An Asymptotic Expansion with Push-Down
of Malliavin Weights

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Abstract

This paper derives asymptotic expansion formulas for option prices and implied volatilities as well as the density of the underlying asset price in multi-dimensional stochastic volatility models. In particular, the integration-by-parts formula in Malliavin calculus and the push-down of Malliavin weights are effectively applied. We provide an expansion formula for generalized Wiener functionals and closed-form approximation formulas in stochastic volatility environment. In addition, we present applications of the general formula to expansions of option prices for the shifted log-normal model with stochastic volatility. Moreover, with some results of Malliavin calculus in jump-type models, we derive an approximation formula for the jump-diffusion model in stochastic volatility environment. Some numerical examples are also shown.

Keywords: Malliavin calculus, Asymptotic expansion, Stochastic volatility, Implied volatility, Shifted log-normal model, Jump-diffusion model, Integration-by-parts, Malliavin weight, Push-down, Malliavin calculus for Poisson processes, Local volatility, Markovian projection

1 Introduction

This paper develops an asymptotic expansion method for generalized Wiener functionals by applying *Malliavin weight* (divergence given by the integration-by-parts formula) and *push-down* (the conditional expectation in Malliavin (1997) and Malliavin-Thalmaier (2006)). As applications, we propose a concrete approximation formula of option prices as well as the density of the underlying asset price in multi-dimensional stochastic volatility models, and then derives a new Taylor expansion formula of the implied volatilities. Moreover, we present applications of the general formula to expansions of option prices for the shifted log-normal model in stochastic volatility environment. Also, combining some results of Malliavin calculus in jump-type models by Bavouzet-Messaoud (2006) with our method, we derive an approximation formula for option prices in the jump-diffusion model with stochastic volatility. To the best of our knowledge, it is the first study with push-down of Malliavin weights for deriving analytical approximation formulas for option prices and implied volatilities in those models. A companion paper, Shiraya-Takahashi-Yamada (2009) applies the method to deriving a concrete approximation formula for valuation of barrier options with discrete monitoring under stochastic volatility models.

Various stochastic volatility models has been proposed for calibration to market prices of options with so called volatility skews and smiles. However, closed-form solutions for option prices in the stochastic volatility environment are rarely found, and hence a large number of studies have been made in order to obtain analytical approximations and efficient numerical schemes for option prices and implied volatilities with stochastic volatility models. Takahashi (1995, 1999) proposed approximation formulas based on the asymptotic expansion method in Watanabe theory (Watanabe (1983, 1984, 1987), Yoshida (1992 a, b)) for valuation of various derivatives including options in stochastic volatility environment. Fournié et al. (1997) provided an expansion result of the second-order partial differential equation which satisfies the uniform ellipticity condition and showed its application to Monte Carlo simulations. Fouque et al. (2000) derived a closed form approximation formula for the fast mean reverting stochastic volatility model using a singular perturbation method, and then discussed the calibration problem. Hagan et al. (2002) introduced the SABR stochastic volatility model and obtained an approximation implied volatility formula. Labordére (2008) generalized SABR model to λ-SABR model and derived an approximation of implied volatilities by applying the heat kernel expansion. Moreover, Gatheral et al. (2009) and Ben Arous-Laurence (2009) presented novel results for this direction. Recently, Antonelli-Scarlatti (2009) gave a Taylor series expansions of option prices with respect to a correlation parameter in a stochastic volatility model. Also, we give more detail comments on some related works in Section 3.3.

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[*]This version contains supplementary results such as an expansion of the local volatility obtained by the so called *Markovian projection*, which are not included in the short version of this paper. All the contents expressed in this research are solely those of the authors and do not represent the view of MTEC Co., Ltd. or any other institutions. The authors are not responsible or liable in any manner for any losses and/or damages caused by the use of any contents in this research.

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The organization of the paper is as follows:the next section derives an asymptotic expansion formula for generalized Wiener functionals after a brief summary of Malliavin calculus necessary for the remaining of the paper. Section 3 applies the general formula to pricing options in stochastic volatility environment and then obtains implied volatilities' expansions. Section 4 presents numerical examples for expansions including calibrations of implied volatilities. As simple applications of our method, Section 5 provides approximation formulas for option prices (or/and implied volatilities) in the shifted log-normal and jump-diffusion models under stochastic volatilities. Section 6 concludes. Appendix gives explicit calculations of push-down of Malliavin weights (coefficients in the expansions), the proof of Proposition 5.2 and an expansion of the local volatility obtained by the so called *Markovian Projection*.

2   Asymptotic Expansion

2.1   Malliavin calculus

This subsection summarizes basic facts on the Malliavin calculus which are necessary for the following discussion.

$$\mathcal{W} = \mathcal{W}^d = C_0([0,T] : \mathbf{R}^d) = \{w : [0,T] \to \mathbf{R}^d; \text{ continuous, } w(0) = 0\}$$

and μ is the Wiener measure. Next, let H be a Hilbert space such that

$$H = \left\{ h \in \mathcal{W}; h_i(t)(i = 1, \cdots, d) \text{ is absolute continuous with respect to } t \text{ and } \sum_{i=1}^{d} \int_0^T |\frac{dh_i(t)}{dt}|^2 dt < \infty \right\}$$

with an inner product $\langle h, h' \rangle_H = \sum_{i=1}^{d} \int_0^T \frac{dh_i(t)}{dt} \frac{dh'_i(t)}{dt} dt$. Then, H is called the Cameron-Martin space.

Define $L^{\infty-}(\mathcal{W})$ as $L^{\infty-}(\mathcal{W}) = \cap_{p<+\infty} L^p(\mathcal{W})$ and a distance on $L^{\infty-}(\mathcal{W})$ as $d_{L^{\infty-}(\mathcal{W})}(F_1, F_2) = \sum_{j=1}^{\infty} 2^{-j}(\min\{\|F_1 - F_2\|_{L^j}, 1\})$, where $\| \cdot \|_{L^p}$ denotes the L^p-norm in $(\mathcal{W}, \mu)$. Given a separable Hilbert space G, let $L^p(\mathcal{W} : G)$ denote the space of measurable maps from $\mathcal{W}$ to G such that $\|f\|_G \in L^p(\mathcal{W})$. The same definition is made for $L^{\infty-}(\mathcal{W} : G)$.

Then, consider the space

$$\mathbf{D}_1^p(\mathcal{W} : G) = \left\{ F \in L^p(\mathcal{W}, G) : \text{there exists } DF \in L^p(\mathcal{W} : H \otimes G) \text{ such that for } h \in H, \lim_{\epsilon \to 0} \frac{1}{\epsilon}[F(w + \epsilon h) - F(w)] = \langle DF, h \rangle_{H \otimes G} \right\}$$

Here, DF is called the (Malliavin) derivative of F. Due to a canonical identification of the Hilbert space $L^2(\mathcal{W} : H \otimes G)$ and $L^2([0,T] \times \mathcal{W} : G)$, the Malliavin derivative DF may be considered as a stochastic process $\{D_t F = (D_{t,1}F, \cdots, D_{t,d}F) : t \in [0,T]\}$ such that $\langle DF, h \rangle_{H \otimes G} = \sum_{i=1}^{d} \int_0^T (D_{t,i}F)(\frac{dh_i(t)}{dt}) dt$.

A norm $\mathbf{D}_1^p(\mathcal{W} : G)$ is given by $\|F\|_{\mathbf{D}_1^p(\mathcal{W}:G)} = \|F\|_{L^p(\mathcal{W}:G)} + \|DF\|_{L^p(\mathcal{W}:H\otimes G)}$. Also, $\mathbf{D}_1^\infty(\mathcal{W} : G)$ is defined by $\mathbf{D}_1^\infty(\mathcal{W} : G) := \cap_{1 \le p < +\infty} \mathbf{D}_1^p(\mathcal{W} : G)$, and a distance on $\mathbf{D}_1^\infty(\mathcal{W} : G)$ is given by $d_{\mathbf{D}_1^\infty(\mathcal{W}:G)}(F_1, F_2) = \sum_{j=1}^{\infty} 2^{-j}(\min\{\|F_1 - F_2\|_{\mathbf{D}_1^j(\mathcal{W}:G)}, 1\})$.

For $r \ge 2$ $(r \in \mathbf{N})$, we introduce the spaces:

$$\mathbf{D}_r^p(\mathcal{W} : G) = \{F \in \mathbf{D}_{r-1}^p(\mathcal{W} : G) : DF \in \mathbf{D}_{r-1}^p(\mathcal{W} : H \otimes G)\}$$

with $\|F\|_{\mathbf{D}_r^p(\mathcal{W}:G)} = \|F\|_{\mathbf{D}_{r-1}^p(\mathcal{W}:G)} + \|D^{r-1}F\|_{\mathbf{D}_1^p(H^{\otimes(r-1)}\otimes G)}$. We also define $\mathbf{D}_0^p(\mathcal{W} : G)$ as $\mathbf{D}_0^p(\mathcal{W} : G) = L^p(\mathcal{W} : G)$.

If $G = \mathbf{R}^n$, We denote $\mathbf{D}_r^p(\mathcal{W})$ as $\mathbf{D}_r^p(\mathcal{W} : G)$.

Some properties of these spaces are the following; $\mathbf{D}_{r'}^{p'}(\mathcal{W}) \subset \mathbf{D}_r^p(\mathcal{W})$, $r' \le r$, and $p' \le p$. The dual space of $(\mathbf{D}_r^q(\mathcal{W}))$, $(\mathbf{D}_r^q(\mathcal{W}))^*$ is given by $(\mathbf{D}_r^q(\mathcal{W}))^* = \mathbf{D}_{-r}^p(\mathcal{W})$, with $p^{-1} + q^{-1} = 1$, $r \ge 0$.

Furthermore, define the space $\mathbf{D}_\infty(\mathcal{W}) = \cap_{p \ge 1, r \ge 0} \mathbf{D}_r^p(\mathcal{W})$. Then, $\mathbf{D}_\infty(\mathcal{W})$ is a complete metric space under a metric, $d_{\mathbf{D}_\infty(\mathcal{W})}(F_1, F_2) = \sum_{p,r=1}^{\infty} \eta_{p,r}(\min\{\|F_1 - F_2\|_{\mathbf{D}_r^p}, 1\})$ where $\eta_{p,r} > 0$ such that $\sum_{p,r}^{\infty} \eta_{p,r} < \infty$. Note that this topology on $\mathbf{D}_\infty(\mathcal{W})$ is independent of the choice of the sequence $\{\eta_{p,r}\}$. We call $F \in \mathbf{D}_\infty(\mathcal{W})$ the smooth functional in the sense of Malliavin.

Given $Z = (Z_1(w), \cdots, Z_d(w)) \in \mathbf{D}_1^p(\mathcal{W} : H)$, there exists $D_i^*(Z_i) \in L^p(\mathcal{W})$, $i = 1, \cdots, d$ such that $E[\int_0^T D_{t,i}F(w)Z_i(w)dt] = E[F(w)D_i^*(Z_i(w))]$ for all $F \in \mathbf{D}_1^\infty(\mathcal{W})$. Then, define $D^*(Z) := \sum_{i=1}^{d} D_i^*(Z_i(w))$. So, there exists $C_p > 0$ such that $\|D^*(Z)\|_{L^p} \le C_p \|Z\|_{\mathbf{D}_1^p(\mathcal{W}:H)}$. We call $D^*(Z)$ the divergence of Z.

We also introduce the notation $D_Z F$ such that $D_Z F := \sum_{i=1}^{d} \int_0^T D_{t,i}F(w)Z_i(w)dt$. Then, we have $E[D_Z F] = E[FD^*(Z)]$. Thus, $D^*(FZ) = FD^*(Z) - D_Z F$ is obtained. (e.g. Proposition 1.16 in Malliavin-Thalmaier (2006))

Definition 2.1 *Let $F = (F_1, \cdots, F_n) \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ be the n-dimensional smooth functional, we call F a non-degenerate in the sense of Malliavin if the Malliavin covariance matrix $\{\sigma_F^{ij}\}_{1 \le i,j \le n}$*

$$\sigma_F^{ij} := \langle DF_i, DF_j \rangle_H = \sum_{k=1}^{d} \int_0^T (D_{t,k}F_i(w))(D_{t,k}F_j(w))dt \tag{1}$$

is invertible a.s. and

$$(\det \sigma_F)^{-1} \in L^{\infty-}(\mathcal{W}).$$

Theorem 2.1 *Let $F \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ be a n-dimensional non-degenerate in the sense of Malliavin and $G \in \mathbf{D}_\infty(\mathcal{W})$. Then, for $\varphi \in C_b^1(\mathbf{R}^n)$,*

$$E[\partial_i \varphi(F)G] \quad = \quad E[\varphi(F)D^*(\sum_{j=1}^n G\gamma_{ij}^F DF^j)] \tag{2}$$

where $(\gamma_{ij}^F)_{1 \le i,j \le n}$ is the inverse matrix of Malliavin covariance of F.

(Proof) See Lemma III.5.2. of Malliavin (1997). □

Theorem 2.2 *Let $F \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ be a non-degenerate functional. F has a smooth density $p^F \in \mathcal{S}(\mathbf{R}^n)$ where $\mathcal{S}(\mathbf{R}^n)$ denotes the space of all infinitely differentiable functions $f : \mathbf{R}^n \mapsto \mathbf{R}$ such that for any $k \ge 1$, and for any multi-index $\beta \in \{1, \cdots, n\}^j$ one has $\sup_{x \in \mathbf{R}^n} |x|^k |\partial_\beta f(x)| < \infty$. (i.e. $\mathcal{S}(\mathbf{R}^n)$ is the Schwartz space and $\mathcal{S}'(\mathbf{R}^n)$ is its dual.)*

(Proof) See Theorem III.5.1. of Malliavin (1997). □

Definition 2.2 *Consider the space $\mathbf{D}_{-\infty}(\mathcal{W}) = \cup_{p \ge 1, r \ge 0} \mathbf{D}_{-r}^p(\mathcal{W})$, that is, the dual of $\mathbf{D}_\infty$. We call $F \in \mathbf{D}_{-\infty}(\mathcal{W})$ a distribution on the Wiener space. We define the duality form on $\mathbf{D}_{-\infty} \times \mathbf{D}_\infty$, $(F, G) \mapsto \langle F, G \rangle_{\mathbf{D}_{-\infty} \times \mathbf{D}_\infty} = E[FG] \in \mathbf{R}$. We call this duality form the generalized expectation.*

2.2 Asymptotic Expansion for Expectation of Generalized Wiener Functionals

Let $F \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ be a non-degenerate functional, and ν and p^F be the law and the density of F, respectively; that is, $\nu(dx) = \mu \circ F^{-1}(dx) = p^F(x)dx$. Also, we define the range O as $O := \{x : p^F(x) > 0\} \subset \mathbf{R}^n$.

By Malliavin (1997) and Malliavin-Thalmaier (2006), the conditional expectation of $g \in L^p(\mathcal{W}, \mu)$ conditioned by a set $\{w : F(w) = x\}$ in σ-field $\sigma(F)$, $E[g|F = x]$ gives a map,

$$E^F : L^p(\mathcal{W}, \mu) \ni g \mapsto E[g|F = x] \in L^p(O, \nu). \tag{3}$$

For multi-index $\alpha^{(k)} = (\alpha_1, \cdots, \alpha_k)$, we define the iterated Malliavin weight. The Malliavin weight $H_{\alpha^{(k)}}$ is recursively defined as follows: for $G \in \mathbf{D}_\infty$,

$$H_{\alpha^{(k)}}(F, G) = H_{(\alpha_k)}(F, H_{\alpha^{(k-1)}}(F, G)), \tag{4}$$

where

$$H_{(l)}(F, G) = D^* \left(\sum_{i=1}^n G\gamma_{li}^F DF_i \right). \tag{5}$$

Here, $\gamma^F = \{\gamma_{ij}^F\}_{1 \le i,j \le n}$ denotes the inverse matrix of the Malliavin covariance matrix of F.

Watanabe (1983, 1984) introduced the distribution on Wiener space as composition of a non-degenerate map F by a Schwartz distribution T. The next theorem restates the result of Watanabe (1984) in terms of Malliavin (1997) and Malliavin-Thalmaier (2006).

Theorem 2.3 *1. Let $\mathcal{S}'$ be the Schwartz distributions. There exists a map*

$$(E^F)^* : \mathcal{S}' \ni T \mapsto T \circ F \in \tilde{\mathbf{D}}_{-\infty} := \cup_{s \ge 0} \cap_{q \ge 1} \mathbf{D}_{-s}^q \subset \mathbf{D}_{-\infty}. \tag{6}$$

$(E^F)^*$ is called the lifting up of T.

2. The conditional expectation defines a map

$$E^F : \mathbf{D}_\infty \ni G \mapsto E^F[G] \in \mathcal{S}(O), \tag{7}$$

where $\mathcal{S}(O)$ stands for the Schwartz space of the rapidly decreasing functions on $O = \{x : p^F(x) > 0\} \subset \mathbf{R}^n$. We call this map the push down of G.

3. The following duality formula is obtained for almost all $x \in O$:

$$\langle (E^F)^* T, G \rangle_{\mathbf{D}_{-\infty} \times \mathbf{D}_\infty} = \langle T, E^F[G] \rangle_{p^F(x)dx} \tag{8}$$

where $\langle \cdot, \cdot \rangle_{p^F(x)dx}$ is defined as follows: for almost all $x \in O$,

$$\langle T, E^F[G] \rangle_{p^F(x)dx} := \int_{\mathbf{R}^n} T(x)E[G|F = x]p^F(x)dx. \tag{9}$$

(Proof)
In this proof, we apply the discussions of Watanabe (1984), Malliavin (1997), Malliavin-Thalmaier (2006) and Nualart (2006).

1. Given $T \in \mathcal{S}'$, there exists $T_n \in \mathcal{S}$ such that $T_n \to T$ in $\mathcal{S}'$, i.e,

$$\|(1 - \Delta + |x|^2)^{-m}T_n - (1 - \Delta + |x|^2)^{-m}T\|_\infty \to 0, \qquad n \to \infty, \tag{10}$$

where $\|f\|_\infty = \sup_{x \in \mathbf{R}^n} |f(x)|$, $\Delta = \frac{1}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. By the Malliavin integration-by-parts formula, we can estimate as follows; for $p^{-1} + q^{-1} = 1$,

$$
\begin{aligned}
\|T_n(F) - T_{n'}(F)\|_{\mathbf{D}^q_{-2m}} &= \sup_{G \in \mathbf{D}^p_{2m}, \|G\|_{\mathbf{D}^p_{2m}} \leq 1} |E[T_n(F)G] - E[T_{n'}(F)G]| \\
&= \sup_{G \in \mathbf{D}^p_{2m}, \|G\|_{\mathbf{D}^p_{2m}} \leq 1} \left| E[(1 - \Delta + |x|^2)^{-m}T_n(F)\pi^F_{(2m)}] - E[(1 - \Delta + |x|^2)^{-m}T_{n'}(F)\pi^F_{(2m)}] \right| \\
&\leq \sup_{G \in \mathbf{D}^p_{2m}, \|G\|_{\mathbf{D}^p_{2m}} \leq 1} \|(1 - \Delta + |x|^2)^{-m}T_n - (1 - \Delta + |x|^2)^{-m}T_{n'}\|_\infty \|\pi^F_{(2m)}\|_{L^1} \\
&\leq C\|(1 - \Delta + |x|^2)^{-m}T_n - (1 - \Delta + |x|^2)^{-m}T_{n'}\|_\infty \to 0,
\end{aligned}
\tag{11}
$$

as $n, n' \to \infty$, where $\pi^F_{(2m)} \in \mathbf{D}_\infty$ and $C := \sup_{G \in \mathbf{D}^p_{2m}, \|G\|_{\mathbf{D}^p_{2m}} \leq 1} \|\pi^F_{(2m)}\|_{L^1} < \infty$. Then $(T_n(F))_{n \in \mathbf{N}}$ is a Cauchy sequence in $\mathbf{D}_{-\infty}$ and thus there exists $(E^F)^*T = T(F) \in \mathbf{D}_{-\infty}$: a composite functional $T(F)$ is uniquely determined.

2. Given $G \in \mathbf{D}_\infty$. For any multi-index $s = (s_1, \cdots, s_k)$, For any $\varphi \in C_b^{|s|}(\mathbf{R}^n)$, by push-down and then the integration by parts formula on $\mathbf{R}^n$ or the integration by parts formula on $\mathcal{W}$ and then push-down, we obtain

$$E[\partial_s^{|s|}\varphi(F)G] = (-1)^{|s|} \int_{\mathbf{R}^n} \varphi(x)\partial_s^{|s|}\{E^{F=x}[G]p^F(x)\}dx = \int_{\mathbf{R}^n} \varphi(x)E^{F=x}[\pi_s^F]p^F(x)dx, \tag{12}$$

where $\pi_s^F = H_s(F, G) \in \mathbf{D}_\infty$. It implies that

$$(-1)^{|s|}\partial_s^{|s|}\{E^{F=x}[G]p^F(x)\} = E^{F=x}[\pi_s^F]p^F(x), \tag{13}$$

where

$$E^{F=x}[\pi_s^F] \in L^p(O, \nu).$$

We define O_ϵ as $O_\epsilon = \{x \in \mathbf{R}^n : p^F(x) > \epsilon\}$. Therefore,

$$(-1)^{|s|}\partial_s^{|s|}\{E^{F=x}[G]p^F(x)\} \in L^p(O_\epsilon, dx), \tag{14}$$

for all s, which implies $E^{F=x}[G]p^F(x) \in C^\infty(O)$. As $p^F(x) \in C^\infty(O)$ where $C^\infty(O)$ stands for the set of real-valued C^∞-functions on O,

$$(p^F(x))^{-1}\{E^{F=x}[G]p^F(x)\} = E^{F=x}[G] \in C^\infty(O). \tag{15}$$

Note also that the conditional expectation has the following expression:

$$E^{F=x}[G]p^F(x) = E[\mathbf{1}_{\{F>x\}}H_{(1,\cdots,n)}(F, G)]. \tag{16}$$

Thus, for all $k \in \mathbf{N}$ and for all $j = 1, \cdots, n$, if $x_j > 0$,

$$
\begin{aligned}
\sup_{x \in \mathbf{R}^d, x_j > 0} & x_j^{2k}\partial_s^{|s|}\{E^{F=x}[G]p^F(x)\} \\
&= \sup_{x \in \mathbf{R}^d, x_j > 0} x_j^{2k}|E[\mathbf{1}_{F>x}H_s(F, H_{(1,\cdots,n)}(F, G))]| \\
&\leq E[|F_j|^{2k}|H_s(F, H_{(1,\cdots,n)}(F, G))|] \\
&< \infty,
\end{aligned}
\tag{17}
$$

if $x_j < 0$, we can derive a similar estimate. These facts imply $E^{F=x}[G] \in \mathcal{S}(O)$.

3. Hence, because there exists $T_n \in \mathcal{S}(\mathbf{R}^n)$, $n \in \mathbf{N}$ for $T \in \mathcal{S}'(\mathbf{R}^n)$ such that $T_n \to T$ in $\mathcal{S}'(\mathbf{R}^n)$, we have for almost all $x \in O$,

$$\langle T_n, E^F[G]\rangle_{p^F(x)dx} = E[T_n(F)G] \to \langle (E^F)^*T, G\rangle_{\mathbf{D}_{-\infty} \times \mathbf{D}_\infty} = \langle T, E^F[G]\rangle_{p^F(x)dx}, \tag{18}$$

as $n \to \infty$.

$\square$

The next theorem presents an asymptotic expansion formula for the expectation of generalized Wiener functionals.

Theorem 2.4 *Consider a family of smooth Wiener functionals $F^\epsilon = (F_1^\epsilon, \cdots, F_n^\epsilon) \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$ such that F^ϵ has an asymptotic expansion in $\mathbf{D}_\infty$ and satisfies the uniformly non-degenerate condition:*

$$\limsup_{\epsilon \downarrow 0} \|(\det \sigma_{F^\epsilon})^{-1}\|_{L^p} < \infty, \ \text{for all } p < \infty. \tag{19}$$

Then, for a Schwartz distribution $T \in S'(\mathbf{R}^n)$, we have an asymptotic expansion in $\mathbf{R}$:

$$\left| E[T(F^\epsilon)] - \left\{ \int_{\mathbf{R}^n} T(x) p^{F^0}(x) dx + \sum_{j=1}^N \epsilon^j \int_{\mathbf{R}^n} T(x) E\left[\overset{(j)}{\sum_k} H_{\alpha^{(k)}}(F^0, \prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}) | F^0 = x \right] p^{F^0}(x) dx \right\} \right| = O(\epsilon^{N+1}), \tag{20}$$

where p^{F^0} is the density of F^0, and $F_i^{0,k} := \frac{1}{k!} \frac{d^k}{d\epsilon^k} F_i^\epsilon|_{\epsilon=0}$, $k \in \mathbf{N}$, $i = 1, \cdots, n$. Also, $\alpha^{(k)}$ denotes a multi-index, $\alpha^{(k)} = (\alpha_1, \cdots, \alpha_k)$ and

$$\overset{(j)}{\sum_k} \equiv \sum_{k=1}^j \sum_{\beta_1 + \cdots + \beta_k = j, \beta_i \geq 1} \sum_{\alpha^{(k)} \in \{1, \cdots, n\}^k} \frac{1}{k!}.$$

Moreover, Malliavin weight $H_{\alpha^{(k)}}$ is recursively defined as follows:

$$H_{\alpha^{(k)}}(F, G) = H_{(\alpha_k)}(F, H_{\alpha^{(k-1)}}(F, G)), \tag{21}$$

where

$$H_{(l)}(F, G) = D^*\left(\sum_{i=1}^n G \gamma_{li}^F DF_i \right). \tag{22}$$

Here, $\gamma^F = \{\gamma_{ij}^F\}_{1 \leq i,j \leq n}$ denotes the inverse matrix of the Malliavin covariance matrix of F.

(*Proof*)

We use α as an abbreviation of $\alpha^{(k)}$ in the proof. Under the uniformly non-degenerate condition of $F^\epsilon \in \mathbf{D}_\infty(\mathcal{W} : \mathbf{R}^n)$, the lifting up of $T \in \mathcal{S}'(\mathbf{R}^n)$, $(E^{F^\epsilon})^*T$, has the asymptotic expansion in distributions on the Wiener space $\mathbf{D}_{-\infty}$, *i.e.* for $N \in \mathbf{N}$, there exists $s \in \mathbf{N}$ *s.t.*

$$\left\| (E^{F^\epsilon})^*T - \{T \circ F^0 + \sum_{j=1}^N \epsilon^j \overset{(j)}{\sum_k} (\partial_\alpha^k T) \circ F^0 \prod_{l=1}^k F_{\alpha_l}^{0,\beta_l} \} \right\|_{\mathbf{D}_{-s}^q} = O(\epsilon^{N+1}), \quad \epsilon \in (0,1], q < \infty \tag{23}$$

Then, there exists an asymptotic expansion of $\langle (E^{F^\epsilon})^*T, \mathbf{1} \rangle_{\mathbf{D}_{-\infty} \times \mathbf{D}_\infty}$. The push-down of the divergence are computed as follows:

$$\begin{aligned}
\left\langle \partial_\alpha^k T(F^0), \prod_{l=1}^k F_{\alpha_l}^{0,\beta_l} \right\rangle_{\mathbf{D}_{-\infty} \times \mathbf{D}_\infty} &= \left\langle T(F^0), H_\alpha(F^0, \prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}) \right\rangle_{\mathbf{D}_{-\infty} \times \mathbf{D}_\infty} \\
&= \left\langle T, E^{F^0}[H_\alpha(F^0, \prod_{l=1}^k F_{\alpha_l}^{0,\beta_l})] \right\rangle_{p^{F^0}(x)dx} \\
&= \int_{\mathbf{R}^n} T(x) E[H_\alpha(F^0, \prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}) | F^0 = x] p^{F^0}(x) dx. \tag{24}
\end{aligned}$$

$\square$

Corollary 2.1 *The density $p^{F^\epsilon}(y)$ is expressed as following asymptotic expansion with the push-down of Malliavin weights:*

$$p^{F^\epsilon}(y) = p^{F^0}(y) + \sum_{j=1}^m \epsilon^j E\left[\overset{(j)}{\sum_k} H_{\alpha^{(k)}}(F^0, \prod_{l=1}^k F_{\alpha_l}^{0,\beta_l}) | F^0 = y \right] p^{F^0}(y) + O(\epsilon^{m+1}), \tag{25}$$

where $p^{F^0}(y)$ is the density of F^0.

(*Proof*)

Take a delta function $\delta_y \in \mathcal{S}'$ in the theorem above. $\square$

3 Asymptotic Expansion in Multi-Dimensional Stochastic Volatility Model

This section applies the general formula in the previous section to pricing options in a stochastic volatility model and then obtains a new implied volatility's expansion formula.

3.1 Asymptotic Expansion of Option Prices

This subsection proves a basic result on an asymptotic expansion for a stochastic volatility model.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ be a filtered probability space and $W = \{(W_{1,t}, \cdots, W_{d,t}) : 0 \leq t \leq T\}$ be a d-dimensional Brownian motion with respect to $(\mathcal{F}_t)_{t \in [0,T]}$. We consider the following stochastic volatility model (n-dimensional volatility factor).

$$
\begin{aligned}
dS_t^{(\epsilon)} &= rS_t^{(\epsilon)}dt + V(\sigma_t^{(\epsilon)})S_t^{(\epsilon)}dW_{1,t}, \\
d\sigma_t^{(\epsilon)} &= A_0(\sigma_t^{(\epsilon)})dt + \epsilon A(\sigma_t^{(\epsilon)})dW_t, \\
S_0^{(\epsilon)} &= S_0^{(0)} = s > 0,\ \sigma_0^{(\epsilon)} = \sigma_0^{(0)} = \sigma \in \mathbf{R}_+^n,
\end{aligned}
\tag{26}
$$

where $V \in C_b^\infty(\mathbf{R}^n \to \mathbf{R})$, $A_0 \in C_b^\infty(\mathbf{R}^n \to \mathbf{R}^n)$, $A \in C_b^\infty(\mathbf{R}^n \to \mathbf{R}^{n \times d})$, $r > 0$ and $\epsilon \in [0,1]$. Note that ϵ is the volatility of volatility parameter. $(\sigma_t^{(0)})_{t \in [0,T]}$ is a deterministic process and satisfies an ordinary differential equation,

$$
d\sigma_t^{(0)} = A_0(\sigma_t^{(0)})dt.
\tag{27}
$$

Next, we impose the following condition which gives the non-degeneracy of the Malliavin covariance of $S_T^{(\epsilon)}$ at $\epsilon = 0$.

Assumption 3.1 *For some $s \in [0,T]$, $V(\sigma_s^{(0)}) \neq 0$.*

We define the logarithmic process of $(S_t^{(\epsilon)})_{t \in [0,T]}$ as

$$
X_t^{(\epsilon)} = \log\left(\frac{S_t^{(\epsilon)}}{s}\right).
$$

Let $p^{SV}(y)$ be the density of the underlying asset of the stochastic volatility model and $C^{SV}(T, K)$ and $P^{SV}(T, K)$ be the call and the put option prices under the stochastic volatility with maturity T and strike price K.

Also let $p^{BS}(y)$ is the log-normal density of the Black-Scholes model, *i.e.*,

$$
p^{BS}(y) := \frac{1}{y\sqrt{2\pi \int_0^T V(\sigma_t^{(0)})^2 dt}} \exp\left(-\frac{1}{2\int_0^T V(\sigma_t^{(0)})^2 dt}\left(\log\left(\frac{y}{s}\right) - r\right)^2\right).
$$

$C^{BS}(T, K)$ and $P^{BS}(T, K)$ denote the Black-Scholes formula of the call and the put options with maturity T and strike price K, *i.e.*,

$$
\begin{aligned}
C^{BS}(T, K) &:= sN(d_1) - Ke^{-rT}N(d_2), \\
P^{BS}(T, K) &:= Ke^{-rT}N(-d_2) - sN(-d_1),
\end{aligned}
\tag{28}
$$

where

$$
\begin{aligned}
n(x) &:= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \\
N(x) &:= \int_{-\infty}^x n(y)dx, \\
d_1 &:= \frac{\log\left(\frac{s}{K}\right) + rT}{\bar{\sigma}\sqrt{T}} + \frac{1}{2}\bar{\sigma}\sqrt{T}, \\
d_2 &:= \frac{\log\left(\frac{s}{K}\right) + rT}{\bar{\sigma}\sqrt{T}} - \frac{1}{2}\bar{\sigma}\sqrt{T},
\end{aligned}
\tag{29}
$$

with

$$
\bar{\sigma} := \left(\frac{1}{T}\int_0^T V(\sigma_t^{(0)})^2 dt\right)^{1/2}.
\tag{30}
$$

Let $\sigma_{S_T^{(0)}}$ be the Malliavin (co)variance of $S_T^{(\epsilon)}$ at $\epsilon = 0$, *i.e.*,

$$
\sigma_{S_T^{(0)}} = \|D_1 S_T^{(0)}\|_H^2 = (S_T^{(0)})^2 \int_0^T V(\sigma_s^{(0)})^2 ds,
\tag{31}
$$

We introduce the expressions;

$$Z(t) \quad := \quad \frac{D_{t,1}S_T^{(0)}}{\|D_1 S_T^{(0)}\|_H^2} = \frac{S_T^{(0)}V(\sigma_t^{(0)})}{(S_T^{(0)})^2 \int_0^T V(\sigma_u^{(0)})^2 du},$$

$$S_t^{(k)} \quad = \quad \frac{1}{k!}\frac{\partial^k}{\partial \epsilon^k} S_t^{(\epsilon)}|_{\epsilon=0},$$

$$\Psi^{\beta_1,\cdots,\beta_k}(t) \quad = \quad \prod_{i=1}^k S_t^{(\beta_i)}, \tag{32}$$

where $\beta_l \geq 1$ satisfy

$$\sum_{l=1}^k \beta_l = j, \quad j \in \mathbf{N}, \quad 1 \leq k \leq j.$$

Given Z above, for $\Psi \in \mathbf{D}_\infty$, $D^*(Z) - D_Z : \mathbf{D}_\infty \to \mathbf{D}_\infty$ is expressed as

$$(D^*(Z) - D_Z) \circ \Psi = \Psi \int_0^T Z(t)dW_{1,t} - \int_0^T D_{t,1}\Psi Z(t)dt. \tag{33}$$

Then, we have the following result.

Theorem 3.1 *Under the stochastic volatility model (26), we have asymptotic expansions of the density and the option prices as follows:*

$$p^{SV}(y) \quad = \quad p^{BS}(y) + \sum_{j=1}^N \epsilon^j E[\pi_j | S_T^{(0)} = y]p^{BS}(y) + O(\epsilon^{N+1}),$$

$$C^{SV}(T,K) \quad = \quad C^{BS}(T,K,\bar{\sigma}) + \sum_{j=1}^N \epsilon^j \int_{\mathbf{R}_+} e^{-rT}(y-K)^+ E[\pi_j | S_T^{(0)} = y]p^{BS}(y)dy + O(\epsilon^{N+1}),$$

$$P^{SV}(T,K) \quad = \quad P^{BS}(T,K,\bar{\sigma}) + \sum_{j=1}^N \epsilon^j \int_{\mathbf{R}_+} e^{-rT}(K-y)^+ E[\pi_j | S_T^{(0)} = y]p^{BS}(y)dy + O(\epsilon^{N+1}), \tag{34}$$

where π_j is the jth-order Malliavin weight, i.e.

$$\pi_j \quad = \quad \sum_{k=1}^j \sum_{\beta_1+\cdots+\beta_k=j,\beta_i\geq 1} \frac{1}{k!}H_k(S_T^{(0)}, \Psi^{\beta_1,\cdots,\beta_k}(T)) \in \mathbf{D}_\infty,$$

with

$$H_1(S_T^{(0)}, \Psi^{\beta_1,\cdots,\beta_k}(T)) \quad = \quad \Psi^{\beta_1,\cdots,\beta_k}(T) \int_0^T Z(s)dW_{1,s} - \int_0^T D_{s,1}\Psi^{\beta_1,\cdots,\beta_k}(T)Z(s)ds,$$

$$H_k(S_T^{(0)}, \Psi^{\beta_1,\cdots,\beta_k}(T)) \quad = \quad (D^*(Z) - D_Z)^k \circ \Psi^{\beta_1,\cdots,\beta_k}. \tag{35}$$

(*Proof*)
Let $(Y_t)_t$ be the solution of the following stochastic differential equation;

$$dY_j^i(t) \quad = \quad \sum_{l=1}^d \sum_{k=1}^n a_{k,l}^i(s)Y_j^k(s)dW_t^l + \sum_{k=1}^n b_k^i(s)Y_j^k(s)ds, \tag{36}$$

$$Y_j^i(0) \quad = \quad \delta_j^i, \tag{37}$$

where

$$a_{k,l}^i(s) \quad = \quad \partial_k A_l^i(\sigma_s^{(\epsilon)}), \tag{38}$$

$$b_k^i(s) \quad = \quad \partial_k A_0^i(\sigma_s^{(\epsilon)}), \tag{39}$$

and δ_j^i is the Kronecker's delta. Let $D_{s,k}$, $k = 1,\cdots,d$ be the Malliavin derivative acting on the Brownian motion $W_{k,t}$. Then, the Malliavin covariance of $S_T^{(\epsilon)}$ is given by;

$$\sigma_{S_T^{(\epsilon)}} \quad = \quad \sum_{i=1}^d \int_0^T (D_{s,i}S_T^{(\epsilon)})^2 ds$$

$$= \quad (S_T^{(\epsilon)})^2 \sum_{i=1}^d \int_0^T (D_{s,i}X_T^{(\epsilon)})^2 ds, \tag{40}$$

where

$$
\begin{aligned}
D_{s,1}X_T^{(\epsilon)} &= V(\sigma_s^{(\epsilon)}) - \epsilon \sum_{i=1}^{n} \int_s^T V(\sigma_u^{(\epsilon)}) \partial_i V(\sigma_u^{(\epsilon)}) D_{s,1}\sigma_{i,u}^{(\epsilon)} du \\
&\quad + \epsilon \sum_{i=1}^{n} \int_s^T \partial_i V(\sigma_u^{(\epsilon)}) D_{s,1}\sigma_{i,u}^{(\epsilon)} dW_{1,u} \\
D_{s,k}X_T^{(\epsilon)} &= -\epsilon \sum_{i=1}^{n} \int_s^T V(\sigma_u^{(\epsilon)}) \partial_i V(\sigma_u^{(\epsilon)}) D_{s,k}\sigma_{i,u}^{(\epsilon)} du \\
&\quad + \epsilon \sum_{i=1}^{n} \int_s^T \partial_i V(\sigma_u^{(\epsilon)}) D_{s,k}\sigma_{i,u}^{(\epsilon)} dW_{1,u}.
\end{aligned}
\tag{41}
$$

Here, $\partial_i V(\sigma_u^{(\epsilon)}) = \frac{\partial V(x)}{\partial x_i}\big|_{x=\sigma_u^{(\epsilon)}}$ and for $s \leq u$,

$$
D_{s,k}\sigma_{i,u}^{(\epsilon)} = \sum_{l,j=1}^{n} Y_l^i(u) Y^{-1}(s)_j^l A_k^j(\sigma_s^{(\epsilon)}).
\tag{42}
$$

Assumption 3.1 yields the nondegeneracy of $S_T^{(0)}$;

$$
\|\sigma_{S_T^{(0)}}^{-1}\|_{L^p} < \infty, \ for\ all\ p < \infty.
\tag{43}
$$

Then, the similar argument to Takahashi-Yoshida (2004), we have an asymptotic expansion in $\mathbf{D}_{-\infty}$;

$$
\delta_y(S_T^{(\epsilon)}) = \delta_y(S_T^{(0)}) + \sum_{i=1}^{N} \epsilon^j \sum_{k=1}^{j} \sum_{\beta_1+\cdots+\beta_k=j,\beta_i\geq 1} \frac{1}{k!} \partial^k \delta_y(S_T^{(0)}) \Psi^{\beta_1,\cdots,\beta_k}(T).
\tag{44}
$$

$$
e^{-rT}(S_T^{(\epsilon)}-K)^+ = e^{-rT}(S_T^{(0)}-K)^+ + \sum_{i=1}^{N} \epsilon^j \sum_{k=1}^{j} \sum_{\beta_1+\cdots+\beta_k=j,\beta_i\geq 1} \frac{1}{k!} e^{-rT} \partial^k (S_T^{(0)}-K)^+ \Psi^{\beta_1,\cdots,\beta_k}(T).
\tag{45}
$$

By the integration by parts formula,

$$
\begin{aligned}
\left\langle \partial^k \delta_y(S_T^{(0)}), \Psi^{\beta_1,\cdots,\beta_k}(T) \right\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty} &= \left\langle \delta_y(S_T^{(0)}), H_k(S_T^{(0)}, \Psi^{\beta_1,\cdots,\beta_k}(T)) \right\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty} \\
&= \left\langle \delta_y, E^{S_T^{(0)}}[H_k(S_T^{(0)}, \Psi^{\beta_1,\cdots,\beta_k}(T))] \right\rangle_{p^{BS}(x)dx} \\
&= E[H_k(S_T^{(0)}, \Psi^{\beta_1,\cdots,\beta_k}(T))|S_T^{(0)}=y]p^{BS}(y),
\end{aligned}
\tag{46}
$$

where

$$
\partial^k \delta_y(S_T^{(0)}) = \frac{\partial^k}{\partial x^k} \delta_y(x)\big|_{x=S_T^{(0)}}.
$$

Similarly, we have

$$
\left\langle e^{-rT} \partial^k (S_T^{(0)}-K)^+, \Psi^{\beta_1,\cdots,\beta_k}(T) \right\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty} = \int_{\mathbf{R}_+} e^{-rT}(y-K)^+ E[H_k(S_T^{(0)}, \Psi^{\beta_1,\cdots,\beta_k}(T))|S_T^{(0)}=y]p^{BS}(y)dy,
\tag{47}
$$

where

$$
\partial^k \left(S_T^{(0)}-K\right)^+ = \frac{\partial^k}{\partial x^k}(x-K)^+\big|_{x=S_T^{(0)}}.
$$

Then, we obtain asymptotic expansion formulas of the density and the option prices;

$$
p^{SV}(y) = p^{BS}(y) + \sum_{j=1}^{N} \epsilon^j E[\pi_j|S_T^{(0)}=y]p^{BS}(y) + O(\epsilon^{N+1}).
\tag{48}
$$

$$
C^{SV}(T,K) = C^{BS}(T,K,\bar\sigma) + \sum_{j=1}^{N} \epsilon^j \int_{\mathbf{R}_+} e^{-rT}(y-K)^+ E[\pi_j|S_T^{(0)}=y]p^{BS}(y)dy + O(\epsilon^{N+1}),
\tag{49}
$$

$$
P^{SV}(T,K) = P^{BS}(T,K,\bar\sigma) + \sum_{j=1}^{N} \epsilon^j \int_{\mathbf{R}_+} e^{-rT}(K-y)^+ E[\pi_j|S_T^{(0)}=y]p^{BS}(y)dy + O(\epsilon^{N+1}).
\tag{50}
$$

The Malliavin weights are computed by the iterated Skorohod integrals.

$$
\begin{aligned}
H_1\left(S_T^{(0)}, \Psi^{\beta_1,\cdots,\beta_k}\right) &= D^*\left(\Psi^{\beta_1,\cdots,\beta_k}\sigma_{S_T^{(0)}}^{-1}DS_T^{(0)}\right), \\
&= (D^*(Z) - D_Z)\circ\Psi^{\beta_1,\cdots,\beta_k} \\
&= \Psi^{\beta_1,\cdots,\beta_k}(T)\int_0^T Z(s)dW_{1,s} - \int_0^T D_{s,1}\Psi^{\beta_1,\cdots,\beta_k}(T)Z(s)ds, \quad (51) \\
H_k(S_T^{(0)}, \Psi^{\beta_1,\cdots,\beta_k}(T)) &= H_1(S_T^{(0)}, H_{k-1}(S_T^{(0)}, \Psi^{\beta_1,\cdots,\beta_k}(T))) \\
&= (D^*(Z) - D_Z)^k \circ \Psi^{\beta_1,\cdots,\beta_k}. \quad (52)
\end{aligned}
$$

□

3.2 Implied Volatility Expansion

This subsection derives an asymptotic expansion formula for the implied volatility in the stochastic volatility model considered in the previous subsection, where we obtained an approximation formula of a call option:

$$
C^{SV}(T,K) = C^{BS}(T,K,\bar{\sigma}) + \epsilon C_1 + \epsilon^2 C_2 + \epsilon^3 C_3 + O(\epsilon^4), \quad (53)
$$

where

$$
C_i = \int_{\mathbf{R}} e^{-rT}\left(se^{y+rT-\frac{1}{2}\Sigma} - K\right)^+ E\left[\pi_i | \int_0^T V(\sigma_t^{(0)})dW_{1,t} = y\right]\frac{1}{\sqrt{2\pi\Sigma}}e^{-\frac{1}{2\Sigma}y^2}dy, \qquad i = 1,2,3. \quad (54)
$$

We obtain an asymptotic expansion formula of the implied volatility around $\bar{\sigma} = \left(\frac{1}{T}\int_0^T V(\sigma_t^{(0)})^2 dt\right)^{1/2}$.

Theorem 3.2 *Under the stochastic volatility model (26), an asymptotic expansion of the implied volatility is given by*

$$
\sigma^{IV}(T,K) = \bar{\sigma} + \epsilon\frac{C_1}{C_\sigma^{BS}(\bar{\sigma})} + \epsilon^2\left\{\frac{C_2}{C_\sigma^{BS}(\bar{\sigma})} - \frac{1}{2}\frac{C_1^2}{C_\sigma^{BS}(\bar{\sigma})^3}C_{\sigma\sigma}^{BS}(\bar{\sigma})\right\}
$$
$$
+\epsilon^3\left\{\frac{C_3}{C_\sigma^{BS}(\bar{\sigma})} - \left(\frac{C_1}{C_\sigma^{BS}(\bar{\sigma})^2}\right)\left\{\frac{C_2}{C_\sigma^{BS}(\bar{\sigma})} - \frac{1}{2}\frac{C_1^2}{C_\sigma^{BS}(\bar{\sigma})^3}C_{\sigma\sigma}^{BS}(\bar{\sigma})\right\}C_{\sigma\sigma}^{BS}(\bar{\sigma}) - \frac{1}{3!}\frac{C_1^3}{C_\sigma^{BS}(\bar{\sigma})^4}C_{\sigma\sigma\sigma}^{BS}(\bar{\sigma})\right\} + O(\epsilon^4),
$$
$$
\quad (55)
$$

where $C_\sigma^{BS}(\bar{\sigma})$, $C_{\sigma\sigma}^{BS}(\bar{\sigma})$, $C_{\sigma\sigma\sigma}^{BS}(\bar{\sigma})$ are derivatives of the Black-Scholes formula with respect to the volatility, i.e.,

$$
\begin{aligned}
C_\sigma^{BS}(\bar{\sigma}) &:= \frac{\partial}{\partial\sigma}C^{BS}|_{\sigma=\bar{\sigma}} = s\sqrt{T}n(d_1), \\
C_{\sigma\sigma}^{BS}(\bar{\sigma}) &:= \frac{\partial^2}{\partial\sigma^2}C^{BS}|_{\sigma=\bar{\sigma}} = \frac{s\sqrt{T}}{\bar{\sigma}}n(d_1)d_1 d_2, \\
C_{\sigma\sigma\sigma}^{BS}(\bar{\sigma}) &:= \frac{\partial^3}{\partial\sigma^3}C^{BS}|_{\sigma=\bar{\sigma}} = \frac{s\sqrt{T}}{\bar{\sigma}^2}n(d_1)\{d_1^2 d_2^2 - d_1 d_2 - d_1^2 - d_2^2\}. \quad (56)
\end{aligned}
$$

(*Proof*)
Suppose that an implied volatility is expanded as;

$$
\sigma^{IV}(T,K) = \bar{\sigma} + \epsilon\sigma_1 + \epsilon^2\sigma_2 + \epsilon^3\sigma_3 + O(\epsilon^4). \quad (57)
$$

Then we have

$$
\begin{aligned}
C^{BS}(T,K,\sigma^{IV}(T,K)) &= C^{BS}(T,K,\bar{\sigma}) + \epsilon C_\sigma^{BS}(\bar{\sigma})\sigma_1 + \epsilon^2\left\{C_\sigma^{BS}(\bar{\sigma})\sigma_2 + \frac{1}{2}C_{\sigma\sigma}^{BS}(\bar{\sigma})(\sigma_1)^2\right\} \\
&+ \epsilon^3\left\{\sigma_3 C_\sigma^{BS}(\bar{\sigma}) + \sigma_1\sigma_2 C_{\sigma\sigma}^{BS}(\bar{\sigma}) + \frac{1}{3!}\sigma_1^3 C_{\sigma\sigma\sigma}^{BS}(\bar{\sigma})\right\} + O(\epsilon^4). \quad (58)
\end{aligned}
$$

By the definition of the implied volatility in the stochastic volatility, *i.e.*, $C^{SV}(T,K) = C^{BS}(T,K,\sigma^{IV}(T,K))$, the approximation terms of the implied volatility are given by

$$
\begin{aligned}
\sigma_1 &= \frac{C_1}{C_\sigma^{BS}(\bar{\sigma})}, \\
\sigma_2 &= \left\{\frac{C_2}{C_\sigma^{BS}(\bar{\sigma})} - \frac{1}{2}\frac{C_1^2}{C_\sigma^{BS}(\bar{\sigma})^3}C_{\sigma\sigma}^{BS}(\bar{\sigma})\right\}, \\
\sigma_3 &= \frac{C_3}{C_\sigma^{BS}(\bar{\sigma})} - \left(\frac{C_1}{C_\sigma^{BS}(\bar{\sigma})^2}\right)\left\{\frac{C_2}{C_\sigma^{BS}(\bar{\sigma})} - \frac{1}{2}\frac{C_1^2}{C_\sigma^{BS}(\bar{\sigma})^3}C_{\sigma\sigma}^{BS}(\bar{\sigma})\right\}C_{\sigma\sigma}^{BS}(\bar{\sigma}) - \frac{1}{3!}\frac{C_1^3}{C_\sigma^{BS}(\bar{\sigma})^4}C_{\sigma\sigma\sigma}^{BS}(\bar{\sigma}). \quad (59)
\end{aligned}
$$

□

3.3 Comments on Related Works

Fournié et al. (1999) applied Malliavin calculus to efficient Monte Carlo estimators for computing Greeks of options in the Black-Scholes framework, *e.g.* for $dS_t = rS_t dt + \sigma^{(\epsilon)} S_t dW_t$ and $\sigma^{(\epsilon)} = \sigma + \epsilon$,

$$\frac{\partial}{\partial \sigma} E[e^{-rT}(S_T - K)^+] = \frac{\partial}{\partial \epsilon} E[e^{-rT}(S_T^{(\epsilon)} - K)^+]|_{\epsilon=0} = E[e^{-rT}(S_T^{(0)} - K)^+ \pi]. \tag{60}$$

The estimator π is the so-called *Malliavin weight*. Subsequently, a number of papers extended the method. In particular, related to our work, Siopacha and Teichmann (2010) developed strong and weak Taylor methods for a system of stochastic differential equations (SDEs) with perturbations. Especially, the weak Taylor method is based on the integration by parts on the Wiener space, which is a powerful tool for efficient Monte Carlo simulations and is general enough to be applied to multi-dimensional SDEs. As an example, they applied the method to swaptions under a market model of interest rates with one-dimensional Heston-type stochastic volatility and obtained an approximation of the option prices including the expectation of the divergence on the Wiener space. In the last step, they used Monte Carlo simulations for computing the option prices and demonstrated the efficiency of their method comparing with well-known existing Monte Carlo schemes.

Lewis (2000) developed a new method for approximations of European option prices and implied volatilities. In particular, he proposed an expansion with respect to a stochastic volatility parameter, "*vol-of-vol*" (ξ below) under a general one-dimensional time-homogeneous diffusion process of the stochastic volatility, which is considered as a special case of our setup. However, he took a different approach from ours to obtain an approximation: for example, he considered the following generalized Heston model :

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t} S_t dW_{1t}, \\ dv_t &= b(v_t) dt + \xi \eta(v_t)(\rho(v) dW_{1t} + \sqrt{1 - \rho(v)^2} dW_{2t}), \end{aligned} \tag{61}$$

where r and ξ are constants. Also, $W = (W_{1t}, W_{2t})$ is a two-dimensional Brownian motion and $\rho(v)$ stands for the instantaneous correlation between S and v. For valuation of call options, he used the following formula:

$$C(T, K) = S_0 - \frac{Ke^{-rT}}{2\pi} \int_{\frac{i}{2}-\infty}^{\frac{i}{2}+\infty} e^{-ikx} \frac{H(k, v, T)}{k^2 - ik} dk, \tag{62}$$

where $x = \log\left(S/Ke^{-rT}\right)$, and $H($ called "*the fundamental transform*") is the solution to

$$\frac{\partial H}{\partial T} = \frac{1}{2} \xi^2 \eta^2(v) \frac{\partial^2 H}{\partial v^2} + (b(v) - ik\xi\rho(v)\eta(v)\sqrt{v}) \frac{\partial H}{\partial v} - \{(k^2 - ik)/2\} vH, \tag{63}$$

with the initial condition $H(k, V, 0) = 1$. Then, he gave an approximation for a call price showing a scheme for the expansion of H with respect to ξ:

$$H(k, v, T) = H^{(0)}(k, v, T) + \xi H^{(1)}(k, v, T) + \xi^2 H^{(2)}(k, v, T) + \cdots. \tag{64}$$

He also presented an explicit result of the expansion for the following model.

$$dv_t = (\omega - \theta v_t) dt + \xi v_t^\beta (\rho dW_{1t} + \sqrt{1 - \rho^2} dW_{2t}),$$

where ω, θ, β and ρ are some appropriate constants.

This paper applies an asymptotic expansion approach to stochastic volatility models and derive the approximation terms including the Malliavin weights. Then, these weights are transformed into a finite-dimensional integration through the *duality formula* of Malliavin (1997); that is, for $T \in \mathcal{S}'$,

$$E[T(S_T)\pi] = \langle (E^{S_T})^* T, \pi \rangle_{\mathbf{D}_{-\infty} \times \mathbf{D}_\infty} = \langle T, E^{S_T}[\pi] \rangle_{p(x)dx} = \int_{\mathbf{R}} T(x) E[\pi | S_T = x] p(x) dx, \tag{65}$$

where $p(x)$ is the density of $S_T \in \mathbf{D}_\infty$. This formula suggests that an element of the distributions on the Wiener space $\mathbf{D}_{-\infty}$ is the adjoint operator of the conditional expectation. It also shows that the push-down of the Malliavin weights (the conditional expectation of the divergence on the Wiener space) is an element of the Schwartz space $\mathcal{S}$. Thus, applying both the integration by parts and the duality formula, we can obtain analytical approximations for density functions, option prices and implied volatilities.

As a simple example, for $dS_t = rS_t dt + \sigma_t^{(\epsilon)} S_t dW_{1t}$ and $d\sigma_t^{(\epsilon)} = A_0(\sigma_t^{(\epsilon)}) dt + \epsilon A_1(\sigma^{(\epsilon)})(\rho dW_{1t} + \sqrt{1 - \rho^2} dW_{2t})$ for some real-valued functions $A_0(x)$ and $A_1(x)$, we can obtain an approximation for a call option as follows:

$$\begin{aligned} C(T, K) = E[e^{-rT}(S_T^{(\epsilon)} - K)^+] &= E[e^{-rT}(S_T^{(0)} - K)^+] + \epsilon \frac{\partial}{\partial \epsilon} E[e^{-rT}(S_T^{(\epsilon)} - K)^+]|_{\epsilon=0} + O(\epsilon^2) \\ &= E[e^{-rT}(S_T^{(0)} - K)^+] + \epsilon E[e^{-rT}(S_T^{(0)} - K)^+ \pi] + O(\epsilon^2) \\ &= C^{BS}(T, K) + \epsilon \int_{\mathbf{R}} \left(se^{x - \frac{1}{2}\Sigma} - K \right)^+ \vartheta(x) \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{1}{2\Sigma}x^2} dx + O(\epsilon^2), \end{aligned}$$

$$\tag{66}$$

where $C^{BS}(T, K)$ represents a Black-Scholes call price and

$$\vartheta(x) = E[\pi| \int_0^T \sigma_t^{(0)} dW_{1,t} = x] = \zeta \left(\frac{1}{\Sigma^3} x^3 - \frac{1}{\Sigma^2} x^2 - \frac{1}{\Sigma^2} 3x + \frac{1}{\Sigma} \right).$$

Here, ζ and Σ are some constants. (See Corollary A.1. in Appendix A for the detail.) Further, the integral in the last line of (66) can be explicitly given.

Consequently, our method is a natural extension of Fournié et al. (1999) and Siopacha and Teichmann (2010) in a sense that we obtain analytical approximations that do not rely on Monte Carlo simulations by making use of push-down of Malliavin weights. Although Lewis (2000) took a different approach, our method can be regarded as its extension because his models are included in our framework: in fact, Theorem 3.1 and 3.2 show that our method and formulas can be applied to multi-dimensional models including time-inhomogeneous ones in a unified way. That is, we can readily deal with multi-dimensional stochastic volatility models by the same procedure as in one-dimensional ones. A concrete example will be shown in **3.4.2** Double Heston model. In this regard, it seems not an easy task for other analytical approximations such as Lewis (2000), Hagan et al.(2002), Labordere (2008) and Antonelli-Scarlatti (2009) to apply the methods to higher dimensional models.

4 Numerical Examples

4.1 One-dimensional Stochastic Volatility Models

This subsection shows numerical examples for the following stochastic volatility model:

$$\begin{aligned}
dS_t &= (\sigma_t)^\delta S_t dW_{1,t}, \\
d\sigma_t &= \lambda(\theta - \sigma_t)dt + \epsilon(\sigma_t)^\gamma (\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}),
\end{aligned} \tag{67}$$

where all the parameters will be specified later. We set the risk-free interest rate to be zero (i.e. $r = 0$). We also use the following notations;

$$\begin{aligned}
n(x) &:= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \\
N(x) &:= \int_{-\infty}^x n(x)dx, \\
\zeta &:= \rho \int_0^T \delta(\sigma_t^{(0)})^{(\delta-1)} e^{-\lambda t}(\sigma_t^{(0)})^\delta \int_0^t e^{\lambda s}(\sigma_s^{(0)})^{\gamma+\delta}dsdt, \\
\Sigma &:= \int_0^T \left((\sigma_t^{(0)})^\delta \right)^2 dt, \\
\vartheta(x) &:= E\left[\frac{\partial}{\partial \epsilon} S_T^{(\epsilon)}|_{\epsilon=0} \int_0^T \frac{D_{t,1}S_T^{(0)}}{\int_0^T (D_{s,1}S_T^{(0)})^2 ds} dW_{1,t} - \int_0^T D_{s,1} \frac{\partial}{\partial \epsilon} S_T^{(\epsilon)}|_{\epsilon=0} \frac{D_{t,1}S_T^{(0)}}{\int_0^T (D_{s,1}S_T^{(0)})^2 ds} dt| \int_0^T (\sigma_t^{(0)})^\delta dW_{1,t} = x \right] \\
&= \zeta \left(\frac{1}{\Sigma^3} x^3 - \frac{1}{\Sigma^2} x^2 - \frac{1}{\Sigma^2} 3x + \frac{1}{\Sigma} \right), \\
\bar{\sigma} &:= \left(\frac{1}{T} \int_0^T \left((\sigma_t^{(0)})^\delta \right)^2 dt \right)^{1/2}, \\
d_1 &:= \frac{\log \left(\frac{s}{K} \right)}{\bar{\sigma}\sqrt{T}} + \frac{1}{2}\bar{\sigma}\sqrt{T}, \\
d_2 &:= \frac{\log \left(\frac{s}{K} \right)}{\bar{\sigma}\sqrt{T}} - \frac{1}{2}\bar{\sigma}\sqrt{T}, \\
C^{BS} &:= sN(d_1) - KN(d_2), \\
C_1 &:= \int_{\mathbf{R}} \left(se^{x-\frac{1}{2}\bar{\sigma}^2 T} - K \right)^+ \vartheta(x) \frac{1}{\sqrt{2\pi\bar{\sigma}^2 T}} \exp\left(-\frac{1}{2\bar{\sigma}^2 T}x^2 \right) dx, \\
vega &:= s\sqrt{T}n(d_1).
\end{aligned} \tag{68}$$

By using Corollary A.1 in Appendix A, the first order approximations of the asymptotic expansions of the option price and the implied volatility are given by

$$\textbf{Malliavin AE} \quad := \quad C^{BS} + \epsilon C_1,$$

and

$$\textbf{Malliavin IV} \quad := \quad \bar{\sigma} + \epsilon \frac{C_1}{vega}.$$

First, we give the numerical results on the accuracy of our approximation formula for call option prices.

$T = 1$, $\delta = 1/2$, $\gamma = 1/2$, $s = 100$, $(\sigma_0)^{\frac{1}{2}} = 0.3$, $\lambda = 2.0$, $\theta = 0.09$, $\epsilon = 0.1$, $\rho = -0.5$

Strike price	Benchmark	Malliavin AE	Relative error
70	31.5478	31.5496	0.00%
80	23.6382	23.6471	0.04%
90	17.0487	17.0631	0.08%
100	11.8647	11.8816	0.14%
110	7.9947	8.0105	0.20%
120	5.2356	5.2474	0.23%

Benchmark : Heston's Fourier transform solutions.

$T = 5$, $\delta = 1$, $\gamma = 1$, $s = 100$, $\sigma_0 = 0.4$, $\lambda = 0.1$, $\theta = 0.4$, $\epsilon = 0.2$, $\rho = -0.5$

Strike price	Benchmark	Malliavin AE	Relative error
70	46.4585	46.3591	-0.21%
80	41.5476	41.4427	-0.25%
90	37.1770	37.0591	-0.31%
100	33.2892	33.1521	-0.41%
110	29.8338	29.6697	-0.55%
120	26.7639	26.5645	-0.74%

Benchmark : Monte Carlo simulation (1,000,000 trials, 500 time steps).

$T = 10$, $\delta = 1$, $\gamma = 1$, $s = 100$, $\sigma_0 = 0.4$, $\lambda = 0.1$, $\theta = 0.4$, $\epsilon = 0.2$, $\rho = -0.5$

Strike price	Benchmark	Malliavin AE	Relative error
70	55.3842	55.2118	-0.31%
80	51.4141	51.2058	-0.10%
90	47.8183	47.5671	-0.53%
100	44.5521	44.2513	-0.68%
110	41.5785	41.2207	-0.86%
120	38.8648	38.4432	-1.10%

Benchmark : Monte Carlo simulation (3,000,000 trials, 1000 time steps).

Next, we show the numerical results for implied volatilities.

$T = 0.25$, $\delta = 1/2$, $\gamma = 1/2$, $s = 100$, $(\sigma_0)^{\frac{1}{2}} = 0.15$, $\lambda = 4.0$, $\theta = 0.0225$, $\epsilon = 0.1$, $\rho = -0.5$

Strike price	Exact IV	Malliavin IV	Relative error
70	17.25	17.17	-0.46%
80	16.39	16.35	-0.24%
90	15.62	15.63	0.06%
100	14.95	14.98	0.20%
110	14.39	14.40	0.07%
120	13.96	13.86	-0.72%

$T = 0.5$, $\delta = 1/2$, $\gamma = 1/2$, $s = 100$, $(\sigma_0)^{\frac{1}{2}} = 0.4$, $\lambda = 2.5$, $\theta = 0.16$, $\epsilon = 0.1$, $\rho = -0.5$

Strike price	Exact IV	Malliavin IV	Relative error
70	40.63	40.34	-0.71%
80	40.30	40.20	-0.26%
90	40.02	40.07	0.13%
100	39.78	39.96	0.45%
110	39.58	39.85	0.69%
120	39.41	39.76	0.89%

$T = 1$, $\delta = 1/2$, $\gamma = 1/2$, $s = 100$, $(\sigma_0)^{1/2} = 0.2$, $\lambda = 2.0$, $\theta = 0.04$, $\epsilon = 0.1$, $\rho = -0.25$

Strike price	Exact IV	Malliavin IV	Relative error
70	20.68	20.60	-0.40%
80	20.36	20.36	0.00%
90	20.10	20.15	0.26%
100	19.90	19.96	0.32%
110	19.75	19.80	0.23%
120	19.63	19.64	0.06%

4.2 Double Heston Model

Our method is general enough to be applied to multi-dimensional models. For instance, Gatheral (2008) introduced the following double stochastic volatility model;

$$dS_t^{(\epsilon)} = rS_t^{(\epsilon)}dt + \sqrt{v_t^{(\epsilon)}}S_t^{(\epsilon)}dW_{1,t}, \tag{69}$$

$$dv_t^{(\epsilon)} = \kappa(\bar{v}_t^{(\epsilon)} - v_t^{(\epsilon)})dt + \epsilon\sqrt{v_t^{(\epsilon)}}(\rho_1 dW_{1,t} + \sqrt{1-\rho_1^2}dW_{2,t}), \tag{70}$$

$$d\bar{v}_t^{(\epsilon)} = \bar{\kappa}(\theta - \bar{v}_t^{(\epsilon)})dt + \epsilon\sigma_2\sqrt{\bar{v}_t^{(\epsilon)}}(\rho_2 dW_{1,t} + \sqrt{1-\rho_2^2}dW_{3,t}). \tag{71}$$

with $\rho_1, \rho_2 \in [-1, 1]$.

We derive an approximation formula for the double Heston model and show some numerical examples. Let $C^{Double.SV}$ and $\sigma^{IV}(T, K)$ be the call option and the implied volatility under the double Heston model;

$$C^{Double.SV}(T, K) = e^{-rT}E[(S_T^{(\epsilon)} - K)^+],$$
$$C^{BS}(T, K, \sigma^{IV}(T, K)) = C^{Double.SV}(T, K).$$

We have the following approximation formulas for $C^{Double.SV}$ and $\sigma^{IV}(T, K)$.

Proposition 4.1

$$C^{Double.SV}(T, K) = C^{BS}(T, K, \bar{\sigma}) + \epsilon_1 C_{11} + \epsilon_2 C_{12} + O(\epsilon^2), \tag{72}$$

$$\sigma^{IV}(T, K) = \bar{\sigma} + \epsilon_1 \frac{1}{vega}C_{11} + \epsilon_2 \frac{1}{vega}C_{12} + O(\epsilon^2), \tag{73}$$

where $\epsilon_1 = \epsilon$, $\epsilon_2 = \epsilon\sigma_2$,

$$C_{1i} = e^{-rT}\int_{\mathbf{R}} \left(se^{x+rT-\frac{1}{2}\Sigma} - K\right)^+ \eta_i \left(\frac{1}{\Sigma^3}x^3 - \frac{1}{\Sigma^2}x^2 - \frac{1}{\Sigma^2}3x + \frac{1}{\Sigma}\right)\frac{1}{\sqrt{2\pi\Sigma}}e^{-\frac{1}{2\Sigma}x^2}dx \ (i=1.2),$$

$$\eta_1 = \frac{1}{2}\rho_1 \int_0^t e^{-\kappa s}\int_0^s e^{\kappa u}v_u^{(0)}duds,$$

$$\eta_2 = \frac{1}{2}\rho_2 \int_0^t \kappa e^{-\kappa s}\int_0^s e^{\bar{\kappa}u}\sqrt{\bar{v}_u^{(0)}}\int_u^s e^{(\kappa-\bar{\kappa})v}dv\sqrt{v_u^{(0)}}duds,$$

$$\bar{\sigma} = \theta + (v_0^{(0)} - \theta)\frac{(1-e^{-\kappa T})}{\kappa T} + (\bar{v}_0 - \theta)\frac{\kappa}{\kappa-\bar{\kappa}}\left(\frac{(1-e^{-\bar{\kappa}T})}{\bar{\kappa}T} - \frac{(1-e^{-\kappa T})}{\kappa T}\right),$$

$$vega = s\sqrt{T}n(d_1),$$

with $\kappa \neq \bar{\kappa}$.

(Proof)

By the asymptotic expansion with push-down of the Malliavin weights, we have

$$C^{Double.SV}(T, K) = e^{-rT}E[(S_T^{(\epsilon)} - K)^+]$$
$$= e^{-rT}E[(S_T^{(0)} - K)^+] + \epsilon e^{-rT}E\left[(S_T^{(0)} - K)^+ H\left(S_T^{(0,0)}, \frac{\partial}{\partial\epsilon}S_t^{(\epsilon)}|_{\epsilon=0}\right)\right] + O(\epsilon^2)$$
$$= C^{BS}(T, K, \bar{\sigma}) + \epsilon e^{-rT}\int_{\mathbf{R}}(se^{x+rT-\frac{1}{2}\Sigma} - K)^+\vartheta(x)\frac{1}{\sqrt{2\Sigma}}e^{-\frac{1}{2\Sigma}}dx + O(\epsilon^2),$$

where

$$\vartheta(x) = E\left[H\left(S_T^{(0)}, \frac{\partial}{\partial\epsilon}S_t^{(\epsilon)}|_{\epsilon=0}\right) \mid \int_0^T \sqrt{v_s^{(0)}}dW_{1,s} = x\right]. \tag{74}$$

$\frac{\partial}{\partial\epsilon}S_t^{(\epsilon)}|_{\epsilon=0}$ is given by

$$\frac{\partial}{\partial\epsilon}S_t^{(\epsilon)}|_{\epsilon=0} = S_t^{(0,0)}X_t,$$

where

$$X_t = \left(\int_0^t \frac{1}{2\sqrt{v_s^{(0)}}}\frac{\partial}{\partial\epsilon}v_s^{(\epsilon)}|_{\epsilon=0}dW_{1,s} - \frac{1}{2}\int_0^t \frac{\partial}{\partial\epsilon}v_s^{(\epsilon)}|_{\epsilon=0}ds\right).$$

Then, ϑ, the push-down of Malliavin weight is computed as follows;

$$
\begin{aligned}
\vartheta(x) &= E\left[(D^*(Z) - D_Z) \circ \frac{\partial}{\partial \epsilon} S_t^{(\epsilon)}|_{\epsilon=0} \Big| \int_0^T \sqrt{v_s^{(0)}} dW_{1,s} = x\right] \\
&= \left(\frac{x}{\Sigma} - \frac{\partial}{\partial x}\right) \circ E\left[X_T \Big| \int_0^T \sqrt{v_s^{(0)}} dW_{1,s} = x\right] \\
&= (\zeta_1 + \zeta_2)\left(\frac{1}{\Sigma^3} x^3 - \frac{1}{\Sigma^2} x^2 - \frac{1}{\Sigma^2} 3x + \frac{1}{\Sigma}\right),
\end{aligned}
\tag{75}
$$

where

$$
\begin{aligned}
Z &= \frac{D_1 S_T^{(0)}}{\|D_1 S_T^{(0)}\|_H^2}, \\
\zeta_1 &= \frac{1}{2}\rho_1 \int_0^t e^{-\kappa s} \int_0^s e^{\kappa u} v_u^{(0)} du ds, \\
\zeta_2 &= \frac{1}{2}\rho_2 \sigma_2 \int_0^t \kappa e^{-\kappa s} \int_0^s e^{\bar{\kappa} u} \sqrt{\bar{v}_u^{(0)}} \int_u^s e^{(\kappa-\bar{\kappa})v} dv \sqrt{v_u^{(0)}} du ds.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&C^{Double.SV}(T, K) \\
&= C^{BS}(T, K, \bar{\sigma}) + \epsilon e^{-rT} \int_{\mathbf{R}} \left(se^{x+rT-\frac{1}{2}\Sigma} - K\right)^+ (\zeta_1 + \zeta_2)\left(\frac{1}{\Sigma^3} x^3 - \frac{1}{\Sigma^2} x^2 - \frac{1}{\Sigma^2} 3x + \frac{1}{\Sigma}\right) \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{1}{2\Sigma} x^2} dx \\
&\quad + O(\epsilon^2).
\end{aligned}
\tag{76}
$$

By the similar argument as in **3.2**, the implied volatility is expanded as follows:

$$
\begin{aligned}
&\sigma^{IV}(T, K) \\
&= \bar{\sigma} + \epsilon \frac{1}{vega} e^{-rT} \int_{\mathbf{R}} \left(se^{x+rT-\frac{1}{2}\Sigma} - K\right)^+ (\zeta_1 + \zeta_2)\left(\frac{1}{\Sigma^3} x^3 - \frac{1}{\Sigma^2} x^2 - \frac{1}{\Sigma^2} 3x + \frac{1}{\Sigma}\right) \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{1}{2\Sigma} x^2} dx \\
&\quad + O(\epsilon^2).
\end{aligned}
\tag{77}
$$

$\square$

Next, we show the numerical results on the accuracy of our approximation formula for the call option prices under the double Heston model. Also, in the tables below we define **Malliavin AE** and **Malliavin IV** as follows:

$$
\begin{aligned}
\textbf{Malliavin AE} &:= C^{BS} + \epsilon_1 C_{11} + \epsilon_2 C_{12}, \\
\textbf{Malliavin IV} &:= \bar{\sigma} + \epsilon_1 \frac{C_{11}}{vega} + \epsilon_2 \frac{C_{12}}{vega}
\end{aligned}
$$

$T = 1$, $s = 100$, $\sigma_0 = v_0^{1/2} = 0.4$, $\lambda_1 = 0.1$, $\epsilon_1 = 0.1$, $\rho_1 = -0.5$, $\lambda_2 = 1.0$, $\theta = 0.16$, $\epsilon_2 = 0.1$, $\rho_2 = -0.25$

Strike price	Benchmark	Malliavin AE	Relative error
70	33.6198	33.5304	0.27%
80	26.5663	26.4797	0.33%
90	20.6084	20.5439	0.31%
100	15.7241	15.6937	0.19%
110	11.8225	11.8301	-0.06%
120	8.7741	8.8198	-0.52%

Benchmark : Monte Carlo simulation (1,000,000 trials, 500 time steps).

$T = 2.5$, $s = 100$, $\sigma_0 = v_0^{1/2} = 0.4$, $\lambda_1 = 0.1$, $\epsilon_1 = 0.1$, $\rho_1 = -0.5$, $\lambda_2 = 1.0$, $\theta = 0.16$, $\epsilon_2 = 0.1$, $\rho_2 = -0.25$

Strike price	Benchmark	Malliavin AE	Relative error
70	39.3315	39.4345	0.26%
80	33.5871	33.6975	0.33%
90	28.6053	28.7170	0.39%
100	24.3161	24.4228	0.44%
110	20.6430	20.7397	0.47%
120	17.5111	17.5926	0.47%

Benchmark : Monte Carlo simulation (1,000,000 trials, 500 time steps).

4.3 Parameter Identification

It is useful to see if the model parameter can be identified by our implied volatility expansion. For this purpose, we first compute the implied volatility for a stochastic volatility model with some fixed parameters by a numerical method such as Monte Carlo simulation. Next, we estimate the model parameters using our implied volatility expansion. As an example, we use the Heston model;

$$\begin{aligned} dX_t &= -\frac{v_t}{2}dt + \sqrt{v_t}dW_{1,t}, \\ dv_t &= \kappa(\theta - v_t)dt + \epsilon\sqrt{v_t}(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}), \end{aligned} \tag{78}$$

where $\rho \in [-1,1]$. Here, κ and θ are known to be the speed and the level of the mean-reversion parameters, respectively. Also, the parameter $\nu = \epsilon\rho$ is regarded as a skewness parameter.

We compute the implied volatility with $S_0 = e^{X_0} = 100$, $v_0 = 0.04$, $\kappa = 1.5$, $\theta = 0.09$ and $\nu = -0.05$. Then, we obtain data set $\{\sigma^{MC}(T_i, K_{ij})\}_{i,j}$ of the implied volatilities (**Exact IV**). Using our implied volatility expansion and the data set $\{\sigma^{MC}(T_i, K_{ij})\}_{i,j}$, we solve the following minimization problem to estimate model parameters as if they were unknown:

$$\min_{v_0,\kappa,\theta,\nu} \sum_{i=1}^{n}\sum_{j=1}^{m} |\sigma^{MC}(T_i, K_{ij}) - \textbf{Malliavin IV}(T_i, K_{ij}, v_0, \kappa, \theta, \nu)|^2, \tag{79}$$

where

$$\textbf{Malliavin IV}(T, K, v_0, \kappa, \theta, \nu) = \bar{\sigma} + \nu\frac{C_1}{vega}, \tag{80}$$

with

$$\bar{\sigma} = \theta + (v_0^{(0)} - \theta)\frac{(1-e^{-\kappa T})}{\kappa T}, \tag{81}$$

$$C_1 = e^{-rT}\int_{\mathbf{R}} \left(se^{x-\frac{1}{2}\Sigma} - K\right)^+ \eta_1 \left(\frac{1}{\Sigma^3}x^3 - \frac{1}{\Sigma^2}x^2 - \frac{1}{\Sigma^2}3x + \frac{1}{\Sigma}\right)\frac{1}{\sqrt{2\pi\Sigma}}e^{-\frac{1}{2\Sigma}x^2}dx, \tag{82}$$

$$\eta_1 = \frac{1}{2}\int_0^t e^{-\kappa s}\int_0^s e^{\kappa u}v_u^{(0)}du\,ds. \tag{83}$$

As a result, we obtain the optimal parameters $v_0^* = 0.0404$, $\kappa^* = 1.5022$, $\theta^* = 0.0863$ and $\nu^* = -0.0336$, which are close to the true ones given above. Hence, the model parameters seems well identified by our implied volatility expansion for this case.

The fitting results are shown in the following tables, where **Malliavin IV** are computed by these optimal parameters $(v_0^*, \kappa^*, \theta^*, \nu^*) = (0.0404, 1.5022, 0.0863, -0.0336)$.

Monte Carlo parameter versus optimal parameter

Parameters	Exact IV	Malliavin IV
v_0	0.0400	0.0404
κ	1.5000	1.5022
θ	0.0900	0.0863
ν	-0.0500	-0.0336

$T = 0.4$:

Strike price	Exact IV	Malliavin IV	Relative error
70	24.31	24.23	-0.32%
80	23.68	23.66	-0.05%
90	23.17	23.17	-0.01%
100	22.67	22.72	0.23%
110	22.30	22.32	0.09%
120	21.92	21.96	0.17%

$T = 0.5$:

Strike price	Exact IV	Malliavin IV	Relative error
70	24.71	24.74	0.13%
80	24.13	24.17	0.14%
90	23.64	23.66	0.06%
100	23.18	23.20	0.10%
110	22.83	22.79	-0.20%
120	22.48	22.41	-0.33%

4.4 Calibration to Market Data

In this subsection, we show calibration examples using an implied volatility data of Nikkei-225 option as of January 31, 2011, where the strikes are the closest to ATM with maturities 0.20, 0.28, 0.37, 0.62, 0.87, 1.37 and 1.87 years. Moreover, the data for ten different strikes other than ATM with maturity 0.87 year are also used since this kind of data is most available for the maturity. We compare the fitting result of our expansion in the single Heston model with that in the double Heston model. The single Heston model is expressed as

$$
\begin{aligned}
dX_t &= -\frac{v_t}{2}dt + \sqrt{v_t}dW_{1,t}, \\
dv_t &= \kappa(\theta - v_t)dt + \epsilon_1\sqrt{v_t}(\rho_1 dW_{1,t} + \sqrt{1-\rho_1^2}dW_{2,t}).
\end{aligned}
\tag{84}
$$

The double Heston model is expressed as

$$
\begin{aligned}
dX_t &= -\frac{v_t}{2}dt + \sqrt{v_t}dW_{1,t}, \\
dv_t &= \kappa_1(\bar{v}_t - v_t)dt + \epsilon_1\sqrt{v_t}(\rho_1 dW_{1,t} + \sqrt{1-\rho_1^2}dW_{2,t}), \\
d\bar{v}_t &= \kappa_2(\theta - \bar{v}_t)dt + \epsilon_2\sqrt{\bar{v}_t}(\rho_2 dW_{1,t} + \sqrt{1-\rho_2^2}dW_{3,t}),
\end{aligned}
\tag{85}
$$

with $W = (W_1, W_2, W_3)$ is a 3-dimensional Brownian motion. We define the skewness parameters ν_1 and ν_2 as $\nu_1 = \epsilon_1\rho_1$ and $\nu_2 = \epsilon_2\rho_2$. We also define **Malliavin IV 1** and **Malliavin IV 2** as

$$
\textbf{Malliavin IV 1} \ (T, K, v_0, \kappa_1, \theta, \nu_1) \quad := \quad \bar{\sigma}_1 + \nu_1\frac{C_{11}}{vega},
$$

and

$$
\textbf{Malliavin IV 2} \ (T, K, v_0, \bar{v}_0, \kappa_1, \kappa_2, \theta, \nu_1, \nu_2) \quad := \quad \bar{\sigma}_2 + \nu_1\frac{C_{11}}{vega} + \nu_2\frac{C_{12}}{vega}.
$$

where C_{1i} $(i = 1, 2)$ are previously given, and

$$
\begin{aligned}
\bar{\sigma}_1 &= \theta + (v_0^{(0)} - \theta)\frac{(1 - e^{-\kappa T})}{\kappa T}, \\
\bar{\sigma}_2 &= \theta + (v_0^{(0)} - \theta)\frac{(1 - e^{-\kappa T})}{\kappa T} + (\bar{v}_0 - \theta)\frac{\kappa}{\kappa - \bar{\kappa}}\left(\frac{(1 - e^{-\bar{\kappa}T})}{\bar{\kappa}T} - \frac{(1 - e^{-\kappa T})}{\kappa T}\right).
\end{aligned}
$$

To reduce the degree of freedom in **Malliavin IV 2**, we give the condition $v_0 = \bar{v}_0 = \theta$. For the Nikkei-225 data $\{\sigma^{Data}(T_i, K_{ij})\}_{i,j}$, we solve the following minimization problems using our implied volatility expansions.

Optimization for Malliavin IV 1 (Single Heston model)

$$
\min_{v_0, \kappa_1, \theta, \nu_1} \sum_{i=1}^{n}\sum_{j=1}^{m}|\sigma^{Data}(T_i, K_{ij}) - \textbf{Malliavin IV 1} \ (T_i, K_{ij}, v_0, \kappa_1, \theta, \nu_1)|^2.
\tag{86}
$$

Optimization for Malliavin IV 2 (Double Heston model)

$$
\min_{v_0, \kappa_1, \kappa_2, \theta, \nu_1, \nu_2} \sum_{i=1}^{n}\sum_{j=1}^{m}|\sigma^{Data}(T_i, K_{ij}) - \textbf{Malliavin IV 2} \ (T_i, K_{ij}, v_0, \kappa_1, \kappa_2, \theta, \nu_1, \nu_2)|^2.
\tag{87}
$$

Then, we obtain the optimal parameters for the single and double Heston models, as shown in the following two tables.

Parameters	Malliavin IV 1
v_0	0.024
κ_1	2.805
θ	0.0444
ν_1	-0.136

Parameters	Malliavin IV 2
v_0	0.038
κ_1	9.355
θ	0.038
ν_1	-0.195
κ_2	9.070
ν_2	0.580

The total differences (in L^2-sense) between the calculated volatilities and the actual data are given as 2.956% for **Malliavin IV 1** and 2.663% for **Malliavin IV 2**. Hence, the fitting result of the expansion of the double Heston is better than that of the single Heston. Also, the figures of the result for the double Heston model are shown in Figure 1 and 2.
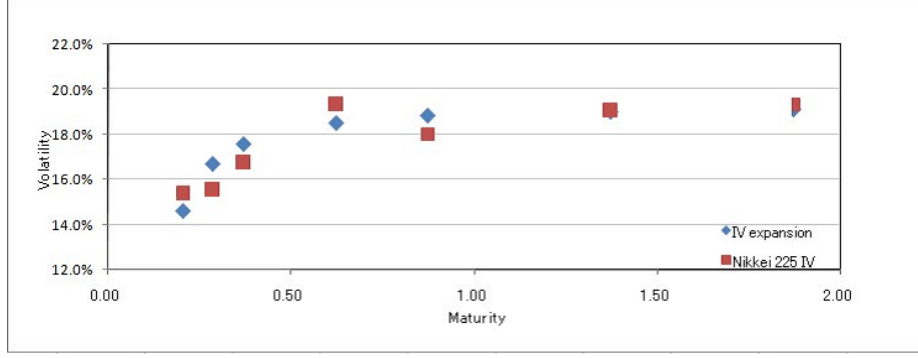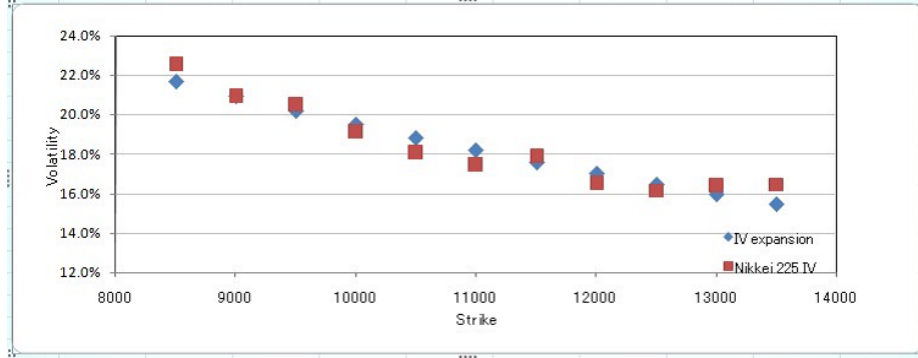
Figure 1: Term Structure

Figure 2: Skew

5 Applications

This section provides approximation formulas for option prices under the shifted log-normal and jump-diffusion models with stochastic volatilities; an expansion of the implied volatility is also given for the jump-diffusion model. Also, for simplicity we set the risk-free interest rate $r = 0$ in this section.

5.1 Shifted Log-normal Model

This subsection derives an approximation formula of the option price in the shifted log-normal model with stochastic volatility:

$$
\begin{aligned}
dS_t^{(\epsilon)} &= V(\sigma_t^{(\epsilon)})(S_t^{(\epsilon)} - \beta)dW_{1,t}; \ S_0^{(\epsilon)} = s > 0 \\
d\sigma_t^{(\epsilon)} &= A_0(\sigma_t^{(\epsilon)})dt + \epsilon A_1(\sigma_t^{(\epsilon)})(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}),
\end{aligned}
\tag{88}
$$

where β is a constant such that $s > \beta$. At $\epsilon = 0$, the option price is given by

$$
C^{BS}(\beta) = \int_{\mathbf{R}} ((s-\beta)e^{x-\frac{1}{2}\Sigma} - (K-\beta))^+ \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{x^2}{2\Sigma}} dx.
\tag{89}
$$

We define the following deterministic process:

$$
\eta_t := \exp\left\{ \int_0^t A_0'(\sigma_u^{(0)})du \right\}.
$$

Then, the following proposition is obtained.

Proposition 5.1 *An asymptotic expansion formula for the shifted log-normal model (88) is given as follows;*

$$
C^{SV}(\beta) = C^{BS}(\beta) + \epsilon C_1(\beta) + O(\epsilon^2),
\tag{90}
$$

where

$$
C_1(\beta) = \int_{\mathbf{R}} ((s-\beta)e^{x-\frac{1}{2}\Sigma} - (K-\beta))^+ \vartheta(x) \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{x^2}{2\Sigma}} dx,
\tag{91}
$$

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with

$$\Sigma = \int_0^T V(\sigma_t^{(0)})^2 dt,$$

$$\vartheta(x) = \zeta\left(\frac{1}{\Sigma^3}x^3 - \frac{1}{\Sigma^2}x^2 - \frac{1}{\Sigma^2}3x + \frac{1}{\Sigma}\right),$$

$$\zeta = \rho\int_0^T \partial V(\sigma_t^{(0)})\eta_t V(\sigma_t^{(0)})\int_0^t \eta_s^{-1}A_1(\sigma_s^{(0)})V(\sigma_s^{(0)})dsdt. \qquad (92)$$

(*Proof*)
We first note that

$$dS_t^{(\epsilon)} = dF_t^{(\epsilon)} = V(\sigma_t^{(\epsilon)})F_t^{(\epsilon)}dW_{1,t}; \ F_0^{(\epsilon)} = s - \beta > 0, \qquad (93)$$

where $F_t^{(\epsilon)} = S_t^{(\epsilon)} - \beta$. Note also that $(S_T^{(\epsilon)} - K)^+ = (F_T^{(\epsilon)} - (K - \beta))^+$. Hence, in stead of the original problem we can consider the option pricing problem with the underlying asset price process $F^{(\epsilon)}$ and strike $K - \beta$. Then, the same argument as in Theorem 3.1 and Remark 3.1 can be applied to computation of the push-down of the Malliavin weight; $\vartheta(x) := E\left[\pi_1 | \int_0^T V(\sigma_t^{(0)})dW_{1,t} = x\right]$, where

$$\pi_1 = H_1(F_T^{(0)}, \Psi^{\beta_1}(T))$$

$$\Psi^{\beta_1}(T) = \frac{\partial}{\partial\epsilon}F_T^{(\epsilon)}|_{\epsilon=0}. \qquad (94)$$

Thus, the result is obtained. □

5.2 Jump-Diffusion with Stochastic Volatility Model

This subsection applies the Malliavin calculus to a jump-diffusion stochastic volatility (SVJ) model. Let $(\Omega, \mathcal{F}, P)$ be a probability space on which we define a Brownian motion $(W_t)_t$, a Poisson process $(N_t)_t$ with intensity λ and i.i.d random variables $(\Delta_j)_{j\in\mathbf{N}}$ such that $\Delta_j \sim N(0,1)$. We will assume that the σ-algebras generated by $(W_t)_t$, $(N_t)_t$, $(\Delta_j)_j$ are independent.

First, we introduce the perturbed stochastic differential equation : For $\epsilon \in [0,1]$,

$$dS_t^{(\epsilon)} = V(\sigma_t^{(\epsilon)})S_{t-}^{(\epsilon)}dW_{1,t} + S_{t-}^{(\epsilon)}(dJ_t - \lambda m dt),$$

$$S_0^{(\epsilon)} = s,$$

$$d\sigma_t^{(\epsilon)} = A_0(\sigma_t^{(\epsilon)})dt + \epsilon A_1(\sigma_t^{(\epsilon)})(\rho dW_{1,t} + \sqrt{1-\rho^2}dW_{2,t}),$$

$$\sigma_0^{(\epsilon)} = \sigma, \qquad (95)$$

where J_t is defined by

$$J_t = \sum_{j=1}^{N(t)}(e^{Y_j} - 1), \qquad (96)$$

with $Y_j = a + b\Delta_j$ and $m = E[e^{Y_j} - 1]$.

The solution of (95) is given by

$$S_T = s\exp\left\{\int_0^T V(\sigma_t^{(\epsilon)})dW_{1,t} - \frac{1}{2}\int_0^T V(\sigma_t^{(\epsilon)})^2 dt\right\}\prod_{j=1}^{N(T)}e^{Y_j}$$

$$= s\exp\left\{\int_0^T V(\sigma_t^{(\epsilon)})dW_{1,t} - \frac{1}{2}\int_0^T V(\sigma_t^{(\epsilon)})^2 dt + \sum_{j=1}^{N(T)}Y_j\right\}. \qquad (97)$$

We define the following notations:

$$X_t^{(\epsilon)} := \log(S_t^{(\epsilon)}/s), \qquad (98)$$

$$\sigma^{BS} := \left(\frac{1}{T}\int_0^T V(\sigma_t^{(0)})^2 dt\right)^{1/2},$$

$$\Sigma_n := \left(\int_0^T V(\sigma_t^{(0)})^2 dt + b^2 n\right),$$

$$\sigma_n := \left(\frac{1}{T}\int_0^T V(\sigma_t^{(0)})^2 dt + \frac{b^2 n}{T}\right)^{1/2} = \left((\sigma^{BS})^2 + (b^2 n/T)\right)^{1/2},$$

$$\hat{r} := \frac{-\lambda m T + (a + b^2/2)n}{T},$$

$$d_n := \frac{1}{\sigma_n \sqrt{T}} \left[\log\left(\frac{s}{K}\right) + \left(\hat{r} + \frac{\sigma_n^2}{2}\right) T \right],$$

$$C_n^{BS}(T, K, \sigma_n) := sN(d_n) - e^{-\hat{r}T} K N(d_n - \sigma_n \sqrt{T}),$$

$$C^M(T, K, \sigma^{BS}) := \sum_{n=0}^{\infty} \frac{(\lambda(1+m)T)^n e^{-\lambda(1+m)T}}{n!} C_n^{BS}(T, K, \sigma_n)$$

$$C^{SVJ}(T, K) := E[(S_T^{(\epsilon)} - K)^+].$$

Moreover, let us call $\sigma^{SVJ.IV}(T, K)$ an implied volatility, which satisfies

$$C^{SVJ}(T, K) = C^M(T, K, \sigma^{SVJ.IV}(T, K)). \tag{99}$$

Then, the following proposition is obtained.

Proposition 5.2 *Under the jump-diffusion stochastic volatility (SVJ) model (95), the call option price $C^{SVJ}(T, K)$ and its implied volatility $\sigma^{SVJ.IV}(T, K)$ are expanded up to the ϵ-order as follows:*

$$C^{SVJ}(T, K) = C^M(T, K, \sigma^{BS}) + \epsilon C_1^M + O(\epsilon^2), \tag{100}$$

$$\sigma^{SVJ.IV}(T, K) = \sigma^{BS} + \epsilon \frac{1}{vega^M} C_1^M + O(\epsilon^2), \tag{101}$$

where

$$C_1^M = \sum_{n=0}^{\infty} \frac{(\lambda T)^n e^{-\lambda T}}{n!} \int_{\mathbf{R}} \left(s e^{x - \lambda m T - \frac{1}{2} \int_0^T V(\sigma_t^{(0)})^2 dt + na} - K \right)^+ \vartheta_n(x) \frac{1}{\sqrt{2\pi\Sigma_n}} e^{-\frac{1}{2\Sigma_n} x^2} dx,$$

$$vega^M = \sum_{n=0}^{\infty} \frac{(\lambda(m+1)T)^n e^{-\lambda(m+1)T}}{n!} \frac{\sigma^{BS}}{\sigma_n} s\sqrt{T} n(d_n), \tag{102}$$

with

$$\vartheta_n(x) = \zeta \left(\frac{1}{\Sigma_n^3} x^3 - \frac{1}{\Sigma_n^2} x^2 - \frac{1}{\Sigma_n^2} 3x + \frac{1}{\Sigma_n} \right),$$

$$\zeta = \rho \int_0^T \partial V(\sigma_t^{(0)}) V(\sigma_t^{(0)}) \eta_t \int_0^t \eta_s^{-1} A_1(\sigma_s^{(0)}) V(\sigma_s^{(0)}) ds dt,$$

$$\eta_t = \exp\left\{ \int_0^t A_0'(\sigma_u^{(0)}) du \right\}. \tag{103}$$

(Proof)
See Appendix C.

6 Conclusion

This paper developed an asymptotic expansion method for generalized Wiener functionals based on the integration-by-parts formula in Malliavin calculus and the push-down of Malliavin weights. As an application, we derived asymptotic expansion formulas for option prices and implied volatilities as well as the density of the underlying asset price in stochastic volatility environment. we also presented numerical examples for expansions up to the first order (ϵ-order). Moreover, we applied the general formula to expansions of option prices for the shifted log-normal model with stochastic volatility. Further, combining some existing results of Malliavin calculus in jump-type models with our method, this paper we presented an approximation formula for the jump-diffusion model in stochastic volatility environment. More detailed numerical experiments and higher-order expansions are our next research topics.

A Computation of Push-down of Malliavin Weights

This section derives the push-down of Malliavin weights for the first and second order approximation terms of the asymptotic expansion of option prices.

We consider the following stochastic volatility model;

$$dS_t^{(\epsilon)} = V(\sigma_t^{(\epsilon)}) S_t^{(\epsilon)} dW_{1,t},$$

$$d\sigma_t^{(\epsilon)} = A_0(\sigma_t^{(\epsilon)}) dt + \epsilon A_1(\sigma_t^{(\epsilon)})(\rho dW_{1,t} + \sqrt{1 - \rho^2} dW_{2,t}),$$

$$S_0^{(\epsilon)} = S_0^{(0)} = s,$$

where $V, A_0, A_1 \in C_b^\infty(\mathbf{R})$, $r \geq 0$, $\rho \in [-1, 1]$ and $\epsilon \in [0, 1]$.

Also we use the following notations;

$$
\begin{aligned}
B_t &= \rho W_{1,t} + \sqrt{1 - \rho^2} W_{2,t}, &&(104)\\
\eta_t &= \exp\left\{\int_0^t A'(\sigma_u^{(0)})du\right\}, \\
\sigma_t^{(\epsilon)} &= \sigma_t^{(0)} + \epsilon\sigma_t^{(1)} + \epsilon^2\sigma_t^{(2)} + O(\epsilon^3), \\
\sigma_t^{(1)} &= \frac{\partial}{\partial\epsilon}\sigma_t^{(\epsilon)}|_{\epsilon=0} = \int_0^t \eta_t\eta_s^{-1}A_1(\sigma_{0s})dB_s, \\
\sigma_t^{(2)} &= \frac{1}{2}\frac{\partial^2}{\partial\epsilon^2}\sigma_t^{(\epsilon)}|_{\epsilon=0} = \int_0^t \partial A_1(\sigma_{0s})\int_0^s \eta_s\eta_u^{-1}A_1(\sigma_{0u})dB_udB_s, \\
v_t^{(0)} &= V(\sigma_t^{(0)}), \\
v_t^{(1)} &= V'(\sigma_{0t})\sigma_t^{(1)}, \\
v_t^{(2)} &= 2V'(\sigma_{0t})\sigma_t^{(2)} + V''(\sigma_{0t})(\sigma_t^{(1)})^2. \\
X_T^{(0)} &= \int_0^T V(\sigma_{0t})dW_{1,t} - \frac{1}{2}\int_0^T V(\sigma_{0t})^2 dt, \\
X_T^{(1)} &= \frac{\partial}{\partial\epsilon}X_T^{(\epsilon)}|_{\epsilon=0} \\
&= \int_0^T v_t^{(1)}dW_{1,t} - \int_0^T V(\sigma_{0t})v_t^{(1)}dt, \\
X_T^{(2)} &= \frac{\partial^2}{\partial\epsilon^2}X_T^{(\epsilon)}|_{\epsilon=0} \\
&= \int_0^T v_t^{(2)}dW_{1,t} - \int_0^T (v_t^{(1)})^2 dt - \int_0^T V(\sigma_{0t})v_t^{(2)}dt.
\end{aligned}
$$

The closed-form approximation of the density and the call option price at $t = 0$ with strike K and maturity T are given by

$$
\begin{aligned}
p^{SV}(y) &= p^{BS}(y) + \epsilon E[\pi_1|S_T^{(0)} = x]p^{BS}(y) + \epsilon^2 E[\pi_2|S_T^{(0)} = x]p^{BS}(y) + O(\epsilon^2), \\
C_0^{SV}(T, K) &= C_0^{BS}(T, K, \bar\sigma) \\
&\quad + \epsilon\int_{\mathbf{R}} (y - K)^+ E[\pi_1|S_T^{(0)} = x]p^{BS}(y)dy \\
&\quad + \epsilon^2\int_{\mathbf{R}} (y - K)^+ E[\pi_2|S_T^{(0)} = x]p^{BS}(y)dy + O(\epsilon^3) \\
&= C_0^{BS}(T, K, \bar\sigma) \\
&\quad + \epsilon\int_{\mathbf{R}} \left(se^{x - \frac{1}{2}\int_0^T V(\sigma_{0s})^2 ds} - K\right)^+ E[\pi_1|\int_0^T V(\sigma_{0t})dW_{1,t} = x]n(\Sigma : x)dx \\
&\quad + \epsilon^2\int_{\mathbf{R}} \left(se^{x - \frac{1}{2}\int_0^T V(\sigma_{0s})^2 ds} - K\right)^+ E[\pi_2|\int_0^T V(\sigma_{0t})dW_{1,t} = x]n(\Sigma : x)dx + O(\epsilon^3).
\end{aligned}
$$

where π_1 and π_2 is the Malliavin weights:

$$
\begin{aligned}
\pi_1 &= H_1\left(S_T^{(0)}, \frac{\partial}{\partial\epsilon}S_T^{(\epsilon)}|_{\epsilon=0}\right), &&(105)\\
\pi_2 &= \frac{1}{2}H_1\left(S_T^{(0)}, \frac{\partial^2}{\partial\epsilon^2}S_T^{(\epsilon)}|_{\epsilon=0}\right) + \frac{1}{2}H_2\left(S_T^{(0)}, \left(\frac{\partial}{\partial\epsilon}S_T^{(\epsilon)}|_{\epsilon=0}\right)^2\right). &&(106)
\end{aligned}
$$

Note first that for $t \in [0, T]$, $D_{t,2}X_T^{(0)} = D_{t,2}S_T^{(0)} = 0$, $D_{t,1}X_T^{(0)} = V(\sigma_t^{(0)})$, $D_{t,1}S_T^{(0)} = S_T^{(0)}D_{t,1}X_T^{(0)} = S_T^{(0)}V(\sigma_t^{(0)})$ and $\|DS_T^{(0)}\|_H^2 = \|D_1 S_T^{(0)}\|_H^2 = (S_T^{(0)})^2\Sigma$ where $\Sigma = \int_0^T V(\sigma_t^{(0)})^2 dt$. Let Z be the function on $L^2[0, T]$ defined by $Z(s) = \frac{D_{s,1}X_T^{(0)}}{\|D_1 X_T^{(0)}\|_H^2} = \frac{V(\sigma_s^{(0)})}{\Sigma}$. Recall also that D_1 and D_1^* are the Malliavin derivative and its adjoint operator (Skorohod integral) for the Brownian motion W_1. Then, the second-order Malliavin weight is computed as follows;

$$
\begin{aligned}
H_1\left(S_T^{(0)}, \frac{\partial^2}{\partial\epsilon^2}S_T^{(\epsilon)}|_{\epsilon=0}\right) &= D_1^*\left(\frac{D_1 S_T^{(0)}S_T^{(0)}\{\frac{\partial^2}{\partial\epsilon^2}X_T^{(\epsilon)}|_{\epsilon=0} + (\frac{\partial}{\partial\epsilon}X_T^{(\epsilon)}|_{\epsilon=0})^2\}}{\|D_1 S_T^{(0)}\|_H^2}\right) \\
&= D_1^*\left(\frac{D_1 S_T^{(0)}S_T^{(0)}\frac{\partial^2}{\partial\epsilon^2}X_T^{(\epsilon)}|_{\epsilon=0}}{\|D_1 S_T^{(0)}\|_H^2}\right) + D_1^*\left(\frac{D_1 S_T^{(0)}S_T^{(0)}(\frac{\partial}{\partial\epsilon}X_T^{(\epsilon)}|_{\epsilon=0})^2}{\|D_1 S_T^{(0)}\|_H^2}\right)
\end{aligned}
$$

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$$= H_1\left(X_T^{(0)}, X_T^{(2)}\right) + H_1\left(X_T^{(0)}, (X_T^{(1)})^2\right)$$

$$= X_T^{(2)}\int_0^T Z(s)dW_{1,s} - \int_0^T D_{s,1}X_T^{(2)}Z(s)ds$$

$$+ (X_T^{(1)})^2\int_0^T Z(s)dW_{1,s} - \int_0^T D_{s,1}(X_T^{(1)})^2 Z(s)ds, \tag{107}$$

$$H_2\left(S_T^{(0)}, \left(\frac{\partial}{\partial\epsilon}S_T^{(\epsilon)}|_{\epsilon=0}\right)^2\right) = H_1\left(S_T^{(0)}, H_1\left(S_T^{(0)}, \left(\frac{\partial}{\partial\epsilon}S_T^{(\epsilon)}|_{\epsilon=0}\right)^2\right)\right)$$

$$= D_1^*\left(\frac{D_1 S_T^{(0)}}{\|D_1 S_T^{(0)}\|_H^2}D_1^*\left(\frac{D_1 S_T^{(0)}(S_T^{(0)}\frac{\partial}{\partial\epsilon}X_T^{(\epsilon)}|_{\epsilon=0})^2}{\|D_1 S_T^{(0)}\|_H^2}\right)\right)$$

$$= D_1^*\left(\frac{D_1 S_T^{(0)}}{\|D_1 S_T^{(0)}\|_H^2}D_1^*\left((S_T^{(0)}X_T^{(1)})^2 Z\right)\right)$$

$$= D_1^*\left(\frac{D_1 S_T^{(0)}}{\|D_1 S_T^{(0)}\|_H^2}\left\{S_T^{(0)}(X_T^{(1)})^2\int_0^T Z(s)dW_{1,s} - \int_0^T D_{s,1}(S_T^{(0)}(X_T^{(1)})^2)Z(s)ds\right\}\right)$$

$$= D_1^*\left(\frac{D_1 S_T^{(0)}}{\|D_1 S_T^{(0)}\|_H^2}\left\{S_T^{(0)}(X_T^{(1)})^2\int_0^T Z(s)dW_{1,s} - \int_0^T (D_{1,s}S_T^{(0)})(X_T^{(1)})^2 Z(s)ds - \int_0^T S_T^{(0)}(D_{s,1}(X_T^{(1)})^2)Z(s)ds\right\}\right)$$

$$= D_1^*\left(\frac{D_1 S_T^{(0)}}{\|D_1 S_T^{(0)}\|_H^2}\left\{S_T^{(0)}(X_T^{(1)})^2\int_0^T Z(s)dW_{1,s} - S_T^{(0)}(X_T^{(1)})^2\int_0^T V(\sigma_s^{(0)})Z(s)ds - S_T^{(0)}\int_0^T (D_{s,1}(X_T^{(1)})^2)Z(s)ds\right\}\right)$$

$$= D_1^*\left(\frac{D_1 X_T^{(0)}}{\|D_1 X_T^{(0)}\|_H^2}\left\{(X_T^{(1)})^2\int_0^T Z(s)dW_{1,s} - \int_0^T (D_{s,1}(X_T^{(1)})^2)Z(s)ds - (X_T^{(1)})^2\right\}\right)$$

$$= H_2\left(X_T^{(0)}, (X_T^{(1)})^2\right) - H_1\left(X_T^{(0)}, (X_T^{(1)})^2\right). \tag{108}$$

Then, we have

$$\pi_2 = \frac{1}{2}H_1\left(X_T^{(0)}, X_T^{(2)}\right) + \frac{1}{2}H_2\left(X_T^{(0)}, (X_T^{(1)})^2\right) \tag{109}$$

$$= \frac{1}{2}H_1\left(X_T^{(0)}, \int_0^T v_t^{(2)}dW_{1,t}\right) - \frac{1}{2}H_1\left(X_T^{(0)}, \int_0^T (v_t^{(1)})^2 dt\right) - \frac{1}{2}H_1\left(X_T^{(0)}, \int_0^T V(\sigma_{0t})v_t^{(2)}dt\right)$$

$$+ \frac{1}{2}H_2\left(X_T^{(0)}, \left(\int_0^T v_t^{(1)}dW_{1,t}\right)^2\right) - H_2\left(X_T^{(0)}, \int_0^T v_t^{(1)}dW_{1,t}\int_0^T V(\sigma_{0t})v_t^{(1)}dt\right) + \frac{1}{2}H_2\left(X_T^{(0)}, \left(\int_0^T V(\sigma_{0t})v_t^{(1)}dt\right)^2\right). \tag{110}$$

In order to compute the expansion coefficient including the push-down of the Malliavin weights, we give the following formula (111) which is modified version of Malliavin (1997) and Malliavin-Thalmaier (2006).

Proposition A.1 *For $\Psi \in \mathbf{D}_\infty(\mathcal{W}:\mathbf{R})$ and $Z = \frac{h}{\|h\|_H^2}$ with $h \in H$,*

$$E[D^*(Z)\Psi - D_Z\Psi|D^*(h) = x] = \frac{x}{\|h\|_H^2}E[\Psi|D^*(h) = x] - \frac{\partial}{\partial x}E[\Psi|D^*(h) = x]. \tag{111}$$

(Proof)
We follow Proposition (8.2) in p.82 of Malliavin (1997).

Let g be a non-degenerate Wiener smooth map, $g \in \mathbf{D}_\infty(\mathcal{W}:\mathbf{R}^n)$ and $z = (z^1, z^2, \cdots, z^n)$ be a vector field on $\mathbf{R}^n$. Suppose also that $Z = (Z^1, Z^2, \cdots, Z^d)$ be a lift of z to the Wiener space (a covering vector field of z);

$$Z^k(t) = \sum_{1 \le s,l \le n}\gamma^{s,l}(D_{t,k}g^s)z^l,$$

where $\gamma^{s,l}$ is the (s,l)-element of the inverse matrix of the Malliavin covariance matrix of g.

Then, Proposition (8.2) in p.82 of Malliavin (1997) says that for $\Psi \in \mathbf{D}_\infty(\mathcal{W}:\mathbf{R})$ and $k(x) = E^{g=x}[\Psi]$ where $E^{g=x}$ denotes the conditional expectation under $g = x$,

$$E^{g=x}[D^*(Z)\Psi - D_Z\Psi] = k(x)E^{g=x}[D^*(Z)] - \partial_z k(x),$$

where $\partial_z k(x) := \langle z, (dk(x)/dx)\rangle$. In our case, set $g = D^*(h)$ and then, $g \in \mathbf{D}_\infty(\mathcal{W}:\mathbf{R})$ (i.e. $n = 1$). Also, let $z \equiv 1$ and hence,

$$Z = \frac{Dg}{\|Dg\|_H^2} = \frac{h}{\|h\|_H^2}.$$

Finally, note that $E^{g=x}[D^*(Z)] = \frac{x}{\|h\|_H^2}$ in our case, and thus the result (111) is obtained. □

Using (111) above, we give the first order approximation formula of the call option price under the stochastic volatility.

Corollary A.1 *The first order approximation of the call price is given by*

$$C^{SV}(T,K) = C^{BS}(T,K,\bar{\sigma}) + \epsilon \int_{\mathbf{R}} \left(s e^{x - \frac{1}{2}\Sigma} - K \right)^+ \vartheta(x) \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{1}{2\Sigma}x^2} dx,$$

(112)

where

$$\vartheta(x) = \zeta \left(\frac{1}{\Sigma^3} x^3 - \frac{1}{\Sigma^2} x^2 - \frac{1}{\Sigma^2} 3x + \frac{1}{\Sigma} \right)$$

(113)

$$\Sigma = \int_0^T V(\sigma_s^{(0)})^2 ds,$$

$$\zeta = \rho \int_0^T V'(\sigma_t^{(0)}) \eta_t V(\sigma_t^{(0)}) \int_0^t \eta_s^{-1} A_1(\sigma_s^{(0)}) V(\sigma_s^{(0)}) ds dt,$$

(114)

with

$$\eta_t = \exp\left\{ \int_0^t A_0'(\sigma_u^{(0)}) du \right\},$$

(115)

where $A_0'(\sigma_u^{(0)}) = \frac{dA_0(x)}{dx}\big|_{x=\sigma_u^{(0)}}$.

(Proof)

By (111) and the formula [1];

$$E\left[\int_0^T \int_0^t h_{2u} dW_{1,u} h_{3t} dW_{1,t} \Big| \int_0^T h_{1v} dW_{1,v} = x \right] = \left(\int_0^T \int_0^t h_{2u} h_{1u} du h_{3t} h_{1t} dt \right) \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right)$$

(116)

for $h_i \in L^2[0,T]$, $i=1,2,3$, The push-down of the Malliavin weight $H_1(X_T^{(0)}, \int_0^T v_t^{(1)} dW_{1,t})$ is obtained as follows.

$$E\left[H_1\left(X_T^{(0)}, \int_0^T v_t^{(1)} dW_{1,t} \right) \Big| \int_0^T V(\sigma_{0s}) dW_{1,s} = x \right]$$

$$= E\left[(D^*(Z) - D_Z) \circ \int_0^T v_t^{(1)} dW_{1,t} \Big| \int_0^T V(\sigma_{0s}) dW_{1,s} = x \right]$$

$$= \left(\frac{x}{\Sigma} - \frac{\partial}{\partial x} \right) \circ E\left[\int_0^T V'(\sigma_{0t}) \int_0^t \eta_t \eta_s^{-1} A_1(\sigma_{0s}) dB_s dW_{1,t} \Big| \int_0^T V(\sigma_{0s}) dW_{1,s} = x \right]$$

$$= \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2} \right) \zeta,$$

(117)

where $\zeta = \rho \int_0^T V'(\sigma_{0t}) \eta_t V(\sigma_{0t}) \int_0^t \eta_s^{-1} A_1(\sigma_{0s}) V(\sigma_{0s}) ds dt$.

By (111) and the formula;

$$E\left[\int_0^T h_{2t} dW_{1,t} \Big| \int_0^T h_{1v} dW_{1,v} = x \right] = \left(\int_0^T h_{2t} h_{1t} dt \right) \frac{x}{\Sigma},$$

(118)

the push-down of the Malliavin weight $H_1\left(X_T^{(0)}, \int_0^T V(\sigma_{0t}) v_t^{(1)} dt \right)$ is obtained as

$$E\left[H_1(X_T^{(0)}, \int_0^T V(\sigma_{0t}) v_t^{(1)} dt) \Big| \int_0^T V(\sigma_{0s}) dW_{1,s} = x \right]$$

$$= E\left[(D^*(Z) - D_Z) \circ \int_0^T V(\sigma_{0t}) V'(\sigma_{0t}) \int_0^t \eta_t \eta_s^{-1} A_1(\sigma_{0s}) dB_s dt \Big| \int_0^T V(\sigma_{0s}) dW_{1,s} = x \right]$$

$$= E\left[(D^*(Z) - D_Z) \circ \int_0^T \eta_s^{-1} A_1(\sigma_{0s}) \int_t^T V(\sigma_{0t}) V'(\sigma_{0t}) \eta_t dt dB_s \Big| \int_0^T V(\sigma_{0s}) dW_{1,s} = x \right]$$

$$= \left(\frac{x}{\Sigma} - \frac{\partial}{\partial x} \right) \circ E\left[\int_0^T \eta_s^{-1} A_1(\sigma_{0s}) \int_t^T V(\sigma_{0t}) V'(\sigma_{0t}) \eta_t dt dB_s \Big| \int_0^T V(\sigma_{0s}) dW_{1,s} = x \right]$$

$$= \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) \zeta.$$

(119)

---

[1] For the derivation and more general results, see Section 3 in Takahashi-Takehara-Toda (2009).

Therefore, we obtain

$$E[\pi_1 | \int_0^T V(\sigma_{0t})dW_{1,t} = x] = \zeta \left(\frac{1}{\Sigma^3} x^3 - \frac{1}{\Sigma^2} x^2 - \frac{1}{\Sigma^2} 3x + \frac{1}{\Sigma} \right). \tag{120}$$

□

A.1 Second Order Approximation

This subsection lists up the results necessary for computing push-down of the Malliavin weights of ϵ^2-order terms, (110).

A.1.1 $H_1(X_T^{(0)}, \int_0^T v_t^{(2)} dW_{1,t})$

$$E[H_1(X_T^{(0)}, \int_0^T v_t^{(2)} dW_{1,t}) | \int_0^T V(\sigma_{0s})dW_{1,s} = x]$$
$$= \left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma^2} \right)(2b_{11} + b_{12}) + \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) b_{13}, \tag{121}$$

where

$$b_{11} = \rho^2 \int_0^T V'(\sigma_{0t})V(\sigma_{0t}) \int_0^t A_1'(\sigma_{0s})\eta_s V(\sigma_{0s}) \int_0^s \eta_u^{-1} A_1(\sigma_{0u})V(\sigma_{0u})dudsdt,$$

$$b_{12} = \int_0^T \left(\int_0^t V(\sigma_{0s})\eta_s^{-1}A_1(\sigma_{0s})ds \right)^2 V''(\sigma_{0t})\eta_t^2 V(\sigma_{0t})dt,$$

$$b_{13} = \int_0^T V''(\sigma_{0t})V(\sigma_{0t}) \left(\int_0^t \eta_t \eta_t^{-1}A_1(\sigma_{0s}) \right)^2 dsdt. \tag{122}$$

A.1.2 $H_1(X_T^{(0)}, \int_0^T (v_t^{(1)})^2 dt)$

$$E[H_1(X_T^{(0)}, \int_0^T (v_t^{(1)})^2 dt) | \int_0^T V(\sigma_{0s})dW_{1,s} = x] = \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2} \right) b_{21} + \frac{x}{\Sigma} b_{22}, \tag{123}$$

where

$$b_{21} = \rho^2 \int_0^T (V'(\sigma_{0t})\eta_t)^2 (\int_0^t \eta_s^{-1}A_1(\sigma_{0s})V(\sigma_{0s})ds)^2 dt,$$

$$b_{22} = \int_0^T (V'(\sigma_{0t})\eta_t)^2 \int_0^t (\eta_s^{-1}A_1(\sigma_{0s}))^2 dsdt. \tag{124}$$

A.1.3 $H_1(X_T^{(0)}, \int_0^T V(\sigma_{0t})v_t^{(2)} dt)$

$$E[H_1(X_T^{(0)}, \int_0^T V(\sigma_{0t})v_t^{(2)} dt) | \int_0^T V(\sigma_{0s})dW_{1,s} = x] = 2 \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2} \right) b_{31} + \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2} \right) b_{32} + \frac{x}{\Sigma} b_{33}, \tag{125}$$

where

$$b_{31} = \rho^2 \int_0^T V'(\sigma_{0t})V(\sigma_{0t}) \int_0^t A_1'(\sigma_{0s})\eta_s V(\sigma_{0s}) \int_0^s \eta_u^{-1} A_1(\sigma_{0u})V(\sigma_{0u})dudsdt,$$

$$b_{32} = \rho^2 \int_0^T (V''(\sigma_{0t})\eta_t^2 V(\sigma_{0t}))(\int_0^t \eta_s^{-1}A_1(\sigma_{0s})V(\sigma_{0s})ds)^2 dt,$$

$$b_{33} = \int_0^T (V''(\sigma_{0t})\eta_t^2 V(\sigma_{0t})) \int_0^t (\eta_s^{-1}A_1(\sigma_{0s}))^2 dsdt. \tag{126}$$

A.1.4 $H_2(X_T^{(0)}, (\int_0^T v_t^{(1)}dW_{1,t})^2)$

$$E[H_2(X_T^{(0)}, (\int_0^T v_t^{(1)}dW_{1,t})^2)| \int_0^T V(\sigma_{0s})dW_{1,s} = x].$$
$$= \left(\frac{x^6}{\Sigma^6} - \frac{15x^4}{\Sigma^5} + \frac{45x^2}{\Sigma^4} - \frac{15}{\Sigma^3} \right) b_{41} + \left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma} \right) (2b_{42} + 2b_{43} + b_{44}) + \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma^2} \right) b_{45},$$
(127)

where

$$b_{41} = \left(\rho \int_0^T V'(\sigma_{0t})\eta_t V(\sigma_{0t}) \int_0^t \eta_s^{-1} A_1(\sigma_{0s})V(\sigma_{0s})dsdt \right)^2,$$

$$b_{42} = \int_0^T V'(\sigma_{0t})\eta_t V(\sigma_{0t}) \int_0^t V'(\sigma_{0s})\eta_s V(\sigma_{0s}) \int_0^s (\eta_u^{-1} A_1(\sigma_{0u}))^2 dudsdt,$$

$$b_{43} = \rho^2 \int_0^T V'(\sigma_{0t})\eta_t V(\sigma_{0t}) \int_0^t V'(\sigma_{0s})A_1(\sigma_{0s}) \int_0^s V(\sigma_{0u})\eta_u^{-1} A_1(\sigma_{0u})dudsdt,$$

$$b_{44} = \rho^2 \int_0^T (V'(\sigma_{0t})\eta_t)^2 \left(\int_0^t \eta_s^{-1} A_1(\sigma_{0s})V(\sigma_{0s})ds \right)^2 dt,$$

$$b_{45} = \int_0^T (V'(\sigma_{0t})\eta_t)^2 \int_0^t (\eta_s^{-1} A_1(\sigma_{0s}))^2 dsdt.$$
(128)

A.1.5 $H_2(X_T^{(0)}, \int_0^T v_t^{(1)}dW_t \int_0^T V(\sigma_{0t})v_t^{(1)}dt)$

$$E[H_2(X_T^{(0)}, \int_0^T v_t^{(1)}dW_{1,t} \int_0^T V(\sigma_{0t})v_t^{(1)}dt)| \int_0^T V(\sigma_{0t})dW_{1,t} = x]$$
$$= \left(\frac{x^5}{\Sigma^5} - \frac{10x^3}{\Sigma^4} + \frac{15x}{\Sigma^3} \right) b_{51} + \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2} \right) (b_{52} + b_{53}),$$
(129)

where

$$b_{51} = \rho^2 \left(\int_0^T q_{3t}q_{1t} \int_0^t q_{2s}q_{1s}dsdt \right) \left(\int_0^T q_{4u} \left(\int_u^T q_{5s}ds \right) q_{1u}du \right)$$

$$b_{52} = \left(\int_0^T q_{3t}q_{1t} \int_0^t q_{4s}q_{2s} \left(\int_s^T q_{5u}du \right) dsdt \right)$$

$$b_{53} = \left(\rho^2 \int_0^T q_{4t}q_{3t} \left(\int_t^T q_{5u}du \right) \int_0^t q_{2s}q_{1s}dsdt \right),$$

$$q_{1t} = V(\sigma_{0t}),$$

$$q_{2t} = q_{4t} = \eta_t^{-1} A_1(\sigma_{0t}),$$

$$q_{3t} = V'(\sigma_{0t})\eta_t,$$

$$q_{5t} = V'(\sigma_{0t})V(\sigma_{0t})\eta_t.$$
(130)

A.1.6 $H_2(X_T^{(0)}, (\int_0^T V(\sigma_{0t})v_t^{(1)}dt)^2)$

$$E[H_2(X_T^{(0)}, (\int_0^T V(\sigma_{0t})v_t^{(1)}dt)^2)| \int_0^T V(\sigma_{0s})dW_{1,s} = x] = 2\left[\left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma} \right) b_{61} + \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) b_{62} \right],$$
(131)

where

$$b_{61} = \rho^2 \int_0^T q_{3t} \left(\int_0^t q_{2u}q_{1u}du \right) \left(\int_0^t \left(\int_u^t q_{3s}ds \right) q_{2u}q_{1u}du \right) dt$$

$$b_{62} = \int_0^T q_{3t} \int_0^t q_{2u}^2 \left(\int_u^t q_{3s}ds \right) dudt,$$

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$$
\begin{aligned}
q_{1t} &= V(\sigma_{0t}), \\
q_{2t} &= \eta_t^{-1} A_1(\sigma_{0t}), \\
q_{3t} &= V'(\sigma_{0t}) V(\sigma_{0t}) \eta_t.
\end{aligned}
$$

B   Detail of Computation in the Second Order Asymptotic Expansion

This section presents the detail for computation of push-down of Malliavin weights in the second order asymptotic expansion. The results has been already reported in the previous subsection.

B.1 $H_1(X_T^{(0)}, \int_0^T v_{2t} dW_{1,t})$

$$
H_1(X_T^{(0)}, \int_0^T v_{2t} dW_t) = (D^*(Z) - D_Z) \circ \int_0^T v_{2t} dW_{1,t}.
$$

By the formula (111) in Proposition A.1, we have

$$
\begin{aligned}
&E[H_1(X_T^{(0)}, \int_0^T v_{2t} dW_t) | \int_0^T V(\sigma_{0s}) dW_{1,s} = x] \\
&= \left(\frac{x}{\Sigma} - \frac{\partial}{\partial x} \right) \circ E[\int_0^T v_{2t} dW_t | \int_0^T V(\sigma_{0s}) dW_{1,s} = x].
\end{aligned} \tag{132}
$$

Note that

$$
\begin{aligned}
&E[\int_0^T v_{2t} dW_t | \int_0^T V(\sigma_{0s}) dW_{1,s} = x] \\
&= 2E[\int_0^T V'(\sigma_{0t}) \int_0^t A_1'(\sigma_{0s}) \int_0^s \eta_s \eta_u^{-1} A_1(\sigma_{0u}) dB_u dB_s dW_t | \int_0^T V(\sigma_{0s}) dW_{1,s} = x] \\
&\quad + E[\int_0^T V''(\sigma_{0t}) \left(\int_0^t \eta_t \eta_t^{-1} A_1(\sigma_{0s}) dB_s \right)^2 dW_t | \int_0^T V(\sigma_{0s}) dW_{1,s} = x] \\
&= 2 \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2} \right) \left(\rho^2 \int_0^T V'(\sigma_{0t}) V(\sigma_{0t}) \int_0^t A_1'(\sigma_{0s}) \eta_s V(\sigma_{0s}) \int_0^s \eta_u^{-1} A_1(\sigma_{0u}) V(\sigma_{0u}) du ds dt \right) \\
&\quad + \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2} \right) \int_0^T \left(\int_0^t V(\sigma_{0s}) \eta_s^{-1} A_1(\sigma_{0s}) ds \right)^2 V''(\sigma_{0t}) \eta_t^2 V(\sigma_{0t}) dt \\
&\quad + \frac{x}{\Sigma} \int_0^T V''(\sigma_{0t}) V(\sigma_{0t}) \left(\int_0^t \eta_t \eta_t^{-1} A_1(\sigma_{0s}) \right)^2 ds dt \quad \text{(formulas 4.,5. in Appendix E)} \\
&= \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2} \right) (2b_{11} + b_{12}) + \left(\frac{x}{\Sigma} \right) b_{13},
\end{aligned}
$$

where

$$
\begin{aligned}
b_{11} &= \rho^2 \int_0^T V'(\sigma_{0t}) V(\sigma_{0t}) \int_0^t A_1'(\sigma_{0s}) \eta_s V(\sigma_{0s}) \int_0^s \eta_u^{-1} A_1(\sigma_{0u}) V(\sigma_{0u}) du ds dt, \\
b_{12} &= \int_0^T \left(\int_0^t V(\sigma_{0s}) \eta_s^{-1} A_1(\sigma_{0s}) ds \right)^2 V''(\sigma_{0t}) \eta_t^2 V(\sigma_{0t}) dt, \\
b_{13} &= \int_0^T V''(\sigma_{0t}) V(\sigma_{0t}) \left(\int_0^t \eta_t \eta_t^{-1} A_1(\sigma_{0s}) \right)^2 ds dt.
\end{aligned} \tag{133}
$$

Malliavin weight:

$$
\begin{aligned}
&E[H_1(X_T^{(0)}, \int_0^T v_{2t} dW_{1,t}) | \int_0^T V(\sigma_{0s}) dW_{1,s} = x] \\
&= \left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma^2} \right) (2b_{11} + b_{12}) + \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) b_{13}.
\end{aligned} \tag{134}
$$

B.1.1 $H_1(X_T^{(0)}, \int_0^T v_{1t}^2 dt)$

$$H_1(X_T^{(0)}, \int_0^T v_{1t}^2 dt) = (D^*(Z) - D_Z) \circ \int_0^T v_{1t}^2 dt.$$

By the formula (111) in Proposition (A.1), we have

$$E[H_1(X_T^{(0)}, \int_0^T v_{1t}^2 dt)| \int_0^T V(\sigma_{0s})dW_{1,s} = x]$$
$$= \left(\frac{x}{\Sigma} - \frac{\partial}{\partial x}\right) \circ E[\int_0^T v_{1t}^2 dt| \int_0^T V(\sigma_{0s})dW_{1,s} = x]. \tag{135}$$

$$E[\int_0^T v_{1t}^2 dt| \int_0^T V(\sigma_{0s})dW_{1,s} = x] = E[\int_0^T (V'(\sigma_{0t})\eta_t \int_0^t \eta_s^{-1} A_1(\sigma_{0s})dB_s)^2 dt| \int_0^T V(\sigma_{0s})dW_{1,s} = x]$$

$$= \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right)\left(\rho^2 \int_0^T (V'(\sigma_{0t})\eta_t)^2 (\int_0^t \eta_s^{-1} A_1(\sigma_{0s})V(\sigma_{0s})ds)^2 dt\right)$$

$$+ \int_0^T (V'(\sigma_{0t})\eta_t)^2 \int_0^t (\eta_s^{-1} A_1(\sigma_{0s}))^2 ds dt. \text{ (formula 3. in Appendix E)}$$

$$= \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right) b_{21} + b_{22}, \tag{136}$$

where

$$b_{21} = \rho^2 \int_0^T (V'(\sigma_{0t})\eta_t)^2 (\int_0^t \eta_s^{-1} A_1(\sigma_{0s})V(\sigma_{0s})ds)^2 dt,$$

$$b_{22} = \int_0^T (V'(\sigma_{0t})\eta_t)^2 \int_0^t (\eta_s^{-1} A_1(\sigma_{0s}))^2 ds dt. \tag{137}$$

Malliavin weight:

$$E[H_1(X_T^{(0)}, \int_0^T v_{1t}^2 dt)| \int_0^T V(\sigma_{0s})dW_{1,s} = x] = \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right) b_{21} + \frac{x}{\Sigma} b_{22}. \tag{138}$$

B.2 $H_1(X_T^{(0)}, \int_0^T V(\sigma_{0t})v_{2t}dt)$

$$H_1(X_T^{(0)}, \int_0^T V(\sigma_{0t})v_{2t}dt) = (D^*(Z) - D_Z) \circ \int_0^T V(\sigma_{0t})v_{2t}dt.$$

By the formula (111) in Proposition A.1, we have

$$E[H_1(X_T^{(0)}, \int_0^T V(\sigma_{0t})v_{2t}dt)| \int_0^T V(\sigma_{0s})dW_{1,s} = x]$$
$$= \left(\frac{x}{\Sigma} - \frac{\partial}{\partial x}\right) E[\int_0^T V(\sigma_{0t})v_{2t}dt| \int_0^T V(\sigma_{0s})dW_{1,s} = x]. \tag{139}$$

$$E[\int_0^T V(\sigma_{0t})v_{2t}dt| \int_0^T V(\sigma_{0s})dW_{1,s} = x]$$

$$= 2E[\int_0^T V(\sigma_{0t})V'(\sigma_{0t}) \int_0^t A_1'(\sigma_{0s}) \int_0^s \eta_s \eta_u^{-1} A_1(\sigma_{0u})dB_u dB_s dt| \int_0^T V(\sigma_{0s})dW_{1,s} = x]$$

$$+ E[\int_0^T V(\sigma_{0t})V''(\sigma_{0t}) \left(\int_0^t \eta_t \eta_s^{-1} A_1(\sigma_{0s})dB_s\right)^2 dt| \int_0^T V(\sigma_{0s})dW_{1,s} = x]$$

$$= 2\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right)\left(\rho^2 \int_0^T V'(\sigma_{0t})V(\sigma_{0t}) \int_0^t A_1'(\sigma_{0s})\eta_s V(\sigma_{0s}) \int_0^s \eta_u^{-1} A_1(\sigma_{0u})V(\sigma_{0u})du ds dt\right)$$

$$+ \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}\right)\left(\rho^2 \int_0^T (V''(\sigma_{0t})\eta_t^2 V(\sigma_{0t}))(\int_0^t \eta_s^{-1} A_1(\sigma_{0s})V(\sigma_{0s})ds)^2 dt\right)$$

$$+ \int_0^T (V''(\sigma_{0t})\eta_t^2 V(\sigma_{0t})) \int_0^t (\eta_s^{-1} A_1(\sigma_{0s}))^2 ds dt, \text{ (formulas 2., 3. in Appendix E)}$$

$$= \ 2 \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) b_{31} + \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) b_{32} + b_{33}, \tag{140}$$

where

$$b_{31} \ = \ \rho^2 \int_0^T V'(\sigma_{0t}) V(\sigma_{0t}) \int_0^t A_1'(\sigma_{0s})\eta_s V(\sigma_{0s}) \int_0^s \eta_u^{-1} A_1(\sigma_{0u}) V(\sigma_{0u}) du ds dt,$$

$$b_{32} \ = \ \rho^2 \int_0^T (V''(\sigma_{0t})\eta_t^2 V(\sigma_{0t}))(\int_0^t \eta_s^{-1} A_1(\sigma_{0s}) V(\sigma_{0s}) ds)^2 dt,$$

$$b_{33} \ = \ \int_0^T (V''(\sigma_{0t})\eta_t^2 V(\sigma_{0t})) \int_0^t (\eta_s^{-1} A_1(\sigma_{0s}))^2 ds dt. \tag{141}$$

Malliavin weight:

$$E[H_1(X_T^{(0)}, \int_0^T V(\sigma_{0t}) v_{2t} dt) | \int_0^T V(\sigma_{0s}) dW_{1,s} = x] \ = \ 2 \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2} \right) b_{31} + \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2} \right) b_{32} + \frac{x}{\Sigma} b_{33}. \tag{142}$$

B.2.1 $\ H_2(X_T^{(0)}, (\int_0^T v_{1t} dW_{1,t})^2)$

$$E[H_2(X_T^{(0)}, (\int_0^T v_{1t} dW_{1,t})^2) | \int_0^T V(\sigma_{0s}) dW_{1,s} = x]$$

$$= \ \left(\frac{x}{\Sigma} - \frac{\partial}{\partial x} \right)^2 \circ E[(\int_0^T v_{1t} dW_{1,t})^2 | \int_0^T V(\sigma_{0s}) dW_{1,s} = x].$$

$$E[(\int_0^T v_{1t} dW_{1,t})^2 | \int_0^T V(\sigma_{0s}) dW_{1,s} = x]$$

$$= \ E[(\int_0^T V'(\sigma_{0t})\eta_t \int_0^t \eta_s^{-1} A_1(\sigma_{0s}) dB_s dW_{1,t})^2 | \int_0^T V(\sigma_{0s}) dW_{1,s} = x]$$

$$= \ \left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma^2} \right) \left(\rho \int_0^T V'(\sigma_{0t})\eta_t V(\sigma_{0t}) \int_0^t \eta_s^{-1} A_1(\sigma_{0s}) V(\sigma_{0s}) ds dt \right)^2$$

$$+ 2 \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) \left(\int_0^T V'(\sigma_{0t})\eta_t V(\sigma_{0t}) \int_0^t V'(\sigma_{0s})\eta_s V(\sigma_{0s}) \int_0^s (\eta_u^{-1} A_1(\sigma_{0u}))^2 du ds dt \right)$$

$$+ 2 \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) \left(\rho^2 \int_0^T V'(\sigma_{0t})\eta_t V(\sigma_{0t}) \int_0^t V'(\sigma_{0s}) A_1(\sigma_{0s}) \int_0^s V(\sigma_{0u})\eta_u^{-1} A_1(\sigma_{0u}) du ds dt \right)$$

$$+ \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) \left(\rho^2 \int_0^T (V'(\sigma_{0t})\eta_t)^2 \left(\int_0^t \eta_s^{-1} A_1(\sigma_{0s}) V(\sigma_{0s}) ds \right)^2 dt \right)$$

$$+ \int_0^T (V'(\sigma_{0t})\eta_t)^2 \int_0^t (\eta_s^{-1} A_1(\sigma_{0s}))^2 ds dt \text{ (formula 7. in Appendix E)}$$

$$= \ \left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma^2} \right) b_{41} + 2 \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) b_{42} + 2 \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) b_{43} + \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) b_{44} + b_{45}. \tag{143}$$

where

$$b_{41} \ = \ \left(\rho \int_0^T V'(\sigma_{0t})\eta_t V(\sigma_{0t}) \int_0^t \eta_s^{-1} A_1(\sigma_{0s}) V(\sigma_{0s}) ds dt \right)^2,$$

$$b_{42} \ = \ \int_0^T V'(\sigma_{0t})\eta_t V(\sigma_{0t}) \int_0^t V'(\sigma_{0s})\eta_s V(\sigma_{0s}) \int_0^s (\eta_u^{-1} A_1(\sigma_{0u}))^2 du ds dt,$$

$$b_{43} \ = \ \rho^2 \int_0^T V'(\sigma_{0t})\eta_t V(\sigma_{0t}) \int_0^t V'(\sigma_{0s}) A_1(\sigma_{0s}) \int_0^s V(\sigma_{0u})\eta_u^{-1} A_1(\sigma_{0u}) du ds dt,$$

$$b_{44} \ = \ \rho^2 \int_0^T (V'(\sigma_{0t})\eta_t)^2 \left(\int_0^t \eta_s^{-1} A_1(\sigma_{0s}) V(\sigma_{0s}) ds \right)^2 dt,$$

$$b_{45} \ = \ \int_0^T (V'(\sigma_{0t})\eta_t)^2 \int_0^t (\eta_s^{-1} A_1(\sigma_{0s}))^2 ds dt. \tag{144}$$

Malliavin weight:

$$E[H_2(X_T^{(0)}, (\int_0^T v_{1t}dW_{1,t})^2)| \int_0^T V(\sigma_{0s})dW_{1,s} = x].$$

$$= \left(\frac{x^6}{\Sigma^6} - \frac{15x^4}{\Sigma^5} + \frac{45x^2}{\Sigma^4} - \frac{15}{\Sigma^3}\right) b_{41} + \left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma}\right)(2b_{42} + 2b_{43} + b_{44}) + \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma^2}\right)b_{45}.$$

(145)

B.3 $\quad H_2(X_T^{(0)}, \int_0^T v_{1t}dW_t \int_0^T V(\sigma_{0t})v_{1t}dt)$

$$E[H_2(X_T^{(0)}, \int_0^T v_{1t}dW_{1,t} \int_0^T V(\sigma_{0t})v_{1t}dt)| \int_0^T V(\sigma_{0s})dW_{1,s} = x]$$

$$= \left(\frac{x}{\Sigma} - \frac{\partial}{\partial x}\right)^2 \circ E[\int_0^T v_{1t}dW_{1,t} \int_0^T V(\sigma_{0t})v_{1t}dt| \int_0^T V(\sigma_{0s})dW_{1,s} = x].$$

$$E[\int_0^T v_{1t}dW_{1,t} \int_0^T V(\sigma_{0t})v_{1t}dt| \int_0^T V(\sigma_{0s})dW_{1,s} = x]$$

$$= \rho^2 \left(\int_0^T q_{3t}q_{1t} \int_0^t q_{2s}q_{1s}dsdt\right)\left(\int_0^T q_{4u}\left(\int_u^T q_{5s}ds\right)q_{1u}du\right)\left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right)$$

$$+ \left(\int_0^T q_{3t}q_{1t} \int_0^t q_{4s}q_{2s}\left(\int_s^T q_{5u}du\right)dsdt\right.$$

$$+ \rho^2 \int_0^T q_{4t}q_{3t}\left(\int_t^T q_{5u}du\right)\int_0^t q_{2s}q_{1s}dsdt\right)\frac{x}{\Sigma} \text{ (formula 6'. in Appendix E)}$$

$$= \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right)b_{51} + \frac{x}{\Sigma}(b_{52} + b_{53}),$$

(146)

where

$$q_{1t} = V(\sigma_{0t}),$$
$$q_{2t} = q_{4t} = \eta_t^{-1}A_1(\sigma_{0t}),$$
$$q_{3t} = V'(\sigma_{0t})\eta_t,$$
$$q_{5t} = V'(\sigma_{0t})V(\sigma_{0t})\eta_t,$$
$$b_{51} = \rho^2 \left(\int_0^T q_{3t}q_{1t} \int_0^t q_{2s}q_{1s}dsdt\right)\left(\int_0^T q_{4u}\left(\int_u^T q_{5s}ds\right)q_{1u}du\right)$$
$$b_{52} = \left(\int_0^T q_{3t}q_{1t} \int_0^t q_{4s}q_{2s}\left(\int_s^T q_{5u}du\right)dsdt\right)$$
$$b_{53} = \left(\rho^2 \int_0^T q_{4t}q_{3t}\left(\int_t^T q_{5u}du\right)\int_0^t q_{2s}q_{1s}dsdt\right).$$

(147)

Malliavin weight:

$$E[H_2(X_T^{(0)}, \int_0^T v_{1t}dW_{1,t} \int_0^T V(\sigma_{0t})v_{1t}dt)| \int_0^T V(\sigma_{0t})dW_{1,t} = x]$$

$$= \left(\frac{x^5}{\Sigma^5} - \frac{10x^3}{\Sigma^4} + \frac{15x}{\Sigma^3}\right)b_{51} + \left(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}\right)(b_{52} + b_{53}).$$

(148)

B.3.1 $\quad H_2(X_T^{(0)}, (\int_0^T V(\sigma_{0t})v_{1t}dt)^2)$

$$E[H_2(X_T^{(0)}, (\int_0^T V(\sigma_{0t})v_{1t}dt)^2)| \int_0^T V(\sigma_{0s})dW_{1,s} = x]$$

$$= \left(\frac{x}{\Sigma} - \frac{\partial}{\partial x}\right)^2 \circ E[(\int_0^T V(\sigma_{0t})v_{1t}dt)^2| \int_0^T V(\sigma_{0s})dW_{1,s} = x].$$

$$E[(\int_0^T V(\sigma_{0t})v_{1t}dt)^2 | \int_0^T V(\sigma_{0s})dW_{1,s} = x]$$

$$= E[(\int_0^T V'(\sigma_{0t})\eta_t V(\sigma_{0t}) \int_0^t \eta_s^{-1}A_1(\sigma_{0s})dB_s dt)^2 | \int_0^T V(\sigma_{0s})dW_{1,s} = x]$$

$$\left\{ 2\rho^2 \int_0^T q_{3t} \left(\int_0^t q_{2u}q_{1u}du \right) \left(\int_0^t \left(\int_u^t q_{3s}ds \right) q_{2u}q_{1u}du \right) dt \right\} \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right)$$

$$+ \quad 2\int_0^T q_{3t} \int_0^t q_{2u}^2 \left(\int_u^t q_{3s}ds \right) dudt \quad \text{(formula 3'. in Appendix E)}$$

$$= \quad 2\left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) b_{61} + 2b_{62}, \tag{149}$$

where

$$q_{1t} = V(\sigma_{0t}),$$
$$q_{2t} = \eta_t^{-1}A_1(\sigma_{0t}),$$
$$q_{3t} = V'(\sigma_{0t})V(\sigma_{0t})\eta_t,$$
$$b_{61} = \rho^2 \int_0^T q_{3t} \left(\int_0^t q_{2u}q_{1u}du \right) \left(\int_0^t \left(\int_u^t q_{3s}ds \right) q_{2u}q_{1u}du \right) dt$$
$$b_{62} = \int_0^T q_{3t} \int_0^t q_{2u}^2 \left(\int_u^t q_{3s}ds \right) dudt \tag{150}$$

Malliavin weight:

$$E[H_2(X_T^{(0)}, (\int_0^T V(\sigma_{0t})v_{1t}dt)^2) | \int_0^T V(\sigma_{0s})dW_{1,s} = x] = 2\left[\left(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma} \right) b_{61} + \left(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma} \right) b_{62} \right]. \tag{151}$$

C   Proof of Proposition 5.2

Let C_1^M denote the coefficient of ϵ-order in the asymptotic expansion of C^{SVJ} around $\epsilon = 0$:

$$C_1^M = \frac{\partial}{\partial \epsilon} E\left[(S_T^{(\epsilon)} - K)^+ \right] |_{\epsilon=0}$$

$$= E\left[\partial_x \left\{ \left(S_T^{(0)} - K \right)^+ \right\} \frac{\partial}{\partial \epsilon} S_T^{(\epsilon)} |_{\epsilon=0} \right]$$

$$= \sum_{n=0}^{\infty} E\left[\partial_x \left\{ \left(S_T^{(0)} - K \right)^+ \right\} \frac{\partial}{\partial \epsilon} S_T^{(\epsilon)} |_{\epsilon=0} \mathbf{1}_{\{N(T)=n\}} \right], \tag{152}$$

where

$$\partial_x \left\{ \left(S_T^{(0)} - K \right)^+ \right\} = \frac{\partial}{\partial x}(x - K)^+ |_{x=S_T^{(0)}}.$$

Next, by the integration by parts formula including the jump amplitudes (e.g. Bavouzet and Messaoud (2006)), the right hand side of the last equality in the above equation is expressed as

$$C_1^M = \sum_{n=0}^{\infty} E\left[\left(S_T^{(0)} - K \right)^+ H_{1,n} \mathbf{1}_{\{N(T)=n\}} \right], \tag{153}$$

where $H_{1,n}$ is the Malliavin weight on $\{N(T)=n\}$, which will be given below:

Let D_0 be the Malliavin derivative with respect to Brownian motion for W_1 and D_i be the Malliavin derivative with respect to the jump amplitudes Δ_i, $i = 1, \cdots, n$. Then, on $\{N(T)=n\}$,

$$D_{0,t}S_T^{(0)} = S_T^{(0)}D_{0t}X_T^{(0)} = S_T^{(0)}V(\sigma_t^{(0)}), \ t \in [0, T],$$
$$D_i S_T^{(0)} = S_T^{(0)}D_i X_T^{(0)} = S_T^{(0)}b, \ i = 1, \cdots, n, \tag{154}$$

$$X_T^{(0)} = \int_0^T V(\sigma_t^{(0)})dW_{1,t} - \frac{1}{2}\int_0^T V(\sigma_t^{(0)})^2 dt + \sum_{j=1}^n Y_j,$$

where we use $D_i X_T^{(0)} = b$ (e.g. see p.280 in Bavouzet-Messaoud (2006)) as well as $D_{0,t}X_T^{(0)} = V(\sigma_t^{(0)})$.

Then, the Malliavin covariance of $S_T^{(0)}$, $\sigma_{S_T^{(0)}}$ is given by

$$
\begin{aligned}
\sigma_{S_T^{(0)}} &= \int_0^T \left(D_{0,t} S_T^{(0)}\right)^2 dt + \sum_{i=1}^n \left(D_i S_T^{(0)}\right)^2 \\
&= (S_T^{(0)})^2 \int_0^T V(\sigma_t^{(0)})^2 dt + (S_T^{(0)})^2 b^2 n \\
&= (S_T^{(0)})^2 \Sigma_n,
\end{aligned}
\tag{155}
$$

where

$$
\Sigma_n := \int_0^T V(\sigma_t^{(0)})^2 dt + (S_T^{(0)})^2 b^2 n,
$$

and

$$
\sum_{i=1}^n \left(D_i S_T^{(0)}\right)^2 = (S_T^{(0)})^2 (b^2 n)
$$

is the contribution by the jump part. (e.g. See p.282 in Bavouzet-Messaoud (2006).)

Using the Skorohod integral for the Brownian motion and the jump amplitude, on $\{N(T) = n\}$ we can compute the Malliavin weight $H_{1,n}$ as follows;

$$
\begin{aligned}
H_{1,n} &= \sum_{i=0}^n D_i^* \left(\frac{\frac{\partial}{\partial \epsilon} S_T^{(\epsilon)}|_{\epsilon=0} D_i S_T^{(0)}}{\sigma_{S_T^{(0)}}} \right) = \sum_{i=0}^n D_i^* \left(\frac{S_T^{(0)} X_T^{(1)} S_T^{(0)} D_i X_T^{(0)}}{(S_T^{(0)})^2 \Sigma_n} \right) \\
&= \frac{1}{\Sigma_n} \sum_{i=0}^n D_i^* \left(X_T^{(1)} D_i X_T^{(0)} \right) \\
&= \frac{1}{\Sigma_n} \left(X_T^{(1)} D_0^*(D_0 X_T^{(0)}) - \int_0^T (D_{0,t} X_T^{(1)})(D_{0,t} X_T^{(0)}) dt \right) \\
&\quad + \frac{1}{\Sigma_n} \sum_{i=1}^n \left((X_T^{(1)} D_i^*(D_i X_T^{(0)}) - (D_i X_T^{(1)})(D_i X_T^{(0)}) \right),
\end{aligned}
\tag{156}
$$

where each D_i^* $(i = 0, 1, \cdots, n)$ is the adjoint operator of D_i $(i = 0, 1, \cdots, n)$ and

$$
\begin{aligned}
X_T^{(1)} &= \frac{\partial}{\partial \epsilon} X_T^{(\epsilon)}|_{\epsilon=0} \\
&= \int_0^T V'(\sigma_{0t}) \int_0^t \eta_t \eta_s^{-1} A_1(\sigma_{0s}) dB_s dW_{1,t} - \int_0^T V(\sigma_{0t}) V'(\sigma_{0t}) \int_0^t \eta_t \eta_s^{-1} A_1(\sigma_{0s}) dB_s dt,
\end{aligned}
$$

with

$$
B_t = \rho W_{1,t} + \sqrt{1 - \rho^2} W_{2,t}.
$$

Note that $D_0^*(D_0 X_T^{(0)})$ is expressed by the usual Itô integral,

$$
D_0^*(D_0 X_T^{(0)}) = \int_0^T V(\sigma_t^{(0)}) dW_{1,t}.
$$

Recall that Δ_i is Gaussian random variable $N(0, 1)$. Hence, from p.280 in Bavouzet-Messaoud (2006), $D_i^*(D_i X_T^{(0)})$ for $i = 1, \cdots, n$ is given by

$$
D_i^*(D_i X_T^{(0)}) = D_i^*(b) = -b \frac{\partial}{\partial \Delta_i} \log p(\Delta_i) = b \Delta_i,
\tag{157}
$$

where

$$
p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.
$$

Because $X_T^{(1)}$ is a stochastic integral driven by the Brownian motion which is independent of Δ_i, we have for $t \in [0, T]$,

$$
\begin{aligned}
D_{0,t} X_T^{(1)} &= V'(\sigma_t^{(0)}) \int_0^t \eta_t \eta_s^{-1} A_1(\sigma_s^{(0)}) dB_s + \rho \eta_t^{-1} A_1(\sigma_t^{(0)}) \int_t^T V'(\sigma_u^{(0)}) \eta_u dW_{1,u} \\
&\quad - \rho \eta_t^{-1} A_1(\sigma_t^{(0)}) \int_t^T V'(\sigma_u^{(0)}) V(\sigma_u^{(0)}) \eta_u du,
\end{aligned}
$$

and for $i = 1, \cdots, n$,

$$D_i X_T^{(1)} = 0.$$

Hence, (156) is expressed as

$$H_{1,n} = \frac{1}{\Sigma_n} \left(X_T^{(1)} \left(\int_0^T V(\sigma_t^{(0)}) dW_{1,t} + \sum_{i=1}^n b\Delta_i \right) - \int_0^T D_{0,t} X_T^{(1)} V(\sigma_t^{(0)}) dt \right). \tag{158}$$

Then,

$$
\begin{aligned}
C_1^M &= \sum_{n=0}^{\infty} E\left[\left(S_T^{(0)} - K \right)^+ H_{1,n} \mathbf{1}_{\{N(T)=n\}} \right] \\
&= \sum_{n=0}^{\infty} \frac{(\lambda T)^n e^{-\lambda T}}{n!} E\left[\left(S_T^{(0)} - K \right)^+ H_{1,n} \right] \\
&= \sum_{n=0}^{\infty} \frac{(\lambda T)^n e^{-\lambda T}}{n!} \int_{\mathbf{R}} \left(s e^{x - \lambda m T - \frac{1}{2} \int_0^T V(\sigma_t^{(0)})^2 dt + na} - K \right)^+ \vartheta_n(x) \frac{1}{\sqrt{2\pi\Sigma_n}} e^{-\frac{1}{2\Sigma_n}x^2} dx,
\end{aligned}
$$
$$\tag{159}$$

where $\vartheta_n(x)$ is the push down of the Malliavin weight $H_{1,n}$;

$$\vartheta_n(x) = E\left[H_{1,n} \Big| \int_0^T V(\sigma_t^{(0)}) dW_{1,t} + \sum_{i=1}^n b\Delta_i = x \right].$$

We easily evaluate $\vartheta_n(x)$ as

$$\vartheta_n(x) = \zeta \left(\frac{1}{\Sigma_n^3} x^3 - \frac{1}{\Sigma_n^2} x^2 - \frac{1}{\Sigma_n^2} 3x + \frac{1}{\Sigma_n} \right). \tag{160}$$

Thus, we obtain (100):

$$C^{SVJ}(T, K) = C^M(T, K, \sigma^{BS}) + \epsilon C_1^M + O(\epsilon^2). \tag{161}$$

Next, let us regard C^M as a function of σ^{BS}:

$$\sigma^{BS} \mapsto C^M(T, K, \sigma^{BS}) = \sum_{n=0}^{\infty} \frac{(\lambda(m+1)T)^n e^{-\lambda(m+1)T}}{n!} C_n^{BS} \left(\left(\left(\sigma^{BS} \right)^2 + (b^2 n/T) \right)^{1/2} \right).$$

Also, the *vega* of C^M with respect to σ^{BS}, $vega^M$ is given by

$$
\begin{aligned}
vega^M &= \frac{\partial}{\partial \sigma^{BS}} C^M(T, K, \sigma^{BS}) \\
&= \sum_{n=0}^{\infty} \frac{(\lambda(m+1)T)^n e^{-\lambda(m+1)T}}{n!} \frac{\sigma^{BS}}{\sigma_n} s\sqrt{T} n(d_n).
\end{aligned}
\tag{162}
$$

Suppose that $\sigma^{SVJ.IV}$ satisfies

$$C^{SVJ}(T, K) = C^M(T, K, \sigma^{SVJ.IV}(T, K)),$$

and that $\sigma^{SVJ.IV}(T, K)$ is expanded around $\epsilon = 0$:

$$\sigma^{SVJ.IV}(T, K) = \sigma^{BS} + \epsilon \cdot \sigma^1 + O(\epsilon^2).$$

Then, we have

$$C^M(\sigma^{SVJ.IV}) = C^M(T, K, \sigma^{BS}) + \epsilon \cdot vega^M \sigma^1 + O(\epsilon^2). \tag{163}$$

Finally, comparing (100) with (163), we obtain

$$\sigma^1 = \frac{C_1^M}{vega^M}, \tag{164}$$

which leads to the result (101). $\square$

D   Local Volatility Expansion

This section provides an approximation formula for local volatilities obtained by the so called *Markovian projection* of the stochastic volatility model. First, note that the Dupire formula of the local volatility in the stochastic volatility model is expressed by Malliavin calculus as follows:

$$\sigma^{LV}(T,K) = E[V(\sigma_T^{(\epsilon)})|S_T^{(\epsilon)} = K] \quad = \quad \frac{\langle \delta_K(S_T^{(\epsilon)}), V(\sigma_T^{(\epsilon)})\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty}}{\langle \delta_K(S_T^{(\epsilon)}), 1\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty}}. \tag{165}$$

Then, we can derive a closed-form approximation of the Dupire formula via the distribution theory on the Wiener space and the following propostion is obtained,

Proposition D.1 *Under the stochastic volatility model (26) with $r = 0$, the local volatility $\sigma^{LV}(T,K)$ defined by $\sigma^{LV}(T,K) = E[V(\sigma_T^{(\epsilon)})|S_T^{(\epsilon)} = K]$ is expanded up to the ϵ^2-order as follows:*

$$\sigma^{LV}(T,K) \quad = \quad \sigma_0^{LV} + \epsilon\sigma_1^{LV} + \epsilon^2\sigma_2^{LV} + O(\epsilon^3), \tag{166}$$

where

$$\begin{aligned}
\sigma_0^{LV} &= V(\sigma_T^{(0)}), \\
\sigma_1^{LV} &= c_1 \frac{h_1(\alpha;\Sigma)}{\Sigma}, \\
\sigma_2^{LV} &= (c_{21} + c_{21} + c_{23} + c_{24} - 3c_{27})\frac{h_2(\alpha;\Sigma)}{\Sigma^2} + (c_{25} + 2c_{27})\frac{h_1(\alpha;\Sigma)}{\Sigma} + c_{26} \\
\Sigma &= \int_0^T V(\sigma_{0t})^2 dt \\
\alpha &= -\log(S_0/K) + \frac{1}{2}\int_0^T V(\sigma_{0t})^2 dt,
\end{aligned} \tag{167}$$

and $h_n(\cdot;\Sigma)$ denotes the Hermite polynomial of degree n, with

$$\begin{aligned}
c_1 &= V'(\sigma_T^{(0)})\rho\left(\int_0^T \eta_T\eta_s^{-1}A_1(\sigma_{0s})V(\sigma_{0s})ds\right), \\
c_{21} &= V'(\sigma_T^{(0)})\rho^2\int_0^T A_1'(\sigma_{0s})V(\sigma_{0s})\eta_s\int_0^s \eta_u^{-1}A_1(\sigma_{0u})V(\sigma_{0u})duds, \\
c_{22} &= \frac{1}{2}V''(\sigma_T^{(0)})\left(\rho\int_0^T \eta_T\eta_s^{-1}A_1(\sigma_{0s})V(\sigma_{0s})ds\right)^2, \\
c_{23} &= V'(\sigma_T^{(0)})\eta_T\left(\rho^2\int_0^T V'(\sigma_t^{(0)})V(\sigma_t^{(0)})\eta_t\int_0^t \eta_s^{-2}A_1(\sigma_s^{(0)})^2 dsdt\right), \\
c_{24} &= V'(\sigma_T^{(0)})\left(\rho^2\eta_T\int_0^T V'(\sigma_t^{(0)})A_1(\sigma_t^{(0)})\int_0^t \eta_s^{-1}A_1(\sigma_s^{(0)})V(\sigma_s^{(0)})dsdt\right), \\
c_{25} &= -\left(\eta_T\int_0^T \eta_s^{-2}A_1V(\sigma_{0s})^2\int_s^T \eta_t V(\sigma_{0t})V'(\sigma_t^{(0)})dtds\right), \\
c_{26} &= \frac{1}{2}V''(\sigma_T^{(0)})\eta_T^2\int_0^T \eta_s^{-2}A_1(\sigma_{0s})^2 ds \\
c_{27} &:= \frac{V'(\sigma_T^{(0)})}{\Sigma}\eta_T\rho^2\left(\int_0^T \eta_s^{-1}A_1(\sigma_{0s})V(\sigma_{0s})ds\right)\left(\int_0^T V'(\sigma_t^{(0)})V(\sigma_t^{(0)})\eta_t\int_0^t \eta_s^{-1}A_1(\sigma_s^{(0)})V(\sigma_s^{(0)})dsdt\right).
\end{aligned} \tag{168}$$

(*Proof*)

First, we define $X_T^{(\epsilon)} = \log\left(\frac{S_T^{(\epsilon)}}{s}\right)$, $S_{kT} := \frac{\partial^k}{\partial\epsilon^k}S_T^{(\epsilon)}|_{\epsilon=0}$, $X_{kT} := \frac{\partial^k}{\partial\epsilon^k}X_T^{(\epsilon)}|_{\epsilon=0}$ for $k \in \mathbf{N}$. Also, we set $r = 0$ for simplisity. The second order expansion of the density is given by the formula.

$$\begin{aligned}
&\langle \delta_K(S_T^{(\epsilon)}), \mathbf{1}\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty} \\
&= p^{BS}(K) + \epsilon E[H_1(S_T^{(0)}, S_{1T})|Y = \alpha]p^{BS}(K) + \epsilon^2\frac{1}{2!}E[H_1(S_T^{(0)}, S_{2T}) + H_2(S_T^{(0)}, S_{1T}^2)|Y = \alpha]p^{BS}(K) + O(\epsilon^3),
\end{aligned} \tag{169}$$

where

$$Y \;=\; \int_0^T V(\sigma_s^{(0)})dW_{1,s},$$

$$\alpha \;=\; -\log(S_0/K) + \frac{1}{2}\int_0^T V(\sigma_{0t})^2 dt.$$

p_1 and p_2 are defined by

$$p_1(K) \;=\; E[H_1(S_T^{(0)}, S_{1T})|Y=\alpha]p^{BS}(K), \tag{170}$$

and

$$p_2(K) \;=\; \frac{1}{2!}E[H_1(S_T^{(0)}, S_{2T}) + H_2(S_T^{(0)}, S_{1T}^2)|Y=\alpha]p^{BS}(K). \tag{171}$$

Note first that $V(\sigma_T^{(\epsilon)})\delta_K(S_T^{(\epsilon)})$ is expanded in $\mathbf{D}_{-\infty}$ as follows:

$$
\begin{aligned}
&V(\sigma_T^{(\epsilon)})\delta_K(S_T^{(\epsilon)}) \\
=\;& V(\sigma_T^{(0)})\delta_K(S_T^{(0)}) + \epsilon\{\delta_K(S_T^{(0)})v_{1T} + V(\sigma_T^{(0)})\partial\delta_K(S_T^{(0)})S_{1T}\} \\
&+\epsilon^2\left\{V(\sigma_T^{(0)})\frac{1}{2!}(\partial\delta_K(S_T^{(0)})S_{2T} + \partial^2\delta_K(S_T^{(0)})S_{1T}^2) + \partial\delta_K(S_T^{(0)})S_{1T}v_{1T} + \frac{1}{2!}\delta_K(S_T^{(0)})v_{2T}\right\} + O(\epsilon^3),
\end{aligned}
\tag{172}
$$

where

$$
\begin{aligned}
v_{1T} &=\; V'(\sigma_T^{(0)})\sigma_{1T}, \\
v_{2T} &=\; 2V'(\sigma_T^{(0)})\sigma_{2t} + V''(\sigma_{0T})\sigma_{1t}^2,
\end{aligned}
\tag{173}
$$

with

$$
\begin{aligned}
\sigma_{1t} &=\; \int_0^t \eta_t\eta_s^{-1}A_1(\sigma_{0s})dB_s, \\
\sigma_{2t} &=\; \int_0^t A_1'(\sigma_{0s})\int_0^s \eta_s\eta_u^{-1}A_1(\sigma_{0u})dB_u dB_s.
\end{aligned}
$$

Then, its expectation is expanded as follows;

$$
\begin{aligned}
&\langle\delta_K(S_T^{(\epsilon)}), V(\sigma_T^{(\epsilon)})\rangle_{\mathbf{D}_{-\infty}\times\mathbf{D}_\infty} \\
=\;& V(\sigma_T^{(0)})p^{BS}(K) + \epsilon\{E[v_{1T}|Y=\alpha]p^{BS}(K) + V(\sigma_T^{(0)})E[H_1(S_T^{(0)}, S_{1T})|Y=\alpha]p^{BS}(K)\} \\
&+\epsilon^2\{V(\sigma_T^{(0)})\frac{1}{2!}(E[H_1(S_T^{(0)}, S_{2T}) + H_2(S_T^{(0)}, S_{1T}^2)|Y=\alpha]p^{BS}(K)) \\
&+E[H_1(S_T^{(0)}, S_{1T}v_{1T})|Y=\alpha]p^{BS}(K) + \frac{1}{2!}E[v_{2T}|Y=\alpha]p^{BS}(K)\} + O(\epsilon^3).
\end{aligned}
\tag{174}
$$

g_1 and g_2 are defined by

$$g_1(K) \;=\; E[v_{1T}|Y=\alpha]p^{BS}(K) + V(\sigma_T^{(0)})E[H_1(S_T^{(0)}, S_{1T})|Y=\alpha]p^{BS}(K), \tag{175}$$

and

$$
\begin{aligned}
g_2(K) \;=\;& V(\sigma_T^{(0)})\frac{1}{2!}(E[H_1(S_T^{(0)}, S_{2T})|Y=\alpha]p^{BS}(K) + E[H_2(S_T^{(0)}, S_{1T}^2)|Y=\alpha]p^{BS}(K)) \\
&+E[H_1(S_T^{(0)}, S_{1T}v_{1T})|Y=\alpha]p^{BS}(K) + \frac{1}{2!}E[v_{2T}|Y=\alpha]p^{BS}(K).
\end{aligned}
\tag{176}
$$

Therefore, the Dupire formula in stochastic volatility model is expanded as follows;

$$
\begin{aligned}
&E[V(\sigma_T^{(\epsilon)})|S_T^{(\epsilon)} = K] \\
=\;& \left(V(\sigma_T^{(0)})p^{BS}(K) + \epsilon g_1(K) + \epsilon^2 g_2(K) + O(\epsilon^3)\right)\left(p^{BS}(K) + \epsilon p_1(K) + \epsilon^2 p_2(K) + O(\epsilon^3)\right)^{-1} \\
=\;& \left(V(\sigma_T^{(0)}) + \epsilon\frac{g_1(K)}{p^{BS}(K)} + \epsilon^2\frac{g_2(K)}{p^{BS}(K)} + O(\epsilon^3)\right)\left(1 + \epsilon\frac{p_1(K)}{p^{BS}(K)} + \epsilon^2\frac{p_2(K)}{p^{BS}(K)} + O(\epsilon^3)\right)^{-1} \\
=\;& \left(V(\sigma_T^{(0)}) + \epsilon\frac{g_1(K)}{p^{BS}(K)} + \epsilon^2\frac{g_2(K)}{p^{BS}(K)} + O(\epsilon^3)\right)\left(1 - \epsilon\frac{p_1(K)}{p^{BS}(K)} + \epsilon^2\left(\frac{p_1(K)^2}{p^{BS}(K)^2} - \frac{p_2(K)}{p^{BS}(K)}\right) + O(\epsilon^3)\right). \\
=\;& \sigma_0^{LV} + \epsilon\sigma_1^{LV} + \epsilon^2\sigma_2^{LV} + O(\epsilon^3).
\end{aligned}
\tag{177}
$$

D.1 Computation of σ_1^{LV} and σ_2^{LV} in Proposition D.1

This section shows the detail of the computation of σ_1^{LV} and σ_2^{LV} in Proposition D.1. The formulas for the evaluation of conditional expectations appearing below are summarized in Appendix E.

σ_0^{LV}, σ_1^{LV} and σ_2^{LV} are computed as follows;

$$
\begin{aligned}
\sigma_0^{LV} &= V(\sigma_T^{(0)}), \\
\sigma_1^{LV} &= \frac{g_1(K)}{p^{BS}(K)} - V(\sigma_T^{(0)})\frac{p_1(K)}{p^{BS}(K)} \\
&= E[v_{1T}|Y = \alpha] \\
&= V'(\sigma_T^{(0)})E[\sigma_{1T}|Y = \alpha] = V'(\sigma_T^{(0)})E\left[\int_0^t \eta_t\eta_s^{-1}A_1(\sigma_{0s})dB_s|Y = \alpha\right] \\
&= V'(\sigma_T^{(0)})\left(\rho\eta_T\int_0^T \eta_s^{-1}A_1(\sigma_{0s})V(\sigma_{0s})ds\right)\frac{h_1(\alpha;\Sigma)}{\Sigma}\text{(formula 1 in Appendix E)}, \\
\sigma_2^{LV} &= V(\sigma_T^{(0)})\left(\frac{p_1(K)^2}{p^{BS}(K)^2} - \frac{p_2(K)}{p^{BS}(K)}\right) - \frac{g_1(K)p_1(K)}{p^{BS}(K)^2} + \frac{g_2(K)}{p^{BS}(K)} \\
&= \frac{1}{2}E[v_{2T}|Y = \alpha] - E[v_{1T}|Y = \alpha]E[H_1(S_T^{(0)}, S_{1T})|Y = \alpha] + E[H_1(S_T^{(0)}, S_{1T}v_{1T})|Y = \alpha]
\end{aligned}
\tag{178}
$$

In particular, as for σ_2^{LV} note first that by **A.1**,

$$
\begin{aligned}
E[H_1(S_T^{(0)}, S_{1T})|Y = \alpha] &= E\left[H_1(X_T^{(0)}, \int_0^T v_{1t}dW_{1t})|Y = \alpha\right] - E\left[H_1(X_T^{(0)}, \int_0^T V(\sigma_{0t})v_{1t}dW_{1t})|Y = \alpha\right] \\
&= \left\{\frac{h_3(\alpha, \Sigma)}{\Sigma^3} - \frac{h_2(\alpha, \Sigma)}{\Sigma^2}\right\}a_1,
\end{aligned}
\tag{179}
$$

where $a_1 = \rho\int_0^T V'(\sigma_{0t})\eta_t V(\sigma_{0t})\int_0^t \eta_s^{-1}A_1(\sigma_{0s})V(\sigma_{0s})dsdt$.

Next, note that

$$
H_1(S_T^{(0)}, S_{1T}v_{1T}) = D_1^*(ZX_{1T}v_{1T}) = H_1(X_T^{(0)}, X_{1T}v_{1T}) = (D^*(Z) - D_Z)(X_{1T}v_{1T}).
\tag{180}
$$

Hence, by the formula (111) in Proposition A.1,

$$
E[H_1(S_T^{(0)}, S_{1T}v_{1T})|Y = \alpha] = \left(\frac{\alpha}{\Sigma} - \frac{\partial}{\partial\alpha}\right)E[X_{1T}v_{1T}|Y = \alpha],
\tag{181}
$$

where

$$
\begin{aligned}
X_{1T} &= \frac{\partial}{\partial\epsilon}X_T^{(\epsilon)}|_{\epsilon=0} \\
&= \int_0^T V'(\sigma_{0t})\eta_t\int_0^t \eta_s^{-1}A_1(\sigma_{0s})dB_s dW_{1,t} - \int_0^T V(\sigma_{0t})V'(\sigma_{0t})\eta_t\int_0^t \eta_s^{-1}A_1(\sigma_{0s})dB_s dt \\
&= \int_0^T V'(\sigma_{0t})\eta_t\int_0^t \eta_s^{-1}A_1(\sigma_{0s})dB_s dW_{1,t} - \int_0^T \left(\int_s^T V(\sigma_{0t})V'(\sigma_{0t})\eta_t dt\right)\eta_s^{-1}A_1(\sigma_{0s})dB_s \\
v_{1T} &= V'(\sigma_{0T})\eta_T\int_0^T \eta_s^{-1}A_1(\sigma_{0s})dB_s.
\end{aligned}
$$

Then, $E[X_{1T}v_{1T}|Y = \alpha]$ can be computed by formulas 2. and 6. in Appendix E.

Finally, σ_2^{LV} is evauated as follows:

$$
\begin{aligned}
\sigma_2^{LV} &= V'(\sigma_T^{(0)})\rho^2\int_0^T A_1'(\sigma_{0s})V(\sigma_{0s})\int_0^s \eta_s\eta_u^{-1}A_1(\sigma_{0u})V(\sigma_{0u})duds\frac{h_2(\alpha;\Sigma)}{\Sigma^2} \\
&+ \frac{1}{2}V''(\sigma_T^{(0)})\left\{\left(\rho\int_0^T \eta_T\eta_s^{-1}A_1(\sigma_{0s})V(\sigma_{0s})ds\right)^2\frac{h_2(\alpha;\Sigma)}{\Sigma^2} + \eta_T^2\int_0^T \eta_s^{-2}A_1(\sigma_{0s})^2 ds\right\} \\
&- V'(\sigma_T^{(0)})\rho^2\left(\eta_T\int_0^T \eta_s^{-1}A_1(\sigma_{0s})V(\sigma_{0s})ds\right)\frac{h_1(\alpha;\Sigma)}{\Sigma} \\
&\times \left(\int_0^T V'(\sigma_t^{(0)})\eta_t V(\sigma_t^{(0)})\int_0^t \eta_s^{-1}A_1(\sigma_s^{(0)})V(\sigma_s^{(0)})dsdt\right)\left(\frac{h_3(\alpha;\Sigma)}{\Sigma^3} - \frac{h_2(\alpha;\Sigma)}{\Sigma^2}\right) \\
&+ V'(\sigma_T^{(0)})\rho^2\left(\eta_T\int_0^T \eta_s^{-1}A_1(\sigma_{0s})V(\sigma_{0s})ds\right)\left(\int_0^T V'(\sigma_t^{(0)})\eta_t V(\sigma_t^{(0)})\int_0^t \eta_s^{-1}A_1(\sigma_s^{(0)})V(\sigma_s^{(0)})dsdt\right)\frac{h_4(\alpha;\Sigma)}{\Sigma^4}
\end{aligned}
$$

$$+ \quad V'(\sigma_T^{(0)})\eta_T \left(\int_0^T V'(\sigma_t^{(0)})V(\sigma_t^{(0)}) \int_0^t \eta_t \eta_s^{-2} A_1(\sigma_s^{(0)})^2 ds dt + \rho^2 \int_0^T V'(\sigma_t^{(0)}) A_1(\sigma_t^{(0)}) \int_0^t \eta_s^{-1} A_1(\sigma_s^{(0)}) V(\sigma_s^{(0)}) ds dt \right)$$

$$\times \frac{h_2(\alpha;\Sigma)}{\Sigma^2}$$

$$- \quad V'(\sigma_T^{(0)})\rho^2 \left(\eta_T \int_0^T \eta_s^{-1} A_1(\sigma_{0s}) V(\sigma_{0s}) ds \right) \left(\int_0^T \eta_t V(\sigma_{0t}) V'(\sigma_{0t}) \int_0^t \eta_s^{-1} A_1(\sigma_{0s}) V(\sigma_{0s}) ds dt \right) \frac{h_3(\alpha;\Sigma)}{\Sigma^3}$$

$$- \quad \left(\eta_T \int_0^T \eta_s^{-2} A_1(\sigma_{0s})^2 \int_s^T \eta_t V(\sigma_{0t}) V'(\sigma_{0t}) dt ds \right) \frac{h_1(\alpha;\Sigma)}{\Sigma} \tag{182}$$

$$= \quad V'(\sigma_T^{(0)})\rho^2 \int_0^T A_1'(\sigma_{0s}) V(\sigma_{0s}) \eta_s \int_0^s \eta_u^{-1} A_1(\sigma_{0u}) V(\sigma_{0u}) du ds \frac{h_2(\alpha;\Sigma)}{\Sigma^2}$$

$$+ \frac{1}{2} V''(\sigma_T^{(0)})\eta_T^2 \left\{ \left(\rho \int_0^T \eta_s^{-1} A_1(\sigma_{0s}) V(\sigma_{0s}) ds \right)^2 \frac{h_2(\alpha;\Sigma)}{\Sigma^2} + \int_0^T \eta_s^{-2} A_1(\sigma_{0s})^2 ds \right\}$$

$$+ V'(\sigma_T^{(0)})\eta_T \left(\int_0^T V'(\sigma_t^{(0)}) V(\sigma_t^{(0)}) \eta_t \int_0^t \eta_s^{-2} A_1(\sigma_s^{(0)})^2 ds dt \right) \frac{h_2(\alpha;\Sigma)}{\Sigma^2}$$

$$+ V'(\sigma_T^{(0)})\eta_T \left(\rho^2 \int_0^T V'(\sigma_t^{(0)}) A_1(\sigma_t^{(0)}) \int_0^t \eta_s^{-1} A_1(\sigma_s^{(0)}) V(\sigma_s^{(0)}) ds dt \right) \frac{h_2(\alpha;\Sigma)}{\Sigma^2}$$

$$- \eta_T \left(\int_0^T \eta_s^{-2} A_1(\sigma_{0s})^2 \int_s^T \eta_t V(\sigma_{0t}) V'(\sigma_{0t}) dt ds \right) \frac{h_1(\alpha;\Sigma)}{\Sigma}$$

$$+ c \left(\frac{1}{\Sigma^4} (h_4(\alpha,\Sigma) - h_1(\alpha,\Sigma) h_3(\alpha,\Sigma)) + \frac{1}{\Sigma^3} (h_1(\alpha,\Sigma) h_2(\alpha,\Sigma) - h_3(\alpha,\Sigma)) \right), \tag{183}$$

where

$$c \quad := \quad V'(\sigma_T^{(0)}) \eta_T \rho^2 \left(\int_0^T \eta_s^{-1} A_1(\sigma_{0s}) V(\sigma_{0s}) ds \right) \left(\int_0^T V'(\sigma_t^{(0)}) V(\sigma_t^{(0)}) \eta_t \int_0^t \eta_s^{-1} A_1(\sigma_s^{(0)}) V(\sigma_s^{(0)}) ds dt \right). \tag{184}$$

Note that $h_1(\alpha,\Sigma) = \alpha$, $h_2(\alpha,\Sigma) = \alpha^2 - \Sigma$, $h_3(\alpha,\Sigma) = \alpha^3 - 3\Sigma\alpha$ and $h_4(\alpha,\Sigma) = \alpha^4 - 6\Sigma\alpha^2 + \Sigma^2$. Then,

$$\frac{1}{\Sigma^4} (h_4(\alpha,\Sigma) - h_1(\alpha,\Sigma) h_3(\alpha,\Sigma)) = -\frac{3}{\Sigma^3}(\alpha^2 - \Sigma) = -\frac{3}{\Sigma^3} h_2(\alpha,\Sigma)$$

$$\frac{1}{\Sigma^3} (h_1(\alpha,\Sigma) h_2(\alpha,\Sigma) - h_3(\alpha,\Sigma)) = \frac{2}{\Sigma^2}\alpha = \frac{2}{\Sigma^2} h_1(\alpha,\Sigma). \tag{185}$$

Hence,

$$c \left\{ \frac{1}{\Sigma^4} (h_4(\alpha,\Sigma) - h_1(\alpha,\Sigma) h_3(\alpha,\Sigma)) + \frac{1}{\Sigma^3} (h_1(\alpha,\Sigma) h_2(\alpha,\Sigma) - h_3(\alpha,\Sigma)) \right\} = c \left(\frac{2}{\Sigma^2} h_1(\alpha,\Sigma) - \frac{3}{\Sigma^3} h_2(\alpha,\Sigma) \right). \tag{186}$$

E   Conditional Expectation Formulas for Certain Wiener-Itô Integrals

This section summarizes conditional expectation formulas used for the calculations in the previous sections.

In the following, W is a d-dimensional Brownian motion and $q_i = (\hat{q}_{i1}, \cdots, \hat{q}_{id})'$ where $\hat{q}_i \in L^2[0,T]$, $i = 1, 2, ..., 5$ and x' denotes the transpose of x. $h_n(x;\Sigma)$ denotes the Hermite polynomial of degree n and $\Sigma = \int_0^T |q_{1t}|^2 dt$. For the derivation and more general results, see Section 3 in Takahashi-Takehara-Toda[2009].

1.

$$\mathbf{E} \left[\int_0^T q_{2t}' dW_t | \int_0^T q_{1v}' dW_v = x \right] = \left(\int_0^T q_{2t}' q_{1t} dt \right) \frac{h_1(x;\Sigma)}{\Sigma}$$

2.

$$\mathbf{E} \left[\int_0^T \int_0^t q_{2u}' dW_u q_{3t}' dW_t | \int_0^T q_{1v}' dW_v = x \right] =$$

$$\left(\int_0^T \int_0^t q_{2u}' q_{1u} du q_{3t}' q_{1t} dt \right) \frac{h_2(x;\Sigma)}{\Sigma^2}$$

3.

$$\mathbf{E}\left[\left(\int_0^T q'_{2u}dW_u\right)\left(\int_0^T q'_{3s}dW_s\right)\mid \int_0^T q'_{1v}dW_v = x\right] =$$

$$\left(\int_0^T q'_{2u}q_{1u}du\right)\left(\int_0^T q'_{3s}q_{1s}ds\right)\frac{h_2(x;\Sigma)}{\Sigma^2}$$

$$+\int_0^T q'_{2t}q_{3t}dt$$

3'.

$$\mathbf{E}\left[\left(\int_0^T \left(\int_0^t q_{2s}dB_s\right)q_{3t}dt\right)^2 \mid \int_0^T q_{1v}dW_v = x\right] =$$

$$\left\{2\rho^2\int_0^T q_{3t}\left(\int_0^t q_{2u}q_{1u}du\right)\left(\int_0^t \left(\int_u^t q_{3s}ds\right)q_{2u}q_{1u}du\right)dt\right\}\frac{h_2(x;\Sigma)}{\Sigma^2}$$

$$+\ 2\int_0^T q_{3t}\int_0^t q_{2u}^2\left(\int_u^t q_{3s}ds\right)dudt,$$

where B and W are one-dimensional standard Brownian motion such that $dB_t dW_t = \rho dt$, and $q_i \in L^2[0,T]$, $i = 1,2,3$.

4.

$$\mathbf{E}\left[\int_0^T\int_0^t\int_0^s q'_{2u}dW_u q'_{3s}dW_s q'_{4t}dW_t \mid \int_0^T q'_{1v}dW_v = x\right] =$$

$$\left(\int_0^T q'_{4t}q_{1t}\int_0^t q'_{3s}q_{1s}\int_0^s q'_{2u}q_{1u}dudsdt\right)\frac{h_3(x;\Sigma)}{\Sigma^3}$$

5.

$$\mathbf{E}\left[\int_0^T\left(\int_0^t q'_{2u}dW_u\right)\left(\int_0^t q'_{3s}dW_s\right)q'_{4t}dW_t\mid \int_0^T q'_{1v}dW_v = x\right] =$$

$$\left\{\int_0^T\left(\int_0^t q'_{2u}q_{1u}du\right)\left(\int_0^t q'_{3s}q_{1s}ds\right)q'_{4t}q_{1t}dt\right\}\frac{h_3(x;\Sigma)}{\Sigma^3}$$

$$+\left(\int_0^T\int_0^t q'_{2u}q_{3u}du q'_{4t}q_{1t}dt\right)\frac{h_1(x;\Sigma)}{\Sigma}$$

6.

$$\mathbf{E}\left[\left(\int_0^T\int_0^t q'_{2s}dW_s q'_{3t}dW_t\right)\left(\int_0^T q'_{4u}dW_u\right)\mid \int_0^T q'_{1v}dW_v = x\right] =$$

$$\left(\int_0^T q'_{3t}q_{1t}\int_0^t q'_{2s}q_{1s}dsdt\right)\left(\int_0^T q'_{4u}q_{1u}du\right)\frac{h_3(x;\Sigma)}{\Sigma^3}$$

$$+\left(\int_0^T q'_{3t}q_{1t}\int_0^t q'_{2s}q_{4s}dsdt+\int_0^T q'_{3t}q_{4t}\int_0^t q'_{2s}q_{1s}dsdt\right)\frac{h_1(x;\Sigma)}{\Sigma}$$

6'.

$$\mathbf{E}\left[\left(\int_0^T\int_0^t q_{2s}dB_s q_{3t}dW_t\right)\left(\int_0^T\left(\int_0^t q_{4u}dB_u\right)q_{5t}dt\right)\mid \int_0^T q_{1v}dW_v = x\right] =$$

$$\rho^2\left(\int_0^T q_{3t}q_{1t}\int_0^t q_{2s}q_{1s}dsdt\right)\left(\int_0^T q_{4u}\left(\int_u^T q_{5s}ds\right)q_{1u}du\right)\frac{h_3(x;\Sigma)}{\Sigma^3}$$

$$+\left(\int_0^T q_{3t}q_{1t}\int_0^t q_{4s}q_{2s}\left(\int_s^T q_{5u}du\right)dsdt\right.$$

$$+\rho^2\int_0^T q_{4t}q_{3t}\left(\int_t^T q_{5u}du\right)\int_0^t q_{2s}q_{1s}dsdt\right)\frac{h_1(x;\Sigma)}{\Sigma},$$

where B and W are one-dimensional standard Brownian motion such that $dB_t dW_t = \rho dt$, and $q_i \in L^2[0,T]$, $i = 1,2,3,4,5$.

7.

$$\mathbf{E}\left[\left(\int_0^T \int_0^t q'_{2s} dW_s q'_{3t} dW_t\right)\left(\int_0^T \int_0^r q'_{4u} dW_u q'_{5r} dW_r\right) \mid \int_0^T q'_{1v} dW_v = x\right] =$$

$$\left(\int_0^T q'_{3t} q_{1t} \int_0^t q'_{2s} q_{1s} ds dt\right)\left(\int_0^T q'_{5r} q_{1r} \int_0^r q'_{4u} q_{1u} du dr\right)\frac{h_4(x;\Sigma)}{\Sigma^4}$$

$$+\left\{\int_0^T q'_{3t} q_{1t} \int_0^t q'_{5r} q_{1r} \int_0^r q'_{2u} q_{4u} du dr dt + \int_0^T q'_{5t} q_{1t} \int_0^t q'_{3r} q_{1r} \int_0^r q'_{2u} q_{4u} du dr dt\right.$$

$$+\int_0^T q'_{3t} q_{1t} \int_0^t q'_{2r} q_{5r} \int_0^r q'_{4u} q_{1u} du dr dt + \int_0^T q'_{3t} q_{5t}\left(\int_0^t q'_{2s} q_{1s} ds\right)\left(\int_0^t q'_{4u} q_{1u} du\right) dt$$

$$\left.+\int_0^T q'_{5r} q_{1r} \int_0^r q'_{3u} q_{4u} \int_0^u q'_{2s} q_{1s} ds du dr\right\}\frac{h_2(x;\Sigma)}{\Sigma^2}$$

$$+\int_0^T \int_0^t q'_{2u} q_{4u} du q'_{3t} q_{5t} dt$$

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