\}}{K},$$

which implies that there exists $(\hat{\omega}_k)_{k=1}^K\in\Omega^{K*}(\lambda_K)$ such that

(15) $$\frac{\sharp\{k\in\{1,\cdots,K\}\mid\sigma_k(\hat{\omega}_1,...,\hat{\omega}_K)\neq\hat{\omega}_k\}}{K}>\eta.$$

A strategy $\sigma\in\Sigma$ is said to be *cyclic* for $(\omega_k)_{k=1}^K\in\Omega^K$ if there exist $S\subseteq\{1,\cdots,K\}$ and a one-to-one function $\tau:\{1,...,\#S\}\to S$ such that $2\leq\sharp S\leq K$,

$\omega_s\neq\omega_{s'}$ for all $s\in S$ and $s'\in S\setminus\{s\}$, and

$\sigma_{\tau(l)}(\omega_1,...,\omega_K)=\omega_{\tau(l+1)}$ for all $l\in\{1,...,\#S\}$, where $\sharp S+1=1$.

If σ is not cyclic for $(\hat{\omega}_k)_{k=1}^K$, then the proportion of the tasks for which the agent announces incorrect private signals, i.e., $\frac{\sharp\{k\in\{1,\cdots,K\}\mid\sigma_k(\hat{\omega}_1,...,\hat{\omega}_K)\neq\hat{\omega}_k\}}{K}$, is less than or equal to

(16) $$\sum_{\omega\in\Omega}\left|\frac{\sharp\{k\in\{1,\cdots,K\}\,|\,\hat{\omega}_k=\omega\}}{K}-\frac{B(\omega)}{K}\right|,$$

which is close to zero because of (13). This contradicts (12). Hence, σ must be cyclic for $(\hat{\omega}_k)_{k=1}^{K}$, i.e., there exist $S\subseteq\{1,\cdots,K\}$ and a one-to-one function $\tau:\{1,...,\#S\}\rightarrow S$ such that $2\leq\sharp S\leq K$,

$$\hat{\omega}_s\neq\hat{\omega}_{s'} \text{ for all } s\in S \text{ and } s'\in S\setminus\{s\}, \text{ and}$$

$$\sigma_{\tau(l)}(\hat{\omega}_1,...,\hat{\omega}_K)=\hat{\omega}_{\tau(l+1)} \text{ for all } l\in\{1,...,\#S\}.$$

We specify a strategy $\tilde{\sigma}\in\Sigma$ by

$$\tilde{\sigma}_s(\hat{\omega}_1,...,\hat{\omega}_K)=\hat{\omega}_s \text{ for all } s\in S,$$

$$\tilde{\sigma}_s(\hat{\omega}_1,...,\hat{\omega}_K)=\sigma_s(\hat{\omega}_1,...,\hat{\omega}_K) \text{ for all } s\notin S, \text{ and}$$

$$\tilde{\sigma}(\omega_1,...,\omega_K)=\sigma(\omega_1,...,\omega_K) \text{ for all } (\omega_k)_{k=1}^{K}\neq(\hat{\omega}_k)_{k=1}^{K}.$$

Note that the expected number of the tasks for which the agent lies according to $\tilde{\sigma}$ is less than that according to σ, i.e.,

$$E\left[\frac{\sharp\{k\in\{1,\cdots,K\}\,|\,\tilde{\sigma}_k(\omega_1,...,\omega_K)\neq\omega_k\}}{K}\middle|\tilde{\sigma},\Gamma^*(K)\right]$$

$$<E\left[\frac{\sharp\{k\in\{1,\cdots,K\}\,|\,\sigma_k(\omega_1,...,\omega_K)\neq\omega_k\}}{K}\middle|\sigma,\Gamma^*(K)\right].$$

From Condition 1 and the fact that σ is a best response in $\Gamma^*(K)$, it follows that $\tilde{\sigma}$ is another best response in $\Gamma^*(K)$. This contradicts (14). **Q.E.D.**

Theorems 1 and 2 imply that Condition 1 is necessary and sufficient for virtual implementation. These theorems also imply that whenever a sufficiently large number of tasks are delegated to the agent, all that is required for virtual implementation is to check whether the linking mechanism functions or not. This implies that side payment devices are irrelevant to virtual implementation.

The following proposition shows that we can replace Condition 1 with a more intuitive condition termed as *supermodularity*.[11] This along with Theorem 2 implies that supermodularity is sufficient for the linking mechanisms to virtually implement the

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[11] See Topkis (1979) and Fudenberg and Tirole (1993, Chapter 12) for supermodularity and its related concepts.

social choice function.

Condition 2 (Supermodularity): Ω is an ordered set with $\geq$, and for every $\omega, \omega', \omega'', \omega''' \in \Omega$,

$$u(f(\omega), \omega') + u(f(\omega''), \omega''') \leq u(f(\omega \vee \omega''), \omega' \vee \omega''') + u(f(\omega \wedge \omega''), \omega' \wedge \omega'''),$$

where $\omega \vee \omega' = \max\{\omega, \omega'\}$ and $\omega \wedge \omega' = \min\{\omega, \omega'\}$.

Proposition 3: *Condition 2 implies Condition 1.*

Proof: Consider any $L \in \{2, \cdots, I\}$ and $(\omega(1), \cdots, \omega(L)) \in \Omega^L$ such that

$$\omega(l) \leq \omega(l+1) \text{ for all } l \in \{1, \cdots, L-1\}.$$

From Condition 2, it follows that the right-hand side of (3) is rewritten as

$$u(f(\omega(2)), \omega(1)) + u(f(\omega(1)), \omega(L)) + \sum_{l=2}^{L-1} u(f(\omega(l+1)), \omega(l))$$

$$\leq u(f(\omega(1)), \omega(1)) + u(f(\omega(2)), \omega(L)) + \sum_{l=2}^{L-1} u(f(\omega(l+1)), \omega(l))$$

$$= u(f(\omega(1)), \omega(1)) + u(f(\omega(2)), \omega(L)) + u(f(\omega(3)), \omega(2))$$

$$+ \sum_{l=3}^{L-1} u(f(\omega(l+1)), \omega(l))$$

$$\leq u(f(\omega(1)), \omega(1)) + u(f(\omega(2)), \omega(2)) + u(f(\omega(3)), \omega(L))$$

$$+ \sum_{l=3}^{L-1} u(f(\omega(l+1)), \omega(l)) \ \cdots \leq \sum_{l=1}^{L} u(f(\omega(l)), \omega(l)),$$

which implies Condition 1. **Q.E.D.**

5. Exact Implementation

This section investigates *exact* implementation that requires the value of a social choice function to be realized with certainty, irrespective of which private signal profile the agent observes. The following proposition shows that Condition 1 is necessary and sufficient for exact implementation irrespective of K; therefore, the necessary and sufficient condition is the same for both virtual and exact implementation.

In contrast with virtual implementation, in order to exactly implement a nontrivial social choice function, we have to use *inconstant* side payment devices. If we confine our analysis to mechanisms with constant side payments, the exactly implementable social choice functions f are the only *trivial* ones such that

$$u(f(\omega),\omega) \geq u(f(\omega'),\omega) \text{ for all } \omega \in \Omega \text{ and all } \omega' \in \Omega .$$

Needless to say, linking mechanisms do not function effectively when we require, not only virtual, but also exact, implementation.

Proposition 4: *A social choice function* f *is exactly implementable with respect to* K *if and only if Condition 1 holds.*[12]

Proof: We can apply Theorem 1 proposed by Fan (1956) in the same manner as it was used in D'Aspremont and Gèrard-Varet (1979, Theorem 7). For the complete proof, see Appendix B.

Since Condition 1 does not depend on K, the set of exactly implementable social choice functions is the same irrespective of the number of tasks. From Theorem 1 presented in Fan (1956), it follows that a necessary and sufficient condition for exact implementation is that for every $\mu : \Omega^2 \to R_+ \cup \{0\}$, if $\sum_{\tilde{\omega} \neq \omega \in \Omega} \{\mu(\omega,\tilde{\omega}) - \mu(\tilde{\omega},\omega)\} = 0$ for all $\omega \in \Omega$, then

$$\sum_{\omega \in \Omega} \sum_{\tilde{\omega} \neq \omega \in \Omega} \{u(f(\omega),\omega) - u(f(\tilde{\omega}),\omega)\} \mu(\omega,\tilde{\omega}) \geq 0 .$$

Without loss of generality, we can focus only on the set of functions $\mu : \Omega^2 \to R_+ \cup \{0\}$ such that $\sum_{\omega' \in \Omega} \mu(\omega,\omega') = 1$ for all $\omega \in \Omega$, i.e., the set of mixed strategies in the direct mechanism, where $\mu(\omega,\omega')$ is the probability that the agent announces ω' given that he observes ω. The above condition is equivalent to the condition that for every mixed strategy μ in the direct mechanism, if the frequencies of announcing private signals

[12] The sufficiency part of Proposition 4 and the definition of virtual implementation imply that the Condition 1 is a sufficient condition for virtual implementation *with* side payments. Note that Theorem 2 states that the Condition 1 is a sufficient condition for virtual implementation *without* side payments.

are the same as the probabilities of these signals being observed, i.e.,

$$p(\omega' \mid \mu) \equiv \sum_{\omega\in\Omega} \mu(\omega,\omega') p(\omega) = p(\omega') \quad \text{for all} \quad \omega' \in \Omega ,$$

the ex ante expected payoff with no side payments induced by the dishonest mixed strategy is not greater than that induced by the honest strategy, i.e.,

$$\sum_{\omega\in\Omega} \sum_{\tilde{\omega}\in\Omega} u(f(\tilde{\omega}),\omega)\mu(\omega,\tilde{\omega}) p(\omega) \leq \sum_{\omega\in\Omega} u(f(\omega),\omega) p(\omega) \ .$$

Based on the law of large numbers, this inequality implies that the requirement of truth-telling being a best response is almost satisfied when the number of the tasks K is sufficiently large. Hence, the functioning of the message space restriction in the linking mechanism as a whole incentive scheme parallels that of the incentive payment scheme in the direct mechanism for each task.

6. Multiple Agents

This section generalizes the previous results to a case in which *multiple* n agents who observe their respective private signals are in conflict with each other over their own interests.

6.1. The Model

The following multiple agent model is a direct extension of the single agent model in Section 2. Each agent $i \in \{1,...,n\}$ chooses a profile of K alternatives $(a_{[i],1},...,a_{[i],K}) \in \times_{k=1}^{K} A_{[i],k}$, where $A_{[i],k}$ denotes the finite set of agent $i's$ alternatives for the k-th task, and $a_{[i],k} \in A_{[i],k}$. Agent i observes a profile of K private signals $(\omega_{[i],1},...,\omega_{[i],K}) \in \times_{k=1}^{K} \Omega_{[i],k}$, where $\Omega_{[i],k}$ denotes the finite set of agent $i's$ private signals for the k-th task, and $\omega_{[i],k} \in \Omega_{[i],k}$. Assume that $A_{[i],k} = A_{[i]}$ and $\Omega_{[i],k} = \Omega_{[i]}$ for all $k \in \{1,...,K\}$ and all $i \in \{1,...,n\}$. Let $\sharp\Omega_{[i]} = I_i < \infty$, $A = \times_{i=1}^{n} A_{[i]}$, $\Omega = \times_{i=1}^{n} \Omega_{[i]}$, $a_k = (a_{[1],k},...,a_{[n],k}) \in A$, $\omega = (\omega_{[1]},...,\omega_{[n]}) \in \Omega$, and $\omega_k = (\omega_{[1],k},...,\omega_{[n],k}) \in \Omega$.

A mechanism is defined as $\Gamma = \Gamma(K) \equiv (M,(g,t))$, where $M = \times_{i=1}^{n} M_{[i]}$, $M_{[i]}$ is the finite set of messages for agent i, $t=(t_{[i]})_{i=1}^{n}$, and $t_{[i]}:M\to R$. When the agents observe $(\omega_k)_{k=1}^{K}\in\Omega^K$ and choose $(a_k)_{k=1}^{K}\in A^K$ and the principal chooses $t=(t_{[i]})_{i=1}^{n}$, agent $i's$ payoff is given by $\frac{1}{K}\sum_{k=1}^{K}u_{[i]}(a_k,\omega_k)+t_{[i]}$, where $u_{[i]}:A\times\Omega\to R$.

A *strategy* for agent i is defined as $\sigma_{[i]}:\Omega_{[i]}^{K}\to M_{[i]}$. Let $\Sigma_{[i]}$ denote the set of strategies for agent i. Let $\Sigma=\prod_{i=1}^{n}\Sigma_{[i]}$ and $\sigma=(\sigma_{[i]})_{i=1}^{K}\in\Sigma$. Let

$$v_{[i]}(\sigma,\Gamma)=E[\frac{1}{K}\sum_{k=1}^{K}u_{[i]}(a_k,\omega_k)+t_{[i]}\,|\,\sigma,\Gamma]$$

denote the expected payoff for agent i induced by a strategy profile $\sigma\in\Sigma$ in Γ. A strategy profile $\sigma\in\Sigma$ is said to be a *Nash equilibrium* in Γ if for every $i\in\{1,...,n\}$ and every $\sigma'_{[i]}\in\Sigma_{[i]}$,

$$v_{[i]}(\sigma,\Gamma)\geq v_{[i]}(\sigma'_{[i]},\sigma_{-[i]},\Gamma).$$

A social choice function f is said to be *exactly implementable with respect to* K in terms of Nash equilibrium if there exists a mechanism $\Gamma(K)$ and a Nash equilibrium $\sigma\in\Sigma$ in $\Gamma(K)$ such that

$$g(\sigma(\omega_1,...,\omega_K))(f(\omega_1),...,f(\omega_K))=1 \text{ for all } (\omega_k)_{k=1}^{K}\in\Omega^K.$$

In this section, we require only the *weak* sense of implementation, i.e., do not require the uniqueness of Nash equilibrium outcomes. Moreover, we weaken the Nash equilibrium concept as follows. For each $\varepsilon>0$, a strategy profile $\sigma\in\Sigma$ is said to be a $\varepsilon-$*Nash equilibrium* in $\Gamma(K)$ if for every $i\in\{1,...,n\}$ and every $\sigma'_{[i]}\in\Sigma_{[i]}$,

$$v_{[i]}(\sigma,\Gamma)+\varepsilon\geq v_{[i]}(\sigma'_{[i]},\sigma_{-[i]},\Gamma).$$

By using this weakened concept, we define virtual implementation for the multiple agent case as follows. An infinite sequence of mechanisms $(\Gamma(K))_{K=1}^{\infty}$ is said to *virtually implement a social choice function* f if for every $\eta>0$ and every $\varepsilon>0$, there exists $\bar{K}$ such that for every $K\geq\bar{K}$, there is a $\varepsilon-$Nash equilibrium $\sigma\in\Sigma$ in $\Gamma(K)$ satisfying

$$E\left[\frac{\sharp\{k\in\{1,\cdots,K\}\mid a_k = f(\omega_k)\}}{K}\middle|\sigma,\Gamma(K)\right]\geq 1-\eta .$$[13]

Let $p_{[i]}(\omega_{[i]})\equiv\sum_{\omega_{-[i]}\in\Omega_{-[i]}} p(\omega)$. As a natural extension of Section 4, we define the linking mechanism $\Gamma^*(K)=(M,(g,t))$ as follows. We specify $B_{[i]}=B_{[i]}(\cdot,K):\Omega_{[i]}\to\{0,...,K\}$ such that $\sum_{\omega_{[i]}\in\Omega_{[i]}} B_i(\omega_{[i]},K)=K$, and for every $b_{[i]}:\Omega_{[i]}\to\{0,...,K\}$,

$$\sum_{\omega_{[i]}\in\Omega_{[i]}}\left|\frac{B_{[i]}(\omega_{[i]})}{K}-p_{[i]}(\omega_{[i]})\right|\leq\sum_{\omega_{[i]}\in\Omega_{[i]}}\left|\frac{b_{[i]}(\omega_{[i]})}{K}-p_{[i]}(\omega_{[i]})\right| \text{ if } \sum_{\omega_{[i]}\in\Omega_{[i]}} b_{[i]}(\omega_{[i]})=K .$$

For every $i\in\{1,...,n\}$, let

$$M_{[i],k}=\Omega_{[i]} \text{ for all } k\in\{1,...,K\},$$

$$M_{[i]}=\left\{(m_{[i],k})_{k=1}^K\in\Omega_{[i]}^K\middle|\sharp\left\{k\in\{1,\cdots,K\}\mid m_{[i],k}=\omega_{[i],k}\right\}=B_i(\omega_{[i]}) \textit{ for all } \omega_{[i]}\in\Omega_{[i]}\right\},$$

$$m_{[i],k}\in M_{[i],k}, \text{ and } m_k=(m_{[1],k},...,m_{[n],k})\in\Omega .$$

For every $m\in M$, let

$$g(m)(f(m_1),...,f(m_K))=1 \text{ and } t_{[i]}(m)=0 \text{ for all } i\in\{1,...,n\} .$$

6.2 Results

The following Condition is a direct extension of Condition 1, where each agent is required to satisfy slightly modified versions of the inequalities (3) in which the payoff $u(f(\omega),\omega)$ is replaced with the conditional expected value $E_{[i]}\left[u_{[i]}(f(\omega),\omega)\middle|\omega_{[i]}\right]$.

Condition 3: For every $i\in\{1,...,n\}$, every $L\in\{2,\cdots,I\}$, and every $(\omega_{[i]}(1),\cdots,\omega_{[i]}(L))\in\Omega_{[i]}^L$, if

$$\omega_{[i]}(l)\neq\omega_{[i]}(l') \text{ for all } l\in\{1,\ldots,L\} \text{ and } l'\in\{1,\ldots,L\}\setminus\{l\},$$

[13] In the single agent model, weak implementation secures the same utility level of the agent. In the multi-agent model, our weaker notion of virtual implementation with ε-Bayesian Nash equilibrium does not secure the same utility level for agents.

then

$$\sum_{l=1}^{L} E_{[i]}\left[u_{[i]}(f(\omega_{[i]}(l),\omega_{-[i]}),\omega_{[i]}(l),\omega_{-[i]})\middle|\omega_{[i]}(l)\right]$$
$$\geq \sum_{l=1}^{L} E_{[i]}\left[u_{[i]}(f(\omega_{[i]}(l+1),\omega_{-[i]}),\omega_{[i]}(l),\omega_{-[i]})\middle|\omega_{[i]}(l)\right].$$

The following condition is also a direct extension of Condition 2, where each agent is required to satisfy slightly modified version of supermodularity in which the payoff $u(f(\omega),\omega)$ is replaced with the conditional expected value $E_{[i]}\left[u_{[i]}(f(\omega),\omega)\middle|\omega_{[i]}\right]$.

Condition 4 (Supermodularity): For every $i \in \{1,...,n\}$, $\Omega_{[i]}$ is an ordered set with $\geq$, and for every $\omega_{[i]},\omega'_{[i]},\omega''_{[i]},\omega'''_{[i]} \in \Omega_{[i]}$,

$$E_{[i]}\left[u_{[i]}(f(\omega_{[i]},\omega_{-[i]}),\omega'_{[i]},\omega_{-[i]})\middle|\omega'_{[i]}\right]$$
$$+E_{[i]}\left[u_{[i]}(f(\omega''_{[i]},\omega_{-[i]}),\omega'''_{[i]},\omega_{-[i]})\middle|\omega'''_{[i]}\right]$$
$$\leq E_{[i]}\left[u_{[i]}(f(\omega_{[i]}\vee\omega''_{[i]},\omega_{-[i]}),\omega'_{[i]}\vee\omega'''_{[i]},\omega_{-[i]})\middle|\omega'_{[i]}\vee\omega'''_{[i]}\right]$$
$$+E_{[i]}\left[u_{[i]}(f(\omega_{[i]}\wedge\omega''_{[i]},\omega_{-[i]}),\omega'_{[i]}\wedge\omega'''_{[i]},\omega_{-[i]})\middle|\omega'_{[i]}\wedge\omega'''_{[i]}\right].$$

In the same way as in the previous sections, we can show the following theorem.

Theorem 5: *Suppose that the agents' private signals for each task are independent of each other, i.e.,* $p(\omega)=\prod_{i=1}^{n} p_{[i]}(\omega_{[i]})$ *for all* $\omega\in\Omega$. *Then, the following four properties hold.*

Property 1: *If there exists an infinite sequence of mechanisms* $(\Gamma(K))_{K=1}^{\infty}$ *that virtually implement a social choice function* f, *then Condition 3 holds.*

Property 2: *Under Condition 3,* $(\Gamma^{*}(K))_{K=1}^{\infty}$ *virtually implements* f.

Property 3: *Condition 4 implies Condition 3.*

Property 4: *A social choice function* f *is exactly implementable with respect to* K

if and only if Condition 3 holds.

Properties 3 and 4 are easy to prove because we can directly apply the proofs of Propositions 3 and 4. Note that the definition of virtual implementation in this section is different from that in the single agent case presented in Section 2. This is because we do not require the agents to play their best responses in the *exact* sense. However, even if we replace the original definition in Section 2 with that provided in this section, we can prove Property 1, i.e., the necessity of Condition 3 for virtual implementation, in exactly the same manner as in the proof of Theorem 1. Based on this, we can extend the necessity result obtained in the single agent case to Property 1 for the multiple agent case by simply applying the same logic as that used in the former case.

We need to provide detailed explanations in order to prove Property 2. Let us assume any positive real number $\eta > 0$ sufficiently close to zero and consider a sufficiently large K. Let $\Sigma_{[i]}(\eta, K) \subset \Sigma_{[i]}$ denote the set of agent $i's$ strategies $\sigma_{[i]}$ such that the expected value of the proportion of the tasks for which the agent announces incorrect private signals is less than η, i.e.,

$$E\left[\frac{\sharp\{k \in \{1,\cdots,K\} \mid m_{[i],k} \neq \omega_{[i],k}\}}{K}\middle|\sigma_{[i]}, \Gamma^*(K)\right] < \eta .$$

As in Theorem 2, we can observe that $\Sigma_{[i]}(\eta, K) \subset \Sigma_{[i]}$ is nonempty for a sufficiently large K. Define

$$\varepsilon_{[i]}(\eta, K) \equiv \max_{\sigma_{-[i]} \in \Sigma_{-[i]}(\eta,K)} \left[\max_{\sigma_{[i]} \in \Sigma_{[i]}} E\left[\frac{1}{K}\sum_{k=1}^{K} v_{[i]}(a_k, \omega_k) + t_{[i]} \middle| \sigma, \Gamma^*(K)\right]\right.$$
$$\left. - \max_{\sigma_{[i]} \in \Sigma_{[i]}(\eta,K)} E\left[\frac{1}{K}\sum_{k=1}^{K} u_{[i]}(a_k, \omega_k) + t_{[i]} \middle| \Gamma^*(K), \sigma\right]\right].$$

Note that $\varepsilon_{[i]}(\eta, K)$ approximates the maximum of agent $i's$ gain from deviation. Clearly, there exists a $\max_{i\in\{1,\dots,n\}} \varepsilon_{[i]}(\eta, K) -$Nash equilibrium in $\Gamma^*(K)$. As in the proof of Theorem 2, we can observe that there exists a best response for agent i such that the expected value of the proportion of the tasks for which the agent announces incorrect private signals is close to zero.[14] This implies that we can choose a strategy for agent i in $\Sigma_{[i]}(\eta, K)$ that is nearly a best response and, therefore, we can choose $\varepsilon_{[i]}(\eta, K)$ close to zero. In fact, by choosing η as close to zero and then choosing a sufficiently large K, we can obtain $\varepsilon_{[i]}(\eta, K)$ as close to zero as possible. Thus, we have proved Property 2.

6.3. Remarks

From Property 4, it follows as in Appendix B that we can replace Condition 3 with the condition that for every $i \in \{1,...,n\}$, there exists $r_{[i]} : \Omega_{[i]} \to R$ such that for every $\omega_{[i]} \in \Omega_{[i]}$ and every $\tilde{\omega}_{[i]} \in \Omega_{[i]}$,

$$E_{[i]}\left[u_{[i]}(f(\omega),\omega)\middle|\omega_{[i]}\right] + r_{[i]}(\omega_{[i]}) \geq E_{[i]}\left[u_{[i]}(f(\tilde{\omega}_{[i]},\omega_{-[i]}),\omega)\middle|\omega_{[i]}\right] + r_{[i]}(\tilde{\omega}_{[i]}) .$$

By assuming $r_{[i]}(\omega_{[i]}) \equiv 0$ for all $i \in \{1,...,n\}$, we can verify that for Condition 3, it is sufficient that for every $i \in \{1,...,n\}$, every $\omega_{[i]} \in \Omega_{[i]}$, and every $\tilde{\omega}_{[i]} \in \Omega_{[i]}$,

$$E_{[i]}\left[u_{[i]}(f(\omega),\omega)\middle|\omega_{[i]}\right] \geq E_{[i]}\left[u_{[i]}(f(\tilde{\omega}_{[i]},\omega_{-[i]}),\omega)\middle|\omega_{[i]}\right].$$

This condition is the same as the ex ante efficiency that was introduced by Jackson and Sonnenschein (2007) as the sufficient condition for implementation. In contrast with the present paper, Jackson and Sonnenschein (2007) assumed private values and showed full implementation in that for a sufficiently large K, *every* Nash equilibrium in the linking mechanism virtually induces the value of the ex ante efficient social choice function.

In Theorem 5, we have supposed that the agents' private signals are independent of each other. Even in the case of *correlated* private signals across all the agents, we can prove that Properties 1, 2, 3, and the sufficient part of Property 4. However, if the agents' signals are correlated to each other, the class of exactly implementable social choice functions is wider than the class of social choice functions that are virtually

[14] This does not imply that the expected value of the proportion of the tasks for which the agent announces incorrect private signals is less than η. This is why we cannot use the exact Nash equilibrium in place of $\varepsilon-$Nash equilibrium in the case of multiple agents.

implementable by linking mechanisms. In fact, any social choice function is exactly implementable whenever the probability distribution of the other agents' signal profile conditional on each agent's private signal varies across her signals, i.e.,

$$p_{[i]}(\cdot \mid \omega_{[i]}) \neq p_{[i]}(\cdot \mid \omega'_{[i]}) \text{ for all } \omega_{[i]} \in \Omega_{[i]} \text{ and all } \omega'_{[i]} \in \Omega_{[i]} \setminus \{\omega_{[i]}\},$$

where $p_{[i]}(\omega_{-[i]} \mid \omega_{[i]}) \equiv \dfrac{p(\omega)}{\sum_{\omega'_{-[i]} \in \Omega_{-[i]}} p(\omega'_{-[i]}, \omega_{[i]})}$. Needless to say, this sufficient condition is extremely weak. See Crèmer and McLean (1985, 1988), Matsushima (1990, 2007), Aoyagi (1998), Chung (1999), and others. In this case, the incentive wage scheme for each agent depends on the other agents' announcements as well as on his own announcement. This implies that whether each agent should be punished or rewarded is crucially dependent on the *whistle-blowing* of the other agents. Thus, even though the linking mechanism is a potentially powerful tool to incentivize agents in the case of correlated private signals, the drawback of this mechanism as compared with incentive wage schemes is that whistle-blowing is never effective without side payments.

6.4. Macro Shock

Throughout this paper, we have assumed that the private signals were drawn independently across all the tasks. However, by just adding a *prior message stage* in a simple way, the linking mechanism does function effectively even if the private signals are *correlated across all the tasks*.[15]

Consider a situation in which there exist *three or more* agents. Suppose that there exists a *macro* shock $\theta \in \Theta$ on which the probability distribution of ω_k for each task $h \in \{1,...,K\}$ and the social choice function are dependent. We denote $p(\omega) = p(\omega \mid \theta)$, $p_{[i]}(\omega_{[i]}) = p_{[i]}(\omega_{[i]} \mid \theta)$, and $f(\omega) = f(\omega, \theta)$. Here, we assume that Θ is a finite set, and for every $i \in \{1,...,n\}$, every $\theta \in \Theta$, and every $\theta' \in \Theta \setminus \{\theta\}$,

(17) $$p_{[i]}(\cdot \mid \theta) \neq p_{[i]}(\cdot \mid \theta').$$

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[15] This section handles with a special case of correlation, i.e., the signals are correlated through an unobservable macro shock. We figure out how the linking mechanism works in this special case here.

In order to be able to apply the appropriate linking mechanism, the principal needs to know the true macro shock θ. However, the principal and the agents both *cannot* observe this shock.

As we have already known, with a sufficiently large K, it is almost certain, based on the law of large numbers, that the realized proportion of the tasks for which an agent observes each private signal is almost the same as the probability of his observing this signal for a single task. This along with (17) implies that almost certainly each agent can infer the macro shock *correctly* from the observed private signals for all the tasks.

With three or more agents, the principal can incentivize the agents to tell of what they know about the macro shock to the best of their abilities as follows. The principal requires each agent to announce about the macro shock. If more than a half of the agents announce the same macro shock $\tilde{\theta} \in \Theta$, the principal will apply the linking mechanism associated with $p(\cdot) = p(\cdot \,|\, \tilde{\theta})$ and $f(\omega) = f(\omega, \tilde{\theta})$. If there is no such $\tilde{\theta}$, the principal will apply some fixed mechanism. Hence, announcing about the macro shock honestly is nearly a best response for each agent if the other agents announce honestly, because his announcement does not much influence which mechanism the principal will apply. This implies that truth-telling about the macro shock is described as an epsilon-Nash equilibrium strategy.

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Appendix A

We will show that in the proof of Theorem 1, we can assume without loss of generality that $\Gamma(K)$ is symmetric. Suppose that $\Gamma(K)$ is not symmetric. For each permutation π on $\{1,...,K\}$, we define g^{π} and t^{π} in ways that for each $m \in M$,

$$g^{\pi}(m^{\pi})(a_1^{\pi},...,a_K^{\pi}) = g(m)(a_1,...,a_K) \quad \text{for all } (a_k)_{k=1}^{K} \in A^K \text{, and}$$

$$t^{\pi}(m^{\pi}) = t(m) \ .$$

Let $\Gamma^{\pi} = (M, g^{\pi}, t^{\pi})$. We define $\sigma^{\pi} \in \Sigma$ by $\sigma^{\pi}(\omega_1^{\pi},...,\omega_K^{\pi}) = m^{\pi}$, where we denote $m = \sigma(\omega_1,...,\omega_K)$. Note that for every $\sigma \in \Sigma$ and every $(\omega_k)_{k=1}^{K} \in \Omega^K$,

$$E\left[\frac{1}{K}\sum_{k=1}^{K} u(a_k,\omega_k) + t \,\middle|\, \sigma, \Gamma, (\omega_k)_{k=1}^{K}\right] = E\left[\frac{1}{K}\sum_{k=1}^{K} u(a_k,\omega_k) + t \,\middle|\, \sigma^{\pi}, \Gamma^{\pi}, (\omega_k^{\pi})_{k=1}^{K}\right].$$

Since the truthful strategy $\hat{\sigma}$ is a best response in Γ and $\hat{\sigma}^{\pi} = \hat{\sigma}$, it follows from (4) that $\hat{\sigma}$ is a best response in Γ^{π}, and

$$\lim_{K\to\infty} E\left[\frac{\sharp\{k \in \{1,\cdots,K\} \mid a_k = f(\omega_k)\}}{K} \,\middle|\, \hat{\sigma}^{\pi}, \Gamma^{\pi}\right] = 1 .$$

We define a symmetric mechanism $\bar{\Gamma} = (M, \bar{g}, \bar{t})$ by

$$\bar{g} \equiv \sum_{\pi\in\Pi} \frac{1}{K!} g^{\pi} \quad \text{and} \quad \bar{t} \equiv \sum_{\pi\in\Pi} \frac{1}{K!} t^{\pi} ,$$

where Π denotes the set of permutations on $\{1,...,K\}$. Clearly, $\hat{\sigma}$ is a best response in $\bar{\Gamma}$, and

$$\lim_{K\to\infty} E\left[\frac{\sharp\{k\in\{1,\cdots,K\}\mid a_k = f(\omega_k)\}}{K}\middle|\hat{\sigma},\bar{\Gamma}\right]=1\,.$$

Hence, we can assume without loss of generality that $\Gamma(K)$ is symmetric.

Appendix B

We show the complete proof of Proposition 4. Choose K arbitrarily. Suppose that f is exactly implementable with respect to K, i.e., there exist $\Gamma(K)$ and a best response $\sigma^* \in \Sigma$ in $\Gamma(K)$ such that

(B-1) $$g(\sigma^*(\omega_1,...,\omega_K))(f(\omega_1),...,f(\omega_K)) = 1 \quad \text{for all} \quad \omega^K \in \Omega^K\,.$$

Consider any K' such that $K' = yK + z$ for some positive integer y and some integer $z \in \{0,...,K-1\}$. By using $\Gamma(K)$ in a set of y for the first yK tasks, we can construct a mechanism $\Gamma(K')$ such that there exists a best response that induces the values of f for the first yK tasks. This implies that there exists an infinite sequence of mechanisms that virtually implements f. This along with Theorem 1 implies that Condition 1 is necessary for exact implementation.

Next, we will prove the sufficiency. We merely need to show that Condition 1 is sufficient in the case of $K = 1$, because if this is true, we can exactly implement the social choice function irrespective of K by simply using $\Gamma(1)$ in a set of K for all tasks. Thus, it is sufficient to verify whether or not there exists a side payment function $r:\Omega \to R$ such that

$$u(f(\omega),\omega) + r(\omega) \geq u(f(\tilde{\omega}),\omega) + r(\tilde{\omega}) \quad \text{for all} \quad \omega\in\Omega \quad \text{and all} \quad \tilde{\omega}\in\Omega\,.$$

Using Theorem 1 proposed by Fan (1956) as it is used in D'Aspremont and Gèrard-Varet (1979, Theorem 7), we can show that a necessary and sufficient condition for the existence of such an r is that for every $\mu:\Omega^2 \to R_+ \bigcup \{0\}$, if

(B-2) $$\sum_{\tilde{\omega}\neq\omega\in\Omega} \{\mu(\omega,\tilde{\omega}) - \mu(\tilde{\omega},\omega)\} = 0 \quad \text{for all} \quad \omega\in\Omega\,,$$

then

(B-3) $$\sum_{\omega\in\Omega}\sum_{\tilde{\omega}\neq\omega\in\Omega} \{u(f(\omega),\omega) - u(f(\tilde{\omega}),\omega)\}\mu(\omega,\tilde{\omega}) \geq 0\,.$$

For every $L \in \{2,\ldots,I\}$, an L-tuple of private signals $(\omega(1),\ldots,\omega(L)) \in \Omega^L$ is said to be a *cycle* if $\mu(\omega(l),\omega(l+1)) > 0$ and $\omega(l) \neq \omega(l')$ for all $l \in \{1,\ldots,L\}$ and all $l' \in \{1,\ldots,L\} \setminus \{l\}$.

Suppose Condition 1 that for every $L \in \{2,\cdots,I\}$ and every $(\omega(1),\cdots,\omega(L))$, if $\omega(l) \neq \omega(l') \in \Omega$ for all $l \in \{1,\ldots,L\}$ and all $l' \in \{1,\ldots,L\} \setminus \{l\}$, then

(B-4) $$\sum_{l=1}^{L} u(f(\omega(l)),\omega(l)) \geq \sum_{l=1}^{L} u(f(\omega(l+1)),\omega(l)).$$

Evidently, we can choose $\mu = \mu(1) : \Omega^2 \to R_+ \bigcup \{0\}$, $\omega(1) \in \Omega$, and $\omega(2) \in \Omega \setminus \{\omega(1)\}$ satisfying (B-2) and $\mu(\omega(1),\omega(2)) > 0$. If $\mu(\omega(2),\omega(1)) > 0$ holds, then $(\omega(1),\omega(2))$ is a cycle. If $\mu(\omega(2),\omega(1)) = 0$, it follows from (B-2) that we can choose a private signal $\omega(3) \in \Omega \setminus \{\omega(1),\omega(2)\}$ such that $\mu(\omega(2),\omega(3)) > 0$.

Choose a positive integer l arbitrarily. Suppose that $(\omega(1),...,\omega(l-1))$ satisfies

$\omega(l') \neq \omega(l'')$ for all $l' \in \{1,\ldots,l-1\}$ and $l'' \in \{1,\ldots,l-1\} \setminus \{l'\}$,

$\mu(\omega(l'),\omega(l'+1)) > 0$ for all $l' \in \{1,\ldots,l-2\}$, and

$\mu(\omega(l'),\omega(l'')) = 0$ for all $l' \in \{2,\ldots,l-2\}$ and $l'' \in \{1,\ldots,l'-1\}$.

If there exists $l' \in \{1,\ldots,l-2\}$ such that $\mu(\omega(l-1),\omega(l')) > 0$, then $(\omega(l'),...,\omega(l-1))$ is a cycle. If there exists no such l', it follows from (B-2) that we can choose a private signal $\omega(l) \in \Omega \setminus \{\omega(1),...,\omega(l-1)\}$ such that $\mu(\omega(l-1),\omega(l)) > 0$.

Since $\sharp\Omega = I$ is finite, by continuing the above step, we can determine $l \in \{2,...,I\}$ and $l' \in \{1,....,l-1\}$ such that $(\omega(l'),...,\omega(l))$ is a cycle. By replacing l' and l with 1 and L, respectively, we denote this cycle by $C(1) \equiv (\omega(1),\ldots,\omega(L))$.

Let $\xi(1) \equiv \min_{l \in \{1,\ldots,L\}} \mu(\omega(l),\omega(l+1))$. Specify $\eta(1) : \Omega^2 \to R$ such that

$\eta(1)(\omega(l),\omega(l+1)) = \xi(1)$ for all $l \in \{1,...,L\}$,

and for every $(\omega,\omega') \in \Omega^2$, if there exists no $l \in \{1,...,L\}$ such that $(\omega,\omega') = (\omega(l),\omega(l+1))$, then $\eta(1)(\omega,\omega') = 0$. From (B-4), it follows that

$$\sum_{(\omega,\omega') \in \Omega^2} \{u(f(\omega),\omega) - u(f(\omega'),\omega)\}\eta(1)(\omega,\omega') \geq 0.$$

We define $\mu(2):\Omega^2 \to R$ by $\mu(2) \equiv \mu(1) - \eta(1)$. From (B-2) and the definition of $\eta(1)$, it follows that

$$\mu(2)(\omega,\omega') \geq 0 \quad \text{for all} \quad \omega \in \Omega \quad \text{and all} \quad \omega' \in \Omega, \text{ and}$$

$$\sum_{\tilde{\omega} \neq \omega \in \Omega} \{\mu(2)(\omega,\tilde{\omega}) - \mu(2)(\tilde{\omega},\omega)\} = 0 \quad \text{for all} \quad \omega \in \Omega .$$

If $\mu(2)(\omega,\tilde{\omega}) = 0$ for all $\omega \in \Omega$ and all $\tilde{\omega} \in \Omega \setminus \{\omega\}$, the inequality (B-3) holds for $\mu = \mu(2)$, i.e.,

$$\sum_{\omega \in \Omega} \sum_{\tilde{\omega} \neq \omega \in \Omega} \{u(f(\omega),\omega) - u(f(\tilde{\omega}),\omega)\} \mu(2)(\omega,\tilde{\omega}) \geq 0 .$$

If $\mu(2)(\omega,\tilde{\omega}) > 0$ for some $\omega \in \Omega$ and some $\tilde{\omega} \in \Omega \setminus \{\omega\}$, we can construct a cycle $C(2)$ and $\mu(3)$ as we did in $C(1)$ and $\mu(2)$.

By continuing the above step, we can determine a positive integer q, $\eta(q'):\Omega^2 \to R_+ \bigcup \{0\}$ for each $q' \in \{1,\ldots,q-1\}$, and $\mu(q):\Omega^2 \to R_+ \bigcup \{0\}$ such that

$$\sum_{\omega \in \Omega} \sum_{\tilde{\omega} \in \Omega \setminus \{\omega\}} \{u(f(\omega),\omega) - u(f(\tilde{\omega}),\omega)\} \mu(q)(\omega,\tilde{\omega}) \geq 0 ,$$

for every $q' \in \{1,\ldots,q-1\}$,

$$\sum_{(\omega,\omega') \in \Omega^2} \{u(f(\omega),\omega) - u(f(\omega'),\omega)\} \eta(q')(\omega,\omega') \geq 0 \text{ and } \quad \mu = \sum_{q'=1}^{q-1} \eta(q') + \mu(q) .$$

These imply (B-3). Thus, we have proved Proposition 4.