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Improving the Rank-Adjusted Anderson-Rubin Test with Many Instruments and Persistent Heteroscedasticity

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Abstract

Anderson and Kunitomo (2007) have developed the likelihood ratio criterion, which is called the Rank-Adjusted Anderson-Rubin (RAAR) test, for testing the coefficients of a structural equation in a system of simultaneous equations in econometrics against the alternative hypothesis that the equation of interest is identified. It is related to the statistic originally proposed by Anderson and Rubin (1949, 1950), and also to the test procedures by Kleibergen (2002) and Moreira (2003). We propose a modified procedure of RAAR test, which is suitable for the cases when there are many instruments and the disturbances have persistent heteroscedasticities.

Key Words

Structural Equation, Likelihood Ratio Criterion, RAAR test, Modified RAAR test Many Instruments, Persistent Heteroscedasticity.

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1. Introduction

Anderson and Kunitomo (2007) have developed a likelihood ratio test for a hypothesis about the coefficients of one structural equation in a set of simultaneous equations, which is called the *Rank-Adjusted Anderson-Rubin* (RAAR) test. The null hypothesis is that the vector of coefficients is a specified vector; the alternative hypothesis is that the structural equation is identified. They have derived the limiting distribution of the RAAR test under the *standard model* and a model of *many instruments* situation. The limiting distribution of -2 times the logarithm of the likelihood ratio criterion is often chi-square with degrees of freedom equal to one less than the number of coefficients specified in the null hypothesis when the disturbance terms are homoscedastic. The problem of testing a null hypothesis on the coefficients of the structural equation has been studied by many econometricians since Anderson and Rubin (1949). See Moreira (2003) and Andrews, Moreira and Stock (2006) for a recent review of these studies. When there are many instruments and/or the disturbance terms are heteroscedastic, the distribution of the RAAR test may not be a chi-square. Then there is an important question how to extend the existing testing procedures in such cases, which may be useful for practical applications.

The main purpose of this paper is to propose a new way of improving the RAAR test, which may be called the MRAAR test. We show that the MRAAR test statistic improves the asymptotic properties of the RAAR test and many other testing procedures even when there are many instruments including the cases of many weak instruments and the disturbances have persistent heteroscedasticity. The particular type heteroscedasticity with many instruments has been recently discussed by Hausman, J., W. Newey, T. Woutersen, J. Chao and N. Swanson (2007), and Kunitomo (2008) called it the *Persistent Heteroscedasticity.* Also we expect that the resulting procedure based on the MRAAR test should have good asymptotic properties because the MRAAR test is essentially the same as the likelihood ratio test under the standard situation.

In Section 2 we state the structural equation model and the alternative testing procedures of unknown parameters in simultaneous equation models with possibly many instruments. Then in Section 3 we develop a new way of improving the RAAR test procedure and discuss its asymptotic properties. We also relate our test statistic to the testing procedures developed by Kleibergen (2002) and Moreira (2003). In Section 4 we shall discuss possible extensions and in Section 5 we shall report the finite sample properties of the null distributions of the MRAAR test and other test statistics based on a set of Monte Carlo experiments. Then some brief concluding remarks will be given in Section 6. The mathematical derivations will be given in Section 7.

2. Tests in Structural Equation Models with Possibly Many Instruments

Let a single linear structural equation be

$$y_{1i} = \boldsymbol{\beta}_2^{'} \boldsymbol{y}_{2i} + \boldsymbol{\gamma}_1^{'} \boldsymbol{z}_{1i} + u_i \quad (i = 1, \cdots, n), \tag{2.1}$$

where y_{1i} and $\boldsymbol{y}_{2i}$ are a scalar and a vector of G_2 endogenous variables, respectively, and $\boldsymbol{z}_{1i}$ is a vector of K_1 (included) exogenous variables in (2.1), $\boldsymbol{\gamma}_1$ and $\boldsymbol{\beta}_2$ are $K_1 \times 1$ and $G_2 \times 1$ vectors of unknown parameters, and u_i are mutually independent disturbance terms with $\mathcal{E}(u_i) = 0$ and $\mathcal{E}(u_i^2 | \mathbf{z}_i^{(n)}) = \sigma_i^2$ $(i = 1, \cdots, n)$. We assume that (2.1) is one equation in a system of $1 + G_2$ endogenous variables $\boldsymbol{y}_i^{'} = (y_{1i}, \boldsymbol{y}_{2i}^{'})^{'}$ and $\mathbf{Y} = (\mathbf{y}_1^{(n)}, \mathbf{Y}_2^{(n)})$ is an $n \times (1 + G_2)$ vector observations of endogenous variables. We consider

$$\mathbf{Y}_2^{(n)} = \mathbf{\Pi}_{2n}^{(z)} + \mathbf{V}_2 \ , \tag{2.2}$$

where $\mathbf{\Pi}_{2n}^{(z)} = (\boldsymbol{\pi}_{2i}^{'}(\mathbf{z}_i^{(n)}))$ is an $n \times G_2$ matrix, each row $\boldsymbol{\pi}_{2i}^{'}(\mathbf{z}_i^{(n)})$ depends on a $K_n \times 1$ vector $\mathbf{z}_i^{(n)}$, $\mathbf{V}_2^{(n)}$ is an $n \times G_2$ matrix, $\mathbf{v}_1^{(n)} = \mathbf{u} + \mathbf{V}_2^{(n)} \boldsymbol{\beta}_2$, and $\mathbf{V} = (\mathbf{v}_1^{(n)}, \mathbf{V}_2^{(n)})$. Then we can write

$$\mathbf{Y} = \mathbf{\Pi}_n^{(z)} + \mathbf{V}, \tag{2.3}$$

where $\mathbf{\Pi}_n^{(z)} = (\mathbf{Z}_1 \boldsymbol{\gamma}_1 + \mathbf{\Pi}_{2n}^{(z)} \boldsymbol{\beta}_2, \mathbf{\Pi}_{2n}^{(z)})$ is an $n \times (1 + G_2)$ matrix, $\mathbf{Z} = (\mathbf{Z}_1, \mathbf{Z}_{2n}) = (\mathbf{z}_i^{(n)'})$ is an $n \times K_n$ matrix of $K_1 + K_{2n}$ instruments $(\mathbf{z}_i^{(n)} = (\mathbf{z}_{1i}^{'}, \mathbf{z}_{2i}^{(n)'})^{'})$, $\mathbf{V} = (\mathbf{v}_i^{'})$ is an

$n \times (1 + G_2)$ matrix of disturbances with $\mathcal{E}(\mathbf{v}_i | \mathbf{z}_i^{(n)}) = \mathbf{0}$ and

$$\mathcal{E}(\mathbf{v}_i \mathbf{v}_i^{'} | \mathbf{z}_i^{(n)}) = \mathbf{\Omega}_i = \begin{bmatrix} \omega_{11.i} & \boldsymbol{\omega}_{2.i}^{'} \\ \boldsymbol{\omega}_{2.i} & \mathbf{\Omega}_{22.i} \end{bmatrix} . \tag{2.4}$$

The vector of K_n $(= K_1 + K_{2n})$ instruments $\mathbf{z}_i^{(n)}$ satisfies the orthogonal condition $\mathcal{E}[u_i \boldsymbol{z}_i^{(n)}] = \mathbf{0}$ $(i = 1, \cdots, n)$. The relation between (2.1) and (2.2) gives $u_i = (1, -\boldsymbol{\beta}_2^{'})\mathbf{v}_i$ and

$$\sigma_i^2 = (1, -\boldsymbol{\beta}_2^{'})\mathbf{\Omega}_i \begin{pmatrix} 1 \\ -\boldsymbol{\beta}_2 \end{pmatrix} = \boldsymbol{\beta}^{'} \mathbf{\Omega}_i \boldsymbol{\beta} , \tag{2.5}$$

where $\boldsymbol{\beta}^{'} = (1, -\boldsymbol{\beta}_2^{'})$. Since our main interest is the application to micro-econometric data, we impose the condition

$$\frac{1}{n} \sum_{i=1}^{n} \mathbf{\Omega}_i \xrightarrow{p} \mathbf{\Omega} \tag{2.6}$$

and $\mathbf{\Omega}$ is a positive definite (constant) matrix. Then

$$\frac{1}{n} \sum_{i=1}^{n} \sigma_i^2 \xrightarrow{p} \sigma^2 \ (= \boldsymbol{\beta}^{'} \mathbf{\Omega} \boldsymbol{\beta} > 0) . \tag{2.7}$$

Define the $(1 + G_2) \times (1 + G_2)$ matrices by

$$\mathbf{G} = \mathbf{Y}^{'} \mathbf{Z}_{2.1} \mathbf{A}_{22.1}^{-1} \mathbf{Z}_{2.1}^{'} \mathbf{Y} , \tag{2.8}$$

and

$$\mathbf{H} = \mathbf{Y}^{'} \left(\mathbf{I}_n - \mathbf{Z}(\mathbf{Z}^{'}\mathbf{Z})^{-1}\mathbf{Z}^{'} \right) \mathbf{Y} , \tag{2.9}$$

where $\mathbf{Z}_{2.1} = \mathbf{Z}_{2n} - \mathbf{Z}_1 \mathbf{A}_{11}^{-1} \mathbf{A}_{12}$, $\mathbf{A}_{22.1} = \mathbf{Z}_{2.1}^{'} \mathbf{Z}_{2.1}$ and

$$\mathbf{A} = \begin{pmatrix} \mathbf{Z}_1^{'} \\ \mathbf{Z}_{2n}^{'} \end{pmatrix} (\mathbf{Z}_1, \mathbf{Z}_{2n}) = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \tag{2.10}$$

is a nonsingular matrix (a.s.).

The RAAR test developed by Anderson and Kunitomo (2007) is that the null hypothesis that $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta}_0$ (i.e. $\boldsymbol{\beta}^{'} = (1, -\boldsymbol{\beta}_2^{(0)})$ is rejected if

$$\frac{1 + \dfrac{\hat{\boldsymbol{\beta}}^{'} \mathbf{G} \hat{\boldsymbol{\beta}}}{\hat{\boldsymbol{\beta}}^{'} \mathbf{H} \hat{\boldsymbol{\beta}}}}{1 + \dfrac{\boldsymbol{\beta}_0^{'} \mathbf{G} \boldsymbol{\beta}_0}{\boldsymbol{\beta}_0^{'} \mathbf{H} \boldsymbol{\beta}_0}} < c(K_{2n}, q_n) , \tag{2.11}$$

where $\hat{\boldsymbol{\beta}}$ is the solution of

$$\left[\frac{1}{n}\mathbf{G} - \frac{1}{q_n}\lambda_n\mathbf{H}\right]\hat{\boldsymbol{\beta}}_{LIML} = \mathbf{0} \tag{2.12}$$

and λ_n is the smallest root of $|(1/n)\mathbf{G} - l(1/q_n)\mathbf{H}| = 0$ ($q_n = n - K_n$ and $c(K_{2n}, q_n)$ is a constant to be chosen).

As an influential study, Anderson and Rubin (1949) proposed the Anderson-Rubin (AR) test, which is to reject H_0 if

$$\frac{\boldsymbol{\beta}_0'\mathbf{G}\boldsymbol{\beta}_0}{\boldsymbol{\beta}_0'\mathbf{H}\boldsymbol{\beta}_0} > \frac{K_{2n}}{q_n}F_{K_{2n},q_n}(\epsilon) \ , \tag{2.13}$$

where $F_{K_{2n},q_n}(\epsilon)$ denotes the $1-\epsilon$ significance point of the F-distribution with K_{2n} and q_n degrees of freedom.

We call the left-hand side of (2.12) the *Rank-Adjusted Anderson-Rubin* (RAAR) criterion. Moreira (2003) arrived at a statistic which is similar by a somewhat different route and proposed a simulation based test procedure, which is the *conditional likelihood statistic*.

3 The modified RAAR tests

3.1 Modifying the RAAR statistic for Many Instruments with Persistent Heteroscedasticity

We consider the situation that there are many instruments. Anderson, Kunitomo and Masushita (2007) have discussed the estimation problem of the structural equation of interest with many instruments under a set of assumptions. The basic conditions for many instruments which Anderson et al. (2007) have used are

$$\frac{K_{2n}}{n} \longrightarrow c \ \ (0 \leq c < 1), \tag{3.1}$$

$$\frac{1}{d_n^2}\mathbf{\Pi}_{2n}^{(z)'}\mathbf{P}_{2.1}\mathbf{\Pi}_{2n}^{(z)} \stackrel{p}{\longrightarrow} \mathbf{\Phi}_{22.1} \tag{3.2}$$

as $d_n \stackrel{p}{\to} \infty$ $(n \to \infty)$, where $\mathbf{P}_{2.1} = (p_{ij}^{(2.1)}) = \mathbf{Z}_{2.1}\mathbf{A}_{22.1}^{-1}\mathbf{Z}_{2.1}'$ and $\mathbf{\Phi}_{22.1}$ is a nonsingular constant matrix. In the following analysis we mainly consider the standard case when $d_n^2 = n$.

When $c = 0$ and the disturbances are homoscedastic, Anderson and Kunitomo (2007) have shown that the limiting null distribution of the RAAR test is the χ^2-distribution with G_2 degrees of freedom under a set of standard conditions. When $0 < c < 1$ and/or the disturbances are heteroscedastic, however, it is not necessarily χ^2-distribution. The main reason why the RAAR test does not necessarily have standard properties when the disturbances are heteroscedastic with many instruments is the presence of incidental parameters with many instruments and the effects of possible correlation between the conditional covariance $\mathbf{\Omega}_i$ and $p_{ii}^{(2.1)}$ $(i = 1, \cdots, n)$. This prevents from satisfying the *Weak Heteroscedasticity* condition

$$\text{(3.3)} \qquad (\mathbf{WH}) \qquad \operatorname*{plim}_{n\to\infty} \left[\frac{1}{n} \sum_{i=1}^{n} p_{ii}^{(2.1)} \mathbf{\Omega}_i - c\,\mathbf{\Omega} \right] = \mathbf{O} \,.$$

If this condition is not satisfied, we say that the disturbance terms have the *Persistent Heteroscedasticity* condition as (**PH**).

Let $\mathbf{P}_Z = (p_{ij}^{(n)}) = \mathbf{Z}(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'$, $\mathbf{Q}_Z = (q_{ij}^{(n)}) = \mathbf{I}_n - \mathbf{P}_Z$ and $\mathbf{P}_{Z_1} = \mathbf{Z}_1(\mathbf{Z}_1'\mathbf{Z}_1)^{-1}\mathbf{Z}_1'$ be $n \times n$ projection matrices. Then we can utilize the relations $\mathbf{P}_{2.1} = (\mathbf{I}_n - \mathbf{P}_{Z_1})\mathbf{P}_Z(\mathbf{I}_n - \mathbf{P}_{Z_1})$ and $\mathbf{Q}_Z = (\mathbf{I}_n - \mathbf{P}_{Z_1})(\mathbf{I}_n - \mathbf{P}_Z)(\mathbf{I}_n - \mathbf{P}_{Z_1})$. We construct $\mathbf{P}_M = (p_{ij}^{(m)})$ and $\mathbf{Q}_M = (q_{ij}^{(m)}) = \mathbf{I}_n - \mathbf{P}_M$ such that $p_{ij}^{(m)} = p_{ij}^{(n)}$ $(i \neq j)$, $p_{ii}^{(m)} - K_{2n}/n \xrightarrow{p} 0$ $(i, j = 1, \cdots, n)$ and

$$\text{(3.4)} \qquad \operatorname*{plim}_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[p_{ii}^{(m)} - c \right]^2 = 0 \,.$$

Then we define two $(1 + G_2) \times (1 + G_2)$ matrices by

$$\text{(3.5)} \qquad \mathbf{G}_M = \mathbf{Y}'\mathbf{P}_M^*\mathbf{Y}$$

and

$$\text{(3.6)} \qquad \mathbf{H}_M = \mathbf{Y}'\mathbf{Q}_M^*\mathbf{Y} \,,$$

where $\mathbf{P}_M^* = (p_{ij}^*) = (\mathbf{I}_n - \mathbf{P}_{Z_1})\mathbf{P}_M(\mathbf{I}_n - \mathbf{P}_{Z_1})$ and $\mathbf{Q}_M^* = (q_{ij}^*) = (\mathbf{I}_n - \mathbf{P}_{Z_1})\mathbf{Q}_M(\mathbf{I}_n - \mathbf{P}_{Z_1})$.

By using $\mathbf{G}_M$ and $\mathbf{H}_M$, the MRAAR test procedure is defined by the null hypoithesis

$H_0 : \boldsymbol{\beta} = \boldsymbol{\beta}_0$ is rejected if

$$
\text{(3.7)} \qquad \text{MRAAR}_n = (-1)(n - K_n) \log \left[\frac{1 + \dfrac{\hat{\boldsymbol{\beta}}_M^{'} \mathbf{G}_M \hat{\boldsymbol{\beta}}_M}{\hat{\boldsymbol{\beta}}_M^{'} \mathbf{H}_M \hat{\boldsymbol{\beta}}_M}}{1 + \dfrac{\boldsymbol{\beta}_0^{'} \mathbf{G}_M \boldsymbol{\beta}_0}{\boldsymbol{\beta}_0^{'} \mathbf{H}_M \boldsymbol{\beta}_0}} \right] > c^*(K_{2n}, q_n) ,
$$

where $c^*(K_{2n}, q_n)$ is a constant, $\hat{\boldsymbol{\beta}}_M$ is the modified LIML (MLIML) estimator defined by the solution of

$$
\text{(3.8)} \qquad \left[\frac{1}{n} \mathbf{G}_M - \frac{1}{q_n} \lambda_n \mathbf{H}_M \right] \hat{\boldsymbol{\beta}}_M = \mathbf{0}
$$

and λ_n is the smallest root of $|(1/n)\mathbf{G}_M - l(1/q_n)\mathbf{H}_M| = 0$.

The choice of $c^*(K_{2n}, q_n)$ will be discussed in Section 3.3. The testing procedure and the MRAAR test statistic can be often very close to the RAAR testing procedure; they are exactly the same in the case when there are (orthogonal) 1 or -1 dummy instrumental variables such that $(1/n)\mathbf{A}_{22.1} = \mathbf{I}_{K_2}$ and $p_{ii}^{(n)} = K_n/n$.

3.2 Asymptotic Properties of the MRAAR test

We shall investigate the asymptotic properties of the MRAAR test statistic when there are many instruments. One of the attractive features of the MRAAR statistic is that it satisfies (3.4), which leads to (**WH**) with $p_{ij}^{(n)}$ $(i = 1, \cdots, n)$ under (2.6) within the LIML estimation. (See Section 4 of Anderson et al. (2007) and Kunitomo (2008).) This condition plays an important role for the asymptotic variance of the LIML estimator, which is free from the form of higher order moments of disturbances. It also makes the limiting distribution of the MRAAR statistic to have a simple form. Then we have the representation of the limiting distribution of the MRAAR statistic as Theorem 1 and the proof will be given in Section 7.

Theorem 1 : Let $\mathbf{z}_i^{(n)}, i = 1, 2, \cdots, n$, be a set of $K_n \times 1$ vectors $(K_n = K_1 + K_{2n}, n > 2)$. Let $\mathbf{v}_i, i = 1, 2, \cdots, n$, be a set of $(1 + G_2) \times 1$ mutually independent random vectors such that $\mathcal{E}(\mathbf{v}_i | \mathbf{z}_i^{(n)}) = \mathbf{0}$, $\mathcal{E}(\mathbf{v}_i \mathbf{v}_i^{'} | \mathbf{z}_i^{(n)}) = \boldsymbol{\Omega}_i$ (a.s.) is a function of $\mathbf{z}_i^{(n)}$, say,

$\mathbf{\Omega}_i[n, \mathbf{z}_i^{(n)}]$ and $\mathcal{E}(\|\mathbf{v}_i\|^4)$ are bounded. For (2.1) and (2.2), suppose (2.6), (3.1),

$$\frac{1}{n}\max_{1\le i\le n}\|\boldsymbol{\pi}_{2i}(\mathbf{z}_i^{(n)})\|^2 \xrightarrow{p} 0 \tag{3.9}$$

and

$$\frac{1}{n}\mathbf{\Pi}_{2n}^{(z)'}(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{\Pi}_{2n}^{(z)} \xrightarrow{p} \mathbf{\Phi}_M^* \tag{3.10}$$

is a positive definite matrix as $n \to \infty$ ($q_n \to \infty$), where $(1/n)\mathbf{\Pi}_{2n}^{(z)'}\mathbf{P}_M^*\mathbf{\Pi}_{2n}^{(z)} \xrightarrow{p} \mathbf{\Phi}_{1M}^*$, $(1/q_n)\mathbf{\Pi}_{2n}^{(z)'}\mathbf{Q}_M^*\mathbf{\Pi}_{2n}^{(z)} \xrightarrow{p} \mathbf{\Phi}_{2M}^*$, $\mathbf{\Pi}_{2n}^{(z)} = (\boldsymbol{\pi}_{2i}(\mathbf{z}_i^{(n)'}))$ and $c_* = c/(1-c)$.

(i) Then

$$\mathrm{MRAAR}_n - \mathrm{MRAAR}_{1n}^* \xrightarrow{p} 0\ , \tag{3.11}$$

and

$$\mathrm{MRAAR}_{1n}^* = \frac{1}{\sigma_0^2}\mathbf{u}'(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\left[\mathbf{\Pi}_{2n}^{(z)} + \mathbf{W}_2\right]\left[\mathbf{\Pi}_{2n}^{(z)'}(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{\Pi}_{2n}^{(z)}\right]^{-1} \times \left[\mathbf{\Pi}_{2n}^{(z)'} + \mathbf{W}_2'\right](\mathbf{P}_n^* - c_*\mathbf{Q}_n^*)\mathbf{u}\ , \tag{3.12}$$

where $\sigma_0^2 = \boldsymbol{\beta}_0'\mathbf{\Omega}\boldsymbol{\beta}_0$ and an $n \times G_2$ matrix $\mathbf{W}_2 = (\mathbf{w}_{2i}')$ is defined by $\mathbf{w}_{2i} = \mathbf{v}_{2i} - u_i(\mathbf{0}, \mathbf{I}_{G_2})\mathbf{\Omega}\boldsymbol{\beta}_0/\sigma_0^2$ $(i = 1, \cdots, n)$.

(ii) Furthermore (3.12) can be decomposed as

$$\mathrm{MRAAR}_{1n}^* = \mathbf{\Lambda}_{1n} + \mathbf{\Lambda}_{2n} + 2\mathbf{\Lambda}_{3n}\ , \tag{3.13}$$

where

$$\mathbf{\Lambda}_{1n} = \frac{1}{\sigma_0^2}\mathbf{u}'(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{\Pi}_{2n}^{(z)}\left[\mathbf{\Pi}_{2n}^{(z)'}(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{\Pi}_{2n}^{(z)}\right]^{-1}\mathbf{\Pi}_{2n}^{(z)'}(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{u},$$

$$\mathbf{\Lambda}_{2n} = \frac{1}{\sigma_0^2}\mathbf{u}'(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{W}_2\left[\mathbf{\Pi}_{2n}^{(z)'}(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{\Pi}_{2n}^{(z)}\right]^{-1}\mathbf{W}_2'(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{u},$$

$$\mathbf{\Lambda}_{3n} = \frac{1}{\sigma_0^2}\mathbf{u}'(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{W}_2\left[\mathbf{\Pi}_{2n}^{(z)'}(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{\Pi}_{2n}^{(z)}\right]^{-1}\mathbf{\Pi}_{2n}^{(z)'}(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{u}\ .$$

(iii) If $c = 0$, then $\mathbf{\Lambda}_{2n} = o_p(1)$ and $\mathbf{\Lambda}_{3n} = o_p(1)$.

In the standard case when K_n is a fixed number and the disturbances are homoscedastic, it has been known that the limiting distribution of the RAAR statistic is χ^2 with G_2 degrees of freedom under H_0. More generally, in that case under the local alternative

(3.14) $$\boldsymbol{\beta} = \boldsymbol{\beta}_0 + \frac{1}{\sqrt{n}}\boldsymbol{\zeta} \ , \ \boldsymbol{\zeta} = \begin{bmatrix} 0 \\ -\boldsymbol{\zeta}_2 \end{bmatrix} ,$$

the limiting distribution of the likelihood ratio statistic is the noncentral χ^2 with G_2 degrees of freedom and the noncentrality

(3.15) $$\boldsymbol{\xi} = \text{plim}_{n\to\infty}\boldsymbol{\zeta}_2' \left[\frac{1}{n}\boldsymbol{\Pi}_{2n}^{(z)'}\mathbf{P}_{2.1}\boldsymbol{\Pi}_{2n}^{(z)}\right]\boldsymbol{\zeta}_2 \ .$$

Anderson and Kunitomo (2007) have extended this result slightly when there are incidental parameters and $c = 0$. In this case the power function of the RAAR test attains the asymptotic bound as the standard situation.

Corollary 1 : Assume that $\max_{1\leq i\leq n}|\sigma_i^2 - \sigma_0^2| \xrightarrow{p} 0$ as $n \to \infty$. Then the limiting distribution of MRAAR_n under H_0 is χ^2 with G_2 degrees of freedom if $c = 0$.

In the more general situation three cases can be considered. We have already investigated the first case of $d_n = O_p(n^{1/2})$ and $K_{2n} = O(n)$. Anderson et al. (2007) have given the asymptotic covariance of the LIML estimator under alternative assumptions, which is useful for investigating the limiting behavior of the MRAAR statistic. The second case is the standard large sample asymptotics, which corresponds to the cases of $d_n = O_p(n^{1/2+\delta})$ $(\delta > 0)$, or $d_n = O_p(n^{1/2})$ and $K_{2n}/n = o(1)$. In this case we have the standard χ^2 as the limiting distribution under the null hypothesis. The third case occurs when $d_n = o_p(n^{1/2})$ and $\sqrt{n}/d_n^2 \to 0$, which may correspond to the case of many weak instruments. In this case

(3.16) $$\frac{d_n^2}{n}\text{MRAAR}_n \xrightarrow{d} \boldsymbol{\Lambda}_2(c) ,$$

where $[d_n^2/n]\boldsymbol{\Lambda}_{2n}(c) \xrightarrow{d} \boldsymbol{\Lambda}_2(c)$ and $\boldsymbol{\Lambda}_2(c)$ follows a weighted χ^2 distribution depending on c.

The second and third terms of (3.13) have often less impacts on the limiting distribution of the MRAAR statistic because the second and the third terms are dominated by the first term of MRAR_{1n}^* due to the effects of n and the noncentrality.

In the case of *many weak instruments*, however, the approximation of the standard χ^2 distribution could be poor even if the disturbances are homoscedastic when the noncentrality is relatively small.

3.3 Modifying the RAAR testing procedure

There are alternative ways to use Theorem 1 to construct the testing procedure for $H_0: \ \boldsymbol{\beta} = \boldsymbol{\beta}_0$. One possible method is to use the simulated distribution of an approximate random variable of the MRAAR statistic introduced. For this purpose we utilize

$$
\begin{aligned}
(3.17) \qquad & (\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)(\boldsymbol{\Pi}_{2n}^{(z)} + \mathbf{W}_2) \\
= \ & (\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)(\boldsymbol{\Pi}_n^{(z)} + \mathbf{V}) \times \left[\mathbf{I}_{G_2+1} - \frac{\boldsymbol{\beta}_0\boldsymbol{\beta}_0^{'}\boldsymbol{\Omega}}{\boldsymbol{\beta}_0^{'}\boldsymbol{\Omega}\boldsymbol{\beta}_0}\right]\begin{bmatrix} \mathbf{0}^{'} \\ \mathbf{I}_{G_2} \end{bmatrix} .
\end{aligned}
$$

When u_i $(i = 1, \cdots, n)$ are independently distributed, the random variable in (3.13) can be further approximated by

$$
(3.18) \qquad \mathrm{MRAAR}_{2n}^* = \mathbf{X}_n^{'}\boldsymbol{\Xi}_n\mathbf{X}_n ,
$$

where

$$
\begin{aligned}
\mathbf{X}_n &= [\boldsymbol{\Psi}_M^*]^{-1/2}\frac{1}{\sqrt{n}}\left[\boldsymbol{\Pi}_{2n}^{(z)'} + \mathbf{W}_2^{'}\right](\mathbf{P}_n^* - c_*\mathbf{Q}_n^*)\mathbf{u} , \\
\boldsymbol{\Xi}_n &= \left[\sigma_0^{-2}\boldsymbol{\Psi}_M^*\right]^{1/2}\left[\frac{1}{n}\boldsymbol{\Pi}_{2n}^{(z)'}(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\boldsymbol{\Pi}_{2n}^{(z)}\right]^{-1}\left[\sigma_0^{-2}\boldsymbol{\Psi}_M^*\right]^{1/2}
\end{aligned}
$$

and

$$
\boldsymbol{\Psi}_M^* = \frac{1}{n}\boldsymbol{\Pi}_{2n}^{(z)'}\mathbf{P}_n^{**}\boldsymbol{\Sigma}_n\mathbf{P}_n^{**}\boldsymbol{\Pi}_{2n}^{(z)} + \frac{1}{n}\sum_{i,j=1}^{n}\left[u_i^2\mathbf{w}_{2j}\mathbf{w}_{2j}^{'} + \mathbf{w}_{2i}u_i\mathbf{w}_{2j}^{'}u_j\right]\left[p_{ij}^* - c_*q_{ij}^*\right]^2 ,
$$

which is given in Lemma 2 of Section 6, where $\boldsymbol{\Sigma}_n = (\mathrm{diag} u_i^2)$.
Since the limiting distribution of $\mathbf{X}_n$ is $N_{G_2}(\mathbf{0}, \mathbf{I}_{G_2})$ and the rank of $\boldsymbol{\Xi}_n$ is G_2, the limiting distribution of MRAAR_{2n}^* under H_0 is χ^2 when the disturbances are homoscedastic and $c = 0$. In the more general case under H_0 when u_i $(i = 1, \cdots, n)$ are independently distributed it can be re-expressed as

$$
(3.19) \qquad \mathrm{MRAAR}_{2n}^* = \sum_{i=1}^{G_2} a_{in}X_i^{*2} ,
$$

where a_{in} $(i = 1, \cdots, G_2)$ are the non-zero characteristic roots of $\mathbf{\Xi}_n^*$ and X_i^* $(i = 1, \cdots, G_2)$ are independently distributed as $N(0,1)$.

When u_i $(i = 1, \cdots, n)$ are independently distributed but they are possibly heteroscedastic, we need a further modification because each elements of $\mathbf{u}$ and $\mathbf{W}_2$ are not necessarily (asymptotically) independent even in the asymptotic sense. One simple way of modification is to use

$$\hat{\mathbf{\Lambda}}_n^* = \sigma_0^{-2} \left[\frac{1}{n} \hat{\mathbf{\Pi}}_{2n}^{(z)'} \mathbf{P}_n^{**} \hat{\mathbf{\Pi}}_{2n}^{(z)} \right]^{-1} \hat{\mathbf{\Psi}}_M^* \ , \tag{3.20}$$

where $\mathbf{P}_n^{**} = \mathbf{P}_M^* - c_* \mathbf{Q}_M^*$,

$$\hat{\mathbf{\Psi}}_M^* = \frac{1}{n} \hat{\mathbf{\Pi}}_{2n}^{(z)'} \mathbf{P}_n^{**} \hat{\mathbf{\Sigma}}_n \mathbf{P}_n^{**} \hat{\mathbf{\Pi}}_{2n}^{(z)} + \frac{1}{n} \sum_{i,j=1}^{n} \left[\hat{u}_i^2 \hat{\mathbf{w}}_{2j} \hat{\mathbf{w}}_{2j}^{'} + \hat{\mathbf{w}}_{2i} \hat{u}_i \hat{\mathbf{w}}_{2j}^{'} \hat{u}_j \right] \left[p_{ij}^* - c_* q_{ij}^* \right]^2 , \tag{3.21}$$

$\hat{\mathbf{\Pi}}_{2n}^{(z)}$ and $\hat{\mathbf{\Sigma}}_n = (\text{diag } \hat{u}_i^2)$ are the estimates of $\mathbf{\Pi}_{2n}^{(z)}$ and $\mathbf{\Sigma}_n = (\text{diag } \sigma_i^2)$, respectively, and $\hat{\mathbf{w}}_{2i}$ and $\hat{u}_i$ $(i = 1, \cdots, n)$ are the residuals under the null hypothesis (or the residuaals of the MLIML estimation). As a simple method, we can take $\hat{\mathbf{\Pi}}_{2n}^{(z)'} \mathbf{P}_n^{**} \hat{\mathbf{\Pi}}_{2n}^{(z)} = \mathbf{Y}_2' \mathbf{P}_n^{**} \mathbf{Y}_2$ and $\hat{\mathbf{\Pi}}_{2n}^{(z)'} \mathbf{P}_n^{**} \hat{\mathbf{\Sigma}}_n \mathbf{P}_n^{**} \hat{\mathbf{\Pi}}_{2n}^{(z)} = \mathbf{Y}_2' \mathbf{P}_n^{**} \hat{\mathbf{\Sigma}}_n \mathbf{P}_n^{**} \mathbf{Y}_2$.

When u_i and $\mathbf{w}_{2i}$ are heteroscedastic,

$$\mathcal{E}[\mathbf{w}_{2i} u_i] = (\mathbf{0}, \mathbf{I}_{G_2}) \left[\mathbf{\Omega}_i - \frac{\sigma_i^2}{\sigma_0^2} \mathbf{\Omega} \right] \boldsymbol{\beta} \tag{3.22}$$

are not necessarily zero vectors and then we need the second term in order to estimate (3.27). Then we can evaluate numerically [1] the distribution function of

$$\text{MRAAR}_{3n}^* = \sum_{i=1}^{G_2} \lambda_{in} X_i^2 \ , \tag{3.23}$$

where λ_{in} are the characteristic roots of $\hat{\mathbf{\Lambda}}_n$ and X_i $(i = 1, \cdots, G_2)$ are independently distributed as $N(0,1)$.

When $c = 0$ and the disturbances are homoscedastic, we take $\mathbf{P}_n^{**} = \mathbf{P}_M^* = \mathbf{P}_{2.1}$ and then (3.12) can be written as

$$\text{MRAAR}_{4n}^* = \frac{1}{\sigma_0^2} \boldsymbol{\beta}_0' \mathbf{Y}' \mathbf{P}_{2.1} \mathbf{Y}_2 \left[\mathbf{Y}_2' \mathbf{P}_{2.1} \mathbf{Y}_2 \right]^{-1} \mathbf{Y}_2' \mathbf{P}_{2.1} \mathbf{Y} \boldsymbol{\beta}_0 \ , \tag{3.24}$$

[1] We may use the Monte Carlo experiments to obtain the finite sample distribution given the data.

which is quite similar to the statistic used by Kleibergen (2002). Thus (3.12) could be regarded as its extension (or a modification) to the case of many instruments and the heteroscedastic disturbances.

Because of the form (3.24), the resulting testing procedure based on the finite sample (conditional) distribution of (3.18) could be interpreted as an extension of the conditional likelihood ratio (CRL) approach proposed by Moreira (2003) to the many weak instrument situation. When $u_i \sim N(0, \sigma_0^2)$ $(i = 1, \cdots, n)$ and $c = 0$, (3.24) follows the χ^2-distribution with G_2 degrees of freedom exactly. Thus the above testing procedure could be also regarded as an extension of the standard likelihood ratio and the RAAR test procedures.

4. An Extension

The RAAR test and MARRA test discussed in Section 3 can be extended to some directions. Let

$$\mathbf{G}_M^* = \begin{pmatrix} \mathbf{Z}_1' \\ \mathbf{Y}' \end{pmatrix} \mathbf{P}_M (\mathbf{Z}_1, \mathbf{Y}) \tag{4.1}$$

and

$$\mathbf{H}_M^* = \begin{pmatrix} \mathbf{Z}_1' \\ \mathbf{Y}' \end{pmatrix} \mathbf{Q}_M (\mathbf{Z}_1, \mathbf{Y}) \tag{4.2}$$

be $(K_1 + 1 + G_2) \times (K_1 + 1 + G_2)$ matrices and $\mathbf{P}_M = (p_{ij}^{(m)})$ and $\mathbf{Q}_M = (q_{ij}^{(m)})$ are defined in Section 3.1.

We set the true vector $\boldsymbol{\theta}_0' = (-\boldsymbol{\gamma}_1^{(0)'}, 1, -\boldsymbol{\beta}_2^{(0)})$. For the null hypothesis H_0' : $\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_1^{(0)}$ (a specified vector), $\boldsymbol{\beta}_2 = \boldsymbol{\beta}_2^{(0)}$ (a specified vector) is rejected if

$$\mathrm{MRAAR}_n = (-1)(n - K_n) \log \left[\frac{1 + \dfrac{\hat{\boldsymbol{\theta}}' \mathbf{G}_M^* \hat{\boldsymbol{\theta}}}{\hat{\boldsymbol{\theta}}' \mathbf{H}_M^* \hat{\boldsymbol{\theta}}}}{1 + \dfrac{\boldsymbol{\theta}_0' \mathbf{G}_M^* \boldsymbol{\theta}_0}{\boldsymbol{\theta}_0' \mathbf{H}_M^* \boldsymbol{\theta}_0}} \right] > c^{**}(K_{2n}, q_n) , \tag{4.3}$$

where $\hat{\boldsymbol{\theta}} = (-\hat{\boldsymbol{\gamma}}_1', 1, -\hat{\boldsymbol{\beta}}_2)'$ is the the solution of

$$\left[\frac{1}{n} \mathbf{G}_M^* - \frac{1}{q_n} \lambda_n \mathbf{H}_M^* \right] \hat{\boldsymbol{\theta}}_M = \mathbf{0} \tag{4.4}$$

and λ_n is the smallest root of $|(1/n)\mathbf{G}_M^* - l(1/q_n)\mathbf{H}_M^*| = 0$.
Then the null distribution of the MRAAR statistic can be developed as Theorem 1. The proof is similar to Theorem 1 and it is omitted.

Theorem 2 : Suppose the assumptions of Theorem 1 hold. Under the null hypothesis $H_0^{'}$, as $n \to \infty$ ($q_n \to \infty$),

$$\text{MRAAR}_n - \text{MRAAR}_{5n}^* \xrightarrow{p} 0 \ , \tag{4.5}$$

and

$$\begin{aligned}\text{MRAAR}_{5n}^* &= \frac{1}{\sigma_0^2}\mathbf{u}^{'}\left[(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)(\mathbf{\Pi}_{*n}^{(z)} + \mathbf{W})\right]\left[\mathbf{\Pi}_{*n}^{(z)'}(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{\Pi}_{*n}^{(z)}\right]^{-1} \\ &\quad \times \left[(\mathbf{\Pi}_{*n}^{(z)} + \mathbf{W})^{'}(\mathbf{P}_n^* - c_*\mathbf{P}_n^*)\right]\mathbf{u} \ ,\end{aligned} \tag{4.6}$$

where $\mathbf{\Pi}_{*n}^{(z)} = (\mathbf{Z}_1, \mathbf{\Pi}_{2n}^{(z)})$ and $\mathbf{W} = (\mathbf{O}, \mathbf{W}_2)$.

The null distribution of the MRAAR statistic in this case can be generated as H_0. When $c = 0$, $\mathbf{P}_n^{**} = \mathbf{P}_M^* = \mathbf{P}_{2.1}$ and $\mathbf{\Sigma}_n = \sigma^2\mathbf{I}_n$, the limiting distribution is χ^2 with $K_1 + G_2$ degrees of freedom because u_i and $\mathbf{w}_{2i}$ $(i = 1, \cdots, n)$ are asymptotically uncorrelated in this situation.

More generally, the extension of the MRAAR test to any hypothesis for the subset of parameter vector $\boldsymbol{\theta} = (-\boldsymbol{\gamma}_1^{'}, 1, -\boldsymbol{\beta}_2^{'})$ can be constructed in the same way.

5. On Finite Sample Null Distributions of the MRAAR test and other statistics

We have investigated the finite sample null distribution of the RAAR statistic and the MRAAR statistic based on a set of Monte Carlo experiments in a systematic way. We have used the numerical estimation procedure for the cumulative distribution function (cdf) based on the simulation and we have enough numerical accuracy in most cases. See Anderson et al. (2008) for the details of the numerical computation method. The key parameters in figures and tables are

K_2 (or K_{2n}), $n - K$ (or $n - K_n$), $\alpha = [\omega_{22}/|\mathbf{\Omega}|^{1/2}](\beta_2 - \omega_{12}/\omega_{22})$ ($\mathbf{\Omega} = (\omega_{ij})$) and $\delta^2 = \mathbf{\Pi}_{22}^{(n)\prime}\mathbf{A}_{22.1}\mathbf{\Pi}_{22}^{(n)}/\omega_{22}$. (See Anderson et al. (2008) for the detail of these notations.) In Tables 1-6 we have given the null-distributions or the null-sizes of the MRAAR test statistic. For comparison we also present the null-distributions or the null-size of each test statistics including t-type test, RAAR, Kleibergen's test, Anderson-Rubin test, Moreira's Conditional Likelihood test.

Because Hausman et al. (2007) have investigated a particular case of persistent heteroscedasticity, we have also tried to reproduce their Monte Carlo experiments. Hence in some tables we have given the null-distributions or the null-size of the t-type test given by Hausman et al. (2007) and the MRAAR test we have developed in their setting. Because they used the notation slightly different from ours, we denote the noncentral parameter μ_H^2 $(= \delta^2)$ and the correlation coefficient ρ $(= -\alpha/\sqrt{1+\alpha^2})$. In addition to these test statistics, we have given the null-distributions of each test statistics including t-type test, RAAR, Kleibergen's test, Anderson-Rubin test, Moreira's Conditional Likelihood test when the disturbances have the persistent heteroscedasticity.

When the noncentrality is large or moderate as in Tables 1 and 2, the differences of significance levels are not large among the test statistics we have discussed. When the noncentrality is small as in Tables 3-6, however, the differences become significant with or without heteroscedasticities. From these tables we have found that the null-distribution or the null-sizes of the MRAAR test is quite robust against the cases of many instruments, many weak instruments as well as the persistent heteroscedasticities of disturbances.

There is a natural question on the power comprison of the MRAAR test and the t-type test [2] when there are many instruments and the disturbances have persistent heteroscedasticity. We have found that the empirical sizes of two statistics are often similar, but also we have found that the RAAR test has a better power property against the t-type statistic in some situation when we need a two-sided

[2] See Matsushita (2006) and Anderson et al. (2008) for some details of t-statistic in the many instruments situation.

t-test in particular. In this respect, we only illustrate this situation by showing the power functions of two test statistics as Figure 1 in Appendix. We are currently investigating the conditions when we have this phenomenon.

6. Concluding Remarks

In this paper, we propose a particular modification of the RAAR test procedure for many weak instruments and the heteroscedastic disturbances. When there are many instruments and the disturbances have persistent heteroscedasticity, it might be argued that the RAAR test does not necessarily have desirable asymptotic properties as the likelihood ratio test in the standard asymptotic theory. However, as we have shown that a simple modification of the RAAR test procedure based on the MRAAR statistic gives a reasonable way of testing hypothesis on coefficients of structural equation when there are many instruments and the disturbances are heteroscedastic. We have shown that our test procedure could be interpreted as the extensions of test statistics developed by Kleibergen (2002) and Moreira (2002) to the cases of many instruments and the persistent heteroscedasticity.

As a preliminary study the MRAAR test has the reasonable power property. The finite sample properties of the MRAAR test and other statistics including the t-type statistic are currently under a further investigation.

7 Mathematical Derivations

In this section we give the proof of *Theorem 1.* The method of proof is basically a modification of the arguments in Section 6 of Kunitomo (2008). Some of the details are omitted.

Proof of Theorem 1 : We set the true coefficient vector as $\boldsymbol{\beta}_0$. The MRAAR statistic in (3.7) can be approximately the same as

$$(7.1)\quad LR_1 = (-n)\left[\frac{\hat{\boldsymbol{\beta}}^{'}\frac{1}{n}\mathbf{G}_M\hat{\boldsymbol{\beta}}}{\hat{\boldsymbol{\beta}}^{'}\frac{1}{q_n}\mathbf{H}_M\hat{\boldsymbol{\beta}}} - \frac{\boldsymbol{\beta}_0^{'}\frac{1}{n}\mathbf{G}_M\boldsymbol{\beta}_0}{\boldsymbol{\beta}_0^{'}\frac{1}{q_n}\mathbf{H}_M\boldsymbol{\beta}_0}\right]$$

$$= \;(-\frac{n}{D})\left\{2(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0)^{'}\left[\frac{1}{n}\mathbf{G}_M\boldsymbol{\beta}_0-\frac{1}{q_n}\mathbf{H}_M\boldsymbol{\beta}_0\left(\frac{N_0}{D_0}\right)\right]\right.$$
$$\left.+(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0)^{'}\left[\frac{1}{n}\mathbf{G}_M-\frac{1}{q_n}\mathbf{H}_M\left(\frac{N_0}{D_0}\right)\right](\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0)\right\}+o_p(1)\;,$$

where $N_0=(1/n)\boldsymbol{\beta}_0^{'}\mathbf{G}_M\boldsymbol{\beta}_0$, $D_0=(1/q_n)\boldsymbol{\beta}_0^{'}\mathbf{H}_M\boldsymbol{\beta}_0$, $D=(1/q_n)\hat{\boldsymbol{\beta}}^{'}\mathbf{H}_M\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\beta}}$ is the modified LIML estimator of $\boldsymbol{\beta}$ by defining the LIML estimation with $\mathbf{G}_M$ and $\mathbf{H}_M$. We use the fact that $D-D_0=o_p(1)$ under H_0 and

$$\text{(7.2)}\quad \sqrt{n}\left[\frac{1}{n}\mathbf{G}_M\boldsymbol{\beta}_0-\frac{1}{q_n}\mathbf{H}_M\boldsymbol{\beta}_0\left(\frac{N_0}{D_0}\right)\right]$$
$$=\left[\mathbf{I}_{G_2+1}-\frac{\boldsymbol{\Omega}\boldsymbol{\beta}_0\boldsymbol{\beta}_0^{'}}{\boldsymbol{\beta}_0^{'}\boldsymbol{\Omega}\boldsymbol{\beta}_0}\right]\left[\sqrt{n}(\frac{1}{n}\mathbf{G}_M-\mathbf{G}_M^{(0)})\boldsymbol{\beta}_0-\sqrt{cc_*}\sqrt{q_n}(\frac{1}{q_n}\mathbf{H}_M-\mathbf{H}_M^{(0)})\boldsymbol{\beta}_0\right]\;,$$

where $\mathbf{G}_M^{(0)}=\text{plim}\frac{1}{n}\mathbf{G}_M$ and $\mathbf{H}_M^{(0)}=\text{plim}\frac{1}{q_n}\mathbf{H}_M$.

We first prepare the following lemma.

Lemma 1 : As $n\to\infty$ and $q_n\to\infty$,

$$\text{(7.3)}\qquad \frac{1}{n}\mathbf{V}^{'}\mathbf{P}_M^{*}\mathbf{V}\xrightarrow{p}c\,\boldsymbol{\Omega}$$

and

$$\text{(7.4)}\qquad \frac{1}{q_n}\mathbf{V}^{'}\mathbf{Q}_M^{*}\mathbf{V}\xrightarrow{p}\boldsymbol{\Omega}\;.$$

Proof : Because $\{\mathbf{v}_i\}$ are mutually independent and we have (2.6), we need to investigate the diagonal parts of $\mathbf{P}_M^{*}$ and $\mathbf{Q}_M^{*}$. We use the relation

$$\begin{aligned}\frac{1}{n}\text{tr}(\mathbf{P}_M^{*}) &= \frac{1}{n}\text{tr}(\mathbf{I}_n-\mathbf{P}_{Z_1})\mathbf{P}_M(\mathbf{I}_n-\mathbf{P}_{Z_1})\\ &= \frac{1}{n}\text{tr}(\mathbf{I}_n-\mathbf{P}_{Z_1})\mathbf{P}_Z(\mathbf{I}_n-\mathbf{P}_{Z_1})+\frac{1}{n}\text{tr}(\mathbf{I}_n-\mathbf{P}_{Z_1})(\mathbf{P}_M-\mathbf{P}_Z)(\mathbf{I}_n-\mathbf{P}_{Z_1})\\ &= \frac{1}{n}\text{tr}(\mathbf{P}_Z-\mathbf{P}_{Z_1})+\frac{1}{n}\text{tr}(\mathbf{P}_M-\mathbf{P}_Z)(\mathbf{I}_n-\mathbf{P}_{Z_1})\;.\end{aligned}$$

Since $\text{tr}(\mathbf{P}_Z-\mathbf{P}_M)=o_p(1)$, $|p_{ii}|\le 1$ $(i=1,\cdots,n)$, $K_n/n<1$ and

$$|\text{tr}\,[(\mathbf{P}_Z-\mathbf{P}_M)\mathbf{P}_{Z_1}]\,|=|\text{tr}\left[(\text{diag}(p_{ii}-\frac{K_n}{n}))\mathbf{P}_{Z_1}\right]|\le 2K_1\;,$$

we find that $(1/n)\text{tr}(\mathbf{P}_M^{*})\to c$ as $n\to\infty$ and we have (7.3). Also we use the relation

$$\frac{1}{q_n}\text{tr}(\mathbf{I}_n-\mathbf{P}_{Z_1})\mathbf{Q}_M(\mathbf{I}_n-\mathbf{P}_{Z_1})$$

$$
\begin{aligned}
&= \frac{1}{q_n}\mathrm{tr}\,[\mathbf{I}_n - \mathbf{P}_Z] + \frac{1}{q_n}\mathrm{tr}(\mathbf{I}_n - \mathbf{P}_{Z_1})(\mathbf{P}_Z - \mathbf{P}_M)(\mathbf{I}_n - \mathbf{P}_{Z_1}) \\
&= 1 + \frac{1}{q_n}\mathrm{tr}(\mathbf{P}_Z - \mathbf{P}_M)(\mathbf{I}_n - \mathbf{P}_{Z_1}) \\
&= 1 + \frac{1}{q_n}\mathrm{tr}(\mathbf{P}_Z - \mathbf{P}_M) - \frac{1}{q_n}\mathrm{tr}(\mathbf{P}_Z - \mathbf{P}_M)\mathbf{P}_{Z_1} .
\end{aligned}
$$

Then by using a similar argument we have (7.4). **Q.E.D.**

The next argument is a modification of Kunitomo (2008) and thus the development should be only sketchy to avoid some duplication. By substituting the relation $\mathbf{Y} = \mathbf{\Pi}_n^{(z)} + \mathbf{V}$ into $\mathbf{G}_M$ and $\mathbf{H}_M$, we have

$$
(7.5) \qquad \frac{1}{n}\mathbf{G}_M = \frac{1}{n}\left[\begin{pmatrix} \boldsymbol{\beta}_2^{(0)'} \\ \mathbf{I}_{G_2} \end{pmatrix}\mathbf{\Pi}_{2n}^{(z)'} + \mathbf{V}'\right]\mathbf{P}_M^*\left[\left(\boldsymbol{\beta}_2^{(0)}, \mathbf{I}_{G_2}\right)\mathbf{\Pi}_{2n}^{(z)} + \mathbf{V}\right]
$$

$$
\xrightarrow{p} \mathbf{G}_0 = \mathbf{B}'\mathbf{\Phi}_{1M}^*\mathbf{B} + c\,\mathbf{\Omega} ,
$$

and

$$
(7.6) \qquad \frac{1}{q_n}\mathbf{H}_M = \frac{1}{q_n}\left[\begin{pmatrix} \boldsymbol{\beta}_2^{(0)'} \\ \mathbf{I}_{G_2} \end{pmatrix}\mathbf{\Pi}_{2n}^{(z)'} + \mathbf{V}'\right]\mathbf{Q}_M^*\left[\left(\boldsymbol{\beta}_2^{(0)}, \mathbf{I}_{G_2}\right)\mathbf{\Pi}_{2n}^{(z)} + \mathbf{V}\right]
$$

$$
\xrightarrow{p} \mathbf{H}_0 = \mathbf{B}'\mathbf{\Phi}_{2M}^*\mathbf{B} + \mathbf{\Omega}^* ,
$$

where $\mathbf{B} = (\boldsymbol{\beta}_2^{(0)}, \mathbf{I}_{G_2})$ and and $\boldsymbol{\beta}_0 = (1, -\boldsymbol{\beta}_2^{(0)'})'$. Then (3.8) implies $\lambda_n \xrightarrow{p} c$ and $N_0/D_0 \xrightarrow{p} c$ because of the rank conditon on $\mathbf{\Pi}_n^{(z)}$ and (3.10). Then $\hat{\boldsymbol{\beta}}_M \xrightarrow{p} \boldsymbol{\beta}_0$ and

$$
(7.7) \qquad \frac{1}{n}\mathbf{G}_M - \frac{N_0}{D_0}\mathbf{H}_M \xrightarrow{p} \mathbf{B}'\left[\mathrm{plim}\frac{1}{n}\mathbf{\Pi}_{2n}^{(z)'}(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{\Pi}_{2n}^{(z)}\right]\mathbf{B} .
$$

Define $\mathbf{G}_1$, $\mathbf{H}_1$, λ_{1n}, and $\mathbf{b}_1$ by $\mathbf{G}_1 = \sqrt{n}(\frac{1}{n}\mathbf{G}_M - \mathbf{G}_0)$, $\mathbf{H}_1 = \sqrt{q_n}(\frac{1}{q_n}\mathbf{H}_M - \mathbf{H}_0)$, $\lambda_{1n} = \sqrt{n}(\lambda_n - c)$ and $\mathbf{b}_1 = \sqrt{n}(\hat{\boldsymbol{\beta}}_M - \boldsymbol{\beta}_0)$. From (3.8), we have

$$
[\mathbf{G}_0 - c\,\mathbf{H}_0]\boldsymbol{\beta}_0 + \frac{1}{\sqrt{n}}[\mathbf{G}_1 - \lambda_{1n}\mathbf{H}_0]\boldsymbol{\beta}_0 + \frac{1}{\sqrt{n}}[\mathbf{G}_0 - c\,\mathbf{H}_0]\mathbf{b}_1 + \frac{1}{\sqrt{q_n}}[-c\mathbf{H}_1]\boldsymbol{\beta}_0 = o_p(\frac{1}{\sqrt{n}})
$$

and then for $\hat{\boldsymbol{\beta}}_M' = (1, -\hat{\boldsymbol{\beta}}_{2M}')$ and $\boldsymbol{\beta}' = (1, -\boldsymbol{\beta}_2^{(0)'})$,

$$
(7.8) \qquad \mathbf{B}'\mathbf{\Phi}_M^*\sqrt{n}\left[\hat{\boldsymbol{\beta}}_{2.LI} - \boldsymbol{\beta}_2\right] = (\mathbf{G}_1 - \lambda_{1n}\mathbf{H}_0 - \sqrt{cc_*}\mathbf{H}_1)\,\boldsymbol{\beta}_0 + o_p(1) .
$$

By multiplying (7.5) from the left by $\boldsymbol{\beta}_0^{'} = (1, -\boldsymbol{\beta}_2^{'})$, we find

$$\lambda_{1n} = \frac{\boldsymbol{\beta}_0^{'}(\mathbf{G}_1 - \sqrt{cc_*}\mathbf{H}_1)\boldsymbol{\beta}_0}{\boldsymbol{\beta}^{'}\boldsymbol{\Omega}\boldsymbol{\beta}_{\;0}} + o_p(1) \; .$$

Also by mutilying (7.6) on the left by $(\mathbf{0}, \mathbf{I}_{G_2})$ and substituting λ_{1n}, we have

$$(7.9)\quad \sqrt{n}\left[\hat{\boldsymbol{\beta}}_{2.LI} - \boldsymbol{\beta}_2^{(0)}\right] = \boldsymbol{\Phi}_M^{*-1}(\mathbf{0}, \mathbf{I}_{G_2})\left[\mathbf{I}_{1+G_2} - \frac{\boldsymbol{\Omega}\boldsymbol{\beta}_0\boldsymbol{\beta}_0^{'}}{\boldsymbol{\beta}_0^{'}\boldsymbol{\Omega}\boldsymbol{\beta}_{\;0}}\right](\mathbf{G}_1 - \sqrt{cc_*}\mathbf{H}_1)\boldsymbol{\beta}_0 + o_p(1).$$

By using the relation $\mathbf{V}\boldsymbol{\beta}_0 = \mathbf{u}$, we have the representation

$$(\mathbf{G}_1 - \sqrt{cc_*}\mathbf{H}_1)\boldsymbol{\beta}_0 = \frac{1}{\sqrt{n}}\boldsymbol{\Pi}_{2n}^{(z)'}(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{u} + \frac{1}{\sqrt{n}}\left[\mathbf{V}^{'}(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\right]\mathbf{u} \; .$$

Then (7.9) is rewritten as

$$\begin{aligned}(7.10)\qquad & \sqrt{n}\left[\hat{\boldsymbol{\beta}}_{2.LI} - \boldsymbol{\beta}_2\right] \\ = \; & \boldsymbol{\Phi}_M^{*-1}\frac{1}{\sqrt{n}}\boldsymbol{\Pi}_{2n}^{(z)'}(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{u} \\ & +\boldsymbol{\Phi}_M^{*-1}\frac{1}{\sqrt{n}}[\mathbf{0}, \mathbf{I}_{G_2}]\left[\mathbf{I}_{1+G_2} - \frac{\boldsymbol{\Omega}\boldsymbol{\beta}_0\boldsymbol{\beta}_0^{'}}{\boldsymbol{\beta}_0^{'}\boldsymbol{\Omega}\boldsymbol{\beta}_0}\right]\mathbf{V}^{'}(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{u} + o_p(1) \\ = \; & \boldsymbol{\Phi}_M^{*-1}\frac{1}{\sqrt{n}}\boldsymbol{\Pi}_{2n}^{(z)'}(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{u} + \boldsymbol{\Phi}_M^{*-1}\sqrt{c}\frac{1}{\sqrt{K_{2n}}}\mathbf{W}_2^{'}(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{u} \; ,\end{aligned}$$

where

$$\mathbf{W}_2^{'} = (\mathbf{0}, \mathbf{I}_{G_2})\left[\mathbf{I}_{1+G_2} - \frac{\boldsymbol{\Omega}\boldsymbol{\beta}_0\boldsymbol{\beta}_0^{'}}{\boldsymbol{\beta}_0^{'}\boldsymbol{\Omega}\boldsymbol{\beta}_0}\right]\mathbf{V}^{'} \; .$$

Thus we have obtained the asymptotic distribution of $\hat{\boldsymbol{\beta}}_{2M}$, which can be summarized as the following Lemma 2. By using (7.1), (7.2) and (7.9), LR_1 is asymptotically equivalent to

$$(7.11)\; \mathrm{LR}_2 = (\frac{1}{\sigma_0^2})\left[\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)\right]^{'}\left[\frac{1}{n}\mathbf{G}_M - \frac{1}{q_n}\mathbf{H}_M\left(\frac{N_0}{D_0}\right)\right]\left[\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)\right] + o_p(1) \; .$$

Then by substituting (7.10) into LR_2 and $\sqrt{n}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)^{'} = [0, -\sqrt{n}(\hat{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_2^{(0)}]$, finally we obtain (3.12) by using Lemma 2 below.

Q.E.D.

In our derivation of this section we need the asymptotic distribution of the MLIML estimator with many instruments when there are persistent heteroscedasticity, which is summarized in the next lemma. The proof is silimar to the one in Kunitomo (2008) and it is omitted.

Lemma 2 : Let $\mathbf{z}_i^{(n)}, i = 1, 2, \cdots, n$, be a set of $K_n \times 1$ vectors ($K_n = K_1 + K_{2n}, n > 2$). Let $\mathbf{v}_i, i = 1, 2, \cdots, n$, be a set of $G_* \times 1$ independent random vectors such that $\mathcal{E}(\mathbf{v}_i|\mathbf{z}_i^{(n)}) = \mathbf{0}$, $\mathcal{E}(\mathbf{v}_i\mathbf{v}_i'|\mathbf{z}_i^{(n)}) = \mathbf{\Omega}_i$ (a.s.) is a function of $\mathbf{z}_i^{(n)}$ and $\mathcal{E}(\|\mathbf{v}_i\|^4)$ are bounded. Suppose (2.7), (3.1), (3.9) and

$$\frac{1}{n}\mathbf{\Pi}_{2n}^{(z)\prime}(\mathbf{P}_M^* - c_*\mathbf{Q}_M^*)\mathbf{\Pi}_{2n}^{(z)} \xrightarrow{p} \mathbf{\Phi}_M^* \tag{7.12}$$

is a positive definite matrix as $n \to \infty$ and $q_n \to \infty$. Then

$$\sqrt{n}\left[\hat{\boldsymbol{\beta}}_{2.MLI} - \boldsymbol{\beta}_2\right] \xrightarrow{d} N(\mathbf{0}, \mathbf{\Phi}_M^{*-1}\mathbf{\Psi}_M^*\mathbf{\Phi}_M^{*-1}) \tag{7.13}$$

where

$$\mathbf{\Psi}_M^* = \mathbf{\Psi}_{1M}^* + \mathbf{\Psi}_{2M}^* \ , \tag{7.14}$$

$$\begin{aligned}
\mathbf{\Psi}_{1M}^* &= \text{plim}\frac{1}{n}\sum_{i,j,k=1}^{n} \boldsymbol{\pi}_{2i}(\mathbf{z}_i^{(n)})[p_{ij}^* - c_*q_{ij}^*]\sigma_j^2[p_{jk}^* - c_*q_{jk}^*]\boldsymbol{\pi}_{2k}(\mathbf{z}_k^{(n)})' \ , \\
\mathbf{\Psi}_{2M}^* &= \text{plim}\frac{1}{n}\sum_{i,j=1}^{n}\left[\sigma_i^2\mathcal{E}(\mathbf{w}_{2j}\mathbf{w}_{2j}'|\mathbf{z}_j^{(n)}) + \mathcal{E}(\mathbf{w}_{2i}u_i|\mathbf{z}_i^{(n)})\mathcal{E}(\mathbf{w}_{2j}'u_j|\mathbf{z}_j^{(n)})\right]\left[p_{ij}^* - c_*q_{ij}^*\right]^2 ,
\end{aligned}$$

provided that $\mathbf{\Psi}_{1M}^*$ and $\mathbf{\Psi}_{2M}^*$ converge in probability as $n \to \infty$, and $\mathbf{w}_{2i} = \mathbf{v}_{2i} - u_i(\mathbf{0}, \mathbf{I}_{G_2})\mathbf{\Omega}\boldsymbol{\beta}_0/\sigma^2$ ($i = 1, \cdots, n$).

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APPENDIX : Tables

In Tables 1-6 the null-distributions or the null-sizes of alternative test statistics are given. They include the t (t_{LIML}) test, the RAAR test, Anderson-Rubin (AR) test, Kleibergen (K) test, Moreira (CLR) test, Hausman's t test (t_{HLIML}) and the modified RAAR (mRAAR) test. In Figure 1 we give the empirical power functions of the RAAR test and the t-type test in a particular case.

Table 1: Empirical sizes of statistics that test $H_0 : \beta = \beta_0$ with $n - K = 100$, $K_2 = 3$ and homoscedastic errors

			t_{LIML}	$RAAR$	AR	K	CLR	t_{HLIML}	$mRAAR$
$\delta^2 = 100$	$\alpha = 0.5$	0.10	0.098	0.103	0.107	0.100	0.102	0.107	0.104
		0.05	0.051	0.053	0.055	0.051	0.051	0.056	0.051
		0.01	0.011	0.012	0.011	0.010	0.012	0.014	0.011
	$\alpha = 1$	0.10	0.101	0.108	0.109	0.107	0.107	0.110	0.107
		0.05	0.050	0.056	0.059	0.054	0.055	0.057	0.055
		0.01	0.015	0.011	0.015	0.011	0.013	0.018	0.012
$\delta^2 = 30$	$\alpha = 0.5$	0.10	0.094	0.113	0.106	0.104	0.102	0.096	0.103
		0.05	0.047	0.059	0.055	0.052	0.053	0.052	0.051
		0.01	0.014	0.014	0.014	0.012	0.013	0.016	0.012
	$\alpha = 1$	0.10	0.089	0.110	0.110	0.104	0.107	0.097	0.105
		0.05	0.053	0.058	0.059	0.054	0.056	0.058	0.054
		0.01	0.021	0.014	0.013	0.012	0.014	0.024	0.011

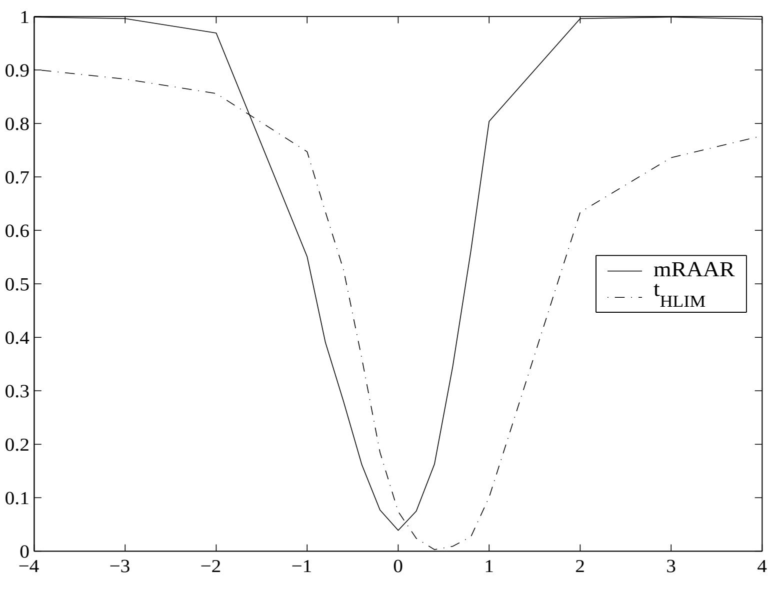

Figure 1: Power of tests: $n - K = 100, K_2 = 10, \alpha = 1, \delta^2 = 10$

Table 2: Empirical sizes of statistics that test $H_0 : \beta = \beta_0$ with $n - K = 100$, $K_2 = 10$ and homoscedastic errors

			t_{LIML}	$RAAR$	AR	K	CLR	t_{HLIML}	$mRAAR$
$\delta^2 = 100$	$\alpha = 0.5$	0.10	0.107	0.116	0.112	0.101	0.103	0.109	0.104
		0.05	0.055	0.060	0.062	0.052	0.053	0.057	0.052
		0.01	0.014	0.013	0.016	0.011	0.012	0.013	0.011
	$\alpha = 1$	0.10	0.108	0.115	0.120	0.107	0.108	0.111	0.109
		0.05	0.058	0.064	0.067	0.057	0.059	0.060	0.059
		0.01	0.016	0.015	0.016	0.012	0.014	0.017	0.012
$\delta^2 = 50$	$\alpha = 0.5$	0.10	0.115	0.131	0.117	0.105	0.106	0.103	0.104
		0.05	0.062	0.072	0.065	0.054	0.055	0.053	0.052
		0.01	0.015	0.018	0.016	0.011	0.013	0.015	0.011
	$\alpha = 1$	0.10	0.109	0.120	0.118	0.104	0.107	0.102	0.108
		0.05	0.058	0.065	0.066	0.055	0.056	0.058	0.052
		0.01	0.020	0.014	0.017	0.011	0.011	0.020	0.009

Table 3: Empirical sizes of statistics that test $H_0 : \beta = \beta_0$ with $n = 500$, $K = 5$ and homoscedastic errors

			t_{LIML}	$RAAR$	AR	K	CLR	t_{HLIML}	$mRAAR$
$\mu_H^2 = 10$	$\rho = 0.3$	0.10	0.085	0.152	0.099	0.095	0.098	0.072	0.105
		0.05	0.041	0.088	0.049	0.049	0.050	0.034	0.051
		0.01	0.007	0.022	0.010	0.010	0.010	0.006	0.010
	$\rho = 0.8$	0.10	0.112	0.125	0.104	0.104	0.104	0.106	0.106
		0.05	0.084	0.068	0.055	0.056	0.056	0.076	0.053
		0.01	0.043	0.017	0.009	0.012	0.013	0.038	0.011
$\mu_H^2 = 3$	$\rho = 0.3$	0.10	0.076	0.245	0.100	0.103	0.104	0.065	0.138
		0.05	0.033	0.152	0.050	0.053	0.051	0.028	0.072
		0.01	0.003	0.046	0.011	0.009	0.011	0.003	0.015
	$\rho = 0.8$	0.10	0.189	0.177	0.099	0.099	0.100	0.156	0.101
		0.05	0.152	0.102	0.051	0.049	0.050	0.118	0.048
		0.01	0.091	0.028	0.010	0.009	0.010	0.068	0.010

Table 4: Empirical sizes of statistics that test $H_0 : \beta = \beta_0$ with $n = 500$, $K = 5$ and heteroscedastic errors

			t_{LIML}	$RAAR$	AR	K	CLR	t_{HLIML}	$mRAAR$
$\mu_H^2 = 10$	$\rho = 0.3$	0.10	0.124	0.280	0.271	0.162	0.210	0.075	0.105
		0.05	0.068	0.193	0.182	0.096	0.134	0.034	0.055
		0.01	0.016	0.082	0.076	0.029	0.051	0.006	0.011
	$\rho = 0.8$	0.10	0.104	0.156	0.170	0.126	0.133	0.114	0.101
		0.05	0.077	0.090	0.099	0.069	0.074	0.080	0.051
		0.01	0.040	0.025	0.030	0.017	0.020	0.036	0.011
$\mu_H^2 = 3$	$\rho = 0.3$	0.10	0.090	0.397	0.260	0.174	0.237	0.063	0.122
		0.05	0.044	0.295	0.172	0.106	0.156	0.028	0.061
		0.01	0.008	0.145	0.062	0.036	0.061	0.005	0.011
	$\rho = 0.8$	0.10	0.162	0.224	0.165	0.131	0.150	0.185	0.122
		0.05	0.127	0.147	0.092	0.074	0.087	0.144	0.067
		0.01	0.080	0.050	0.026	0.018	0.024	0.084	0.018

Table 5: Empirical sizes of statistics that test $H_0 : \beta = \beta_0$ with $n = 500$, $K = 20$ and homoscedastic errors

			t_{LIML}	$RAAR$	AR	K	CLR	t_{HLIML}	$mRAAR$
$\mu_H^2 = 20$	$\rho = 0.3$	0.10	0.184	0.258	0.108	0.108	0.108	0.083	0.114
		0.05	0.103	0.177	0.055	0.050	0.056	0.042	0.056
		0.01	0.031	0.070	0.012	0.012	0.013	0.010	0.012
	$\rho = 0.8$	0.10	0.126	0.161	0.109	0.104	0.105	0.096	0.100
		0.05	0.091	0.095	0.059	0.054	0.054	0.065	0.051
		0.01	0.049	0.029	0.014	0.010	0.013	0.032	0.011
$\mu_H^2 = 10$	$\rho = 0.3$	0.10	0.199	0.362	0.106	0.109	0.113	0.087	0.138
		0.05	0.120	0.273	0.055	0.056	0.055	0.044	0.074
		0.01	0.038	0.135	0.012	0.012	0.012	0.011	0.018
	$\rho = 0.8$	0.10	0.171	0.216	0.107	0.103	0.105	0.125	0.103
		0.05	0.135	0.139	0.053	0.053	0.055	0.090	0.053
		0.01	0.082	0.053	0.012	0.011	0.013	0.045	0.010

Table 6: Empirical sizes of statistics that test $H_0 : \beta = \beta_0$ with $n = 500$, $K = 20$ and heteroscedastic errors

			t_{LIML}	$RAAR$	AR	K	CLR	t_{HLIML}	$mRAAR$
$\mu_H^2 = 20$	$\rho = 0.3$	0.10	0.238	0.393	0.328	0.159	0.229	0.089	0.103
		0.05	0.140	0.306	0.226	0.098	0.151	0.044	0.054
		0.01	0.042	0.176	0.090	0.031	0.060	0.012	0.011
	$\rho = 0.8$	0.10	0.083	0.199	0.182	0.131	0.141	0.098	0.098
		0.05	0.058	0.126	0.104	0.076	0.078	0.067	0.047
		0.01	0.029	0.043	0.029	0.019	0.022	0.032	0.010
$\mu_H^2 = 10$	$\rho = 0.3$	0.10	0.229	0.534	0.329	0.162	0.266	0.087	0.130
		0.05	0.136	0.450	0.225	0.098	0.184	0.046	0.069
		0.01	0.047	0.295	0.089	0.032	0.075	0.013	0.014
	$\rho = 0.8$	0.10	0.111	0.276	0.178	0.147	0.162	0.137	0.112
		0.05	0.089	0.196	0.103	0.084	0.093	0.103	0.055
		0.01	0.057	0.081	0.028	0.022	0.028	0.058	0.013