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# Asymptotic Expansion Approaches in Finance: Applications to Currency Options

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# Asymptotic Expansion Approaches in Finance: Applications to Currency Options *

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**Abstract**

This chapter presents a basic of the methodology so-called an asymptotic expansion approach, and applies this method to approximation of prices of currency options with a libor market model of interest rates and stochastic volatility models of spot exchange rates. The scheme enables us to derive closed-form approximation formulas for pricing currency options even with high flexibility of the underlying model; we do not model a foreign exchange rate's variance such as in Heston [27], but its volatility that follows a general time-inhomogeneous Markovian process. Further, the correlations among all the factors such as domestic and foreign interest rates, a spot foreign exchange rate and its volatility, are allowed. At the end of this chapter some numerical examples are provided and the pricing formula is applied to the calibration of volatility surfaces in the JPY/USD option market.

## 1. Introduction

In this chapter we present a brief review of an asymptotic expansion approach for the evaluation problems in finance and give approximation schemes for currency options under stochastic volatility processes of spot exchange rates in stochastic interest rates environment as important applications of this methodology. In particular, we use models of volatility processes, not variance processes such as in [27], and apply a libor market model developed by Brace, Gatarek and Musiela [7] and Miltersen, Sandmann and Sondermann [53] to modeling term structures of interest rates. Moreover, the correlations among all the factors such as domestic and foreign interest rates, a spot foreign exchange rate and its volatility, are allowed.

Currency options with maturities beyond one year become common in global currencies' markets and even smiles or skews for those maturities are frequently observed. Because it is well known that the effects of interest rates become more substantial in longer maturities, we have to take term structure models into account for the currency

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options. Further, stochastic volatility models and/or jump components of foreign exchange rates are necessary for calibration of smiles and skews. As for term structure models, market models become popular in matured interest rates markets since calibrations of caps, floors and swaptions are required and market models are regarded as most useful.

Hence, development of a model with stochastic volatilities and/or jumps of exchange rates and with a libor market model of interest rates is inevitable. Moreover, a closed-form formula is desirable in practice especially for calibrations since they are very time consuming by numerical methods such as Monte Carlo simulation. Because it is too difficult to obtain an exact closed-form formula, we derive closed-form approximation formulas by an asymptotic expansion approach where a volatility of a spot exchange rate follows a general time-inhomogeneous Markovian process, and domestic and foreign interest rates are generated by a libor market model.

Here is the literature on currency options: Garman and Kohlhagen [19] and Grabbe [22] started research for currency options based on a contingent claim analysis; the framework of Black and Scholes [6], Merton [51] and Black [5] was directly applied to pricing currency options. [22]'s formula also included the case of stochastic interest rates following Gaussian processes though he did not specify the processes explicitly. Rumsey [66] and Melino and Turnbull [50] developed models under the deterministic interest rates assumption.

Amin and Jarrow [3] and Hilliard, Madura and Tucker [28] derived formulas of currency options with Gaussian stochastic interest rates; in particular, [3] combined term structure models under the framework of Heath, Jarrow and Morton(HJM) [23] with currency options.

Amin and Bodurtha [2] and Takahashi and Tokioka [83] gave numerical solutions to price currency American options with stochastic interest rates by lattice methods; [2] used HJM [23] models and [83] applied Hull and White [29], [30] term structure models. Dempster and Hutton [13] considered terminable (Bermudan) differential swaps with Gaussian interest rates models by using the partial differential equations(PDE) approach.

Schlögl [70] extended market models to a cross-currency framework. He did not take stochastic volatilities into account and focus on cross currency derivatives such as differential swaps and options on differential swaps as examples; currency options were not considered. Mikkelsen [52] considered cross-currency options with market models of interest rates and deterministic volatilities of spot exchange rates by simulation. Piterbarg [61] developed a model for cross-currency derivatives such as Power-Reverse-Dual-Currency(PRDC) swaps with calibration to currency options; neither market models nor stochastic volatility models were used.

Our asymptotic expansion approach have been applied to a broad class of Itô processes appearing in finance. It started with pricing average options; Kunitomo and Takahashi [38] derived a first order approximation and Yoshida [91] applied an asymptotic expansion method developed in statistics for stochastic processes. Takahashi [74], [75] presented second or third order schemes for pricing various options in a general Markovian setting with a constant interest rate. [39] provided approximation formulas for pricing bond options and average options on interest rates in term structure models of HJM [23] which is not necessarily Markovian.

Moreover, Takahashi and Yoshida [85], [86] extended the method to dynamic portfolio problems in a general Markovian setting and proposed a new variance reduction scheme of Monte Carlo simulation with an asymptotic expansion. For mathematical validity of the method based on Watanabe [89] in the Malliavin calculus, see Chapter 7 of Malliavin and Thalmaier [48], Yoshida [90], Kunitomo and Takahashi [40] and Takahashi and Yoshida [85], [86].

Other applications and extensions of asymptotic expansions to numerical problems in finance are found as follows: Kawai [34], Kobayashi,Takahashi and Tokioka [36], Takahashi and Saito [77], L*ü*tkebohmert [42], [43], Kunitomo and Takahashi [41], Kunitomo and Kim [37], Muroi [55], Takahashi [76], Matsuoka,Takahshi and Uchida [49], Takahashi and Uchida[84], and Takahashi and Takehara [78], [79], [80]. Moreover, the computation scheme necessary for actual evaluation of the asymptotic expansion in a general setting is given by Takahashi, Takehara and Toda [81].

The organization of this chapter is as follows: First, after some preliminaries of mathematics in Section 2., we present the framework of an asymptotic expansion in Section 3.1. in a general model. Second, Section 4. describes a basic structure of our cross-currency model as the particular setting. Then, Section 5. applies the asymptotic expansion approach to the evaluation problem in two different ways. Finally, Section 6. shows numerical examples. Some proofs, computation scheme, and concrete expressions in propositions and theorems are omitted due to limitation of space and will be found mainly in [78], [80] and [81].

# 2. Preliminary Mathematics

We shall first prepare the fundamental results including *Theorem 2.3* of Watanabe [89]. The theory by [89] on the Malliavin Calculus and Theorem 2.2 of Yoshida [90], [91] are the fundamental ingredients to show the validity of our asymptotic expansion method.

For our purpose, we shall freely use the notations by Ikeda and Watanabe [31] as a standard textbook. The interested readers should see Watanabe [88], [89], Ikeda and Watanabe [31], Yoshida [90], [91], Shigekawa [72] or Nualart [58].

## 2.1. Some Notations and Definitions

Let $\boldsymbol{W}$ be the $r$-dimensional Wiener space, which is a Banach space consisting of the totality of continuous functions $w : [0, T] \to R^r$ $(w(0) = 0)$ with the topology induced by the norm $\| w \| = \max_{0\leq t\leq T} |w(t)|$ . Let also $\boldsymbol{H}$ be the Cameron-Martin subspace of $\boldsymbol{W}$, where $h(t) = (h^j(t)) \in \boldsymbol{H}$ is in $\boldsymbol{W}$ and is absolutely continuous on $[0, T]$ with square integrable derivative $\dot{h}(t)$ endowed with the inner product defined by

$$< h_1, h_2 >_H = \sum_{j=1}^{r} \int_0^T \dot{h}_1^j(s)\dot{h}_2^j(s)ds \ . \tag{1}$$

We shall use the notation of the $H-$norm as $|h|_H^2 =< h, h >_H$ for any $h \in \boldsymbol{H}$. A function $f : \boldsymbol{W} \mapsto \boldsymbol{R}$ is called a polynomial functional if there exist $n \in \boldsymbol{N}$, $h_1, h_2, \cdots, h_n \in \boldsymbol{H}$ and a real polynomial $p(x_1, x_2, \cdots, x_n)$ of n-variables such that $f(w) = p([h_1](w), [h_2](w), \cdots, [h_n](w))$ for $h_i = (h_i^j) \in \boldsymbol{H}$, where

$$[h_i](w) = \sum_{j=1}^{r} \int_0^T \dot{h}_i^j dw^j \tag{2}$$

are defined in the sense of Itô's stochastic integrals.

The standard $L_p$-norm of $\boldsymbol{R}$-valued Wiener functional $F$ is defined by $\|F\|_p = (\int_W |F|^p P(dw))^{1/p}$ . Also a sequence of the norms of $\boldsymbol{R}$-valued Wiener functional $F$ for any $s \in \boldsymbol{R}$, and $p \in (1, \infty)$ is defined by

$$\|F\|_{p,s} = \|(I - \mathcal{L})^{s/2} F\|_p \ , \tag{3}$$

where $\mathcal{L}$ is the Ornstein-Uhlenbeck operator and $\| \cdot \|_p$ is the $\boldsymbol{L}_p$-norm. The O-U operator in (3) means that $(I - \mathcal{L})^{s/2} F = \sum_{n=0}^{\infty} (1 + n)^{s/2} J_n F$ , where $J_n$ are the projection operators in the Wiener's homogeneous chaos decomposition in $\boldsymbol{L}_2$. They are constructed by the totality of $\boldsymbol{R}$-valued polynomials of degree at most $n$ denoted by $\boldsymbol{P}_n$ .

Let $\boldsymbol{P}(= \boldsymbol{P}(\boldsymbol{R}))$ denote the totality of $\boldsymbol{R}$-valued polynomials on the Wiener space $(\boldsymbol{W}, P)$. Then $\boldsymbol{P}$ is dense in $\boldsymbol{L}_p$ and can be extended to the totality of smooth functionals $\boldsymbol{S}(= \boldsymbol{S}(\boldsymbol{R}))$ (the $\boldsymbol{R}$-valued $C^\infty$ functions with derivatives of polynomial growth orders). Then we can construct the Banach space $\boldsymbol{D}_p^s$ as the completion of $\boldsymbol{P}$ with respect to $\| \cdot \|_{p,s}$. The dual space of $\boldsymbol{D}_p^s$ is the $\boldsymbol{D}_q^{-s}$ where $s \in \boldsymbol{R}$, $p > 1$, and $1/p + 1/q = 1$. Set

$$\begin{aligned}
&\boldsymbol{D}^\infty = \cap_{s>0, 1<p<+\infty} \boldsymbol{D}_p^s, \\
&\boldsymbol{D}^{-\infty} = \cup_{s>0, 1<p<+\infty} \boldsymbol{D}_p^{-s}, \\
&\tilde{\boldsymbol{D}}^\infty = \cap_{s>0} \cup_{1<p<+\infty} \boldsymbol{D}_p^s, \\
&\text{and} \\
&\tilde{\boldsymbol{D}}^{-\infty} = \cup_{s>0} \cap_{1<p<+\infty} \boldsymbol{D}_p^{-s}.
\end{aligned}$$

More generally, for a separable Hilbert space $\boldsymbol{E}$, a function $f : \boldsymbol{W} \mapsto \boldsymbol{E}$ is called a polynomial functional if there exist $n \in \boldsymbol{N}$, $h_1, h_2, \cdots, h_n \in \boldsymbol{H}$ and real polynomials $p_i(x_1, x_2, \cdots, x_n)$ of n-variables such that

$$f(w) = \sum_{i=1}^{d} p_i([h_1](w), [h_2](w), \cdots, [h_n](w)) e_i$$

for some $d \in \boldsymbol{N}$, where $e_1, \cdots, e_d \in \boldsymbol{E}$. The totality of $\boldsymbol{E}-$valued polynomial functions and the totality of $\boldsymbol{E}-$valued smooth functionals are denoted by $\boldsymbol{P}(\boldsymbol{E})$ and $\boldsymbol{S}(\boldsymbol{E})$, respectively. Then, similar arguments as above hold and we can construct the spaces $\boldsymbol{D}_p^s(\boldsymbol{E})(s \in \boldsymbol{R},\ p \in (1, \infty))$, $\boldsymbol{D}^\infty(\boldsymbol{E})$, $\boldsymbol{D}^{-\infty}(\boldsymbol{E})$, $\tilde{\boldsymbol{D}}^\infty(\boldsymbol{E})$ and $\tilde{\boldsymbol{D}}^{-\infty}(\boldsymbol{E})$. We call elements in $\mathbf{D}_p^s(\boldsymbol{E})$, $s \geq 0$ Wiener functionals and elements in $\mathbf{D}_p^s(\boldsymbol{E})$, $s < 0$ generalized Wiener functionals.

For $F \in \boldsymbol{P}$ and $h \in \boldsymbol{H}$, the derivative of $F$ in the direction of $h$ is defined by

$$D_h F(w) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{F(w + \varepsilon h) - F(w)\} \ . \tag{4}$$

Then for $F \in \boldsymbol{P}$ and $h \in \boldsymbol{H}$ there exists $DF \in \boldsymbol{P}(\boldsymbol{H} \otimes \boldsymbol{R})$ such that $D_h F(w) =< DF(w), h >_H$, where $< \cdot >_H$ is the inner product of $\boldsymbol{H}$ and $DF$ is called the $H-$derivative of $F$. Also for $F \in \boldsymbol{S}$ there exists a unique $DF \in \boldsymbol{S}(\boldsymbol{H} \otimes \boldsymbol{R})$ . By extending the above construction for $\boldsymbol{P}$ to $\boldsymbol{S}(\boldsymbol{E})$, there exists $DF \in \boldsymbol{S}(\boldsymbol{H} \otimes \boldsymbol{E})$ such that $D_h F(w) =< DF(w), h >_H$.
By repeating this procedure, we can sequentially define the $k-$th order $H-$derivative $D^k F \in \boldsymbol{S}(\boldsymbol{H}^{\otimes k} \otimes \boldsymbol{E})$ for $k \geq 1$ and it is known that the norm $\| \cdot \|_{p,s}$ is equivalent to the norm $\sum_{k=0}^{s} \|D^k \cdot \|_p$. In particular, for $F = (F^i) \in \boldsymbol{D}_p^1(\boldsymbol{R}^d)$, we define the Malliavin-covariance by

$$\sigma_{MC}(F) = (< DF^i(w), DF^j(w) >_H) \ ( i, j = 1, \cdots, d) \ . \tag{5}$$

The coupling

$$_{D^{-\infty}(E)} < \Phi, F >_{D^{\infty}(E)}, \ \Phi \in \boldsymbol{D}^{-\infty}(\boldsymbol{E}), \ F \in \boldsymbol{D}^{\infty}(\boldsymbol{E})$$

is denoted by $\mathbf{E}[< \Phi, F >_E]$ and it is called a *generalized expectation*. In particular, $1 \in \boldsymbol{D}^{\infty}$ where 1 is the functional identically equal to 1. Hence, for $\Phi \in \mathbf{D}^{-\infty}$, $_{D^{-\infty}} < \Phi, 1 >_{D^{\infty}}$ is called the generalied expectation of $\Phi$ and is denoted by the usual notation $\mathbf{E}[\Phi]$ because it is compatible when $\Phi \in \cup_{1<p<\infty} \boldsymbol{L}_p$.

## 2.2. Definitions and Existence of Asymptotic Expansions

Let $X^{(\varepsilon)}(w) = (X^{(\varepsilon),i}(w))$ $(i = 1, \cdots, d \ ; \varepsilon \in (0,1])$ be a Wiener functional with a parameter $\varepsilon$. Then we need to define the asymptotic expansion of $X^{(\varepsilon)}(w)$ with respect to $\varepsilon$ in the proper mathematical sense. For $k > 0$ , $X^{(\varepsilon)}(w) = O(\varepsilon^k)$ in $\boldsymbol{D}_p^s(\boldsymbol{E})$ as $\varepsilon \downarrow 0$ means that

$$\limsup_{\varepsilon \downarrow 0} \frac{\| X^{(\varepsilon)} \|_{p,s}}{\varepsilon^k} < +\infty \ . \tag{6}$$

If for all $p > 1, s > 0$ and every $k = 1, 2, \cdots$,

$$X^{(\varepsilon)}(w) - (g_1 + \varepsilon g_2 + \cdots + \varepsilon^{k-1} g_k) = O(\varepsilon^k) \tag{7}$$

in $\boldsymbol{D}_p^s(\boldsymbol{E})$ as $\varepsilon \downarrow 0$, then we say that $X^{(\varepsilon)}(w)$ has an asymptotic expansion :

$$X^{(\varepsilon)}(w) \sim g_1 + \varepsilon g_2 + \cdots \tag{8}$$

in $\boldsymbol{D}^{\infty}(\boldsymbol{E})$ as $\varepsilon \downarrow 0$ with $g_1, g_2, \cdots \in \boldsymbol{D}^{\infty}(\boldsymbol{E})$.
Also if for every $k = 1, 2, \cdots$, there exists $s > 0$ such that, for all $p > 1$, $X^{(\varepsilon)}(w), g_1, g_2, \cdots \in \boldsymbol{D}_p^{-s}(\boldsymbol{E})$ and

$$X^{(\varepsilon)}(w) - (g_1 + \varepsilon g_2 + \cdots + \varepsilon^{k-1} g_k) = O(\varepsilon^k) \tag{9}$$

in $\boldsymbol{D}_p^{-s}(\boldsymbol{E})$ as $\varepsilon \downarrow 0$, then we say that $X^{(\varepsilon)}(w) \in \tilde{\boldsymbol{D}}^{-\infty}(\boldsymbol{E})$ has an asymptotic expansion:

$$X^{(\varepsilon)}(w) \sim g_1 + \varepsilon g_2 + \cdots \tag{10}$$

in $\tilde{\boldsymbol{D}}^{-\infty}(\boldsymbol{E})$ as $\varepsilon \downarrow 0$ with $g_1, g_2, \cdots \in \tilde{\boldsymbol{D}}^{-\infty}(\boldsymbol{E})$ .

Let $\mathcal{S}(\boldsymbol{R}^d)$ be the real Schwartz space of rapidly decreasing $\mathbf{C}^{\infty}$-functions on $\mathbf{R}^d$ and $\mathcal{S}^{'}(\boldsymbol{R}^d)$ be its dual space that is the space of the Schwartz tempered distributions. Also $X^{(\varepsilon)} \in \boldsymbol{D}^{\infty}(\boldsymbol{R}^d)$ is said to be non-degenerate (in the sense of Malliavin) if for any $p > 1$ the Malliavin-covariance of $X^{(\varepsilon)}$ satisfies

$$\sup_{\varepsilon \in (0,1]} \mathbf{E}[\left(\det[\sigma_{MC}(X^{(\varepsilon)})]\right)^{-p}] < \infty. \tag{11}$$

Suppose that $X^{(\varepsilon)} \in \boldsymbol{D}^{\infty}(\boldsymbol{R}^d)$ satisfies the nondegeneracy condition (11). Then, it has been known that every Schwartz tempered distribution $T(x)$ on $\mathbf{R}^d$ can be lifted up or pulled-back to a generalized Wiener functional $T \circ X^{(\varepsilon)}$ (denoted by $T(X^{(\varepsilon)})$) in $\tilde{\mathbf{D}}^{-\infty}$ under the Wiener map: $w \in \boldsymbol{W} \mapsto X^{(\varepsilon)}(w) \in \mathbf{R}^d$. Since $T \circ X^{(\varepsilon)} \in \tilde{\mathbf{D}}^{-\infty}$, it can act on any test functional in $\tilde{\mathbf{D}}^{\infty}$, which is much larger than $\mathbf{D}^{\infty}$.

With these formulations and notations we are ready to state *Theorem 2.3* of [89].

**Theorem 1** [ *Theorem 2.3 of Watanabe [89]* ] *: Let* $\{X^{(\epsilon)}(w); \epsilon \in (0,1]\}$ *be a family of elements in* $\mathbf{D}^{\infty}(\mathbf{R}^d)$ *such that it has the asymptotic expansion:*

$$X^{(\epsilon)}(w) \sim g_1 + \epsilon g_2 + \cdots \ \ \textit{in } \mathbf{D}^{\infty}(\mathbf{R}^d) \textit{ as } \epsilon \downarrow 0$$

*with* $g_i \in \mathbf{D}^{\infty}(\mathbf{R}^d)$*,* $i = 1, 2, \cdots$ *and satisfies*

$$\limsup_{\epsilon \downarrow 0} \|(\det \sigma_{MC}(X^{(\epsilon)}))^{-1}\|_p < \infty \ \ \textit{for all } 1 < p < \infty \tag{12}$$

*where* $\sigma_{MC}(X^{(\epsilon)}) = (\sigma^{ij}(X^{(\epsilon)}))$ *is the Malliavin covariance of* $X^{(\epsilon)}(w)$*:*
$\sigma^{ij}(X^{(\epsilon)}) =< DX^{(\epsilon),i}(w), DX^{(\epsilon),j}(w) >_H$*. Let* $T \in \mathcal{S}'(\mathbf{R}^{\Gamma})$*. Then,* $\Phi(\epsilon, w) = T \circ X^{(\epsilon)}(w)$ *has the asymptotic expansion in* $\tilde{\mathbf{D}}^{-\infty}$ *(and a fortiori in* $\mathbf{D}^{-\infty}$*):*

$$\Phi(\epsilon, w) \sim \phi_1 + \epsilon\phi_2 + \cdots \ \ \textit{in } \tilde{\mathbf{D}}^{-\infty} \textit{ as } \epsilon \downarrow 0$$

*and* $\phi_i \in \tilde{\mathbf{D}}^{-\infty}$*,* $i = 0, 1, \cdots$ *are determined by the formal Taylor expansion:*

$$\begin{aligned} \Phi(\epsilon, w) &= T(g_1 + [\epsilon g_2 + \epsilon^2 g_3 + \cdots]) \\ &= \sum_{\alpha} \frac{1}{\alpha!}(D_\alpha T) \circ g_1[\epsilon g_2 + \epsilon^2 g_3 + \cdots]^{\alpha} \\ &= \phi_1 + \epsilon\phi_2 + \cdots \end{aligned}$$

*where (i) the summation is taken over all multi-indices and (ii) for every multi-index* $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$ *and* $a = (a_1, a_2, \cdots, a_d) \in \mathbf{R}^d$*, we set as usual*

$$\alpha! = \alpha_1!\alpha_2! \cdots \alpha_d! \textit{ and } a^{\alpha} = a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_d^{\alpha_d}.$$

[90], [91] provided so called "the truncated version" of this theorem. His result is very important from viewpoint of applications because in his version checking the non-degeneracy of $X^{(\epsilon)}(w)$ when $\epsilon = 0$ is enough, which is usually much easier than in the original one. Moreover, he also derived conditional expectation formulas up to the second order that are very useful to obtain explicit approximations. See [90], [91] for the detail.

# 3. An Asymptotic Expansion Approach

## 3.1. An Asymptotic Expansion in a General Markovian Setting

Let $(W, P)$ be the $r$-dimensional Wiener space. We consider a $d$-dimensional diffusion process $X^{(\epsilon)} = \{X_t^{(\epsilon)} = (X_t^{(\epsilon),1}, \cdots, X_t^{(\epsilon),d})\}$ which is the solution to the following stochastic differential equation:

$$\begin{aligned} dX_t^{(\epsilon),i} &= V_0^i(X_t^{(\epsilon)}, \epsilon)dt + \epsilon V^i(X_t^{(\epsilon)})dW_t \quad (i = 1, \cdots, d) \\ X_0^{(\epsilon)} &= x_0 \in \mathbf{R}^d \end{aligned} \tag{13}$$

where $W = (W^1, \cdots, W^r)$ is a $r$-dimensional standard Wiener process, and $\epsilon \in (0, 1]$ is a known parameter.

Suppose that $V_0 = (V_0^1, \cdots, V_0^d) : \mathbf{R}^d \times (0, 1] \mapsto \mathbf{R}^d$ and $V = (V^1, \cdots, V^d)$: $\mathbf{R}^d \mapsto \mathbf{R}^d \otimes \mathbf{R}^r$ satisfy some regularity conditions.(e.g. $V_0$ and $V$ are smooth functions with bounded derivatives at any order.)

Next, suppose that a function $g:\mathbf{R}^d \mapsto \mathbf{R}$ to be smooth and all derivatives have polynomial growth orders. Then, $g(X_T^{(\epsilon)})$ has its asymptotic expansion;

$$g(X_T^{(\epsilon)}) \sim g_{0T} + \epsilon g_{1T} + \cdots$$

in $L^p$ for every $p > 1$(or in $\mathbf{D}^\infty$) as $\epsilon \downarrow 0$. $g_{nT} \in \mathbf{D}^\infty(n = 0,1,\cdots)$, the coefficients in the expansion, can be obtained by formal Taylor's formula and represented based on multiple Wiener-Ito integrals.

Let $A_{kt} = \frac{\partial^k X_t^{(\epsilon)}}{\partial \epsilon^k}|_{\epsilon=0}$ and $A_{kt}^i$, $i = 1,\cdots,d$ denote the $i$-th elements of $A_{kt}$. In particular, $A_{1t}$ is represented by

$$A_{1t} = \int_0^t Y_t Y_u^{-1}\left(\partial_\epsilon V_0(X_u^{(0)},0)du + V(X_u^{(0)})dW_u\right) \tag{14}$$

where $Y$ denotes the solution to the differential equation;

$$dY_t = \partial V_0(X_t^{(0)},0)Y_t dt;\ Y_0 = I_d.$$

Here, $\partial V_0$ denotes the $d\times d$ matrix whose $(j,k)$-element is $\partial_k V_0^j = \frac{\partial V_0^j(x,\epsilon)}{\partial x_k}$, $V_0^j$ is the $j$-th element of $V_0$, and $I_d$ denotes the $d\times d$ identity matrix.

For $k\geq 2$, $A_{kt}^i$, $i = 1,\cdots,d$ is recursively determined by the following:

$$\begin{aligned} A_{kt}^i &= \int_0^t \partial_\epsilon^k V_0^i(X_s^{(0)},0)ds \\ &+ \sum_{l=1}^k \frac{k!}{l!(k-l)!}\sum_{\beta=1}^l \sum_{\vec{l}_\beta\in L_{\beta,l}} \int_0^t \frac{1}{\beta!}\sum_{d_1,\cdots,d_\beta=1}^d \partial_{d_1,\cdots,d_\beta}^\beta \partial_\epsilon^{k-l} V_0^i(X_s^{(0)},0)\prod_{j=1}^\beta A_{l_j,s}^{d_j}ds \\ &+ \sum_{\beta=1}^k \sum_{\vec{l}_\beta\in L_{\beta,k-1}} \int_0^t \frac{1}{\beta!}\sum_{\alpha=1}^r \sum_{d_1,\cdots,d_\beta=1}^d \partial_{d_1,\cdots,d_\beta}^\beta V_\alpha^i(X_s^{(0)})\prod_{j=1}^\beta A_{l_j,s}^{d_j}dW_s^\alpha, \end{aligned} \tag{15}$$

where $\partial_\epsilon^l = \frac{\partial^l}{\partial \epsilon^l}$, $\partial_{d_1,\cdots,d_\beta}^\beta = \frac{\partial^\beta}{\partial x_{d_1}\cdots\partial x_{d_\beta}}$ and

$$L_{\beta,k} = \left\{\vec{l}_\beta = (l_1,\cdots,l_\beta); l_j\geq 0(j=1,\cdots,\beta), \sum_{j=1}^\beta l_j = k\right\}.$$

Then, $g_{0T}$ and $g_{1T}$ can be written as

$$\begin{aligned} g_{0T} &= g(X_T^{(0)}), \\ g_{1T} &= \sum_{i=1}^d \partial_i g(X_T^{(0)})A_{1T}^i. \end{aligned}$$

For $n\geq 2$, $g_{nT}$ is expressed as follows:

$$g_{nT} = \sum_{\vec{s}\in S_n}\left(\frac{n!}{s_1!\cdots s_n!}\right)\prod_{l=1}^n\left(\frac{1}{l!}\right)^{s_l}\sum_{\vec{p}^{s_l}\in P_{s_l}}\left(\frac{s_l!}{p_1^{s_l}!\cdots p_d^{s_l}!}\right)\partial_1^{p_1^{s_l}}\cdots\partial_d^{p_d^{s_l}} g(X_T^{(0)})\prod_{i=1}^d\left(A_{lT}^i\right)^{p_i^{s_l}} \tag{16}$$

where

$$\begin{aligned} S_n &:= \left\{\vec{s} = (s_1,\cdots,s_n);\ s_l\geq 0(l=1,\cdots,n),\ \sum_{l=1}^n l s_l = n\right\}, \\ P_s &:= \left\{\vec{p}^s = (p_1^s,\cdots,p_d^s);\ p_i^s\geq 0(i=1,\cdots,d),\ \sum_{i=1}^d p_i^s = s\right\}. \end{aligned}$$

Next, normalize $g(X_T^{(\epsilon)})$ to

$$G^{(\epsilon)} = \frac{g(X_T^{(\epsilon)}) - g_{0T}}{\epsilon}$$

for $\epsilon \in (0,1]$. Then,

$$G^{(\epsilon)} \sim g_{1T} + \epsilon g_{2T} + \cdots$$

in $L^p$ for every $p > 1$(or in $\mathbf{D}^\infty$). Moreover, let

$$\hat{V}(x,t) = (\partial g(x))^{'}[Y_T Y_t^{-1} V(x)]$$

and make the following assumption:

$$\text{(Assumption 1)} \quad \Sigma_T = \int_0^T \hat{V}(X_t^{(0)},t)\hat{V}(X_t^{(0)},t)^{'} dt > 0.$$

Note that $g_{1T}$ follows a normal distribution with variance $\Sigma_T$; the denstiy function of $g_{1T}$ denoted by $f_{g_{1T}}(x)$ is given by

$$f_{g_{1T}}(x) = \frac{1}{\sqrt{2\pi\Sigma_T}} \exp\left(-\frac{x^2}{2\Sigma_T}\right).$$

Hence, Assumption 1 means that the distribution of $g_{1T}$ does not degenerate. In application, it is easy to check this condition in most cases. Hereafter, Let $\mathcal{S}$ be the real Schwartz space of rapidly decreasing $\mathbf{C}^\infty$-functions on $\mathbf{R}$ and $\mathcal{S}'$ be its dual space that is the space of the Schwartz tempered distributions. Next, take $\Phi \in \mathcal{S}'$. Then, by Watanabe theory([89], [90]) $\Phi(G^{(\epsilon)})$ has an asymptotic expansion in $\tilde{\mathbf{D}}^{-\infty}$(a fortiori in $\mathbf{D}^{-\infty}$) as $\epsilon \downarrow 0$. In other words, the expectation of $\Phi(G^{(\epsilon)})$ is expanded around $\epsilon = 0$ as follows: For $M = 0, 1, 2, \cdots$,

$$\begin{aligned}
\mathbf{E}[\Phi(G^{(\epsilon)})] &= \sum_{j=0}^{M} \epsilon^j \sum_{m=0}^{j} \frac{1}{m!} \mathbf{E}\left[\Phi^{(m)}(g_{1T}) \left\{ \sum_{k \in K_{j,m}} C^{j,m,k} \prod_{n=1}^{j-m+1} g_{(n+1)T}^{k_n} \right\}\right] + o(\epsilon^M) \\
&= \sum_{j=0}^{M} \epsilon^j \sum_{m=0}^{j} \frac{1}{m!} \mathbf{E}\left[\Phi^{(m)}(g_{1T}) \sum_{k \in K_{j,m}} C^{j,m,k} \mathbf{E}\left[X^{j,m,k} \middle| g_{1T}\right]\right] + o(\epsilon^M) \\
&= \sum_{j=0}^{M} \epsilon^j \sum_{m=0}^{j} \frac{1}{m!} \int_{\mathbf{R}} \Phi^{(m)}(x) \sum_{k \in K_{j,m}} C^{j,m,k} \mathbf{E}\left[X^{j,m,k} \middle| g_{1T} = x\right] f_{g_{1T}}(x) dx + o(\epsilon^M) \\
&= \sum_{j=0}^{M} \epsilon^j \sum_{m=0}^{j} \frac{1}{m!} \int_{\mathbf{R}} \Phi(x) \sum_{k \in K_{j,m}} C^{j,m,k} (-1)^m \frac{\partial^m}{\partial x^m} \{\mathbf{E}\left[X^{j,m,k} \middle| g_{1T} = x\right] f_{g_{1T}}(x)\} dx + o(\epsilon^M)
\end{aligned} \tag{17}$$

where $\Phi^{(m)}(g_{1T}) = \frac{\partial^m \Phi(x)}{\partial x^m}\Big|_{x=g_{1T}}$,

$$K_{j,m} = \left\{(k_1, \cdots, k_{j-m+1}); k_n \geq 0, \sum_{n=1}^{j-m+1} k_n = m, \sum_{n=1}^{j-m+1} n k_n = j\right\},$$

$$\begin{aligned}
X^{j,m,k} &= \prod_{n=1}^{j-m+1} g_{(n+1)T}^{k_n}, \\
C^{j,m,k} &= \prod_{n=1}^{j-m+1} \frac{m!}{k_1! \cdots k_{j-m+1}!}.
\end{aligned}$$

As shown under a simple setting in the next subsection, the conditional expectations in (17) can be expressed as linear combinations of a finite number of Hermite polynomials of $g_{1T}$. Then, you can easily implement the differentiation and integration in (17) using the following property of a Gaussian distribution;

$$\frac{\partial}{\partial x}\{H_n(x;\Sigma)f_{g_{1T}}(x)\} = -\frac{1}{\Sigma}H_{n+1}(x;\Sigma)f_{g_{1T}}(x) \tag{18}$$

for $n \geq 0$ where $H_n(x;\Sigma)$ is the $n$-th order Hermite polynomial defined by

$$H_n(x;\Sigma) := (-\Sigma)^n e^{x^2/2\Sigma}\frac{d^n}{dx^n}e^{-x^2/2\Sigma}. \tag{19}$$

These are proven in more general cases by [81] which provides us the methods for actual computation of $\mathbf{E}[X^{j,m,k}|g_{1T} = x]$ and formulas useful for high-order computation.

Thus, once you compute the conditional expectations explicitly, you also have the explicit expansion of $\mathbf{E}[\Phi(G^{(\epsilon)})]$. Moreover, the asymptotic expansion of the probability function of $G^{(\epsilon)}$ can be obtained by letting $\Phi$ be $\delta_x$, the delta function with a mass at $x$.

## 3.2. An Asymptotic Expansion in a Black-Scholes Economy: a simple application

In this subsection, the asymptotic expansion approach described so far is applied to an evaluation problem in a simple Black-Scholes-type economy in order to make a whole procedure in application clearer.

Let $(W, P)$ be a one-dimensional Wiener space. Hereafter $P$ is considered as a risk-neutral equivalent martingale measure and a risk-free interest rate is set to be constantly zero for simplicity. Then, the underlying economy is specified with a ($\mathbf{R}_+$-valued)single risky asset $S^{(\epsilon)} = \{S_t^{(\epsilon)}\}$ satisfying

$$S_t^{(\epsilon)} = S_0 + \epsilon\int_0^t \sigma(S_s^{(\epsilon)}, s)dW_s \tag{20}$$

where $\epsilon \in (0, 1]$ is a constant parameter; $\sigma$: $\mathbf{R}_+^2 \mapsto \mathbf{R}$ satisfies some regularity conditions. We will consider the following pricing problem;

$$V(0, T) = \mathbf{E}[\Phi(S_T^{(\epsilon)})] \tag{21}$$

where $\Phi$ is a payoff function and $\mathbf{E}[\,\cdot\,]$ is an expectation operator under the probability measure $P$. For their rigorous definitions, see Section 2.

Let $A_{kt} = \frac{\partial^k S_t^{(\epsilon)}}{\partial\epsilon^k}|_{\epsilon=0}$. Here we represent $A_{1t}$, $A_{2t}$ and $A_{3t}$ explicitly by

$$\begin{aligned} A_{1t} &= \int_0^t \sigma(S_s^{(0)}, s)dW_s, && (22)\\ A_{2t} &= 2\int_0^t \partial\sigma(S_s^{(0)}, s)A_{1s}dW_s, && (23)\\ A_{3t} &= 3\int_0^t \left(\partial^2\sigma(S_s^{(0)}, s)(A_{1s})^2 + \partial\sigma(S_s^{(0)}, s)(A_{2s})\right)dW_s && (24) \end{aligned}$$

recursively and then $S_T^{(\epsilon)}$ has its asymptotic expansion

$$S_T^{(\epsilon)} = S_0 + \epsilon A_{1T} + \frac{\epsilon^2}{2!} A_{2T} + \frac{\epsilon^3}{3!} A_{3T} + o(\epsilon^3). \tag{25}$$

Note that $S_t^{(0)} = \lim_{\epsilon \downarrow 0} S_t^{(\epsilon)} = S_0$ for all $t$.

Next, normalize $S_T^{(\epsilon)}$ with respect to $\epsilon$ as

$$G^{(\epsilon)} = \frac{S_T^{(\epsilon)} - S_T^{(0)}}{\epsilon}$$

for $\epsilon \in (0, 1]$. Then,

$$G^{(\epsilon)} = A_{1T} + \frac{\epsilon}{2!} A_{2T} + \frac{\epsilon^2}{3!} A_{3T} + o(\epsilon^2) \tag{26}$$

in $L^p$ for every $p > 1$. Here the following assumption is made:

$$\Sigma_T = \int_0^T \sigma^2(S_t^{(0)}, t) dt > 0. \tag{27}$$

Note that $A_{1T}$ follows a normal distribution with mean 0 and variance $\Sigma_T$, and hence this assumption means that the distribution of $A_{1T}$ does not degenerate. It is clear that this assumption is satisfied when $\sigma(S_t^{(0)}, t) > 0$ for some $t > 0$.

Then setting $M = 2$, (17), the expansion of $\mathbf{E}\left[\Phi(G^{(\epsilon)})\right]$, is written as follows in the setting we are considering (hereafter in this section the asymptotic expansion of $\mathbf{E}[\Phi(G^{(\epsilon)})]$ up to the second order will be considered):

$$\begin{aligned}
\mathbf{E}[\Phi(G^{(\epsilon)})] &= \int_{\mathbf{R}} \Phi(x) f_{A_{1T}}(x) dx + \epsilon \int_{\mathbf{R}} \Phi(x)(-1) \frac{\partial}{\partial x} \{\mathbf{E}\left[A_{2T} | A_{1T} = x\right] f_{A_{1T}}(x)\} dx \\
&+ \epsilon^2 \left( \int_{\mathbf{R}} \Phi(x)(-1) \frac{\partial}{\partial x} \{\mathbf{E}\left[A_{3T} | A_{1T} = x\right] f_{A_{1T}}(x)\} dx \right. \\
&\left. + \frac{1}{2} \int_{\mathbf{R}} \Phi(x)(-1)^2 \frac{\partial^2}{\partial x^2} \{\mathbf{E}\left[(A_{2T})^2 | A_{1T} = x\right] f_{A_{1T}}(x)\} dx \right) + o(\epsilon^2).
\end{aligned} \tag{28}$$

where $f_{A_{1T}}(x)$ is a probability density function of $A_{1T}$ following a normal distribution;

$$f_{A_{1T}}(x) := \frac{1}{\sqrt{2\pi\Sigma_T}} \exp\left(-\frac{x^2}{2\Sigma_T}\right). \tag{29}$$

Then, all we have to do to evaluate this expansion is a computation of the conditional expectations in (28).

In the following, it will be shown that $A_{2T}$, $A_{3T}$, $(A_{2T})^2$ can be expressed as summations of a finite number of iterated Itô integrals. First, note that $A_{2T}$ is

$$A_{2T} = 2 \int_0^T \int_0^{t_1} \partial\sigma(S_{t_1}^{(0)}, t_1) \sigma(S_{t_2}^{(0)}, t_2) dW_{t_2} dW_{t_1}, \tag{30}$$

that is a twice-iterated Itô' integral.

Next, by application of Itô's formula to (24) we obtain

$$\begin{aligned}
A_{3T} &= 6\int_0^T\int_0^{t_1}\int_0^{t_2}\partial\sigma(S^{(0)}_{t_1},t_1)\partial\sigma(S^{(0)}_{t_2},t_2)\sigma(S^{(0)}_{t_3},t_3)dW_{t_3}dW_{t_2}dW_{t_1}\\
&+ 6\int_0^T\int_0^{t_1}\int_0^{t_2}\partial^2\sigma(S^{(0)}_{t_1},t_1)\sigma(S^{(0)}_{t_2},t_2)\sigma(S^{(0)}_{t_3},t_3)dW_{t_3}dW_{t_2}dW_{t_1}\\
&+ 3\int_0^T\int_0^{t_1}\partial^2\sigma(S^{(0)}_{t_1},t_1)\sigma^2(S^{(0)}_{t_2},t_2)dt_2dW_{t_1}. \qquad (31)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(A_{2T})^2 &= 16\int_0^T\int_0^{t_1}\int_0^{t_2}\int_0^{t_3}\partial\sigma(S^{(0)}_{t_1},t_1)\partial\sigma(S^{(0)}_{t_2},t_2)\sigma(S^{(0)}_{t_3},t_3)\sigma(S^{(0)}_{t_4},t_4)dW_{t_4}dW_{t_3}dW_{t_2}dW_{t_1}\\
&+ 8\int_0^T\int_0^{t_1}\int_0^{t_2}\int_0^{t_3}\partial\sigma(S^{(0)}_{t_1},t_1)\sigma(S^{(0)}_{t_2},t_2)\partial\sigma(S^{(0)}_{t_3},t_3)\sigma(S^{(0)}_{t_4},t_4)dW_{t_4}dW_{t_3}dW_{t_2}dW_{t_1}\\
&+ 8\int_0^T\int_0^{t_1}\int_0^{t_2}\partial\sigma(S^{(0)}_{t_1},t_1)\partial\sigma(S^{(0)}_{t_2},t_2)\sigma^2(S^{(0)}_{t_3},t_3)dt_3dW_{t_2}dW_{t_1}\\
&+ 8\int_0^T\int_0^{t_1}\int_0^{t_2}\partial\sigma(S^{(0)}_{t_1},t_1)\partial\sigma(S^{(0)}_{t_2},t_2)\sigma(S^{(0)}_{t_2},t_2)\sigma(S^{(0)}_{t_3},t_3)dW_{t_3}dt_2dW_{t_1}\\
&+ 8\int_0^T\int_0^{t_1}\int_0^{t_2}\left(\partial\sigma(S^{(0)}_{t_1},t_1)\right)^2\sigma(S^{(0)}_{t_2},t_2)\sigma(S^{(0)}_{t_3},t_3)dW_{t_3}dW_{t_2}dt_1\\
&+ 4\int_0^T\int_0^{t_1}\left(\partial\sigma(S^{(0)}_{t_1},t_1)\right)^2\sigma^2(S^{(0)}_{t_2},t_2)dt_2dt_1. \qquad (32)
\end{aligned}$$

Moreover, there is a well-known result on conditional expectations of iterated Itô integrals.

**Proposition 1** *Let $J_n(f_n)$ denote the n-times iterated Itô integral of $L^2(\mathbf{T}^n)$-function $f_n$:*

$$J_n(f_n) := \int_0^T\int_0^{t_1}\cdots\int_0^{t_{n-1}} f_n(t_1,\cdots,t_n)dW_{t_n}\cdots dW_{t_2}dW_{t_1}$$

*for $n \geq 1$ and $J_0(f_0) := f_0$(constant).*

*Then, its expectation conditional on $J_1(q) = x$ is given by*

$$\mathbf{E}[J_n(f_n)|J_1(q) = x] = \left(\int_0^T\int_0^{t_1}\cdots\int_0^{t_{n-1}} f_n(t_1,\cdots,t_n)q(t_1)\cdots q(t_n)dt_n\cdots dt_2dt_1\right)\frac{H_n(x;||q||^2_{L^2(\mathbf{T})})}{(||q||^2_{L^2(\mathbf{T})})^n} \qquad (33)$$

*where $\mathbf{T} = [0,T]$ and $t_i \in \mathbf{T}(i = 1,2,\cdots,n)$.*

*(proof)* See [59] or [81].□

Then, thanks to this proposition, the conditional expectations in (28) can be computed as

$$\mathbf{E}[A_{2T}|A_{1T} = x] = \left(2\int_0^T\int_0^{t_1}\partial\sigma(S^{(0)}_{t_1},t_1)\sigma(S^{(0)}_{t_1},t_1)\sigma^2(S^{(0)}_{t_2},t_2)dt_2dt_1\right)\frac{H_2(x;\Sigma_T)}{\Sigma_T^2} \qquad (34)$$

$$\mathbf{E}[A_{3T}|A_{1T} = x] = \Bigg(6\int_0^T\int_0^{t_1}\int_0^{t_2}\partial\sigma(S^{(0)}_{t_1},t_1)\sigma(S^{(0)}_{t_1},t_1)\partial\sigma(S^{(0)}_{t_2},t_2)\sigma(S^{(0)}_{t_2},t_2)\sigma^2(S^{(0)}_{t_3},t_3)dt_3dt_2dt_1$$

$$
\begin{aligned}
&+ \quad 6\int_0^T\int_0^{t_1}\int_0^{t_2}\partial^2\sigma(S_{t_1}^{(0)},t_1)\sigma(S_{t_1}^{(0)},t_1)\sigma^2(S_{t_2}^{(0)},t_2)\sigma^2(S_{t_3}^{(0)},t_3)dt_3dt_2dt_1\Bigg)\frac{H_3(x;\Sigma_T)}{\Sigma_T^3}\\
&+ \quad \left(3\int_0^T\int_0^{t_1}\partial^2\sigma(S_{t_1}^{(0)},t_1)\sigma(S_{t_1}^{(0)},t_1)\sigma^2(S_{t_2}^{(0)},t_2)dt_2dt_1\right)\frac{H_1(x;\Sigma_T)}{\Sigma_T} \qquad (35)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{E}[(A_{2T})^2|A_{1T}=x]\\
= &\Bigg(16\int_0^T\int_0^{t_1}\int_0^{t_2}\int_0^{t_3}\partial\sigma(S_{t_1}^{(0)},t_1)\sigma(S_{t_1}^{(0)},t_1)\partial\sigma(S_{t_2}^{(0)},t_2)\sigma(S_{t_2}^{(0)},t_2)\sigma^2(S_{t_3}^{(0)},t_3)\sigma^2(S_{t_4}^{(0)},t_4)dt_4dt_3dt_2dt_1\\
+ &\ 8\int_0^T\int_0^{t_1}\int_0^{t_2}\int_0^{t_3}\partial\sigma(S_{t_1}^{(0)},t_1)\sigma(S_{t_1}^{(0)},t_1)\sigma^2(S_{t_2}^{(0)},t_2)\partial\sigma(S_{t_3}^{(0)},t_3)\sigma(S_{t_3}^{(0)},t_3)\sigma^2(S_{t_4}^{(0)},t_4)dt_4dt_3dt_2dt_1\Bigg)\frac{H_4(x;\Sigma_T)}{\Sigma_T^4}\\
+ &\Bigg(16\int_0^T\int_0^{t_1}\int_0^{t_2}\partial\sigma(S_{t_1}^{(0)},t_1)\sigma(S_{t_1}^{(0)},t_1)\partial\sigma(S_{t_2}^{(0)},t_2)\sigma(S_{t_2}^{(0)},t_2)\sigma^2(S_{t_3}^{(0)},t_3)dt_3dt_2dt_1\\
+ &\ 8\int_0^T\int_0^{t_1}\int_0^{t_2}\left(\partial\sigma(S_{t_1}^{(0)},t_1)\right)^2\sigma^2(S_{t_2}^{(0)},t_2)\sigma^2(S_{t_3}^{(0)},t_3)dt_3dt_2dt_1\Bigg)\frac{H_2(x;\Sigma_T)}{\Sigma_T^2}\\
+ &\ 4\int_0^T\int_0^{t_1}\left(\partial\sigma(S_{t_1}^{(0)},t_1)\right)^2\sigma^2(S_{t_2}^{(0)},t_2)dt_2dt_1. \qquad (36)
\end{aligned}
$$

Substituting these into (28), we have the asymptotic expansion of $\mathbf{E}\left[\Phi(G^{(\epsilon)})\right]$ up to $\epsilon^2$-order.

Here, at the end of this subsection, we state a brief summary. In the Black-Scholes-type economy, we considered the risky asset $S^{(\epsilon)}$ and evaluate some quantities, expressed as an expectation of the function of the price in the future, such as prices or risk sensitivities of the securities on this asset. First we expanded them around the limit to $\epsilon = 0$ so that we obtained the expansion (28) which contains some conditional expectations. Then, we gave the explicit expressions of these conditional expectation. Finally, substituting computation results into (28), the asymptotic expansion of those quantities was obtained.

Even in applications under more complicated settings such as presented in Section 5. you can follow the procedure in this subsection in the same manner.

# 4. European Currency Options with a Market Model of Interest Rates and Stochastic Volatility Models of Spot Exchange Rates

This section describes the framework of the cross-currency market according to [78] to which our asymptotic expansion approach will be applied in the next section.

Let $(\Omega, \mathcal{F}, \tilde{P}, \{\mathcal{F}_t\}_{0\le t\le T^*<\infty})$ be a complete probability space with filtration satisfying the usual conditions. First we briefly state the basics of European currency options. The payoffs of call and put options with maturity $T \in (0, T^*]$ and strike rate $K > 0$ are expressed as $(S(T) - K)^+$ and $(K - S(T))^+$ respectively where $S(t)$ denotes the spot exchange rate at time $t \ge 0$ and $x^+$ denotes $\max(x, 0)$. For a while we concentrate on the valuation of a call option since the value of a put option can be obtained through the put-call parity or similar method. We also note that the spot exchange rate $S(T)$ can be expressed in terms of a foreign exchange forward(forex forward) rate with the same maturity $T$. That is, $S(T) = F_T(T)$ where $F_T(t)$, $t \in [0,T]$ denotes the time $t$ value of the forex forward rate with maturity $T$. It is well known that the arbitrage-free relation between the forex spot rate and the forex forward rate are given by $F_T(t) = S(t)\frac{P_f(t,T)}{P_d(t,T)}$ where $P_d(t,T)$ and $P_f(t,T)$ denote the time $t$ values of domestic and foreign zero coupon bonds with maturity $T$ respectively.

Hence, our objective is to obtain the present value of the payoff $(F_T(T) - K)^+$. In particular, we need to evaluate:

$$V(0;K,T) \quad = \quad P_d(0,T)\mathbf{E}^P\left[(F_T(T) - K)^+\right] \tag{37}$$

where $V(0;T,K)$ denotes the value of an European call option at time 0 with maturity $T$ and strike rate $K$, and $\mathbf{E}^P[\cdot]$ denotes the expectation operator under EMM(Equivalent Martingale Measure) $P \sim \tilde{P}$ whose associated numeraire is the domestic zero coupon bond maturing at $T$ (we use a term of *the domestic terminal measure* in what follows). Then, the dynamics governing $F_T(T)$ under the domestic terminal measure are necessary for pricing the option. For this objective, a market model and stochastic volatility models possibly with jumps are applied to modeling interest rates' and the spot exchange rate's dynamics respectively.

In the rest of this section, we describe briefly the model to which an asymptotic expansion approach will be applied in the following sections, where some appropriate regularity conditions are implicitly assumed without mentioned.

We first define domestic and foreign forward interest rates as $f_{dj}(t) = \left(\frac{P_d(t,T_j)}{P_d(t,T_{j+1})} - 1\right)\frac{1}{\tau_j}$ and $f_{fj}(t) = \left(\frac{P_f(t,T_j)}{P_f(t,T_{j+1})} - 1\right)\frac{1}{\tau_j}$ respectively, where $j = n(t), n(t)+1, \cdots, N$, $\tau_j = T_{j+1} - T_j$ and $n(t) = \min\{i : t \leq T_i\}$. We also define spot interest rates to the nearest fixing date denoted by $f_{d,n(t)-1}(t)$ and $f_{f,n(t)-1}(t)$ as $f_{d,n(t)-1}(t) = \left(\frac{1}{P_d(t,T_{n(t)})} - 1\right)\frac{1}{(T_{n(t)}-t)}$ and $f_{f,n(t)-1}(t) = \left(\frac{1}{P_f(t,T_{n(t)})} - 1\right)\frac{1}{(T_{n(t)}-t)}$. Finally, we set $T = T_{N+1}$ and will abbreviate $F_{T_{N+1}}(t)$ to $F_{N+1}(t)$ in what follows.

$\mathbf{R}_{++}$-valued processes of domestic forward interest rates under the domestic terminal measure can be specified as; for $j = n(t)-1, n(t), n(t)+1, \cdots, N$,

$$f_{dj}(t) \quad = \quad f_{dj}(0) + \int_0^t f_{dj}(s)\tilde{\gamma}'_{dj}(s)\sum_{i=j+1}^{N}\tilde{g}_{di}(s)ds + \int_0^t f_{dj}(s)\tilde{\gamma}'_{dj}(s)dW_s \tag{38}$$

where $\tilde{g}_{di}(t) := \frac{-\tau_i f_{di}(t)\tilde{\gamma}_{di}(t)}{1+\tau_i f_{di}(t)}$; $x'$ denotes the transpose of $x$, and $W$ is a $D$ dimensional standard Wiener process under the domestic terminal measure; $\tilde{\gamma}_{dj}(t)$ is a function of time-parameter $t$. Similarly, $\mathbf{R}_{++}$-valued processes of foreign ones under the foreign terminal measure are specified as

$$f_{fj}(t) \quad = \quad f_{fj}(0) + \int_0^t f_{fj}(s)\tilde{\gamma}'_{fj}(s)\sum_{i=j+1}^{N}\tilde{g}_{fi}(s)ds + \int_0^t f_{fj}(s)\tilde{\gamma}'_{fj}(s)dW_s^f \tag{39}$$

where $\tilde{g}_{fi}(t) := \frac{-\tau_i f_{fi}(t)\tilde{\gamma}_{fi}(t)}{1+\tau_i f_{fi}(t)}$; $W^f$ is a $D$ dimensional standard Wiener process under the foreign terminal measure and $\tilde{\gamma}_{fj}(t)$ is a function of $t$.

Finally, it is assumed that the spot exchange rate $S(t)$ and its volatility $\tilde{\sigma}(t)$ follow $\mathbf{R}_{++}$-valued stochastic processes as below respectively under the domestic risk neutral measure:

$$S(t) \quad = \quad S(0) + \int_0^t S(s)(r_d(s) - r_f(s))ds + \int_0^t S(s)\tilde{\sigma}(s)\overline{\sigma}' d\hat{W}_s + \int_0^t S(s)d\tilde{A}_s$$

$$\tilde{\sigma}(t) = \tilde{\sigma}(0) + \int_0^t \hat{\mu}(\tilde{\sigma}(s), s)ds + \int_0^t \tilde{\omega}'(\tilde{\sigma}(s), s)d\hat{W}_s \tag{40}$$

where $\hat{W}$ is a $D$ dimensional standard Wiener process under the domestic risk neutral measure and $r_d(t)$ and $r_f(t)$ denote domestic and foreign instantaneous spot interest rates respectively; $\bar{\sigma}$ denotes a $\mathbf{R}^D$-valued constant vector satisfying $\|\bar{\sigma}\| = 1$, and $\tilde{\omega}(x, t)$ is a function of $x$ and $t$. $\tilde{A}$ is some martingale possibly with jumps and independent of $W$(then independent of $W^f$ and $\hat{W}$ as well), which will be restricted to a certain class in Section 5.2.

In the model, the volatility of the forex spot rate process is allowed to be general time-inhomogeneous Markovian while the interest rates' volatilities are specified as a log-normal structure. Note that the correlations' structure among domestic/foreign interest rates, the spot exchange rate and its volatility can be represented through $\tilde{\gamma}_{dj}(t)$, $\tilde{\gamma}_{fj}(t)$, $\bar{\sigma}$ and $\tilde{\omega}(\tilde{\sigma}(t), t)$. It is also noted that our methodology can be applied not only in a Markovian setting but also in a non-Markovian framework as long as the uncertainty is generated by Wiener processes.

Moreover, we have the following well known relations among Wiener processes under different probability measures;

$$\begin{aligned} W_t &= \hat{W}_t - \int_0^t \tilde{\sigma}_{dN+1}(s)ds \\ &= W_t^f + \int_0^t \{\tilde{\sigma}_{fN+1}(s) - \tilde{\sigma}_{dN+1}(s) + \tilde{\sigma}(s)\bar{\sigma}\}ds \end{aligned}$$

where $\tilde{\sigma}_{dN+1}(t)$ and $\tilde{\sigma}_{fN+1}(t)$ are volatilities of the domestic and foreign zero coupon bonds with the maturity $T_{N+1}$, that is,

$$\tilde{\sigma}_{dN+1}(t) := \sum_{i \in J_{N+1}(t)} \tilde{g}_{di}(t), \quad \tilde{\sigma}_{fN+1}(t) := \sum_{i \in J_{N+1}(t)} \tilde{g}_{fi}(t)$$

and $J_{j+1}(t) = \{n(t) - 1, n(t), n(t) + 1, \cdots, j\}$. Because $\gamma_{fj}(t) = 0$ and $\gamma_{dj}(t) = 0$ for all $j$ such that $T_j \leq t$, the set of indices $J_{j+1}(t)$ can be replaced by $\hat{J}_{j+1} := \{0, 1, \cdots, j\}$, which does not depend on $t$.

Using above equations, we can unify expressions of those processes under different measures into ones under the same measure, the domestic terminal measure $P$:

$$\begin{aligned} f_{fj}(t) &= f_{fj}(0) + \int_0^t f_{fj}(s)\tilde{\gamma}'_{fj}(s)\left\{-\sum_{i \in \hat{J}_{j+1}} \tilde{g}_{fi}(s) + \sum_{i \in \hat{J}_{N+1}} \tilde{g}_{di}(s) - \tilde{\sigma}(s)\bar{\sigma}\right\}ds \\ &\quad + \int_0^t f_{fj}(u)\tilde{\gamma}'_{fj}(u)dW_u \end{aligned} \tag{41}$$

$$\tilde{\sigma}(t) = \tilde{\sigma}(0) + \int_0^t \mu(s)ds + \int_0^t \tilde{\omega}'(\tilde{\sigma}(s), s)dW_s \tag{42}$$

where $\mu(t)$ is defined as

$$\mu(t) := \hat{\mu}(\tilde{\sigma}(t), t) + \tilde{\omega}'(\tilde{\sigma}(t), t)\tilde{\sigma}_{dN+1}(t).$$

Since $F_{N+1}(t)$ can be expressed as

$$F_{N+1}(t) = S(t)\frac{P_f(t, T_{N+1})}{P_d(t, T_{N+1})}, \tag{43}$$

we easily notice that it is a martingale under the domestic terminal measure. Consequently, we can obtain its process with application of Itô's formula to (43):

$$
\begin{aligned}
F_{N+1}(t) &= F_{N+1}(0) + \int_0^t \tilde{\sigma}_F'(s)dW_s + \int_0^t F(s)d\tilde{A}_s \qquad (44)\\
\text{where} \quad \tilde{\sigma}_F(t) &:= F_{N+1}(t)\left[\tilde{\sigma}_{fN+1}(t) - \tilde{\sigma}_{dN+1}(t) + \tilde{\sigma}(t)\bar{\sigma}\right]\\
&= F_{N+1}(t)\left[\sum_{j\in \hat{J}_{N+1}}\left\{\frac{-\tau_j f_{fj}(t)\tilde{\gamma}_{fj}(t)}{1+\tau_j f_{fj}(t)} - \frac{-\tau_j f_{dj}(t)\tilde{\gamma}_{dj}(t)}{1+\tau_i f_{dj}(t)}\right\} + \tilde{\sigma}(t)\bar{\sigma}\right] \qquad (45)
\end{aligned}
$$

It is obviously that the process of the forex forward is too complicated to derive the closed-form formula of option prices. Thus, approximation schemes based on an asymptotic expansion will be applied in the following sections.

Despite this difficulty, we here emphasize the generality and importance of our framework investigated in this work: For the stochastic volatility, a general time-inhomogeneous Markovian process is assumed, which is not necessarily classified in the affine model such as in [27]; Any correlation structure can be considered; In addition, we can incorporate a jump process in our model. These settings are flexible enough to capture the complexity of movements of the underlying asset and to calibrate our model to the market with ease even in the severely skewed environment as in a recent JPY-USD market, as shown in Section 6.3.

# 5. Applications of the Asymptotic Expansion Approach to Currency Options

This section applies an asymptotic expansion approach in Section 3.1. to evaluation of currency option prices in the environment described so far. Particularly, first we present a natural application of the method according to [78], which is henceforth called 'a standard scheme,' and then show another application in a somewhat different way, which is called 'a hybrid scheme,' proposed by [80]. The latter is applicable even when the dynamics of the spot forex contains a certain class of jumps.

## 5.1. A Standard Scheme

This subsection presents a standard way of application of an asymptotic expansion to option pricing problem under the setting described in Section 4. Details are sometimes omitted due to limitation of space and found in [78].

First the processes of $f_{dj}^{(\epsilon)}(t)$, $f_{fj}^{(\epsilon)}(t)$, $\sigma^{(\epsilon)}(t)$ and $F_{N+1}^{(\epsilon)}(t)$ under the domestic terminal measure $P$ are redefined in the framework of the asymptotic expansion method as follows;

for $j = n(t)-1, n(t), n(t)+1, \cdots, N$,

$$
\begin{aligned}
f_{dj}^{(\epsilon)}(t) &= f_{dj}(0) + \epsilon^2\int_0^t f_{dj}^{(\epsilon)}(u)\gamma_{dj}'(s)\sum_{i=j+1}^{N} g_{di}^{(\epsilon)}(s)ds + \epsilon\int_0^t f_{dj}^{(\epsilon)}(s)\gamma_{dj}'(s)dW_s \qquad (46)\\
f_{fj}^{(\epsilon)}(t) &= f_{fj}^{(\epsilon)}(0) + \epsilon^2\int_0^t f_{fj}^{(\epsilon)}(u)\gamma_{fj}'(u)\left\{-\sum_{i\in\hat{J}_{j+1}} g_{fi}^{(\epsilon)}(s) + \sum_{i\in\hat{J}_{N+1}} g_{di}^{(\epsilon)}(s) - \sigma^{(\epsilon)}(s)\bar{\sigma}\right\}ds
\end{aligned}
$$

$$+\epsilon\int_0^t f_{fj}^{(\epsilon)}(s)\gamma_{fj}^{'}(s)dW_s \quad (47)$$

$$\sigma^{(\epsilon)}(t) \;=\; \sigma(0)+\int_0^t \mu^{(\epsilon)}(s)ds+\epsilon\int_0^t \omega^{'}(\sigma^{(\epsilon)}(s),s)dW_s \tag{48}$$

$$F_{N+1}^{(\epsilon)}(t) \;=\; F_{N+1}(0)+\epsilon\int_0^t \sigma_F^{(\epsilon)'}(s)dW_s \tag{49}$$

where

$$\begin{aligned}\sigma_F^{(\epsilon)}(t) &:= F_{N+1}^{(\epsilon)}(t)[\sigma_{fN+1}^{(\epsilon)}(t)-\sigma_{dN+1}^{(\epsilon)}(t)+\sigma^{(\epsilon)}(t)\bar{\sigma}]\\ &= F_{N+1}^{(\epsilon)}(t)\left[\sum_{j\in\hat{J}_{N+1}}\left\{\frac{-\tau_j f_{fj}^{(\epsilon)}(t)\gamma_{fj}(t)}{1+\tau_j f_{fj}^{(\epsilon)}(t)}-\frac{-\tau_j f_{dj}^{(\epsilon)}(t)\gamma_{dj}(t)}{1+\tau_i f_{dj}^{(\epsilon)}(t)}\right\}+\sigma^{(\epsilon)}(t)\bar{\sigma}\right]\end{aligned}$$

and

$$g_{di}^{(\epsilon)}(t):=\frac{\tau_i f_{di}^{(\epsilon)}(t)\gamma_{di}(t)}{1+\tau_i f_{di}^{(\epsilon)}(t)},\quad g_{fi}^{(\epsilon)}(t):=\frac{\tau_i f_{fi}^{(\epsilon)}(t)\gamma_{fi}(t)}{1+\tau_i f_{fi}^{(\epsilon)}(t)}.$$

Here, $\tilde{\gamma}_{dj}(t)$, $\tilde{\gamma}_{fj}(t)$, $\tilde{\sigma}(t)$ and $\tilde{\omega}(\sigma^{(\epsilon)}(t),t)$ in the previous section are replaced by $\epsilon\gamma_{dj}(t)$, $\epsilon\gamma_{fj}(t)$, $\epsilon\sigma(t)$, and $\epsilon\omega(\sigma^{(\epsilon)}(t),t)$ respectively. Moreover, in this subsection it is assumed that there is no uncertainty such as jumps except for Wiener processes we have defined(i.e. $\tilde{A}\equiv 0$). Under certain appropriate conditions on $\mu^{(\epsilon)}(t)$ and $\omega(\sigma^{(\epsilon)}(t),t)$, the system of SDEs (46), (47), (48) and (49) have their unique solutions $f_{dj}^{(\epsilon)}(t)$, $f_{fj}^{(\epsilon)}(t)$, $\sigma^{(\epsilon)}(t)$ and $F_{N+1}^{(\epsilon)}(t)$. Note that the limiting processes of these processes are deterministic:

$$\begin{aligned} f_{fj}^{(0)}(t) &:= \lim_{\epsilon\downarrow 0} f_{fj}^{(\epsilon)}(t)=f_{fj}(0),\quad f_{dj}^{(0)}(t):=\lim_{\epsilon\downarrow 0} f_{dj}^{(\epsilon)}(t)=f_{dj}(0)\\ \text{and}\quad \sigma^{(0)}(t) &:= \lim_{\epsilon\downarrow 0}\sigma^{(\epsilon)}(t)=\sigma(0)+\int_0^t \lim_{\epsilon\downarrow 0}\mu^{(\epsilon)}(s)ds.\end{aligned}$$

In what follows, substitution '$\epsilon=0$' into each variable will be frequently used instead of taking its limit as $\epsilon\downarrow 0$. Moreover, the maturity of the option $T_{N+1}$ will be abbreviated as $T$.

Next, substituting $X^{(\epsilon)}=\left(X^{(\epsilon),1},\cdots,X^{(\epsilon),2N+4}\right)=\left(F_{N+1}^{(\epsilon)},\{f_{dj}^{(\epsilon)}\}_{j=0}^N,\{f_{fj}^{(\epsilon)}\}_{j=0}^N,\sigma^{(\epsilon)}\right)$, $g(X_T^{(\epsilon)})=X_T^{(\epsilon),1}=F_{N+1}^{(\epsilon)}(T)$ and $M=2$ into the setting of Section 3.1., we have the following expansion.

**Proposition 2** *The asymptotic expansion of $G_F^{(\epsilon)}=\frac{F_{N+1}^{(\epsilon)}(T)-F_{N+1}(0)}{\epsilon}$ up to $\epsilon^2$-order is given as follows:*

$$G_F^{(\epsilon)}=A_T^{\langle 1\rangle}+\frac{\epsilon}{2!}A_T^{\langle 2\rangle}+\frac{\epsilon^2}{3!}A_T^{\langle 3\rangle}+o(\epsilon^2) \tag{50}$$

*where $A_t^{\langle k\rangle}$, $k=1,2,3$ are obtained by formal Taylor's expansion of $F_{N+1}^{(\epsilon)}(t)$, $\frac{\partial^k F_{N+1}^{(\epsilon)}(t)}{\partial\epsilon^k}|_{\epsilon=0}$, or substitution of $X^{(\epsilon)}=\left(F_{N+1}^{(\epsilon)},\{f_{dj}^{(\epsilon)}\}_{j=0}^N,\{f_{fj}^{(\epsilon)}\}_{j=0}^N,\sigma^{(\epsilon)}\right)$ into (15). For details see [78].*

Note that the first order term $A_T^{\langle 1\rangle} = \int_0^T \sigma_F^{(0)}(s)' dW_s$ follows normal distribution with mean 0 and variance $\Sigma$:

$$\Sigma \quad := \quad \int_0^T \|\sigma_F^{(0)}(s)\|^2 ds. \tag{51}$$

With the expansion of $G_F^{(\epsilon)}$ in Proposition 2, we now focus on pricing options. Hereafter, we will consider a call option with strike rate $K_\epsilon$ where $K_\epsilon$ is defined for some arbitrary $y \in \mathbf{R}$ as $K_\epsilon := F_{N+1}(0) - \epsilon y$. Then, the discounted value of the option is given by

$$\frac{V(0; K_\epsilon, T)}{P_d(0,T)} = \mathbf{E}^P[(F_{N+1}^{(\epsilon)}(T) - K_\epsilon)^+] = \mathbf{E}^P[\epsilon(G_F^{(\epsilon)} + y)^+] \tag{52}$$

Thus, letting $\Phi$ in Section 3.1. be $\Phi(x) = P_d(0,T)\epsilon(x+y)^+$, we obtain the asymptotic expansion of the option price with respect to $\epsilon$ as the following theorem through evaluation of conditional expectations.

**Theorem 2** *We define $K_\epsilon := F_{N+1}(0) - \epsilon y$ for some arbitrary $y \in \mathbf{R}$ and suppose that $\Sigma > 0$.*
*Then an asymptotic expansion of $V(0; T, K_\epsilon)$, the value of the option with strike rate $K_\epsilon$, up to $\epsilon^3$-order is given as follows:*

$$\begin{aligned} V(0; K_\epsilon, T) \quad &= \quad P_d(0,T)\Bigg[\epsilon y \int_{-y}^{\infty} \phi_{0,\Sigma}(x)dx + \epsilon \int_{-y}^{\infty} x\phi_{0,\Sigma}(x)dx \\ &+ \quad \frac{\epsilon^2}{2}\int_{-y}^{\infty} \mathbf{E}^P[A_T^{\langle 2\rangle}|A_T^{\langle 1\rangle} = x]\phi_{0,\Sigma}(x)dx \\ &+ \quad \frac{\epsilon^3}{6}\int_{-y}^{\infty} \mathbf{E}^P[A_T^{\langle 3\rangle}|A_T^{\langle 1\rangle} = x]\phi_{0,\Sigma}(x)dx + \frac{\epsilon^3}{48}(\mathbf{E}^P[(A^{\langle 2\rangle})^2|A_T^{\langle 1\rangle} = x]\phi_{0,\Sigma}(x))_{x=-y}\Bigg] \\ &+ \quad o(\epsilon^3) \end{aligned} \tag{53}$$

*where $\phi_{\mu,\Sigma}(x)$ is defined by*

$$\phi_{\mu,\Sigma}(x) = \frac{1}{\sqrt{2\pi\Sigma}} \exp\left(-\frac{(x-\mu)^2}{2\Sigma}\right). \tag{54}$$

*The conditional expectations appearing in the above equation are eventually expressed as linear combinations of Hermite polynomials;*

$$\mathbf{E}^P[A_T^{\langle 2\rangle}|A_T^{\langle 1\rangle} = x] \quad = \quad C_{2,1}\frac{x}{\Sigma} + C_{2,2}(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}) \tag{55}$$

$$\mathbf{E}^P[A_T^{\langle 3\rangle}|A_T^{\langle 1\rangle} = x] \quad = \quad C_{3,1}\frac{x}{\Sigma} + C_{3,2}(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}) + C_{3,3}(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}) \tag{56}$$

$$\begin{aligned} \mathbf{E}^P[(A_T^{\langle 2\rangle})^2|A_T^{\langle 1\rangle} = x] \quad &= \quad C_{4,0} + C_{4,1}\frac{x}{\Sigma} + C_{4,2}(\frac{x^2}{\Sigma^2} - \frac{1}{\Sigma}) \\ &\quad + C_{4,3}(\frac{x^3}{\Sigma^3} - \frac{3x}{\Sigma^2}) + C_{4,4}(\frac{x^4}{\Sigma^4} - \frac{6x^2}{\Sigma^3} + \frac{3}{\Sigma^2}) \end{aligned} \tag{57}$$

*where $C_{2,1}, C_{2,2}, C_{3,1}, C_{3,2}, C_{3,3}, C_{4,0}, C_{4,1}, C_{4,2}, C_{4,3}$, and $C_{4,4}$ are some constants. Calculation procedures of these quantities are found in Appendix of [78] and those under a more general framework are in [81].*

**Remark 1** *In practice, we are often interested in the accuracy of our formulas for the prices of options whose underlying variables follow the SDEs (38), (41) and (42) with a particular set of parameters such as $\tilde{\gamma}_{dj}(t)$, $\tilde{\gamma}_{fj}(t)$, $\tilde{\sigma}(0)$ , $\mu(t)$ and $\tilde{\omega}(\tilde{\sigma}(t), t)$. From this point of view, given some particular value of $\epsilon$, $\gamma_{dj}(t)$, $\gamma_{fj}(t)$, $\sigma(0)$, $\mu^{(\epsilon)}(t)$ and $\omega(\sigma^{(\epsilon)}(t), t)$ in (46), (47) and (48) should be scaled so that $\epsilon\gamma_{dj}(t) = \tilde{\gamma}_{dj}(t)$, $\epsilon\gamma_{fj}(t) = \tilde{\gamma}_{fj}(t)$, $\epsilon\sigma(0) = \tilde{\sigma}(0)$, $\epsilon\mu^{(\epsilon)}(t) = \mu(t)$ and $\epsilon\omega(\sigma(t), t) = \tilde{\omega}(\tilde{\sigma}(t), t)$ for an arbitrary $t \in [0, T]$. For instance, $\gamma(t)$ is defined as $\gamma(t) := \frac{\tilde{\gamma}(t)}{\epsilon}$ where $\epsilon$ is fixed at a pre-specified constant through our procedure of expansions. Moreover, it can be shown that the approximated prices are unchanged whatever $\epsilon \in (0, 1]$ is taken in evaluation, as long as above conditions are met.*

## 5.2. A Hybrid Scheme

This subsection introduces another 'hybrid' scheme developed by [80]. In this scheme, the option price will be derived via Fourier inversion of the characteristic function(henceforth sometimes called ch.f.) of the log-forward forex. Since the underlying framework of a standard cross-currency model with libor market models we are discussing is too complicated to obtain the closed-form solution of the ch.f., we approximate it with the asymptotic expansion. Moreover, in order to increase accuracy of our method, a certain change of the probability measure and a transformation of variable will be also applied, those are reasons why the method is called 'hybrid'. Finally, the asymptotic expansion will be used as a control variable in Monte Carlo simulations to accelerate their convergence.

### 5.2.1. A Pricing Problem Revisited

In this subsection, we allow existence of the martingale $\tilde{A}$ possibly with jumps and independent of Wiener processes we have defined, which will be somewhat restricted later.

Our objective is to evaluate the following quantity;

$$V(0; K, T) \quad = \quad P_d(0, T) \times \mathbf{E}^P\left[(F_T(T) - K)^+\right]. \tag{58}$$

With a log-price of the forex forward $f_T(t) := ln(\frac{F_T(t)}{F_T(0)})$, (58) can be rewritten as:

$$V(0; K, T) \quad = \quad P_d(0, T) \times F_T(0)\, \mathbf{E}^P\left[(e^{f_T(T)} - e^k)^+\right]$$

where $k := ln(\frac{K}{F_T(0)})$ denotes a log-strike rate. Here we note that $e^{f_T(T)} = F_T(T)$ is a martingale under the domestic terminal measure.

Carr and Madan [10] proposed another expression of option prices as some Fourier inversion of the characteristic function of the logarithm of the underlying asset.

**Proposition 3** *Let $\Phi_T^P(u)$ denote a characteristic function of $f_T(T)$ under $P$. Then, $V(0; K, T)$ is given by:*

$$V(0; K, T) = \mathbf{\Psi}(\Phi_T^P; F_T(0), K, T) \tag{59}$$

*where*

$$\mathbf{\Psi}(\Phi; F, K, T) \quad := \quad P_d(0, T) \times \left[F \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \gamma(u; \Phi) du + (F - K)^+\right], \tag{60}$$

$$\gamma(u; \Phi) \quad := \quad \frac{\Phi(u - i) - 1}{iu(1 + iu)} \;\; and \;\; i := \sqrt{-1}. \tag{61}$$

Then, we need to know the characteristic function of $f_T(T)$ under the domestic terminal measure $P$ for pricing the option. In particular, in our setting the log-forex forward $f_{N+1}(t) = ln(\frac{F_{N+1}(t)}{F_{N+1}(0)})$ follows

$$f_{N+1}(t) \quad = \quad ln(\frac{F_{N+1}(t)}{F_{N+1}(0)}) = Z(t) + A(t) \tag{62}$$

where $Z(t)$ is an exponential-martingale continuous process given by

$$Z(t) \quad = \quad -\frac{1}{2}\int_0^t \|\tilde{\sigma}_Z(s)\|^2\, ds + \int_0^t \tilde{\sigma}_Z^{'}(s) dW_s \tag{63}$$

where

$$\begin{aligned}
\tilde{\sigma}_Z(t) \quad &:= \quad \sum_{j\in\hat{J}_{N+1}} \left(\tilde{g}_{fj}(t) - \tilde{g}_{dj}(t)\right) + \tilde{\sigma}(t)\overline{\sigma} \\
&= \quad \sum_{j\in\hat{J}_{N+1}} \left(\frac{-\tau_j f_{fj}(t)\tilde{\gamma}_{fj}(t)}{1+\tau_j f_{fj}(t)} - \frac{-\tau_j f_{dj}(t)\tilde{\gamma}_{dj}(t)}{1+\tau_i f_{dj}(t)}\right) + \tilde{\sigma}(t)\overline{\sigma}
\end{aligned}$$

and $A(t)$ denotes a continuous or jump process that is an exponential-martingale independent from $Z(t)$ which is directly derived by application of Itô's fornula. Further, we assume that the characteristic function of $A(t)$ is known in closed-form. e.g. $A(t)$ is a compound Poisson process, a variance gamma process, an inverse Gaussian process, a CGMY model or a Lévy process appearing in the Stochastic Skew Model(Carr and Wu [11]).

#### 5.2.2. A Transformation of the Underlying Stochastic Differential Equations

Let $\Phi^P_{N+1}(t,u)$ denote the characteristic function of $f_{N+1}(t)$ under $P$. Then, $\Phi^P_{N+1}(t,u)$ can be decomposed as;

$$\Phi^P_{N+1}(t,u) = \Phi^P_Z(t,u)\Phi^P_A(t,u) \tag{64}$$

where $\Phi^P_Z(t,u)$ and $\Phi^P_A(t,u)$ denote the characteristic functions of $Z(t)$ and $A(t)$ under $P$, respectively.

For evaluation of European currency options, an explicit expression of $\Phi^P_{N+1}(T,u)$ is necessary. However, the process $Z(t)$ is too complicated to obtain the analytical expression of $\Phi^P_Z(T,u)$ (see Section 6.3.2 in Brigo and Mercurio [8] or Section 25.5 in Björk [9] ) while that of $\Phi^P_A(T,u)$ is assumed to be known. Then, later we will suggest to utilize the asymptotic expansion for the approximation of $\Phi^P_Z(T,u)$.

In (63), $Z(t)$, the key process for evaluation of options, has a nonzero drift. Thus, unless we provide the approximation which has not any error in the drift term, even the first moment(i.e. the expectation value) of that approximation will not match the target's. Contrarily, if we can eliminate its drift term by some means, that is the objective process will be a martingale, its first moment can be much easily kept by using a martingale process as an approximation. In this light, here we consider a certain change of measures so that the main objective process of our expansion will be martingale.

For a fixed $u$(an argument of $\Phi_Z^P(T,u)$) we define a new probability measure $Q_u$ on $(\Omega, \mathcal{F}_T)$ with the Radon-Nikodym derivative of

$$\frac{dQ_u}{dP} = \exp\left(-\frac{1}{2}\int_0^T \|\lambda_u(s)\|^2\, ds - \int_0^T \lambda_u'(s)dW_s\right) \tag{65}$$

where

$$\lambda_u(t) := \left((-iu) + i\sqrt{u^2+iu}\right)\tilde{\sigma}_Z(t) = \tilde{h}(u)\tilde{\sigma}_Z(t)$$

and $\tilde{h}(u) := (-iu) + i\sqrt{u^2+iu}$.

Then $\Phi_Z^P(T,u)$, the characteristic function of $Z(T)$ under the measure $P$, is expressed as that of another random variable $\hat{Z}(T)$ under $Q_u$ with a transformation of variable $h(\cdot)$:

$$\begin{aligned}
\Phi_Z^P(T,u) &= \mathbf{E}^P\left[\exp\left(iuZ(T)\right)\right] \\
&= \mathbf{E}^{Q_u}\left[\exp\left(ih(u)\int_0^T \tilde{\sigma}_Z'(s)dW_s^{Q_u}\right)\right] \\
&=: \Phi_{\hat{Z}}^{Q_u}(T, h(u))
\end{aligned} \tag{66}$$

where $\mathbf{E}^{Q_u}[\cdot]$ is an expectation operator under $Q_u$; $W_t^{Q_u} := W_t + \int_0^t \lambda_u'(s)ds$ is now a Wiener process under that measure; $\Phi_{\hat{Z}}^{Q_u}(t,v)$ denotes the characteristic function of $\hat{Z}(t) := \int_0^t \tilde{\sigma}_Z'(s)dW_s^{Q_u}$ under $Q_u$ and $h(u) := \sqrt{u^2+iu}$.

Now, we have the martingale objective process for the approximation. Then, in the following, we will apply the asymptotic expansion method to the underlying system of stochastic differential equations under $Q_u$.

### 5.2.3. Approximating the Characteristic Function by an Asymptotic Expansion

Here, to fit the framework of the asymptotic expansion, the processes of $f_{dj}^{(\epsilon)}(t)$, $f_{fj}^{(\epsilon)}(t)$ and $\sigma^{(\epsilon)}(t)$ in (38), (41) and (42) are again redefined under the measure $Q_u$ with a parameter $\epsilon$ as follows;

for $j = n(t)-1, n(t), n(t)+1, \cdots, N$,

$$\begin{aligned}
f_{dj}^{(\epsilon)}(t) &= f_{dj}(0) + \epsilon^2 \int_0^t f_{dj}^{(\epsilon)}(s)\gamma_{dj}'(s)\sum_{i=j+1}^{N} g_{di}^{(\epsilon)}(s)ds \\
&\quad -\epsilon^2\tilde{h}(u)\int_0^t f_{dj}^{(\epsilon)}(s)\gamma_{dj}'(s)\sigma_Z^{(\epsilon)}(s)ds + \epsilon\int_0^t f_{dj}^{(\epsilon)}(s)\gamma_{dj}'(s)dW_s^{Q_u}
\end{aligned} \tag{67}$$

$$\begin{aligned}
f_{fj}^{(\epsilon)}(t) &= f_{fj}^{(\epsilon)}(0) + \epsilon^2 \int_0^t f_{fj}^{(\epsilon)}(s)\gamma_{fj}'(s)\left\{-\sum_{i\in\hat{J}_{j+1}} g_{fi}^{(\epsilon)}(s) + \sum_{i\in\hat{J}_{N+1}} g_{di}^{(\epsilon)}(s) - \sigma^{(\epsilon)}(s)\overline{\sigma}\right\}ds \\
&\quad -\epsilon^2\tilde{h}(u)\int_0^t f_{fj}^{(\epsilon)}(s)\gamma_{fj}'(s)\sigma_Z^{(\epsilon)}(s)ds + \epsilon\int_0^t f_{fj}^{(\epsilon)}(s)\gamma_{fj}'(s)dW_s^{Q_u}
\end{aligned} \tag{68}$$

and

$$\sigma^{(\epsilon)}(t) = \sigma(0) + \int_0^t \mu^{(\epsilon)}(s)ds - \epsilon^2\tilde{h}(u)\int_0^t \omega'(\sigma^{(\epsilon)}(s),s)\sigma_Z^{(\epsilon)}(s)ds + \epsilon\int_0^t \omega'(\sigma^{(\epsilon)}(s),s)dW_s^{Q_u}. \tag{69}$$

Then $\hat{Z}^{(\epsilon)}(t)$, the analogy of $\hat{Z}(t)$, is given by

$$\hat{Z}^{(\epsilon)}(t) \quad = \quad \epsilon \int_0^t \sigma_Z^{(\epsilon)'}(s) dW_s^{Q_u} \tag{70}$$

where

$$\sigma_Z^{(\epsilon)}(t) \quad := \quad \sum_{j \in \hat{J}_{N+1}} \left( g_{fj}^{(\epsilon)}(t) - g_{dj}^{(\epsilon)}(t) \right) + \sigma^{(\epsilon)}(t) \bar{\sigma}.$$

In a similar manner to the standard method in the previous subsection, we can derive the following asymptotic expansion (for details and concrete expressions of expansion coefficients, see Appendix of [80]).

**Proposition 4** *The asymptotic expansion of $G_{\hat{Z}}^{(\epsilon)} = \frac{1}{\epsilon} \hat{Z}^{(\epsilon)}(T)$ up to $\epsilon^2$ is expressed as follows:*

$$G_{\hat{Z}}^{(\epsilon)} = \hat{G}_T^{Q_u, \langle 1 \rangle} + \frac{\epsilon}{2!} \hat{G}_T^{Q_u, \langle 2 \rangle} + \frac{\epsilon^2}{3!} \hat{G}_T^{Q_u, \langle 3 \rangle} + o(\epsilon^2) \tag{71}$$

*where $\hat{G}_T^{Q_u, \langle k \rangle} := \frac{\partial^k \hat{Z}^{(\epsilon)}(T)}{\partial \epsilon^k} |_{\epsilon=0}$, $k = 1, 2, 3$.*

**Remark 2** *$\hat{G}_T^{Q_u, \langle k \rangle}$ for any $k$ is expressed as a certain (iterated) Itô integral. Since (iterated) Itô integrals always have zero means, the martingale property of $G_{\hat{Z}}^{(\epsilon)}$(and hence $\hat{Z}^{(\epsilon)}(t)$) is kept at any order of this expansion. Especially, the first-order term $\hat{G}_T^{Q_u, \langle 1 \rangle}$ follows a normal distribution with mean $0$ and variance $\Sigma$:*

$$\Sigma := \int_0^{T_{N+1}} \left\| \sigma_Z^{(0)}(s) \right\|^2 ds. \tag{72}$$

*Here it is assumed that $\Sigma > 0$.*

Then, letting $\Phi$ in Section 3.1. be $\Phi(x) = e^{ivx}$ for given $v$, the desired characteristic function can be approximated with the following theorem (for its proof and the concrete expressions of coefficients, again refer to Appendix of [80]).

**Theorem 3** *An asymptotic expansion of $\Phi_{G_{\hat{Z}}}^{Q_u, (\epsilon)}(v)$, the characteristic function of $G_{\hat{Z}}^{(\epsilon)}$ under $Q_u$, is given by*

$$\Phi_{G_{\hat{Z}}}^{Q_u, (\epsilon)}(v) = \left[ 1 + D_2^{Q_u, (\epsilon)}(iv)^2 + D_3^{Q_u, (\epsilon)}(iv)^3 + D_4^{Q_u, (\epsilon)}(iv)^4 + D_5^{Q_u, (\epsilon)}(iv)^5 + D_6^{Q_u, (\epsilon)}(iv)^6 \right] \Phi_{0, \Sigma}(v) + o(\epsilon^2) \tag{73}$$

*where $\Phi_{\mu, \Sigma}(v) := e^{i\mu v - \frac{\Sigma}{2} v^2}$.*

*$D_2^{Q_u, (\epsilon)}, D_3^{Q_u, (\epsilon)}, D_4^{Q_u, (\epsilon)}, D_5^{Q_u, (\epsilon)}$ and $D_6^{Q_u, (\epsilon)}$ are constants for pre-specified $\epsilon$ and $u$. Each subscript corresponds to the order of $(iv)$ in the equation (73).*

**Remark 3** *Rigorously speaking, the specification $\Phi(x) = e^{ivx}$ does not fit in the framework in Section 3.1. Actually, however, the approximate characteristic function obtained by formal expansion of $\mathbf{E}\left[ \exp \left\{ ivG_{\hat{Z}}^{(\epsilon)} \right\} \right]$ is completely the same as that given by the inversion of approximate probability density function in our framework with letting $\Phi$ be $\delta_x$, the delta function with a mass at $x$. For more details about this equivalence, see Section 6 of [81].*

Finally, we provide an approximation formula for valuation of European call options written on $F^{(\epsilon)}_{N+1}(T)$ by direct application of Theorem 3 to Proposition 3.

**Theorem 4** *Let $\hat{V}(0;K,T)$ be an approximated value of $V(0;K,T)$ which denotes the exact value of the option with maturity $T = T_{N+1}$ and strike rate $K$. Then, $\hat{V}(0;K,T)$ is given by:*

$$\hat{V}(0;K,T) \quad := \quad \mathbf{\Psi}(\hat{\Phi}^{(\epsilon)}; F_{N+1}(0), K, T) \tag{74}$$

*where the pricing functional $\mathbf{\Psi}(\,\cdot\,; F,K,T)$ is given in (60), $\hat{\Phi}^{(\epsilon)}(u) := \hat{\Phi}^{Q_u,(\epsilon)}_{G_{\hat{Z}}}(\epsilon h(u)) \times \Phi^P_A(u)$, and $k := ln(\frac{K}{F_{N+1}(0)})$. Here, $\hat{\Phi}^{Q_u,(\epsilon)}_{G_{\hat{Z}}}(v)$ is defined as;*

$$\hat{\Phi}^{Q_u,(\epsilon)}_{G_{\hat{Z}}}(v) = \left[1 + D^{Q_u,(\epsilon)}_2(iv)^2 + D^{Q_u,(\epsilon)}_3(iv)^3 + D^{Q_u,(\epsilon)}_4(iv)^4 + D^{Q_u,(\epsilon)}_5(iv)^5 + D^{Q_u,(\epsilon)}_6(iv)^6\right] \times \Phi_{0,\Sigma}(v)$$

*where $D^{Q_u,(\epsilon)}_2, D^{Q_u,(\epsilon)}_3, D^{Q_u,(\epsilon)}_4, D^{Q_u,(\epsilon)}_5$ and $D^{Q_u,(\epsilon)}_6$ are the coefficients in Theorem 3.*

**Remark 4** *Note that since $h(-i) = 0$ and $A$ is assumed to be an exponential martingale, $\mathbf{E}^P[e^{f^{(\epsilon)}_{N+1}(T_{N+1})}] = \Phi^{P,(\epsilon)}(u)$ is approximated by $\hat{\Phi}^{(\epsilon)}(-i) = \hat{\Phi}^{Q_{-i},(\epsilon)}_{G_{\hat{Z}}}(\epsilon h(-i)) \times \Phi^P_A(-i) = 1$, which means that in our approximation the exponential-martingale property of $f^{(\epsilon)}_{N+1}$ is kept.*

*Especially, when $A \equiv 0$ the first-order approximation of the option price coincides $BS(\Sigma^{\frac{1}{2}}; F_{N+1}(0), K, T)$ which is the Black-Scholes price under the case where the stochastic interest rates and the stochastic volatility would be replaced by (their limiting-)deterministic processes:*

$$BS(\sigma; F, K, T) := P_d(0,T)\left[FN(d_+) - KN(d_-)\right] \tag{75}$$

*where*

$$d_\pm := \frac{ln(F/K) \pm \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad N(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz.$$

*Moreover, in this case($A \equiv 0$), the pricing functional can be modified so that the numerical inversion is stabilized as follows;*

$$V(0;K,T_{N+1}) = \tilde{\mathbf{\Psi}}(\Phi^P_T; F_{N+1}(0), K, T_{N+1}) \tag{76}$$

*where*

$$\tilde{\mathbf{\Psi}}(\Phi; F, K, T) \quad := \quad P_d(0,T) \times \left[F\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\Big(\gamma(u;\Phi) - \gamma(u;\Phi_{BS})\Big)du\right] + BS(\Sigma^{\frac{1}{2}}; F, K, T),$$

*and $\Phi_{BS}(u)$ is the first-order-approximated characteristic function, or equivalently that of the (hypothetical)Gaussian underlying log-forward forex;*

$$\Phi_{BS}(u) := \Phi_{0,\Sigma}(h(u)) = \Phi_{-\frac{1}{2}\Sigma,\Sigma}(u).$$

**Remark 5** *Using these approximation formulas, we can also provide analytical approximations of Greeks of the option, sensitivities of the option price to the factors. Note that our approximation for the underlying characteristic function does not depend upon the initial value of the spot forex. Thus in particular, $\Delta$ and $\Gamma$, the first and* *second derivatives of the option value with respect to $S(0)$ respectively, can be explicitly approximated with ease. For simplicity here we again assume $A \equiv 0$. Then $\hat{\Delta}$ and $\hat{\Gamma}$, the approximations of $\Delta$ and $\Gamma$ respectively, are given by*

$$\begin{aligned}
\hat{\Delta} := P_f(0,T) \quad &\times \quad \left\{ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuk}\left(\gamma(u;\hat{\Phi}^{(\epsilon)}) - \gamma(u;\Phi_{BS})\right)du \right. \\
&\left. \qquad - \frac{1}{2\pi}\int_{-\infty}^{\infty}(-iu)e^{-iuk}\left(\gamma(u;\hat{\Phi}^{(\epsilon)}) - \gamma(u;\Phi_{BS})\right)du \right\} + \Delta_{BS}, \\
\hat{\Gamma} := -\frac{P_f(0,T)}{S(0)} \quad &\times \quad \left\{ \frac{1}{2\pi}\int_{-\infty}^{\infty}(-iu) e^{-iuk}\left(\gamma(u;\hat{\Phi}^{(\epsilon)}) - \gamma(u;\Phi_{BS})\right)du \right. \\
&\left. \qquad - \frac{1}{2\pi}\int_{-\infty}^{\infty}(-iu)^2 e^{-iuk}\left(\gamma(u;\hat{\Phi}^{(\epsilon)}) - \gamma(u;\Phi_{BS})\right)du \right\} + \Gamma_{BS},
\end{aligned}$$

*where $\Delta_{BS}$ and $\Gamma_{BS}$ are the risk sensitivities of the Black-Scholes price $BS(\Sigma^{\frac{1}{2}}; F_{N+1}(0), K, T)$ given by*

$$\Delta_{BS} = P_f(0,T)N'(d_+) \ \ \text{and} \ \ \Gamma_{BS} = \frac{P_f(0,T)}{S(0)\sqrt{\Sigma T}}N'(d_+).$$

*For other risk parameters such as $\Theta$ or Vega, sensitivities of the option price with respect to $t$ and $\sigma(0)$ respectively, their approximations are given in easy ways such as the difference quotient method, which needs few seconds for calculation with our closed-form formula and has satisfactory accuracies.*

#### 5.2.4. A Characteristic-function-based Monte Carlo Simulation with an Asymptotic Expansion

Here we will introduce a Monte Carlo (henceforth sometimes called M.C.) simulation scheme which incorporates the analytically obtained characteristic function. Further, with the asymptotic expansion as a control variable, the variance of this characteristic-function-based(ch.f.-based) M.C. is reduced.

In a usual M.C. procedure, we discretize the stochastic differential equations (38), (41), (42) and (62), and generate $\{f^j\}_{j=1}^M$, $M$ samples of $f_{N+1}^{(\epsilon)}(T)$ (hereafter $F_{N+1}(0)$ will be abbreviated by $F(0)$). Then the approximation for the option value, the discounted average of terminal payoffs, is obtained by;

$$\hat{V}_{MC}^{payoff}(0,M;K,T) := P_d(0,T)\frac{1}{M}\sum_{j=1}^{M}(e^{f^j} - K)^+. \tag{77}$$

On the other hand, via the pricing formula (59) in Proposition 3, the option price can be expressed with the pricing functional $\mathbf{\Psi}(\,\cdot\,; F, K, T)$ substituted the characteristic function of the underlying log-process into:

$$\begin{aligned}
V(0;K,T) &= \mathbf{\Psi}(\Phi^{P,(\epsilon)}; F(0), K, T) \\
\text{where} \ \ \mathbf{\Psi}(\Phi; F, K, T) &= P_d(0,T)\times\left[F\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iuk}\gamma(u;\Phi)du + (F-K)^+\right].
\end{aligned}$$

Since $\Phi^{P,(\epsilon)}(u)$ is defined by $\mathbf{E}^P\left[e^{iuf_{N+1}^{(\epsilon)}(T)}\right] = \mathbf{E}^P\left[e^{iuZ^{(\epsilon)}(T)}\right]\times\mathbf{E}^P\left[e^{iuA(T)}\right]$, the alternative approximation with M.C. can be constructed;

$$\hat{V}_{MC}^{chf}(0,M;K,T) \ := \ \mathbf{\Psi}(\hat{\Phi}_{MC}^P(\,\cdot\,;M); F(0), K, T) \tag{78}$$

$$\hat{\Phi}^P_{MC}(u;M) \quad = \quad \hat{\Phi}^P_{Z,MC}(u;M)\times\Phi^P_A(u) := \left(\frac{1}{M}\sum_{j=1}^{M} e^{iuZ^j}\right)\Phi^P_A(u) \qquad (79)$$

where $\{Z^j\}_{j=1}^M$ are samples of $Z^{(\epsilon)}(T)$. Here it is stressed that in this approximation there does not exist any error caused by M.C. for the (jump or continuous) part $A$.

Further, this ch.f.-based scheme can be much refined through the better estimation for $\Phi_Z^{P,(\epsilon)}(u)$ by M.C., achieved with our asymptotic expansion of the first order. Since $\Phi_Z^{P,(\epsilon)}(u)$ is expressed as $\Phi_{G_{\hat{Z}}}^{Q_u,(\epsilon)}(\epsilon h(u))$, it is done by the approximation of $\Phi_{G_{\hat{Z}}}^{Q_u,(\epsilon)}(\epsilon h(u))$ with M.C.. In what follows in this section, we abbreviate $\epsilon$(or set $\epsilon = 1$) for simplicity and use the notation $g_1 = \hat{G}_T^{Q_u,\langle 1\rangle}$, the first order coefficient of the expansion (71).

Here, in order to avoid the influence appearing in this variance reduction procedure caused by the variable transformation $h(\cdot)$, we use the following relationship

$$\mathbf{E}^{Q_u}\left[e^{ih(u)g_1}\right] = \exp\left(-\frac{1}{2}iu\Sigma\right)\mathbf{E}^{Q_u}\left[e^{iug_1}\right], \qquad (80)$$

i.e. $\Phi^{Q_u}_{g_1}(h(u)) = \exp\left(-\frac{1}{2}iu\Sigma\right)\times\Phi^{Q_u}_{g_1}(u)$. $\Phi^{Q_u}_{g_1}(v)$ is the characteristic function of $g_1$, which is equivalent to $\hat{\Phi}_{G_{\hat{Z}}}^{Q_u,(\epsilon)}(v)$ in Theorem 3 if the expansion were made only up to the first order. This equation can be easily checked with recalling $\Phi^{Q_u}_{g_1}(v) = \Phi_{0,\Sigma}(v) = \exp(-\frac{\Sigma}{2}v^2)$.

Thus on the one hand, the closed-form characteristic function of $g_1$ evaluated at $v = h(u)$ is given by

$$\Phi^{Q_u}_{g_1}(h(u)) = \exp\left(-\frac{1}{2}iu\Sigma\right)\Phi_{0,\Sigma}(u). \qquad (81)$$

But on the other hand, generating samples of $g_1$ following $N(0,\Sigma)$, $\{g^j\}_{j=1}^M$, we can further approximate the right hand side of (80) by

$$\hat{\Phi}^{Q_u}_{g_1,MC}(u;M) := \exp\left(-\frac{1}{2}iu\Sigma\right)\frac{1}{M}\sum_{j=1}^{M}\left(e^{iug^j}\right). \qquad (82)$$

Note that because only the distribution of $g_1$ matters here, we can simulate samples of $\tilde{g}_1 := \int_0^T \sigma_Z^{(0)\prime}(s)dW_s$ following $N(0,\Sigma)$ under $P$ instead of those of $g_1$, not under the measure $Q_u$ but under $P$ as well as other random variables simulated for (79).

Using two functions in (81) and (82), which both are the first-order approximations for $\Phi_Z^{Q_u,(\epsilon)}(h(u))$, define two following estimators for the option price.

$$\hat{V}^{AE}_{ana}(0;K.T) \quad := \quad \mathbf{\Psi}\left(\Phi^{Q_u}_{g_1}(h(\cdot))\times\Phi^P_A;F(0),K,T\right) \qquad (83)$$

$$\hat{V}^{AE}_{MC}(0,M;K,T) \quad := \quad \mathbf{\Psi}\left(\hat{\Phi}^{Q_u}_{g_1,MC}(\,\cdot\,;M)\times\Phi^P_A;F(0),K,T\right) \qquad (84)$$

Finally, using $\Phi^{Q_u}_{g_1}(h(u))$ as a control variable, we can construct the more sophisticated estimator $\hat{V}^{CV}(0,M;K,T)$ for the option price $V(0;K,T)$ as

$$\begin{aligned}\hat{V}^{CV}(0,M;K,T) &:= \hat{V}^{chf}_{MC}(0,M;K,T) + \left(\hat{V}^{AE}_{ana}(0;K,T) - \hat{V}^{AE}_{MC}(0,M;K,T)\right) \qquad (85)\\ &= \mathbf{\Psi}\left(\left\{\hat{\Phi}^P_{Z,MC}(\,\cdot\,;M) + \left(\Phi^{Q_u}_{g_1}(h(\cdot)) - \hat{\Phi}^{Q_u}_{g_1,MC}(\,\cdot\,;M)\right)\right\}\times\Phi^P_A;F(0),K,T\right)\end{aligned}$$

where $T = T_{N+1}$ and

$$\begin{aligned}
\hat{\Phi}^{P}_{Z,MC}(u;M) &= \frac{1}{M}\sum_{j=1}^{M} e^{iuZ^j}, \\
\Phi^{Q_u}_{g_1}(h(u)) &= \exp\left(-\frac{1}{2}iu\Sigma\right)\times\Phi_{0,\Sigma}(u), \\
\hat{\Phi}^{Q_u}_{g_1,MC}(u;M) &= \exp\left(-\frac{1}{2}iu\Sigma\right)\times\frac{1}{M}\sum_{j=1}^{M}\left(e^{iug^j}\right).
\end{aligned}$$

**Remark 6** *Here we note the following fact.*

$$\begin{aligned}
& V(0;K,T) - \hat{V}^{CV}(0,M;K,T) \\
= & \left(V(0;K,T) - \hat{V}^{chf}_{MC}(0,M;K,T)\right) - \left(\hat{V}^{AE}_{ana}(0;K,T) - \hat{V}^{AE}_{MC}(0,M;K,T)\right) \\
= & \boldsymbol{\Psi}\left(\left\{\left(\Phi^{P,(\epsilon)}_{Z} - \hat{\Phi}^{P}_{Z,MC}(\,\cdot\,;M)\right) - \left(\Phi^{Q_u}_{g_1}(h(\cdot)) - \hat{\Phi}^{Q_u}_{g_1,MC}(h(\cdot);M)\right)\right\}\times\Phi^{P}_{A};F(0),K,T\right)
\end{aligned}$$

*where $\Phi^{P,(\epsilon)}_{Z}$ is the exact characteristic function of $Z^{(\epsilon)}(T)$. The former in the first parentheses is the exact characteristic function of $Z^{(\epsilon)}(t)$ and the latter is its approximation by Monte Carlo simulations. Similarly, the former in the second parentheses is the exact one of $g_1$, the first-order expansion for $Z^{(\epsilon)}(t)$, and the latter is its approximation. Thus, in the case where the first and second term in the braces cancel each other out, the error of our hybrid estimator is expected to be small.*

**Remark 7** *We here also summarize the procedures introduced in this section.*

1. *Discretize the processes of $f^{(\epsilon)}_{dj}(t)$, $f^{(\epsilon)}_{fj}(t)$, $\sigma^{(\epsilon)}(t)$ and of $Z^{(\epsilon)}(t)$ under $P$ and generate $\{Z^j\}_{j=1}^{M}$, $M$ samples of $Z^{(\epsilon)}(T)$.*
2. *Also generate $\{\tilde{g}^j\}_{j=1}^{M}$, samples of $\tilde{g}_1 = \int_0^T \sigma^{(0)\prime}_{Z}(s)dW_s$ instead of $g_1$, under $P$ with the same sequence of random numbers used in 1.*
3. *Calculate $\hat{\Phi}^{P}_{Z,MC}(u;M)$ with $\{Z^j\}$ for each $u$, which is the characteristic function of $Z^{(\epsilon)}(T)$ approximated by M.C..*
4. *Similarly calculate $\hat{\Phi}^{Q_u}_{g_1,MC}(u;M)$ with $\{\tilde{g}^j\}$ for each $u$, the approximation for $\Phi^{Q_u}_{g_1}(h(u))$by M.C..*
5. *Using the estimators calculated in 3. and 4., approximate $\Phi^{P,(\epsilon)}_{Z}(u)$ by*

   $$\hat{\Phi}^{P}_{Z,MC}(u;M) + \left(\Phi^{Q_u}_{g_1}(h(u)) - \hat{\Phi}^{Q_u}_{g_1,MC}(u;M)\right)$$

   *where $\Phi^{Q_u}_{g_1}(u)$ is the exact characteristic function of $g_1$ given in closed-form.*
6. *Inverting the estimated characteristic function in 5. via the pricing functional $\boldsymbol{\Psi}(\,\cdot\,;F(0),K,T)$ given in (60), we finally obtain the estimator for the option price with the first-order asymptotic expansion as a control variable.*

# 6. Numerical Examples

This section examines the effectiveness of our methods through some numerical examples. First, the underlying framework is specified clearly. Then, the approximate option prices by our methods are compared to their estimates by Monte Carlo simulations. Moreover, our formula is applied to calibration of volatility surfaces observed in the JPY/USD currency option market. Finally, the examples of the variance reduction by the proposed ch.f.-based Monte Carlo simulations with the asymptotic expansion as a control variable is shown.

## 6.1. Model specification

First of all, the processes of domestic and foreign forward interest rates and of the volatility of the spot exchange rate are specified. We suppose $D = 4$, that is the dimension of the Wiener process is set to be four; it represents the uncertainty in domestic and foreign interest rates, the spot exchange rate, and its volatility. Note that in our framework correlations among all factors are allowed.

Next, we specify a volatility process, not a variance process as in affine-type models, of the spot exchange rate under the domestic risk-neutral measure as follows;

$$\sigma^{(\epsilon)}(t) = \sigma(0) + \kappa \int_0^t (\theta - \sigma^{(\epsilon)}(s))ds + \epsilon\omega' \int_0^t \sqrt{\sigma^{(\epsilon)}(s)}d\hat{W}_s \tag{86}$$

where $\theta$ and $\kappa$ represent the level and speed of its mean-reversion respectively, and $\omega$ denotes a volatility vector on the volatility. In this section the parameters are set as follows; $\epsilon = 1$, $\sigma(0) = \theta = 0.1$, and $\kappa = 0.1$; $\omega = \omega^* \overline{v}$ where $\omega^* = 0.1$ and $\overline{v}$ denotes a four dimensional constant vector given below.

We further suppose that initial term structures of domestic and foreign forward interest rates are flat, and their volatilities also have flat structures and are constant over time: that is, for all $j$, $f_{dj}(0) = f_d$, $f_{fj}(0) = f_f$, $\gamma_{dj}(t) = \gamma_d^* \overline{\gamma}_d 1_{\{t<T_j\}}(t)$ and $\gamma_{fj}(t) = \gamma_f^* \overline{\gamma}_f 1_{\{t<T_j\}}(t)$. Here, $\gamma_d^*$ and $\gamma_f^*$ are constant scalars, and $\overline{\gamma}_d$ and $\overline{\gamma}_f$ denote four dimensional constant vectors. Moreover, given a correlation matrix $\underline{C}$ among all four factors, the constant vectors $\overline{\gamma}_d$, $\overline{\gamma}_f$, $\overline{\sigma}$ and $\overline{v}$ can be determined to satisfy $||\overline{\gamma}_d|| = ||\overline{\gamma}_f|| = ||\overline{\sigma}|| = ||\overline{v}|| = 1$ and $V'V = \underline{C}$ where $V := (\overline{\gamma}_d, \overline{\gamma}_f, \overline{\sigma}, \overline{v})$.

In the following, we consider three different cases for $f_d$, $\gamma_d^*$, $f_f$ and $\gamma_f^*$ as in Table 1. For correlations, four sets of parameters are considered: In the case "Corr.1", all the factors are independent: In "Corr.2", there exists only the correlation of -0.5 between the spot exchange rate and its volatility (i.e. $\overline{\sigma}'\overline{v} = -0.5$) while there are no correlations among the others: In "Corr.3", the correlation between interest rates and the spot exchange rate are allowed while there are no correlations among the others; the correlation between domestic ones and the spot forex is 0.5($\overline{\gamma}_d'\overline{\sigma} = 0.5$) and the correlation between foreign ones and the spot forex is -0.5($\overline{\gamma}_f'\overline{\sigma} = -0.5$). In these three cases, $A(t) \equiv 0$ for simplicity, that is there is assumed no component such as a jump whose characteristic function is available in closed form.

Finally in "Corr.4", correlations among most factors are considered; $\overline{\gamma}_d'\overline{\gamma}_f = 0.3$ between the domestic and foreign interest rates; $\overline{\gamma}_d'\overline{\sigma} = 0.5$, $\overline{\gamma}_f'\overline{\sigma} = -0.5$ between interest rates and the spot forex; and $\overline{\sigma}'\overline{v} = -0.5$ between the spot forex and its volatility. In this case also $A(t)$, a jump component, will be taken into account; $A(t)$ is assumed to be a compensated compound Poisson process with its intensity $\lambda$ and with random jumps following $N(m, s^2)$; $\lambda = 1$, $m = -0.05$ and $s = 0.05$. In this case, the characteristic function of $A(t)$ is given by

$$\Phi_A^P(t, u) = \exp\left(\lambda t\left(e^{imu-\frac{1}{2}s^2u^2} - 1\right) - iu\lambda t\left(e^{m+\frac{1}{2}s^2} - 1\right)\right).$$

It is well known that (both of exact and approximate)evaluation of the long-term options is a hard task in the case with a complex structure of correlations and/or with a jump component, such as "Corr.3" or "Corr.4". In "Corr.4" only the estimates by the hybrid method will be shown.

Lastly, we make another assumption that $\gamma_{dn(t)-1}(t)$ and $\gamma_{fn(t)-1}(t)$, volatilities of the domestic and foreign interest rates applied to the period from $t$ to the next fixing date $T_{n(t)}$, are equal to be zero for arbitrary $t \in [t, T_{n(t)}]$.

Table 1. Initial domestic/foreign forward interest rates and their volatilities

| | $f_d$ | $\gamma_d^*$ | $f_f$ | $\gamma_f^*$ |
|---|---|---|---|---|
| case (i) | 0.05 | 0.2 | 0.05 | 0.2 |
| case (ii) | 0.02 | 0.5 | 0.05 | 0.2 |
| case (iii) | 0.05 | 0.2 | 0.02 | 0.5 |

## 6.2. Examinations of our closed-form approximation formulas

In this subsection the accuracy of each pricing formula is examined. We show numerical examples for evaluation of call options calculated by Monte Carlo (henceforth called M.C.) simulations and by our approximation formulas of the third-order standard or hybrid method with maturities of five or ten years and with different parameters for interest rates and correlations set in the previous subsection. Each estimator based on the M.C. simulations is obtained by 1,000,000 trials with *antithetic variables method*.

Figures 1.-4. show the results in our numerical investigations. Figure 1. reports the differences of estimators by formulas in a five-year maturity and Figure 3. does in a ten-year maturity. In Figure 2. and Figure 4., they are shown in terms of implied volatilities. Differences shown in those figures between the approximations by our formulas and those by M.C. simulations are defined as (the approximate value by asymptotic expansions)-(the estimate by M.C. simulations).

Generally speaking, the third-order hybrid scheme performs better than the standard scheme of the same order: The absolute levels of differences of the hybrid estimators are on average 0.025/0.06% in prices/in implied volatilities for a five-year maturity and 0.100/0.18% for a ten-year maturity; on the other hand, those of the standard estimators are on average 0.039/0.10% for a five-year maturity and 0.177/0.38% for a ten-year maturity. Most of differences of the hybrid ones are less than 0.1/0.2% for five years and 0.25/0.4% for ten years.

Moreover, in "Corr.4" in which most of existing methods for analytical evaluation including our standard method are difficult to be applied, the formula by the hybrid method still works well.

The stability of performances of our methods, even in the complicated settings with many correlated processes and/or with a jump process in addition, can be advantageous in practice.

## 6.3. Calibration to the market

In this subsection, the third order asymptotic expansion formula by the hybrid method which performed better in the previous experiments is applied to calibration of our model parameters to observed volatilities with maturities of five and ten years in the JPY/USD currency option market. Market makers in OTC currency option markets usually provide quotes on Black-Scholes implied volatilities and the moneyness of an option is expressed in terms of Black-Scholes delta, rather than its strike price. We use the data of volatility surfaces on Sep 27, Oct 30 and Dec 07, 2007, after the beginning of the *subprime-loan* crash, which consist of 25 delta put, 10 delta put, at-the-money, 10 delta call, and 25 delta call with their maturities of seven and ten years (these data are provided by Forex Division of Mizuho Corporate Bank, Ltd.). We also construct

Table 2. Comparisons of variances of our estimators, given in terms of ratios to that of a crude M.C.

| Moneyness | 5y | Variance ratio | |
|---|---|---|---|
| | | CIR-type vol. | SSM |
| 1 | Chf/Crude | 1.013 | 0.045 |
| | CV/Crude | 0.161 | 0.007 |
| 1.3 | Chf/Crude | 0.980 | 0.060 |
| | CV/Crude | 0.112 | 0.011 |
| Moneyness | 10y | Variance ratio | |
| | | CIR-type vol. | SSM |
| 1 | Chf/Crude | 1.014 | 0.191 |
| | CV/Crude | 0.207 | 0.035 |
| 1.4 | Chf/Crude | 1.009 | 0.171 |
| | CV/Crude | 0.158 | 0.029 |

domestic/foreign forward interest rates' term structures and volatilities using the data downloaded from Bloomberg on swap rates and cap volatilities in each market.

Tables 3.-4. and Figures 5.-10. show the data on volatility surfaces and our calibrated parameters. In Table 3. and Figures 5.-7., the calibration results to the observed volatility smile for five or ten year, separately. Additionally, Table 4. and Figures 8.-10 shows the result in the joint calibration to the volatilities for five and ten years.

Most of the absolute errors in separate calibration are less than 0.01%, in joint calibration less than 0.3%., which seems small enough for a practical purpose. Consequently, we conclude that our formula is flexible enough for the calibration of observed surfaces, which is a hard task with other time-consuming methods such as numerical ones. We may use the calibrated parameters for valuation of illiquid options and more complicated currency derivatives.

## 6.4. Variance Reduction with the Asymptotic Expansion

Here the convergence of the ch.f.-based Monte Carlo estimator with the asymptotic expansion as a control variable is compared to that of a "crude M.C."(only with the antithetic variables method). In this subsection, the following three estimators are examined; $\hat{V}_{MC}^{payoff}(0, M; T, K)$ in (77) is the standard M.C. estimator which averages the discounted terminal payoffs; $\hat{V}_{MC}^{chf}(0, M; T, K)$ in (78) is obtained via the Fourier inversion of the characteristic function approximated by M.C.; and $\hat{V}^{CV}(0, M; T, K)$ in (85) is the estimator with a use of the first-order asymptotic expansion as a control variable. We apply the antithetic variables method to all estimators.

First, in Table 2., comparisons of their convergences in the same model(indicated by "CIR-type vol.") and the same parameters of "Corr.2" assumed in examples in the previous subsection are shown. It lists up the ratios of variances of $\hat{V}_{MC}^{chf}(0, M; T, K)$ and $\hat{V}^{CV}(0, M; T, K)$ to that of $\hat{V}_{MC}^{payoff}(0, M; T, K)$ with the same 1,000,000 trials: Strictly speaking, we show the variances of a series of these estimators calculated with each 1,000 paths.

The ch.f.-based M.C. seems to have almost the same variance with a crude M.C. in this setting. Contrarily to this, usage of our asymptotic expansion as a control variable for the ch.f.-based M.C. reduces its variances; reducing more for OTM than for ATM in these investigations. They are reduced around to 15% in five years and to 20% in ten years of a crude M.C.'s.

Second, we investigate the convergence of our estimators in the case where the analytically-obtained component $A(t)$ exists. In such cases, their convergences are expected to be faster than those in the previous settings are, since we need not approximate the whole part of the characteristic function of the underlying asset but do by M.C. only the parts without known analytical expressions of their characteristic functions.

In particular, for $A(t)$ we assume the following Stochastic Skew Model(SSM) in [11], which well captures the time-varying behavior of the smiles or skews observed in currency option markets:

$$\begin{aligned} A(t) &= \xi_L(t) + \xi_R(t) \\ &:= \left(-\frac{1}{2}\int_0^t V_L(s)ds + \int_0^t \sqrt{V_L(s)}\vec{\xi}_L' d\hat{W}_s\right) + \left(-\frac{1}{2}\int_0^t V_R(s)ds + \int_0^t \sqrt{V_R(s)}\vec{\xi}_R' d\hat{W}_s\right) \\ V_k(t) &= V_k(0) + \kappa_k \int_0^t (\theta_k - V_k(s))\, ds + \omega_k \int_0^t \sqrt{V_k(s)}\vec{v}_k' d\hat{W}_s, \quad for\ k = L, R \end{aligned}$$

under the domestic risk-neutral measure, where $\xi_L$ and $\xi_R$ are assumed to be independent (hence $\vec{\xi}_L'\vec{\xi}_R = 0$). Further, for the correlations among those components we assume $\vec{\xi}_L'\vec{v}_L < 0$, $\vec{\xi}_R'\vec{v}_R > 0$; $\vec{\xi}_L'\vec{v}_R = \vec{\xi}_R'\vec{v}_L = \vec{v}_L'\vec{v}_R = 0$. These conditions mean that $V_L(V_R)$ correlates to the spot forex negatively(positively) and is independent of the other processes.

This can be regarded as a double Heston-type model which consists of two independent stochastic variance processes correlating to the spot forex in opposite directions. For simplicity, the jump components appearing in the original paper of SSM are omitted here with little loss of generality.

Then the characteristic function of $A(t)$ is given by

$$\begin{aligned} \Phi_A^P(t;u) &= \Phi_{\xi_L}^P(t,u) \times \Phi_{\xi_R}^P(t,u), \\ \Phi_{\xi_k}^P(t,u) &= \left(\cosh\frac{\eta_k t}{2} + \frac{\kappa_k - i\rho_k\omega_k u}{\eta_k}\sinh\frac{\eta_k t}{2}\right)^{-\frac{2\kappa_k\theta_k}{\omega_k^2}} \\ &\quad \times \exp\left(\frac{\kappa_k\theta_k(\kappa_k - i\rho_k\omega_k u)t}{\omega_k^2} - \frac{(u^2 + iu)V_k(0)}{\eta_k \coth\frac{\eta_k t}{2} + \kappa_k - i\rho_k\omega_k u}\right), \end{aligned}$$

where $\eta_k := \sqrt{\omega_k^2(u^2 + iu) + (\kappa_k - i\rho_k\omega_k u)^2}$ and $\rho_k := \vec{\xi}_k'\vec{v}_k$ (See Daffie, Pan and Singleton [14] or Carr and Wu [11] for details).

In the investigations made here, the parameters are set as follows. For interest rates, the parameters of case (ii) in Table 1. are used except for $\vec{\gamma}_d'\vec{\gamma}_f = 0.5$; for the stochastic volatility in $Z(t)$, $\sigma(0) = \theta = \omega^* = 0$ are assumed so as to ensure $\sigma^{(\epsilon)}(t) \equiv 0$, that is the objective $\hat{Z}(t)$ of our asymptotic expansion consists only of the domestic and foreign interest rates. Finally, for SSM, $V_k(0) = \theta_k = 0.0075$, $\kappa_k = 0.5$, $\omega_k = 0.1$ for $k = L, R$; the correlation $\rho_L(\rho_R)$ between $V_L(V_R)$ and the spot forex is assumed to -0.5(0.5). Other correlation parameters among factors are all assumed to be zero. These settings can be interpreted as SSM under a market model of interest rates.

In Table 2., as well as in the examination with the CIR-type volatility model, comparisons of the variances in this setting(indicated by "SSM") are made. Contrary to results in the previous case, in this case the ch.f-based M.C. estimator has the variances of around 5% in five years and 20% in ten years of a crude M.C's, reduced by usage of the analytically solved characteristic function as we expected. The scheme proposed in Section 5.2.4. further cuts down those variances to around 1% in five years and 3% in ten years of the originals. Thus it can be said that, in such cases where the closed characteristic function of a part of the underlying is available, incorporation of this knowledge and our analytical approximation for another part's via the ch.f-based M.C. scheme dramatically accelerates the convergence of M.C. simulations.

In Figures 11-12.,we present these variance-reduction effects. It is stressed that at most 5,000 paths for five years and 50,000 paths for ten years are enough to obtain the accuracy within 0.01.

Figure 1. A comparison of the accuracy of each estimator in prices for a five-year maturity.

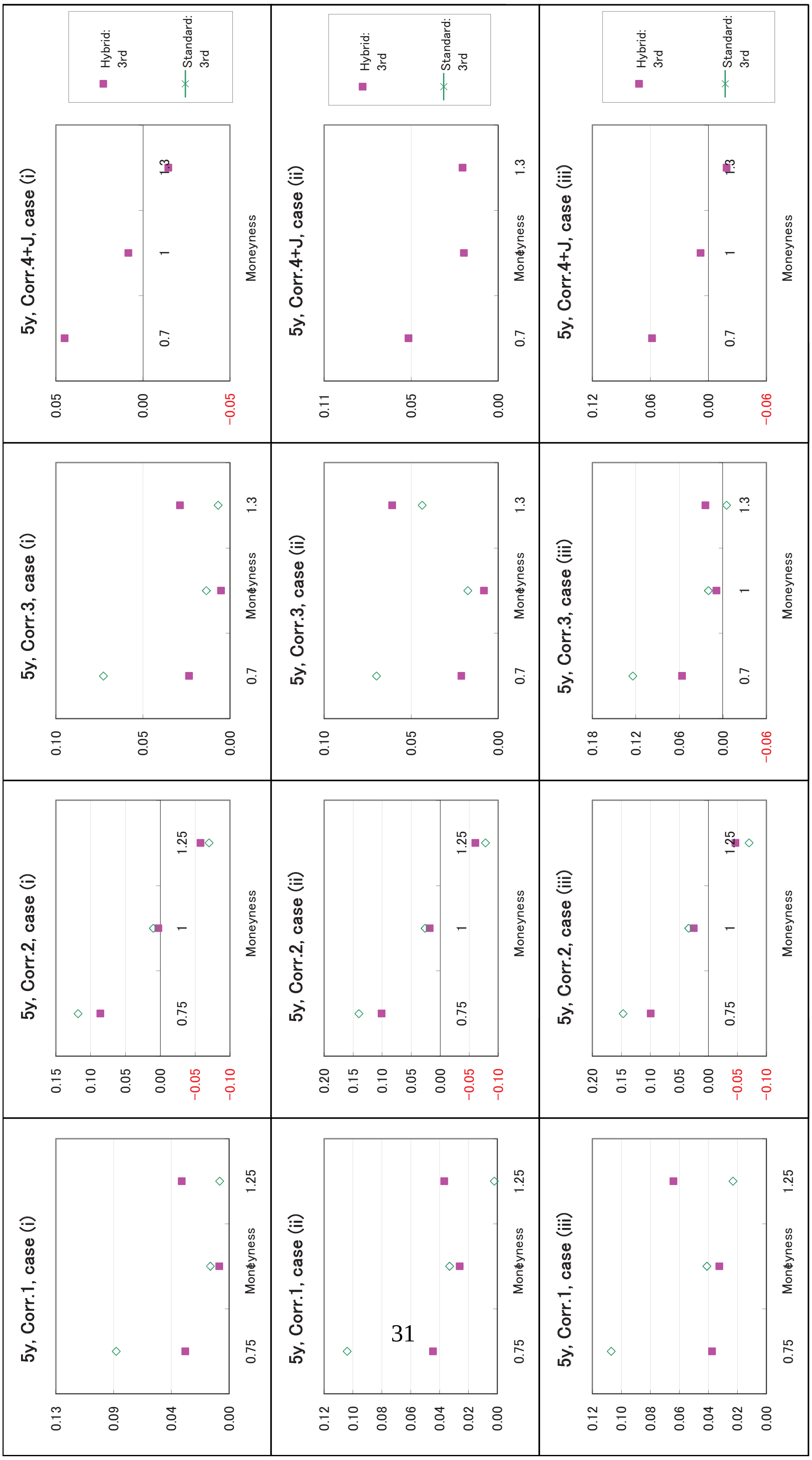


Figure 2. A comparison of the accuracy of each estimator in implied volatilities for a five-year maturity.

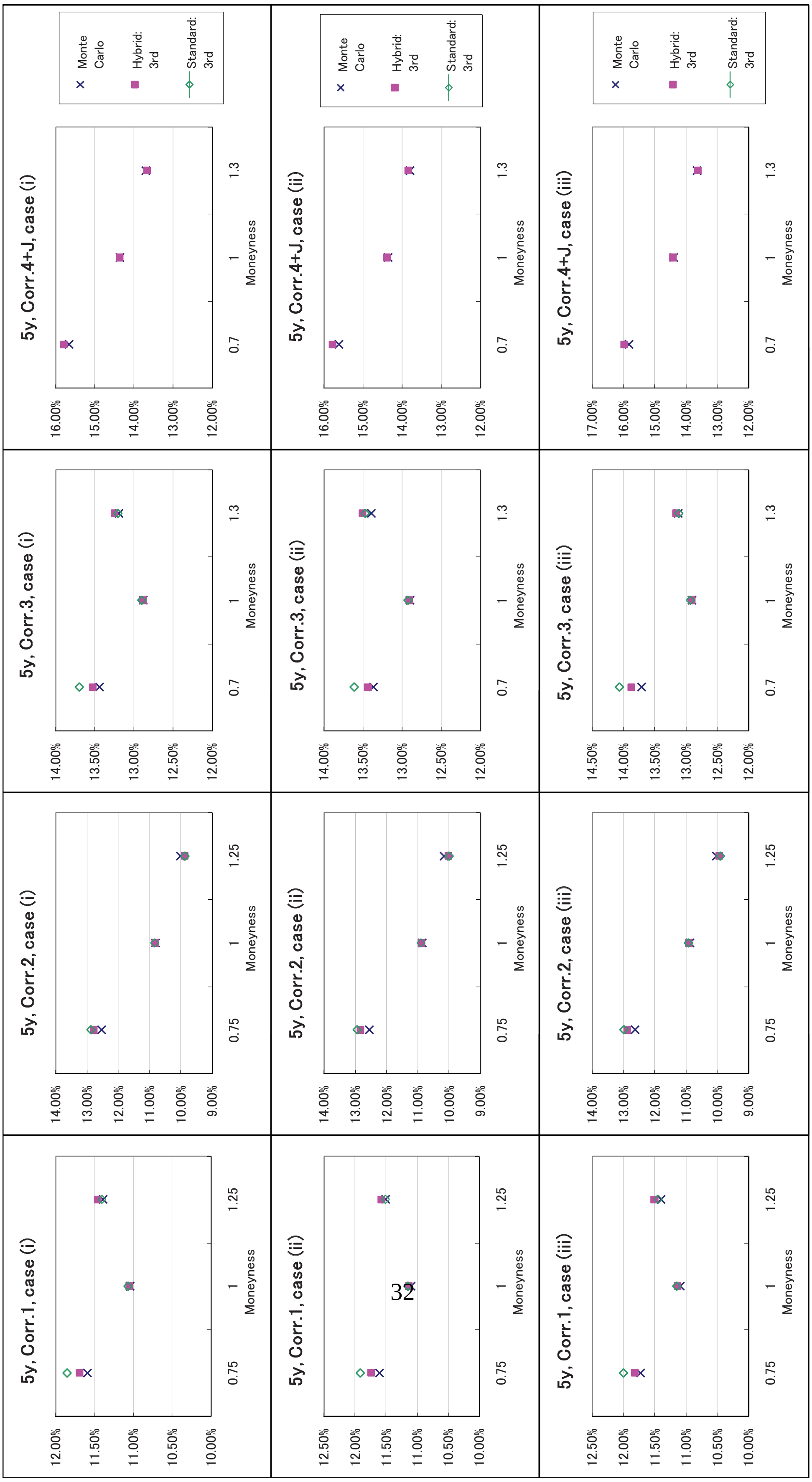


Figure 3. A comparison of the accuracy of each estimator in prices for a ten-year maturity.

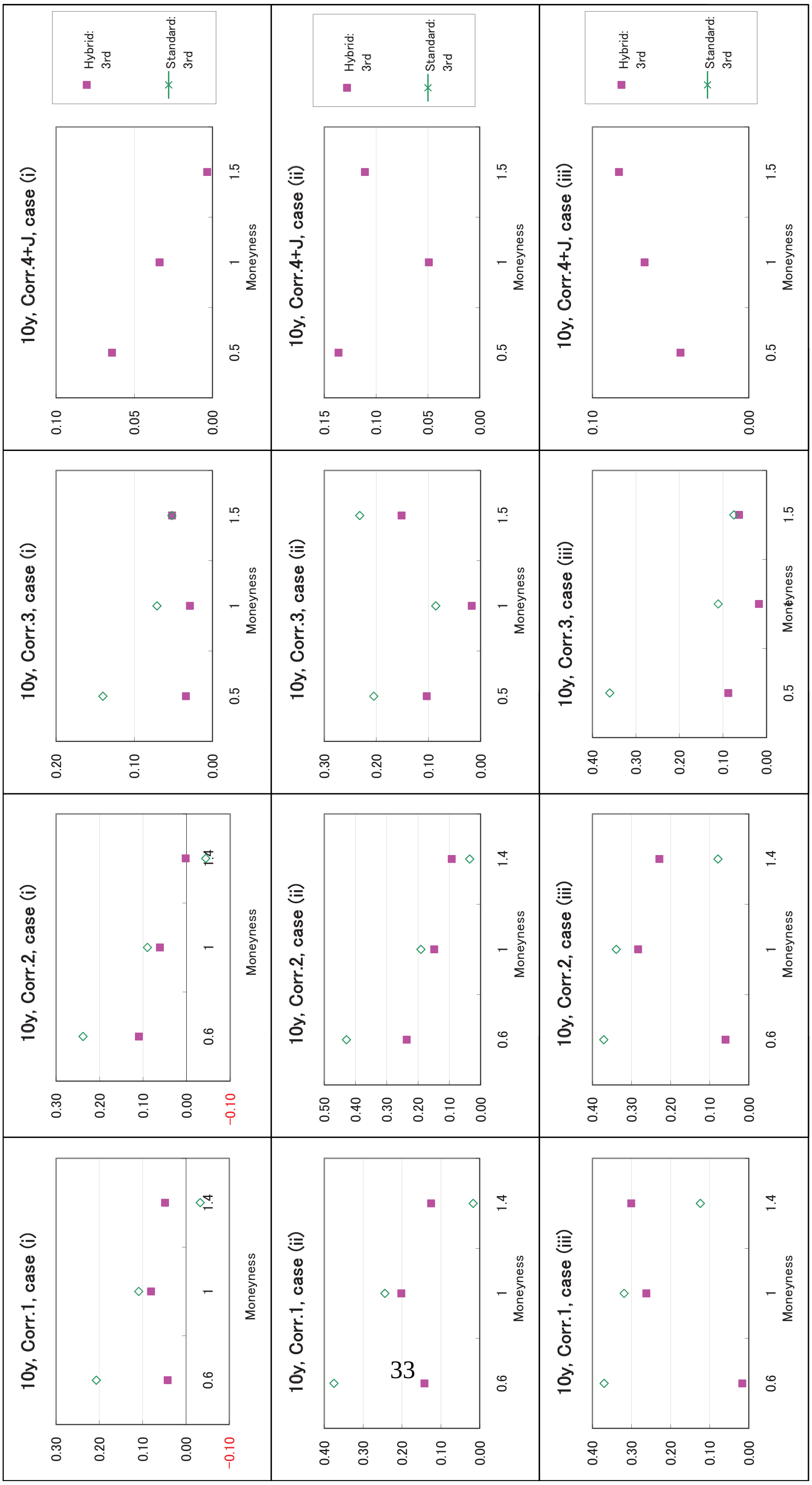


Figure 4. A comparison of the accuracy of each estimator in implied volatilities for a ten-year maturity.

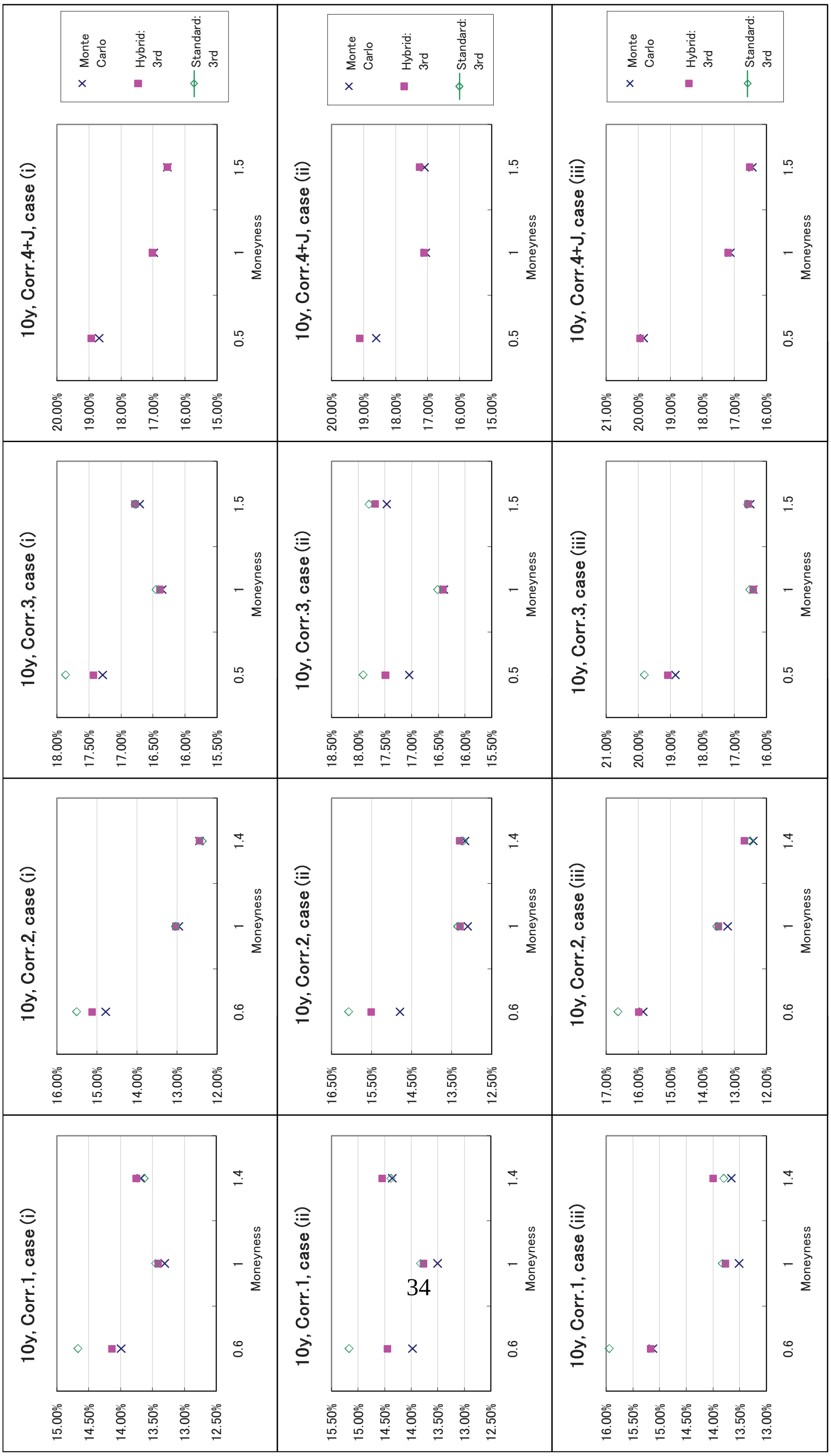


Table 3. A separate calibration to the observed implied volatilities for a five-year and ten-year maturity.

| Date | 07/09/27 | Type | Separate |
|---|---|---|---|

| 5y | 10put | 25put | ATM | 25call | 10call |
|---|---|---|---|---|---|
| Market I.V. | 16.66% | 10.92% | 7.90% | 6.07% | 6.96% |
| Model I.V. | 16.66% | 10.92% | 7.90% | 6.07% | 6.95% |
| differences | 0.00% | -0.01% | 0.00% | 0.00% | -0.01% |

| 10y | 10put | 25put | ATM | 25call | 10call |
|---|---|---|---|---|---|
| Market I.V. | 20.00% | 13.59% | 10.50% | 8.34% | 9.50% |
| Model I.V. | 20.01% | 13.59% | 10.50% | 8.35% | 9.49% |
| differences | 0.01% | -0.00% | 0.00% | 0.01% | -0.02% |

| Date | 07/10/30 | Type | Separate |
|---|---|---|---|

| 5y | 10put | 25put | ATM | 25call | 10call |
|---|---|---|---|---|---|
| Market I.V. | 15.58% | 11.20% | 8.10% | 6.60% | 6.38% |
| Model I.V. | 15.59% | 11.20% | 8.10% | 6.60% | 6.38% |
| differences | 0.01% | 0.00% | 0.00% | -0.00% | 0.00% |

| 10y | 10put | 25put | ATM | 25call | 10call |
|---|---|---|---|---|---|
| Market I.V. | 19.29% | 13.49% | 10.35% | 8.19% | 8.69% |
| Model I.V. | 19.29% | 13.48% | 10.34% | 8.19% | 8.69% |
| differences | -0.00% | -0.01% | -0.01% | 0.00% | 0.00% |

| Date | 07/12/07 | Type | Separate |
|---|---|---|---|

| 5y | 10put | 25put | ATM | 25call | 10call |
|---|---|---|---|---|---|
| Market I.V. | 17.20% | 11.72% | 8.45% | 6.62% | 7.00% |
| Model I.V. | 17.19% | 11.71% | 8.45% | 6.62% | 7.00% |
| differences | -0.01% | -0.01% | 0.00% | 0.00% | 0.00% |

| 10y | 10put | 25put | ATM | 25call | 10call |
|---|---|---|---|---|---|
| Market I.V. | 20.46% | 13.98% | 10.65% | 8.38% | 9.26% |
| Model I.V. | 20.46% | 13.98% | 10.65% | 8.38% | 9.26% |
| differences | -0.00% | 0.00% | 0.00% | -0.00% | 0.00% |

Table 4. A joint calibration to the observed implied volatilities for a five-year and ten-year maturity.

| Date | 07/09/27 | Type | Joint |
|---|---|---|---|

| 5y | 10put | 25put | ATM | 25call | 10call |
|---|---|---|---|---|---|
| Market I.V. | 16.66% | 10.92% | 7.90% | 6.07% | 6.96% |
| Model I.V. | 16.88% | 11.14% | 7.88% | 6.07% | 6.87% |
| differences | 0.22% | 0.22% | -0.02% | 0.00% | -0.09% |
| 10y | 10put | 25put | ATM | 25call | 10call |
| Market I.V. | 20.00% | 13.59% | 10.50% | 8.34% | 9.50% |
| Model I.V. | 19.76% | 13.63% | 10.51% | 8.37% | 9.13% |
| differences | -0.24% | 0.04% | 0.01% | 0.03% | -0.37% |

| Date | 07/10/30 | Type | Joint |
|---|---|---|---|

| 5y | 10put | 25put | ATM | 25call | 10call |
|---|---|---|---|---|---|
| Market I.V. | 15.58% | 11.20% | 8.10% | 6.60% | 6.38% |
| Model I.V. | 15.90% | 11.11% | 8.09% | 6.58% | 6.50% |
| differences | 0.32% | -0.09% | -0.01% | -0.01% | 0.12% |
| 10y | 10put | 25put | ATM | 25call | 10call |
| Market I.V. | 19.29% | 13.49% | 10.35% | 8.19% | 8.69% |
| Model I.V. | 18.99% | 13.57% | 10.35% | 8.19% | 8.69% |
| differences | -0.30% | 0.08% | 0.00% | 0.01% | -0.11% |

| Date | 07/12/07 | Type | Joint |
|---|---|---|---|

| 5y | 10put | 25put | ATM | 25call | 10call |
|---|---|---|---|---|---|
| Market I.V. | 17.20% | 11.72% | 8.45% | 6.62% | 7.00% |
| Model I.V. | 17.40% | 11.77% | 8.45% | 6.68% | 6.94% |
| differences | 0.20% | 0.05% | -0.03% | 0.06% | -0.05% |
| 10y | 10put | 25put | ATM | 25call | 10call |
| Market I.V. | 20.46% | 13.98% | 10.65% | 8.38% | 8.87% |
| Model I.V. | 20.25% | 13.87% | 10.68% | 8.38% | 8.87% |
| differences | -0.21% | -0.11% | 0.03% | 0.00% | -0.39% |

Figure 5. A separate calibration to the observed implied volatilities for a five-year or ten-year maturity on Sep 27, 2007.

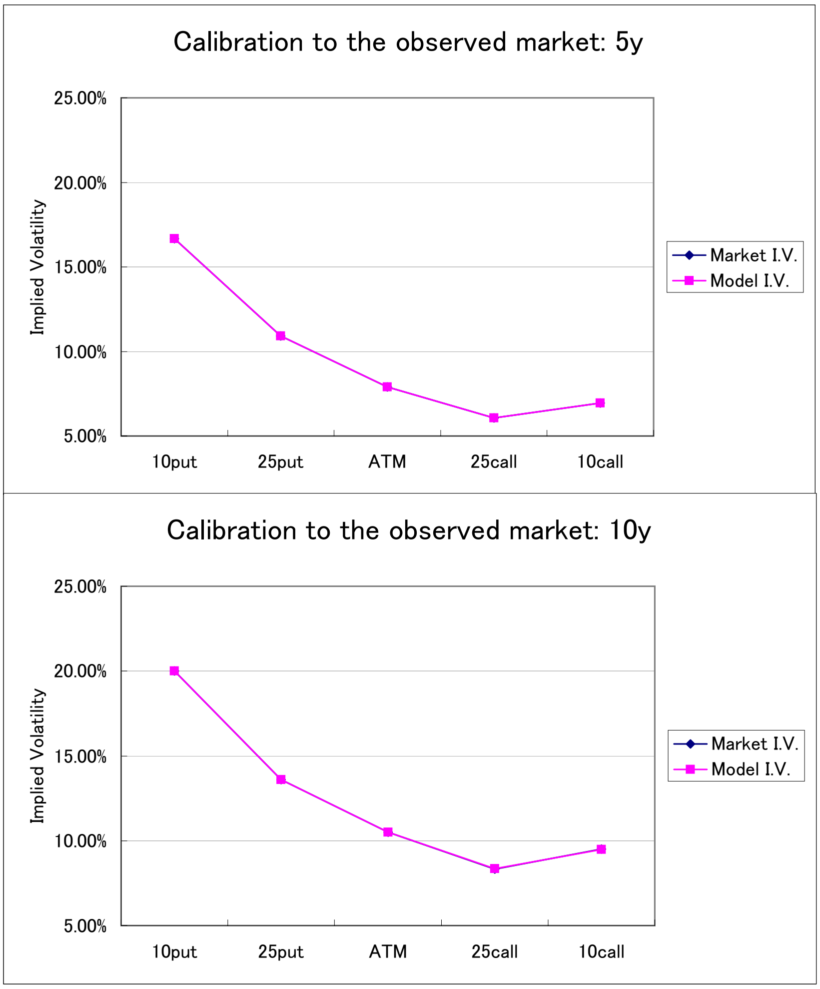


Figure 6. A separate calibration to the observed implied volatilities for a five-year or ten-year maturity on Oct 30, 2007.

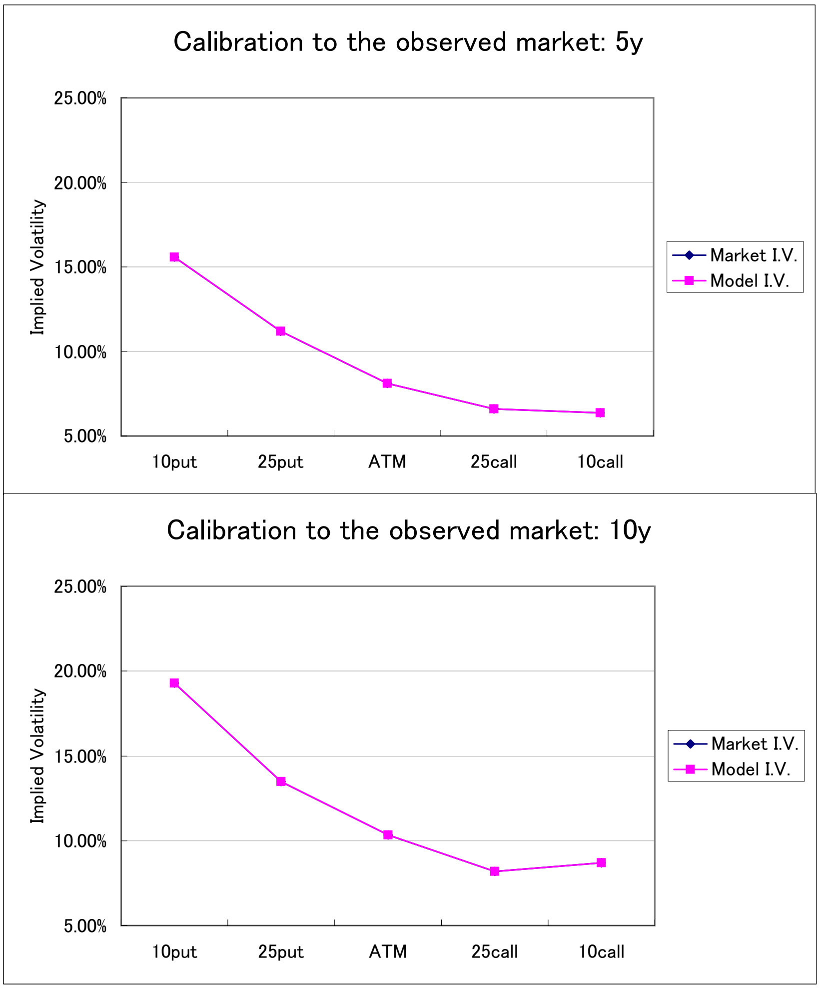


Figure 7. A separate calibration to the observed implied volatilities for a five-year or ten-year maturity on Dec 07, 2007.

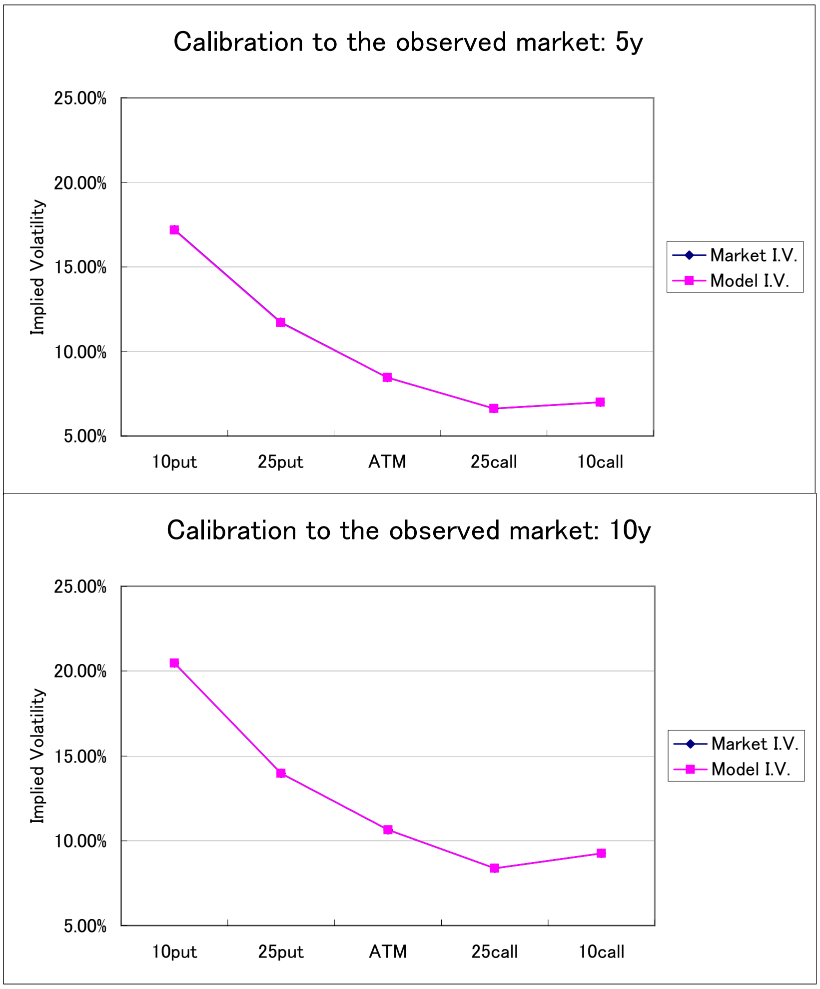


Figure 8. A joint calibration to the observed implied volatilities for a five-year and ten-year maturity on Sep 27, 2007.

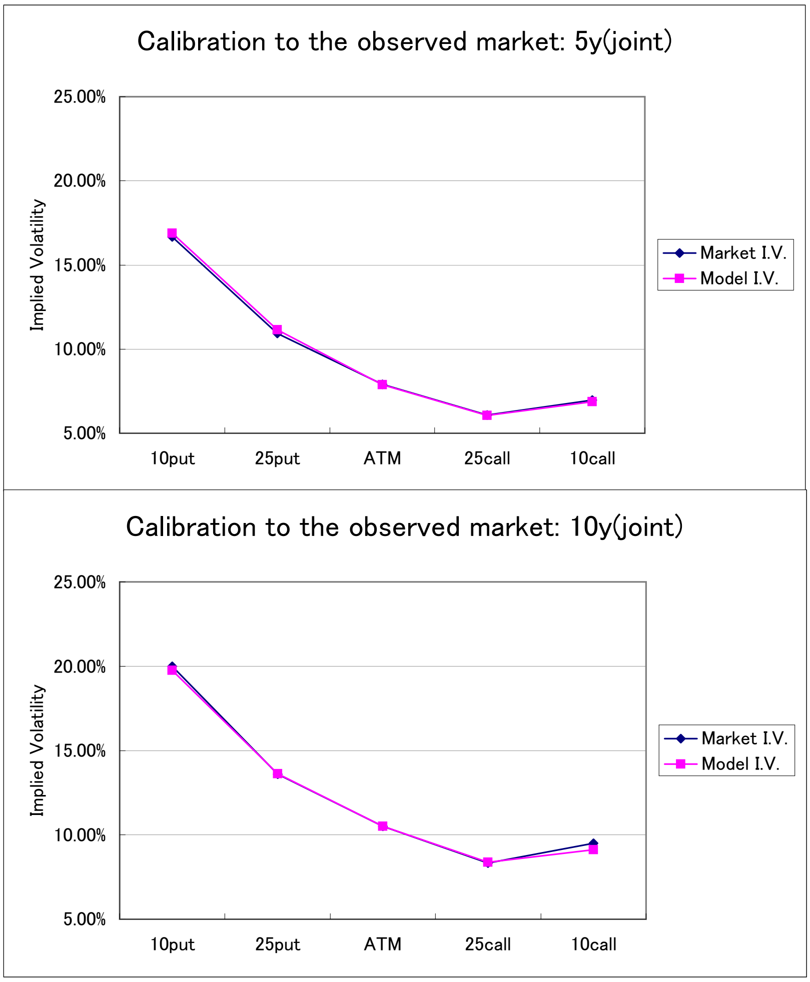


Figure 9. A joint calibration to the observed implied volatilities for a five-year and ten-year maturity on Oct 30, 2007.

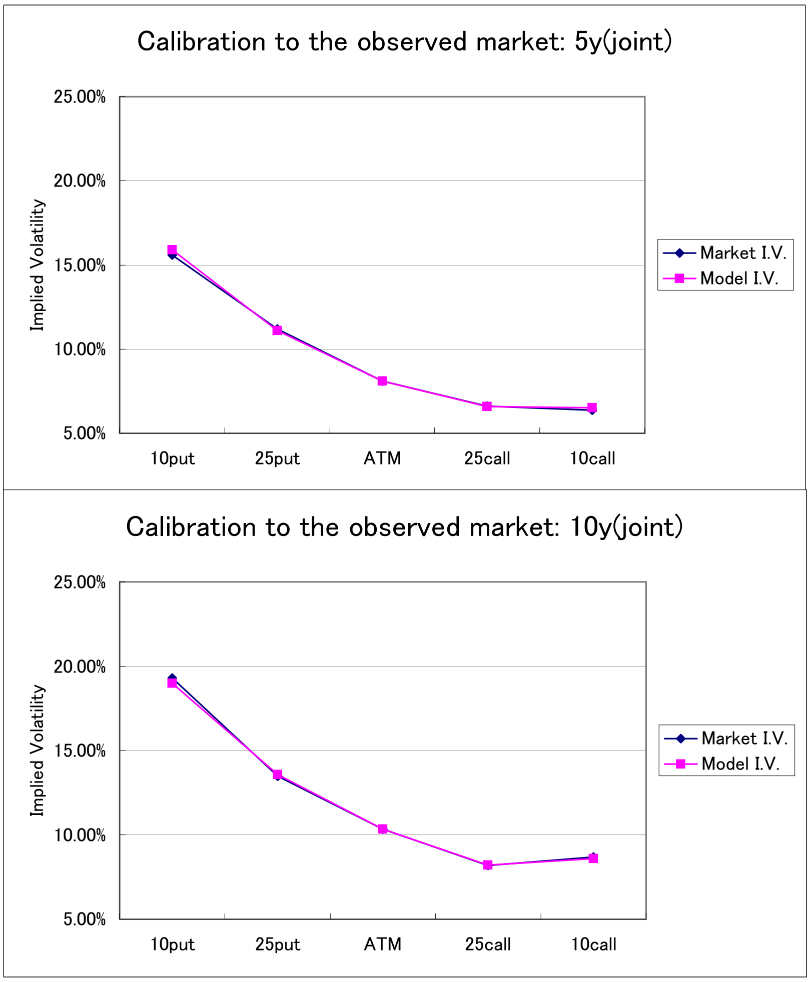


Figure 10. A joint calibration to the observed implied volatilities for a five-year and ten-year maturity on Dec 07, 2007.

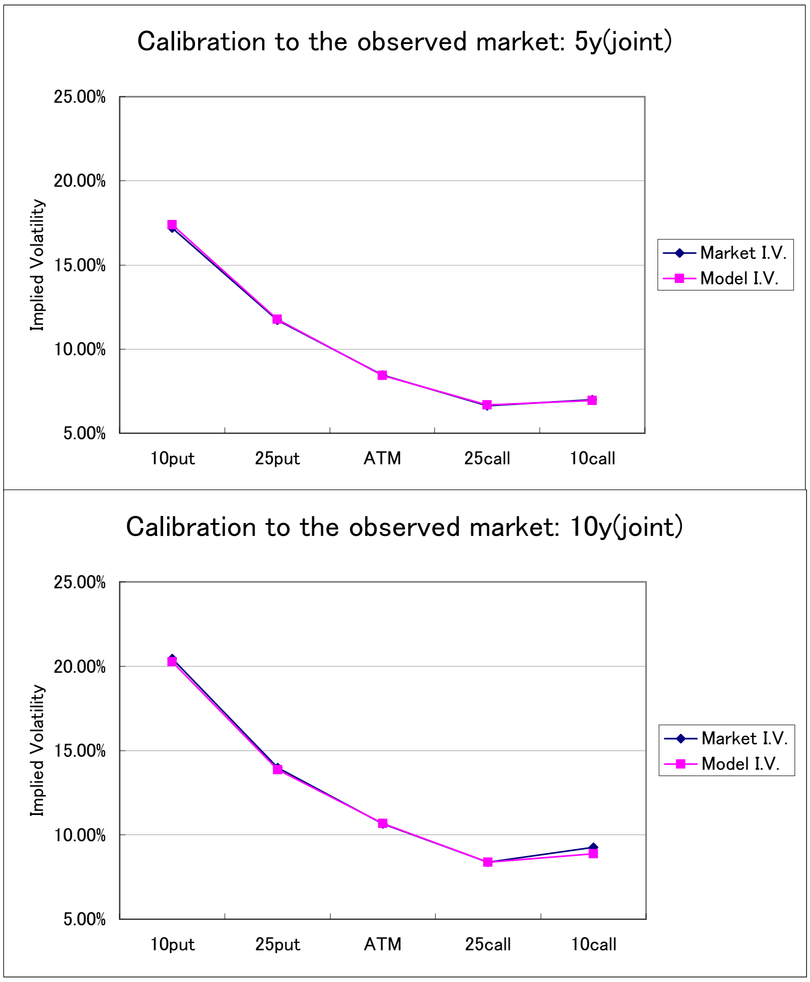


Figure 11. Convergences of the estimators for a five-year maturity in SSM.

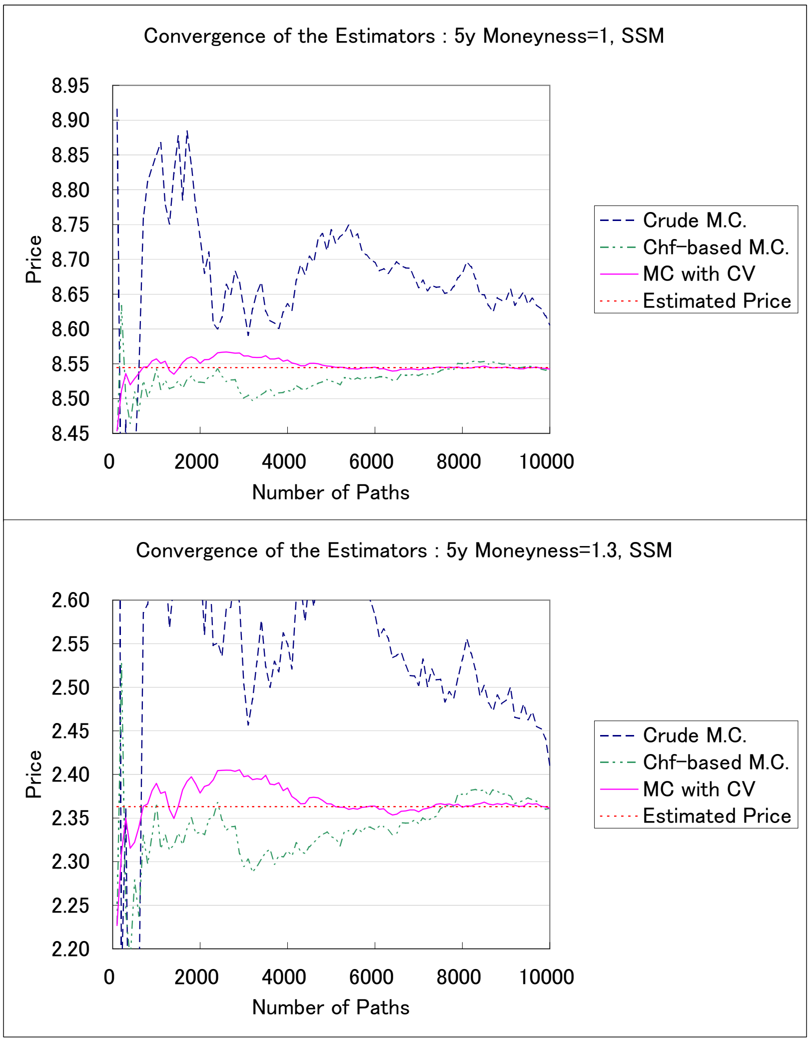
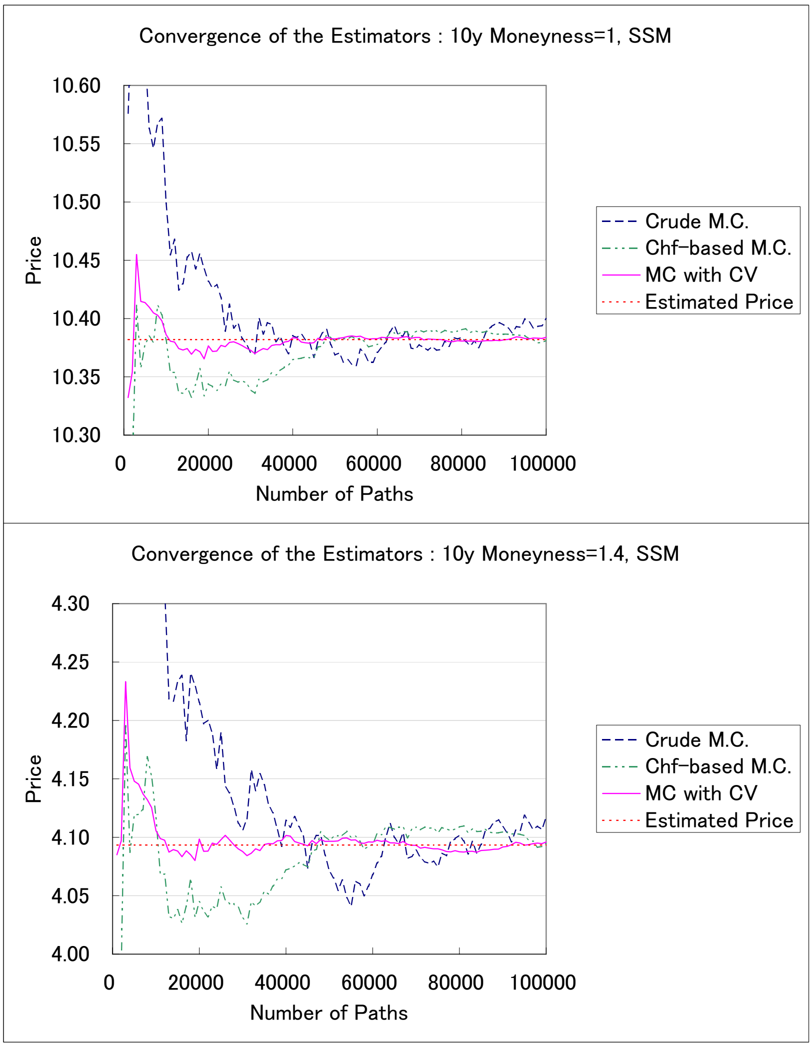


Figure 12. Convergences of the estimators for a ten-year maturity in SSM.



## 7. Conclusion

In this chapter, we proposed approximation formulas based on the asymptotic expansion to evaluate currency options with a libor market model of domestic and foreign interest rates and stochastic volatilities and/or jumping components of spot exchange rates. In particular, the two different approaches were presented; one is the standard approach and the other is the hybrid. Moreover, the variance reduction technique with using the asymptotic expansion as a control variable in the ch.f.-based Mote Carlo simulation was proposed.

We also provided several numerical examples to investigate the accuracy and effectiveness of our methods in approximation of option prices, calibration to the market, and acceleration of simulations: In general, satisfactory accuracy was confirmed in approximating option prices with maturities up to ten years; our formula successfully reconstructed volatility surfaces observed in the recent JPY/USD currency option market; variances in Monte Carlo simulations were reduced maximumly to 1% in a certain setting by the proposed control variable and ch.f.-based simulations.

Finally, our research plans are stated as follows: Similar methods will be applied to valuation and calibration of options with longer maturities; higher order asymptotic expansions or/and different types of expansions may be used; asymptotic expansion formulas will be utilized for extended models where stochastic volatility structures of interest rates are allowed or/and a jump component is added to the volatility process of the spot exchange rate.

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## References

[1] Antonov, A., Misirpashaev, T. and Piterbarg, V.[2007], "Markovian Projection Onto a Heston Model," Working paper.

[2] Amin K.I. and Bodurtha, J.N. [1995], "Discrete-time valuation of American options with stochastic interest rates," *Review of Financial Studies*, Vol. 8, pp. 193-234.

[3] Amin K.I. and Jarrow R. [1991], "Pricing Foreign-Currency Options Under Stochastic Interest-Rates," *Journal of International Money and Finance*, Vol. 10, pp. 310-329.

[4] Bichteler, K., Gravereaux, J.B. and Jacod, J[1987], "Malliavin Calculus for Processes with Jumps," Gordon and Breach Science Publishers.

[5] Black, F. [1976], "The Pricing of Commodity Contracts," *Journal of Financial Economics*, Vol. 3, pp. 167-179.

[6] Black, F and Scholes, M [1973], "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy*, Vol. 81(3), pp. 637-654.

[7] Brace, A., Gatarek, D. and Musiela, M. [1998], "A Multifactor Gauss Markov Implementation of Heath, Jarrow, and Morton," *Mathematical Finance*, Vol. 7, pp. 127-154.

[8] Brigo, D. and Mercurio, F. [2001], *Interest Rate Models: theory and practice - with smile, inflation and credit*, 2nd Edition, Springer.

[9] Björk, T. [2004], *Arbitrage Theory in Continuous Time*, Oxford University Press.

[10] Carr, P. and Madan, D.[1999], "Option Valuation using the Fast Fourier Transform", *Journal of Computational Finance*, Vol. 2(4), pp. 61-73.

[11] Carr, P. and Wu, L.[2007], "Stochastic Skew in Currency Options", *Journal of Financial Economics*, Vol. 86(1), pp.213-247.

[12] Davydov, D. and Linetsky, V.[2003], "Pricing Options on Scalar Diffusions: An Eigenfunction Expansion Approach, "*Operations Research*, Vol. 51, pp. 185-209.

[13] Dempster, M.A.H., and Hutton, J.P. [1997], "Numerical valuation of cross-currency swaps and swaptions," *Mathematics of Derivatives Securities*, pp. 473-503, Cambridge Univ. Press.

[14] Duffie, D., Pan, J. and Singleton, K.[1999], "Transform Analysis and Asset Pricing for Affine Jump-Diffusions," *Econometrica*, Vol. 68, pp. 1343-1376.

[15] Faulhaber, O.[2002], "analytic methods for pricing double barrier options in the presence of stochastic volatility, "Working paper.

[16] Fouque, J.-P., Papanicolaou, G. and Sircar, K. R.[1999], "Financial Modeling in a Fast Mean-reverting Stochastic Volatility Environment, "*Asia-Pacific Financial Markets*, Vol. 6(1), pp.37-48.

[17] Fouque, J.-P., Papanicolaou, G. and Sircar, K. R.[2000], *Derivatives in financial Markets with Stochastic Volatility*, Cambridge University Press.

[18] Fournié, E., Lasry, J.-M. and Touzi, N.[1997], "Monte Carlo Methods for Stochastic Volatility Models," *Numerical Methods in Finance*, pp.146-164, Cambridge University Press.

[19] Garman, M. and Kohlhagen, S. [1983], "Foreign currency options," *Journal of International Money and Finance*, Vol. 2, pp. 231-237.

[20] Glasserman, P.[2003], *Monte Carlo Methods in Financial Engineering*, Springer.

[21] Gorovoi, V. and Linetsky, V.[2004], "Black's Model of Interest Rates as Options, Eigenfunction Expansions and Japanese Interest Rates," *Mathematical Finance*, Vol. 14, pp. 49-78.

[22] Grabbe, J. [1983], "The Pricing of Call and Put Options on Foreign-Exchange," *Journal of International Money and Finance*, Vol. 2, pp. 239-253.

[23] Heath, D., Jarrow, R. and Morton, A.[1992], "Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation," *Econometrica*, Vol. 60, pp. 77-105.

[24] Henry-Labordere, P.[2005a], "A General Asymptotic Implied Volatility for Stochastic Volatility Models," Working Paper.

[25] Henry-Labordere, P.[2005b], "Solvable Local and Stochastic Volatility Models: Supersymmetric Methods in Option Pricing," Working Paper.

[26] Henry-Labordere, P.[2006], "Unifying the BGM and SABR Models: A short Ride in Hyperbolic Geometry," Working Paper.

[27] Heston, S.[1993], “A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options,” *The Review of Financial Studies*, Vol.6, pp. 327-343.

[28] Hilliard, J., Madura, J. and Tucker, A. [1991], “Currency Option Pricing with Stochastic Domestic and Foreign Interest-Rates,” *Journal of Financial and Quantitative Analysis*, Vol. 26, pp. 139-151.

[29] Hull, J. C. and White, A. D. [1990], “Pricing Interest Rate Derivate Securities,” *Review of Financial Studies*, Vol. 3, pp. 573-592.

[30] Hull, J. C. and White, A. D. [1994a], “Numerical Procedures for Implementing Term Structure Models I: Single-factor Models,” *Journal of Derivatives*, Vol. 2(Fall), pp. 7-16.

[31] Ikeda, N. and Watanabe, S. [1989], *Stochastic Differential Equations and Diffusion Processes*, Second Edition, North-Holland/Kodansha, Tokyo.

[32] Karatzas, I. Lehoczky, J. and Shreve, S. [1987], “Optimal Portfolio and Consumption Decisions for a ‘small investor’ on a Finite Horizon,” *SIAM Journal of Control and Optimaization*, Vol. 25, pp. 1157-1186.

[33] Karatzas, I. and Shreve, S. [1998], *Methods of Mathematical Finance*, Springer.

[34] Kawai, A. [2003], “A New Approximate Swaption Formula in the LIBOR Market Model: An Asymptotic Expansion Approach,” *Applied Mathematical Finance*, Vol.10, pp. 49-74.

[35] Kemna, A. G. Z. and Vorst, A. C. F.[1990], *Stochastic Differential Equations and Diffusion Processes*, Second Edition, North-Holland/Kodansha.

[36] Kobayashi, T., Takahashi, A. and Tokioka, N. [2003], “Dynamic Optimality of Yield Curve Strategies, ” *International Review of Finance*, Vol.4, pp. 49-78.

[37] Kunitomo, N. and Kim, Y-J.[2007], “Effects of Stochastic Interest Rates and Volatility on Contingent Claims,” *Japanese Economic Review*, Vol. 58(1), pp. 71-106.

[38] Kunitomo, N. and Takahashi, A. [1992], “Pricing Average Options,” *Japan Financial Review*, Vol.14, pp. 1-20. (In Japanese.)

[39] Kunitomo, N. and Takahashi, A. [2001], “The Aysmptotic Expansion Approach to the Valuation of Interest Rate Contingent Claims,” *Mathematical Finance*, Vol.11, pp. 117-151.

[40] Kunitomo, N. and Takahashi, A. [2003], “On Validity of the Asymptotic Expansion Approach in Contingent Claim Analysis,” *Annals of Applied Probability*, Vol.13(3)

[41] Kunitomo, N. and Takahashi, A. [2004], “Applications of the Asymptotic Expansion Approach based on Malliavin-Watanabe Calculus in Financial Problems,” *Stochastic Processes and Applications to Mathematical Finance*, pp. 195-232, World Scientific.

[42] L*ü*tkebohmert, E. [2004a], “Finite Dimensional Realizations of Interest Rate Models with Jumps and an Asymptotic Expansion for the Black-Scholes Model with Generalized Volatility,” Unpublished Ph.D. Dissertation, Institute for Applied Mathematics, University of Bonn.

[43] L*ü*tkebohmert, E. [2004b], “An Asymptotic Expansion for a Black-Scholes Type Model,” Bull. Sci. math., Vol.128, pp.661-685.

[44] Linetsky, V.[2004a], "Spectral Expansions for Asian (Average Price) Options, " *Operations Research*, Vol. 52, pp. 856-867.

[45] Linetsky, V.[2004b], "The Spectral Decomposition of the Option Value," *International Journal of Theoretical and Applied Finance,* Vol. 7, pp. 337-384.

[46] Linetsky, V.[2004c], "Lookback Options and Diffusion Hitting Times: A Spectral Expansion Approach," *Finance and Stochastics*, Vol. 8, pp. 373-398.

[47] Linetsky, V.[2004d], "The Spectral Representation of Bessel Processes with Drift: Applications in Queueing and Finance," *Journal of Applied Probability*, Vol. 41, pp. 327-344.

[48] Malliavin, P. and Thalmaier, A. [2006], *Stochastic Calculus of Variations in Mathematical Finance,* Springer.

[49] Matsuoka, R., Takahashi, A. and Uchida, Y. [2006], "A New Computational Scheme for Computing Greeks by the Asymptotic Expansion Approach," *Asia-Pacific Financial Markets*, Vol.11, pp. 393-430.

[50] Melino, A. and Turnbull, S. [1991], "The Pricing of Foreign-Currency Options," *Canadian Journal of Economics*, Vol. 24, pp. 251-281.

[51] Merton, R.C. [1973], "Theory of Rational Option Pricing", *Bell Journal of Economics and Management Science*, Vol.4(1), pp. 141-183.

[52] Mikkelsen, P. [2001], "Cross-currency LIBOR market models," Working paper, University of Aarhus.

[53] Miltersen, K., Sandmann, K. and Sondermann, D. [1997], "Closed Form Solutions for Term Structure Derivatives with Long-normal Interest Rates," *Journal of Finance*, Vol. 52, pp. 409-430.

[54] Milshtein, G. and Schoenmakers, J.[2002], "Numerical Construction a Hedging Strategy against the Multi-asset European Claim," *Stochastics and Stochastics Reports*, Vol. 37, pp. 125-157.

[55] Muroi, Y. [2005], "Pricing Contingent Claims with Credit Risk: Asymptotic Expansion Approach," *Finance and Stochastics*, Vol 9(3), pp. 415-427.

[56] Newton, N., J.[1994], "Variance Reduction for Simulated Diffusion," *SIAM Journal on Applied Mathematics*, Vol. 54, pp. 1780-1805.

[57] Ninomiya, S. and Victoir, N. [2006], Weak Approximation of Stochastic Differential Equations and Application to Derivative Pricing, preprint.

[58] Nualart, D. [1995], "The Malliavin Calculus and Related Topics," Springer.

[59] Nualart, D., *Ü*st*ü*nel A. S. and Zalai M. [1988], "On the moments of a multiple Wiener-It*ô* integral and the space induced by the polynomials of the integral," *Stochastics*, Vol 25, 233-340.

[60] Piterbarg, V.[2003], "A Stochastic Volatility Forward Libor Model with a Term Structure of Volatility Smiles," Working paper.

[61] Piterbarg, V.[2005a], "A Multi-currency Model with FX Volatility Skew," Working paper.

[62] Piterbarg, V.[2005b], "Stochastic Volatility Model with Time-dependent Skew," *Applied Mathematical Finances*, Vol. 12(2), pp. 147-185.

[63] Piterbarg, V.[2006], "Markovian Projection Method for Volatility Calibration," Working paper.

[64] Rebonato, R.[2004], "Volatility and Correlation," pp. 427-436, John Wiley & Sons, Ltd.

[65] Robert, C., P. and Casella, G.[2000], *Monte Carlo Statistical Methods*, Springer.

[66] Rumsey, J. [1991], "Pricing Cross-Currency Options," *Journal of Futures Markets*, Vol. 11, pp. 89-93.

[67] Sakamoto, Y., Takada, Y. and Yoshida, N. [2004], "Expansions of the Coverage Probabilities of Prediction Region Based on a Shrinkage Estimator," *Statistics*, Vol. 38, no. 5, 381-390.

[68] Sakamoto, Y. and Yoshida, N. [1996], "Expansion of Perturbed Random Variables Based on Generalized Wiener Functionals," *Journal of Multivariate Analysis*, Vol. 59, no. 1, 34-59.

[69] Sakamoto, Y. and Yoshida, N. [1998], "Asymptotic Expansion of $M$-estimator over Wiener Space," *Statistical Inference for Stochastic Processes*, Vol. 1-1, 85-103.

[70] Schlögl, E.[2002], "A Multi-currency Extension of the Lognormal Interest Rate Market Model," *Finance and Stochastics*, pp. 173-196.

[71] Schoutens, W. [2000], *Stochastic Processes and Orthogonal Polynomials*, volume 146 of *Lecture Notes in Statistics*, Springer-Verlag.

[72] Shigekawa, I. [1998], "Stochastic Analysis," Iwanami(in Japanese).

[73] Siopacha, M. and Teichmann, J.[2007], "Weak and Strong Taylor Methods for Numerical Solutions of Stochastic Differential Equations," Working paper.

[74] Takahashi, A. [1995], "Essays on the Valuation Problems of Contingent Claims," Unpublished Ph.D. Dissertation, Haas School of Business, University of California, Berkeley.

[75] Takahashi, A. [1999], "An Asymptotic Expansion Approach to Pricing Contingent Claims," *Asia-Pacific Financial Markets*, Vol.6, pp. 115-151.

[76] Takahashi, A. [2005], "A Note on Computing Greeks by an Asymptotic Expansion Scheme," Preprint.

[77] Takahashi, A. and Saito, T.[2003], "An Asymptotic Expansion Approach to Pricing American Options," *Monetary and Economic Studies*, Vol. 22, pp. 35-87, Bank of Japan. (In Japanese.)

[78] Takahashi, A. and Takehara, K.[2007], "Pricing Currency Options with a Market Model of Interest Rates under Jump-Diffusion Stochastic Volatility Processes of Spot Exchange Rates," *Asia-Pacific Financial Markets*, Vol.14 , pp. 69-121.

[79] Takahashi, A. and Takehara, K.[2008a], "Fourier Transform Method with an Asymptotic Expansion Approach: an Applications to Currency Options," *International Journal of Theoretical and Applied Finance*, Vol. 11(4), pp. 381-401.

[80] Takahashi, A. and Takehara, K.[2008b], "A Hybrid Asymptotic Expansion Scheme: an Applications to Long-term Currency Options," Working Paper.

[81] Takahashi, A., Takehara, K. and Toda, M.[2009], "Computation in an Asymptotic Expansion Method," Working Paper.

[82] Takahashi, A. Takehara, K. and Yamazaki, A.[2006], "An Asymptotic Expansion Approach to Currency Options with a Market Model of Interest Rates under Stochastic Volatility Processes of Spot Exchange Rates," Working Paper.

[83] Takahashi, A. and Tokioka, N. [1999], “Valuation of Cross-currency Derivatives with Stochastic Interest Rates in a Three-factor Model,” *Gendai Finance*, Vol. 5, pp. 3-16. (In Japanese.)

[84] Takahashi, A. and Uchida, Y. [2006], “New Acceleration Schemes with the Asymptotic Expansion in Monte Carlo Simulation, ”*Advances in Mathematical Economics*, Vol. 8, pp. 411-431.

[85] Takahashi, A. and Yoshida, N.[2004], “An Asymptotic Expansion Scheme for Optimal Investment Problems,” *Statistical Inference for Stochastic Processes*, Vol.7(2), pp. 153-188.

[86] Takahashi, A. and Yoshida, N. [2005], “Monte Carlo Simulation with Asymptotic Method,” *Journal of Japan Statistical Society*, Vol. 35(2), pp. 171-203.

[87] Talay, D. and Tubaro, L.[1990], “Expansion of the Global Error for Numerical Schemes Solving Stochastic Differential Equations,” *Stochastic Analysis and Applications*, Vol. 8, pp. 483-509.

[88] Watanabe, S. [1984], *Lectures on Stochastic Differential Equations and Malliavin Calculus*, Tata Institute of Fundamental Research, Springer-Verlag.

[89] Watanabe, S. [1987], “Analysis of Wiener Functionals (Malliavin Calculus) and its Applications to Heat Kernels,” *The Annals of Probability*, Vol. 15, 1-39.

[90] Yoshida, N. [1992a], ”Asymptotic Expansion for Small Diffusions via the Theory of Malliavin-Watanabe,” *Probability Theory and Related Fields*, Vol. 92, pp. 275-311.

[91] Yoshida, N. [1992b], “Asymptotic Expansions for Statistics Related to Small Diffusions,” *Journal of Japan Statistical Society*, Vol.22, pp. 139-159.