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Human Capital as an Asset Mix and Optimal Life-Cycle Portfolio: An Analytical Solution

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Abstract

This study examines life-cycle optimal consumption and asset allocation in the presence of human capital. Labor income seems like a "money market mutual fund" whose balance in one or two years is predictable but a wide dispersion results after many years, reflecting fluctuations in economic conditions. We use the Martingale method to derive an analytical solution, finding that Merton's well-known "constant-mix strategy" is still true after incorporating human capital from the perspective of "total wealth" management. Moreover, the proportion in risky assets implicit in the agent's human capital is the main factor determining the optimal investment strategy. The numerical examples suggest that young investors should short stocks because their human capital has large market exposure. As they age, however, their human capital becomes "bond-like", and thus they have to hold stocks to achieve optimal overall risk exposure.

Life-cycle investment in asset allocation decisions is not only a key theoretical issue for economists, but also an important practical consideration for households, policymakers and financial advisors who are regularly confronted with the question of how to best invest personal savings in the long run. Defined contribution retirement plans – such as the 401(k) plan in the United States, Japan, and elsewhere – allow households to decide how much of their retirement funds to allocate to different assets, and their rising popularity has sparked renewed interest in life-cycle investment decisions.

When households are deciding on how to invest their savings, age may play an influential role in determining the composition of investment portfolios. For example, Bodie and Treussard (2007) report that almost all of the major Target-date fund (TDF) providers in 2006 followed the well-known rule that the optimal proportion to invest in equities equals 100 minus the person's age. Interestingly, several empirical studies (see, for example, King and Leape (1987), Yoo (1994), Poterba and Samwick (1997)) have reported that younger investors typically have low risky asset holdings. The actual role of one's age in investment decisions has led to a debate over possible recommendations for optimal investment strategies over an investor's lifetime.

The TDF providers' rule is one historic and dominating recommendation. It suggests that, as household investors get older, they should reduce the portion of their portfolio in risky assets because of what is known as the "time-diversification effect." That is, it has been widely believed that a substantial amount of stock investment risk can be eliminated through long-term ownership. Thus, younger investors should place more money into stocks because they have much time ahead and therefore many chances of recovering their losses from stock investment.

Samuelson (1969) and Merton (1969) argues strongly against the existence of this effect by suggesting that investing over longer horizons does not diversify risk away and, as such, the optimal asset mix does not depend on an investor's age. Merton (1969) shows that under some restrictive conditions (such as geometric Brownian motion for the prices of risky assets, constant relative risk aversion utilities, no wage income and so on), a constant-mix portfolio strategy is optimal. Thus, the factors of investment opportunity and risk aversion, but not age, are key in determining life-cycle investment strategy.

Such portfolio choices over the life-cycle must also take into account wage income and its associated risk. Human capital, defined as the present value of future labor income, is increasingly recognized as an important asset class apart from financial assets and which can cause investors to change their allocation in financial assets in a pattern related to the life-cycle. As Merton (2000) argues, a university professor should invest more of his assets in stocks than a stock broker does, since a stock broker has a significant equity exposure through his job while a university professor's wage income may be far less sensitive to market fluctuations. As such, to the extent that the level and risk of the labor income stream change over the life-cycle and portfolio choice depends on these factors, the presence of wage income can provide a rationale for age-varying investment strategies without relying on predictability in asset returns.

Interestingly, most previous studies that incorporate labor income into the model conclude that the TDF providers' advice is correct or that the stock holdings of young agents should be enhanced. As noted above, these academic suggestions are inconsistent with the empirical observations, and is known as the "stock market participation puzzle." Benzoni, Dufresne and Goldstein (2007) recently argues that these models enhance the puzzle because most of them attribute "bond-like" qualities to future labor income. This implies that young investors implicitly hold a large position in risk-free assets through their human capital and therefore it is optimal for them to take a more aggressive position in risky assets to adjust their total portfolio composition. They also developed a model in which the aggregate labor income is cointegrated with dividends. Through this cointegration, the young agent's human capital becomes "stock-like" to generate a hump-shaped life-cycle stock holding that is consistent with the empirical observation.

In this paper, we develop a simpler model to explain why young people do not invest in risky assets. We consider a continuous-time economy on the finite time span $[0, T]$ in which an agent endowed with some initial cash-on-hand and a stochastic wage income flow is interested in maximizing his expected utility of consumption and final wealth. We assume that the agent's investment opportunities are limited to one riskless cash-bond and some risky stocks.

Our work differs significantly from previous literature in two important respects. First, we regard the investor's future labor income as a "money-market mutual fund." Intuitively, an individual may be able to foresee his wages in one or two years, but wages beyond this period are uncertain many years down the road. It is similar to a money-market mutual fund whose balance in one or two years is almost certain but the balance after many years has a wide dispersion reflecting fluctuations in interest rate. To capture this feature of labor income, we specify it as completely deterministic in a very short period but then changes so that the expected labor growth rate follows a mean-reverting stochastic process. As such, our study may resemble the Vasicek model, except that risk is valued in terms of equity returns rather than interest rates.

Second, we use the Martingale method to solve the life-cycle optimal asset allocation problem. As is widely known, the great difficulty in solving problems of this class lies in the non-linearity of the Hamilton-Jacobi-Bellman (HJB) equation. With the exception of a few simpler cases, analytical solutions are basically unavailable. Consequently, most existing studies solve the optimizing problem numerically using the dynamic programming (DP) approach. Motivated by the pursuit of analytical solutions, however, we assume that the market is complete (i.e. the pricing kernel is unique) and use the Martingale method to solve our optimizing problem. As developed by Cox and Huang (1989), this method changes the dynamic problem into a static one so that we can solve the optimal consumption and investment strategies directly by using the classic Lagrange-Kuhn-Tucker approach.

We find that the result of Merton(1969) is still true after incorporating human capital into the model. That is, no matter whether the human capital is "bond-like" or "stock-like," the constant-mix strategy is optimal from the perspective of "total portfolio" management. As such, the allocation of financial wealth has to be closely linked to the quality of another part of one's total portfolio, say, human capital. To corroborate this point, we derive the analytical solution for several cases with different labor income processes including our model and the model of Dufresne. We simulate each case with 10,000 paths and find that the proportion in market portfolio implicit in the agent's human capital is the main factor that heavily affects his stock holdings profile.

The Martingale method also enables us to observe the components of human capital over the life-cycle. When the future flows of labor income are deterministic, human capital has no risk exposure (i.e. it can be seen as a risk-free bond). Thus, the agent should take a more aggressive position in the risky asset when he is young because he has already held a large position in the risk-free asset implicitly. When the future wages follow a geometric Brownian motion, human capital is equally exposed to equity risk at all ages. Consequently, the optimal portfolio strategy becomes an increasing but not hump-shaped function of age when the equity exposure is adequately large. In our model, the fraction of the agent's human capital tied up in the market portfolio is more than 90% at $t = 0$ and then gradually decreases toward 0. Aside from the above, the fraction of cash-bond increases toward 100% as the agent approaches retirement. In this case, we can observe a hump-shaped stock holding profile which is consistent with empirical studies.

Finally, we have to emphasize that even in incomplete markets, the Martingale method is still useful in achieving meaningful economic implications. When the market is incomplete, there exist infinitely many pricing kernels, which are consistent both with no arbitrage conditions and with a family of static budget constraints that have to be considered in order to ensure the feasibility of solutions. Although the Martingale method may no longer seem valid is such a case, it still enables us to get an approximate solution by choosing adequate prices for untradeable risks.

The rest of our paper is organized as follows. In Section 1, we describe the life-cycle consumption and asset allocation problem, and the Martingale method in detail. In Section 2, we derive the optimal consumption and investment strategy for an agent exhibiting constant relative risk aversion. We then provide some numerical examples of optimal strategies for several different labor income processes in Section 3. Next, in Section 4, we use the dynamic programming method to obtain the solution for our model when the market is incomplete. In Section 5, we simulate Dufresne's model and derive analytical solutions. Finally, Section 6 concludes the paper.

1 The Model

Here we present a life-cycle asset allocation model. We consider a continuous-time economy on the finite time span $[0, T]$ in which an agent endowed with some initial cash-on-hand and a stochastic wage income flow is interested in maximizing his expected utility of consumption and final wealth. We assume that the agent's investment opportunities are limited to one riskless cash-bond and some risky stocks.

1.1 Formulation of the Consumption and Investment Problem with Labor Income

Notation:

$(\Omega, \mathcal{F}, P)$: probability space

$w = \{w(t); 0 \leq t \leq T\}$: M-dimentional standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$

$\mathcal{F}_t$: the sigma-field generated by $\{w(s); 0 \leq s \leq t\}$

$\mathbf{F} = \{\mathcal{F}_t; 0 \leq t \leq T\}$: the filtration generated by w

$X = \{X(t); 0 \leq t \leq T\}$ is said to be adapted to $\mathbf{F}$ if $X(t)$ is measurable with respect to $\mathcal{F}_t$, $\forall t \in [0, T]$

$(\Omega \times [0, T], \mathcal{O}, \nu)$: probability space on which are defined process adapted to $\mathbf{F}$, where ν is dealt with as the product measure generated by P and Lebesgue measure

$r(t)$: instantaneous riskless rate at t

$B(t) \triangleq B(0) \exp\left(\int_0^t r(s)\, ds\right)$: price of cash bond at t (with $B(0) \equiv 1$)

Assume there are M stocks in the market and their cum-dividend stock prices, denoted by $S = \{S(t); 0 \leq t \leq T\}$ is an M-dimensional Ito process adapted to $\mathbf{F}$ satisfying

$$S(t) = S(0) + \int_0^t \mu(s)\, ds + \int_0^t \sigma(s)\, dw(s), \forall t \in [0,T], P - a.s. \tag{1.1}$$

where $\mu(t)$ is an $M \times 1$ vector valued process and $\sigma(t)$ is an $M \times M$ matrix valued process. We assume that $\sigma(t)$ is nonsingular $P - a.s.$ for all $t \in [0,T]$ so that the matrix is invertible.

Furthermore, assume that the investor receives an exogeneously-given labor income flow Y until his retirement day T, where $Y = \{Y(t); 0 \leq t \leq T\}$ is an Ito process adapted to $\mathbf{F}$ satisfying

$$Y(t) = Y(0) + \int_0^t \mu_y(s)\, ds + \int_0^t \sigma_y(s)\, dw(s), \forall t \in [0,T], P - a.s. \tag{1.2}$$

where $\sigma_y(t)$ is a $1 \times M$ vector valued process.

Definition 1 A consumption-final wealth pair (c, W_T) is said to be *admissible* if

$$(c, W_T) \in L_+^p(\nu) \times L_+^p(P) \equiv L_+^p(\Omega \times [0,T], \mathcal{O}, \nu) \times L_+^p(\Omega, \mathcal{F}, P), \tag{1.3}$$

where $1 < p < \infty$.

Definition 2 A trading strategy $(\alpha, \theta) \equiv \left\{\left(\alpha(t) \in R, \theta(t) \in R^M\right), t \in [0,T]\right\}$ is said to be *admissible* if

1. there exists consumption-final wealth pair $(c, W_T) \in L_+^p(\nu) \times L_+^p(P)$ such that, $P - a.s.$, $\forall t \in [0,T]$,

$$\begin{aligned}\alpha(t) B(t) + \theta(t)^T S(t) + \int_0^t c(s)\, ds - \int_0^t Y(s)\, ds &= \alpha(0) B(0) + \theta(0)^T S(0) \\ &+ \int_0^t \alpha(s) B(s) r(s)\, ds + \int_0^t \theta(s)^T \mu(s)\, ds \\ &+ \int_0^t \theta(s)^T \sigma(s)\, dw(s) \end{aligned} \tag{1.4}$$

and

2.
$$\alpha(T) B(T) + \theta(T)^T S(T) = W_T, \quad P - a.s. \tag{1.5}$$

Definition 3 A consumption-final wealth pair (c, W) is said to be *marketed with initial wealth W_0 and financed by the trading strategy* (α, θ) if (c, W_T) and (α, θ) satisfy 1. and 2. with $\alpha(0) B(0) + \theta(0)^T S(0) = W_0$.

Let $\mathcal{C}(W_0)$ denote the set of consumption-final wealth pairs (c, W_T) marketed with initial wealth W_0. We consider a life-cycle investor who has a time-additive utility function for consumption, $u(c(t), t)$, a bequest function for final wealth at time T, $V(W_T)$. He is also endowed with an initial wealth $W_0 > 0$ in the form of cash on hand. Thus, the opimization problem of this investor can be written as

$$\sup_{(c, W_T) \in \mathcal{C}(W_0)} E\left[\int_0^T u(c(t), t)\, dt + V(W_T)\right]. \tag{1.6}$$

Note that, instead of modeling the post-retirement days activities explicitly as Benzoni-Dufresne-Goldstein did, we settled the bequest function to capture the saving motive of the investor for his consumption after retirement. For retired investors, the simplest Merton's model can represent their optimal life-cycle planning problem since they will no longer receive wage earnings.

1.2 The Martingale Method

We use the Martingale method to solve this problem. When the market is complete so that we can observe a unique pricing kernel, the Martingale method changes the dynamic problem into a static one and thus we can solve the optimal consumption-final wealth and the optimal asset allocation strategy directly by using the classic Lagrange-Kuhn-Tucker approach. Unlike the dynamic programming approach, this method enables us to avoid estimating the shape of the value function.

Define

$$\kappa(t) \triangleq (\sigma(t))^{-1}(\mu(t) - r(t) S(t)) \in R^M \tag{1.7}$$

$$\eta(t) \triangleq \exp\left\{-\int_0^t \kappa(s)^T dw(s) - \frac{1}{2}\int_0^t \|\kappa(s)\|^2 ds\right\} \in R \tag{1.8}$$

and

$$Q(A) \triangleq \int_A \eta(w, T) P(dw) = E[\eta(T) 1_A] \quad \forall A \in \mathcal{F} \tag{1.9}$$

(i.e., $\eta(T) = \frac{dQ}{dP}$) The probability measure Q is an equivalent martingale measure. Furthermore from Girsanov's Theorem, it is understood that

$$w^*(t) \triangleq w(t) + \int_0^t \kappa(s) ds \tag{1.10}$$

defines a standard M-dimentional Brownian motion under this Q measure.

Let (c, W_T) be financed by (α, θ). Then, the cost of $\{c(s); s \in [t, T]\}$ and W_T at time t turns out to be given by

$$\alpha(t) B(t) + \theta(t)^T S(t) = B(t) E^Q\left[\int_t^T \frac{c(s)}{B(s)} ds - \int_t^T \frac{Y(s)}{B(s)} ds + \frac{W}{B(T)} \mid \mathcal{F}_t\right] \tag{1.11}$$

(See Appendix 1). This leads to the observation by which we can rewrite the dynamic optimizing problem as a static complete market problem:

$$\sup_{(c, W_T) \in L_+^p(\nu) \times L_+^p(P)} E\left[\int_0^T u(c(t), t) dt + V(W_T)\right] \tag{1.12}$$

$$s.t. E\left[\int_0^T \eta(t) \frac{c(t)}{B(t)} dt - \int_0^T \eta(t) \frac{Y(t)}{B(t)} dt + \eta(T) \frac{W_T}{B(T)}\right] = W_0 \tag{1.13}$$

or using the pricing kernel $m(t)$, which is defined by

$$m(t) \triangleq \frac{\eta(t)}{B(t)}, \tag{1.14}$$

the problem can be rewritten as:

$$\sup_{(c, W_T) \in L_+^p(\nu) \times L_+^p(P)} E\left[\int_0^T u(c(t), t) dt + V(W_T)\right] \tag{1.15}$$

$$s.t. E\left[\int_0^T m(t) c(t) dt - \int_0^T m(t) Y(t) dt + m(T) W_T\right] = W_0 \tag{1.16}$$

2 Solving the Problem for a CRRA Utility Function

We assume that the investor's preference over uncertain consumptions and bequests are based upon discounted constant relative risk aversion (CRRA) utility functions, which are

$$u(c(t), t) = e^{-\beta t} \frac{c(t)^{1-\gamma}}{1-\gamma} \tag{2.1}$$

and

$$V(W_T) = \varepsilon e^{-\beta T} \frac{W_T^{1-\gamma}}{1-\gamma} \tag{2.2}$$

where $\gamma\,(\neq 1)$ is the parameter of relative risk aversion and ε captures the relative strength of the utility from the bequest. The first order conditions of optimality are

$$u_c(c(t), t) = 0 \tag{2.3}$$

$$V'(W_T) = 0, \tag{2.4}$$

leading to the conditions

$$e^{-\beta t} c(t)^{-\gamma} = \lambda m(t) \tag{2.5}$$

$$\varepsilon e^{-\beta T} W_T^{-\gamma} = \lambda m(T) \tag{2.6}$$

where λ is the Lagrange multiplier. Define the inverse of $u_c(c,t)$ and $V'(W_T)$ as

$$f(y,t) \triangleq \inf\{c \geq 0, u_c(c,t) \leq y\} \tag{2.7}$$

$$g(y) \triangleq \inf\{W_T \geq 0, V'(W_T) \leq y\}. \tag{2.8}$$

Using these inverse functions, the optimal consumption and final wealth can be written as

$$c^*(t) = f(\lambda m(t), t) = e^{-\frac{\beta}{\gamma}t} (\lambda m(t))^{-\frac{1}{\gamma}} \tag{2.9}$$

$$W_T^* = g(\lambda m(T)) = \varepsilon^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma}T} (\lambda m(T))^{-\frac{1}{\gamma}} \tag{2.10}$$

and λ is found to satisfy

$$\mathcal{H}(\lambda) \triangleq E\left[\int_0^T m(t) f(\lambda m(t), t)\, dt + m(T) g(\lambda m(T))\right] - L_0 = W_0 \tag{2.11}$$

where

$$L(t) \triangleq \frac{1}{m(t)} E_t\left[\int_t^T m(s) Y(s)\, ds\right] \tag{2.12}$$

is the time-t present value of labor income that the investor expects to receive until his retirement. We call this the "Human Capital." Substituting (2.9) and (2.10) into (2.11), we obtain

$$\mathcal{H}(\lambda) = \lambda^{-\frac{1}{\gamma}} E\left[\int_0^T e^{-\frac{\beta}{\gamma}t} m(t)^{1-\frac{1}{\gamma}}\, dt + \varepsilon^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma}T} m(T)^{1-\frac{1}{\gamma}}\right] - L_0. \tag{2.13}$$

Define

$$Q(t) \triangleq E_t\left[\int_t^T e^{-\frac{\beta}{\gamma}(s-t)} \left(\frac{m(s)}{m(t)}\right)^{1-\frac{1}{\gamma}} ds + \varepsilon^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma}(T-t)} \left(\frac{m(T)}{m(t)}\right)^{1-\frac{1}{\gamma}}\right], \tag{2.14}$$

(2.13) can be rewritten as

$$\mathcal{H}(\lambda) = \lambda^{-\frac{1}{\gamma}} Q(0) - L_0 = W_0, \tag{2.15}$$

and consequently

$$\lambda = \left(\frac{W_0 + L_0}{Q_0} \right)^{-\gamma}. \tag{2.16}$$

Insert this λ back into (2.9) and (2.10) again, we obtain

$$c^* (t) = e^{-\frac{\beta}{\gamma} t} m (t)^{-\frac{1}{\gamma}} \left(\frac{W_0 + L_0}{Q_0} \right) \tag{2.17}$$

and

$$W_T^* = \varepsilon^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma} T} m (T)^{-\frac{1}{\gamma}} \left(\frac{W_0 + L_0}{Q_0} \right). \tag{2.18}$$

Since

$$W^* (t) + L (t) = \frac{1}{m (t)} E_t \left[\int_t^T m (s) c^* (s) \, ds + m (T) W_T^* \right], \tag{2.19}$$

substituting $c^* (t)$ and W_T^* into this equation, we found the relation between $W^* (t) + L (t)$ and $W_0 + L_0$ is

$$W^* (t) + L (t) = e^{-\frac{\beta}{\gamma} t} m (t)^{-\frac{1}{\gamma}} Q (t) \left(\frac{W_0 + L_0}{Q_0} \right). \tag{2.20}$$

Thus, we obtain

$$c^* (t) = \left(\frac{W^* (t) + L (t)}{Q (t)} \right). \tag{2.21}$$

The above equation asserts that the optimal consumption at time-t is a fraction of the time-t total wealth (the sum of financial wealth and human capital) of this investor. We can easily find that the result is similar enough to comply with the one from Merton(1969). For next step, we derive the process of $Q (t)$ to make sure that this is very the well-known Merton's result.

2.1 Derivation of the $Q (t)$ Process

We assume that the cum-dividend stock prices follow geometric Brownian motions. That is, define a diagonal matrix

$$I_{S(t)} \triangleq \begin{pmatrix} S_1 (t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S_M (t) \end{pmatrix}, \tag{2.22}$$

$\mu (t)$ and $\sigma (t)$ in (1.1) can be written as

$$\mu (t) = I_{S(t)} \mu \tag{2.23}$$

$$\sigma (t) = I_{S(t)} \sigma \tag{2.24}$$

where μ and σ are constants. Note that, for simplicity, we will not discuss about the interest rate risk in this paper, namely we assume that $r (t)$ is also a constant. Under these assumptions, we can find $\kappa (t)$, defined by (1.7), is a constant $M \times 1$ vector and thus the pricing kernel $\{m (t) : t \in [0, T]\}$ is given by

$$m (t) = \exp \left\{ -\kappa^T w (t) - \left(r + \frac{\|\kappa\|^2}{2} \right) t \right\}. \tag{2.25}$$

Define

$$q_t^s \triangleq E_t \left[\left(\frac{m (s)}{m (t)} \right)^{1 - \frac{1}{\gamma}} \right], \tag{2.26}$$

$Q(t)$ can be rewritten as

$$Q(t) = \int_t^T e^{-\frac{\beta}{\gamma}(s-t)} q_t^s ds + \varepsilon^{\frac{1}{\gamma}} e^{-\frac{\beta}{\gamma}(T-t)} q_t^T. \tag{2.27}$$

Substituting (2.25) into (2.26) and taking expectation, we get

$$q_t^s = \exp\{-\zeta(s-t)\} \tag{2.28}$$

where $\zeta \triangleq \left(1 - \frac{1}{\gamma}\right)\left(r + \frac{\|\kappa\|^2}{2\gamma}\right)$. Substituting (2.28) into (2.27) again, we can finally find

$$Q(t) = \frac{1}{\frac{\beta}{\gamma} + \zeta}\left[1 + \left\{\left(\frac{\beta}{\gamma} + \zeta\right)\varepsilon^{\frac{1}{\gamma}} - 1\right\}\exp\left\{-\left(\frac{\beta}{\gamma} + \zeta\right)(T-t)\right\}\right]. \tag{2.29}$$

This is the optimal consumption rate. We note that it is scaled by the time-t value of total wealth. That is to say, it is optimal for a wage earner to consume a fix but age-dependent fraction of his total wealth. For the case in which there are no wage earnings, $L(t)$ becomes 0. Thus we can find that (2.21) is exactly the result from Merton (1969).

2.2 Derivation of the Optimal Investment Strategy

Here, we determine the optimal investment strategy.

From (2.25), the dynamics of $m(t)$ is

$$dm(t) = -m(t)\, r dt - m(t)\, \kappa^T dw(t). \tag{2.30}$$

Using Ito's lemma, we get the the process of $\{W^*(t) + L(t)\}$ as

$$d(W^*(t) + L(t)) = (\cdot)dt + (W^*(t) + L(t))\frac{1}{\gamma}\kappa^T dw(t). \tag{2.31}$$

We omit the drift term because the only thing we need to derive within the investment strategy is the volatility term. Namely, if the investor takes the optimal investment strategy $\theta^*(t)$ at time t, the process of $\{W^*(t) + L(t)\}$ must be

$$d(W^*(t) + L(t)) = (\cdot)\, dt + \left(\theta^*(t)^T I_{S(t)}\sigma + \sigma_L(t)\right) dw(t) \tag{2.32}$$

where $\sigma_L(t)$ denotes the volatility of the process of $\{L(t)\}$. Since the market is complete, $\sigma_L(t)$ is a $1 \times M$ vector. Thus, we can solve for the optimal strategy through the following equation:

$$\theta^*(t)^T I_{S(t)}\sigma + \sigma_L(t) = (W^*(t) + L(t))\frac{1}{\gamma}\kappa^T, \tag{2.33}$$

And the solution is

$$I_{S(t)}\theta^*(t) + \left(\sigma_L(t)\,\sigma^{-1}\right)^T = (W^*(t) + L(t))\frac{1}{\gamma}\left(\sigma\sigma^T\right)^{-1}(\mu - r\mathbf{1}). \tag{2.34}$$

The first term of the LHS of the equation is the optimal dollar investment to eack stock. We can regard the second term of the LHS as the "implicit" investment included by the human capital of the agent. Therefore, from the above equation, it is understood that if the human capital at each time is viewed as a (time-varying) portfolio of stocks, the optimal "total" dollar investment on each stock must be a fixed fraction of his total portfolio. In other words, Merton's well-known "constant-mix" strategy still holds from the perspective of "total wealth" management.

2.3 Derivation of the $L(t)$ Process

Next, we derive $L(t)$ for three different labor income processes. As noted above, human capital is often the largest, untradable asset in the early part of most people's working life. As such, it can be easily imagined that the nature of $L(t)$ can exercise deep influence on one's investment strategy over the life-cycle. Merton (1971) showed that, in computing the optimal decision rules, the individual with certain labor income has to capitalize all the future flows at the risk-free rate and add it to the current financial wealth. We check this result in Case 1 and then investigate other two cases where the labor income flows are uncertain.

Case 1: Labor Income as a Certain Income Stream

When labor income $\{Y(t) : t \in [0, T]\}$ is nonstochastic, the human capital $L(t)$ is

$$\begin{aligned} L(t) &= \frac{1}{m(t)} E_t \left[\int_t^T m(s) Y(s)\, ds \right] \\ &= \int_t^T E_t \left[\exp\left\{ -\kappa^T (w(s) - w(t)) \right\} \right] \exp\left\{ -\left(r + \frac{1}{2} \|\kappa\|^2 \right)(s-t) \right\} Y(s)\, ds \\ &= \int_t^T \exp(-r(s-t)) Y(s)\, ds. \end{aligned} \tag{2.35}$$

Since the wage is riskless throughout the working lifetime, we can regard the human capital as a risk-free coupon bond and therefore its time-t value is the sum of all future income flows discounted at the risk-free rate. Look back at (2.34), we can find the second term on the LHS disappears and the optimal investment strategy is to invest a fixed fraction of the investor's total wealth in each stock. This complies with Merton's insistence.

Case 2: Labor Income as a Geometric Brownian Motion

For Case 2, we assume the labor income process follows a geometric Brownian motion, that is,

$$\frac{dY(t)}{Y(t)} = \mu_y dt + \sigma_y dw(t) \tag{2.36}$$

where μ_y is a scalar and σ_y is a $1 \times M$ vector. Using Ito's lemma, we get

$$Y(s) = Y(t) \exp\left\{ \left(\mu_y - \frac{1}{2} \|\sigma_y\|^2 \right)(s-t) + \sigma_y (w(s) - w(t)) \right\} \quad \text{for } s \geq t. \tag{2.37}$$

In this case, human capital can be calculated as

$$\begin{aligned} L(t) &= Y(t) E_t \left[\int_t^T \exp\left\{ -\left(r + \frac{1}{2} \|\kappa\|^2 \right)(s-t) - \kappa^T (w(s) - w(t)) \right\} \right. \\ &\qquad \left. \times \exp\left\{ \left(\mu_y - \frac{1}{2} \|\sigma_y\|^2 \right)(s-t) + \sigma_y (w(s) - w(t)) \right\} ds \right] \\ &= \int_t^T \left[Y(t) \exp\left\{ \mu_y (s-t) \right\} \right] \exp\left\{ -(r + \sigma_y \kappa)(s-t) \right\} ds. \end{aligned} \tag{2.38}$$

This equation shows that the human capital $L(t)$ is the expected value of future wage incomes discounted at the rate $r + \sigma_y \kappa$. Note that κ is a vector of risk premium corresponding to each "factor" $dw_i(t)$. Hence, this valuation method is consistent with the Arbitrage Pricing Theory (APT) of Ross (1976). Note also that, since $L(t)$ is linearly related to $Y(t)$, the process of $L(t)$ can be written as $dL(t) = (\cdot)\, dt + \sigma_y L(t)\, dw(t)$. This implies that human capital has "factor exposure" proportional to σ_y and thus is deemed to be a fixed portfolio of stocks.

Case 3: Labor Income with No Short-run Risk

It can hardly be said that describing wages as an Ito process is appropriate. For an individual, the wages he is to receive in many years later intuitively have much uncertainty but the wages in one or two years are to a large extent foreseeable. It is just like a "money market mutual fund" whose balance in one or two years is almost certain but the balance after many years has a wide dispersion reflecting fluctuations of economic conditions. To capture this feature, we specify that the labor income as completely deterministic in a very short period. But the expected labor growth rate follows a mean-reverting stochastic process. This resembles the Vasicek model except the risk is equity risk rather than interest rate risk. That is, define $y(t) \triangleq \log Y(t)$,

$$dy(t) = \mu_y(t)\, dt \tag{2.39}$$

and the drift term $\mu_y(t)$ follows a mean-reverting Brownian motion,

$$d\mu_y(t) = \theta\left(\alpha_\mu(t) - \mu_y(t)\right) dt + \sigma_\mu dw(t) \tag{2.40}$$

where σ_μ is a constant $1 \times M$ vector, $\alpha_\mu(t)$ represents time-t average level of short-term growth rate toward which $\mu_y(t)$ reverts, and θ represents the speed of reversion. By integration, we get

$$y(s) = y(t) + \int_t^s \mu_y(u)\, du \quad \text{for } s \geq t \tag{2.41}$$

where

$$\mu_y(u) = \exp(\theta(t-u))\mu_y(t) + \int_t^u \theta \exp(\theta(v-u))\alpha_\mu(v)\, dv + \int_t^u \exp(\theta(v-u))\sigma_\mu dw(v) \quad \text{for } t \leq u \leq s \tag{2.42}$$

Substituting (2.42) into (2.41), we obtain

$$\begin{aligned} y(s) &= y(t) + \frac{\mu_y(t)}{\theta}\{1-\exp(-\theta(s-t))\} + \int_t^s \{1-\exp(-\theta(s-v))\}\alpha_\mu(v)\, dv \\ &\quad + \frac{1}{\theta}\int_t^s \{1-\exp(-\theta(s-v))\}\sigma_\mu dw(v) \end{aligned} \tag{2.43}$$

Hence,

$$\begin{aligned} Y(s) &= Y(t)\exp\left\{\frac{\mu_y(t)}{\theta}\{1-\exp(-\theta(s-t))\} + \int_t^s \{1-\exp(-\theta(s-v))\}\alpha_\mu(v)\, dv \right. \\ &\quad \left. + \frac{1}{\theta}\int_t^s \{1-\exp(-\theta(s-v))\}\sigma_\mu dw(v)\right\}. \end{aligned} \tag{2.44}$$

Thus, the human capital at time-t can be written as

$$\begin{aligned} L(t) &= Y(t)\, E_t\left[\int_t^T \exp\left\{-\kappa^T \int_t^s dw(v) - \left(r + \frac{\|\kappa\|^2}{2}\right)(s-t)\right\}\right. \\ &\quad \times \exp\left\{\frac{\mu_y(t)}{\theta}\{1-\exp(-\theta(s-t))\} + \int_t^s \{1-\exp(-\theta(s-v))\}\alpha_\mu(v)\, dv\right\} \\ &\quad \left. \times \exp\left\{\frac{1}{\theta}\int_t^s \{1-\exp(-\theta(s-v))\}\sigma_\mu dw(v)\right\} ds\right] \\ &= Y(t)\int_t^T \exp\left\{\frac{\mu_y(t)}{\theta}\{1-\exp(-\theta(s-t))\} + \int_t^s \{1-\exp(-\theta(s-v))\}\alpha_\mu(v)\, dv - \left(r + \frac{\|\kappa\|^2}{2}\right)(s-t)\right. \\ &\quad \left. + \frac{1}{2}\int_t^s \left\|\kappa^T - \frac{\sigma_\mu}{\theta}\{1-\exp(-\theta(s-v))\}\right\|^2 dv\right\} ds. \end{aligned} \tag{2.45}$$

Define
$$R\left(t,T,\mu_y\left(t\right)\right) \triangleq \int_t^T a\left(t,s\right)\exp\left\{\frac{\mu_y\left(t\right)}{\theta}\left\{1-\exp(-\theta\left(s-t\right))\right\}\right\}ds \tag{2.46}$$
where
$$\begin{aligned} a\left(t,s\right) &\triangleq \exp\left\{\int_t^s\left\{1-\exp(-\theta\left(s-v\right))\right\}\alpha_\mu\left(v\right)dv-\left(r+\frac{\left\|\kappa\right\|^2}{2}\right)\left(s-t\right)\right. \\ &\left.+\frac{1}{2}\int_t^s\left\|\kappa^T-\frac{\sigma_\mu}{\theta}\left\{1-\exp(-\theta\left(s-v\right))\right\}\right\|^2dv\right\}, \end{aligned} \tag{2.47}$$
we can rewrite (2.45) as
$$L\left(t\right)=Y\left(t\right)R\left(t,T,\mu_y\left(t\right)\right). \tag{2.48}$$

To specify the optimal investment strategy, we have to in advance specify the time-t volatility of human capital $\sigma_L\left(t\right)$. Taking a logarithm of (2.48), the change of $\log L\left(t\right)$ can be divided into two parts, that is:
$$d\log L\left(t\right)=d\log Y\left(t\right)+d\log R\left(\mu_y\left(t\right),t\right). \tag{2.49}$$
Since $d\log Y\left(t\right)$ is instantaneously nonstochastic,
$$d\log L\left(t\right)=(\cdot)dt+d\log R\left(\mu_y\left(t\right),t\right). \tag{2.50}$$
Applying Ito's lemma, we get
$$\begin{aligned} dR\left(\mu\left(t\right),t\right) &= \frac{\partial R\left(\mu\left(t\right),t\right)}{\partial t}dt+\frac{\partial R\left(\mu\left(t\right),t\right)}{\partial\mu\left(t\right)}d\mu\left(t\right) \\ &= (\cdot)dt+\left\{\int_t^T\frac{a\left(t,s\right)}{\theta}\left\{1-\exp(-\theta\left(s-t\right))\right\}\right. \\ &\quad\left.\times\exp\left\{\frac{\mu_y\left(t\right)}{\theta}\left\{1-\exp(-\theta\left(s-t\right))\right\}\right\}dsd\mu_y\left(t\right)\right\}. \end{aligned} \tag{2.51}$$
Thus, the log-human capital at time-t is deemed to have the stochastic differecial equation:
$$d\log L\left(t\right)=(\cdot)dt+\frac{\int_t^T\frac{a(t,s)}{\theta}\left\{1-\exp(-\theta\left(s-t\right))\right\}\exp\left\{\frac{\mu_y(t)}{\theta}\left\{1-\exp(-\theta\left(s-t\right))\right\}\right\}ds}{\int_t^T a\left(t,s\right)\exp\left\{\frac{\mu_y(t)}{\theta}\left\{1-\exp(-\theta\left(s-t\right))\right\}\right\}ds}\sigma_\mu dw\left(t\right). \tag{2.52}$$
Define
$$\Lambda\left(t,T,\mu_y\left(t\right)\right) \triangleq \frac{\int_t^T\frac{a(t,s)}{\theta}\left\{1-\exp(-\theta\left(s-t\right))\right\}\exp\left\{\frac{\mu_y(t)}{\theta}\left\{1-\exp(-\theta\left(s-t\right))\right\}\right\}ds}{\int_t^T a\left(t,s\right)\exp\left\{\frac{\mu_y(t)}{\theta}\left\{1-\exp(-\theta\left(s-t\right))\right\}\right\}ds}, \tag{2.53}$$
the volatility of the human capital can be written as
$$\sigma_L\left(t\right)=\Lambda\left(t,T,\mu_y\left(t\right)\right)L\left(t\right)\sigma_\mu. \tag{2.54}$$
Finally, from (2.34), the dollar investment embedded in the human capital is
$$\Lambda\left(t,T,\mu_y\left(t\right)\right)L\left(t\right)\left(\sigma_\mu\sigma^{-1}\right), \tag{2.55}$$
and therefore the optimal dollar investment from financial asset at time-t is
$$\left(W^*\left(t\right)+L\left(t\right)\right)\frac{1}{\gamma}\left(\sigma\sigma^T\right)^{-1}\left(\mu-r\mathbf{1}\right)-\Lambda\left(t,T,\mu_y\left(t\right)\right)L\left(t\right)\left(\sigma_\mu\sigma^{-1}\right). \tag{2.56}$$
We use the Monte Carlo simulation to examine the economic implications of our model, which are represented in the next section.

3 Simulation Results

In this section, we use the analytical solutions of a numerical example to study the implications of our model. We consider a wage earner who enters to the job market at age 20, is scheduled to retire at age 65. Thus the optimal asset allocation problem is parameterized over $T = 45$ years. We basically use the same parameter settings in accordance with Dufresne et al. (2007), the details of which are given in Table 1.

Parameter	r	μ	σ	β	γ	ε
Value	0.01	0.07	0.16	0.04	5	8^5

Table 1. The Parameters.

We also assume that only two securities are available for this agent: a risk-free cash bond and a market portfolio whose expected return and volatility are given by μ and σ in Table 1. The agent is endowed with \$150,000 of cash-on-hand at $t = 0$ and receives \$15,000 as his initial annual income. We simulate 10,000 paths for each case. Note that we do not impose any constraints on stock holdings. Consquently, it can be over 100% or below 0% (i.e. the agent is allowed to take short positions). For clearness, however, we only map the value within 0% and 100% in figures presented below.

Case 1: Labor Income as a Certain Income Stream

When the future wages are estimable with certainty, human capital is a riskless asset whose value is the sum of all the future income flows capitalized at the risk-free rate. To reproduce the deterministic labor income profile that Dufresne used (presented in Figure 1) we assume that his income follows a deterministic process:

$$\frac{dY(t)}{Y(t)} = (\alpha_1 + \alpha_2 t)\, dt \tag{3.1}$$

where $\alpha_1 = 0.0728$ and $\alpha_2 = -0.0024$.

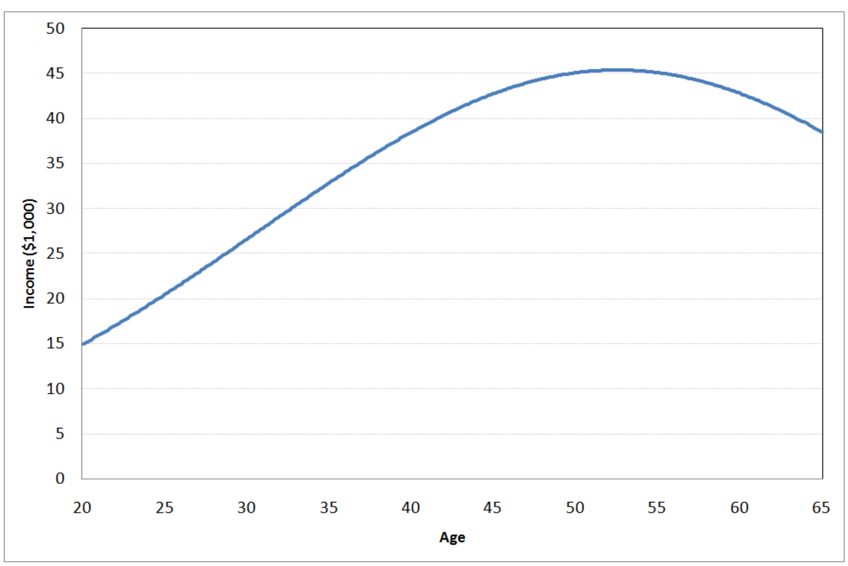

Figure 1. Labor Income Profile.

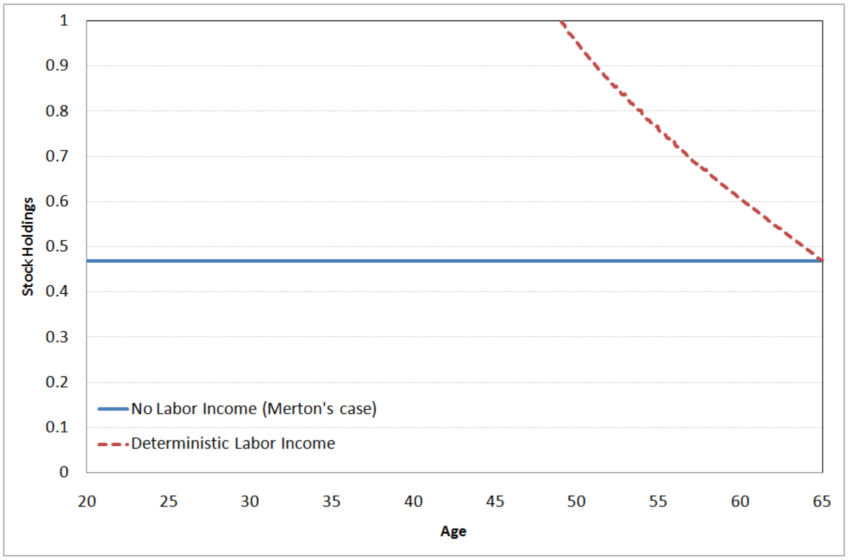

Figure 2. Optimal Stock Holdings.

The solid line of Figure 2 denotes the Merton's solution, which is $\frac{(\mu-r)}{\gamma\sigma^2} \fallingdotseq 0.47$ throughout life-cycle. The broken line depicts the optimal stock holdings of an agent whose labor income is given by (3.1). As much previous literature asserts, when the future flows of labor income are riskless, the agent should take a more aggressive position in the risky asset when he is young because he has already held a large position in the risk-free asset implicitly. As he ages, however, his human capital declines toward 0, as such his optimal stock holding also decreases toward Merton's result. We also find that for an agent with less initial cash-on-hand, the optimal stock weight sticks to 100% for a longer period.

Case 2: Labor Income as a Geometric Brownian Motion

When the future wages follow a geometric Brownian motion, human capital is the sum of all the future flows discounted at the risk-adjusted rate determined by the market. Here, we assume that the log-wage is perfectly correlated with the return on the market portfolio. In this case, human capital also shows a perfect correlation with the stock market, namely,

$$Corr\left(d\log Y\left(t\right), d\log S\left(t\right)\right) = Corr\left(d\log L\left(t\right), d\log S\left(t\right)\right) = 1.0, \tag{3.2}$$

and the equity weight in human asset is given by $\frac{\sigma_y}{\sigma}$ for all $t \in [0, T]$. We can see in Figure 3 that human capital is a hump-shaped function of age. This is because the labor income itself has a hump-shaped profile, and thus the higher income occuring at older ages will be discounted heavily when the agent is young.

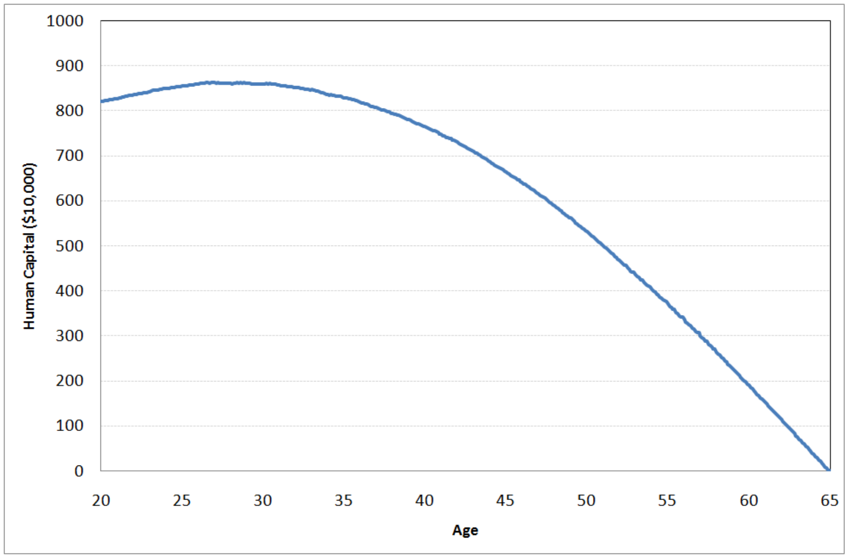

Figure 3. Human Capital.

We make mapping in Figure 4 to emphasize that when wage incomes follow a geometric Brownian motion, the composition of human capital does not make any change throughout one's lifetime.

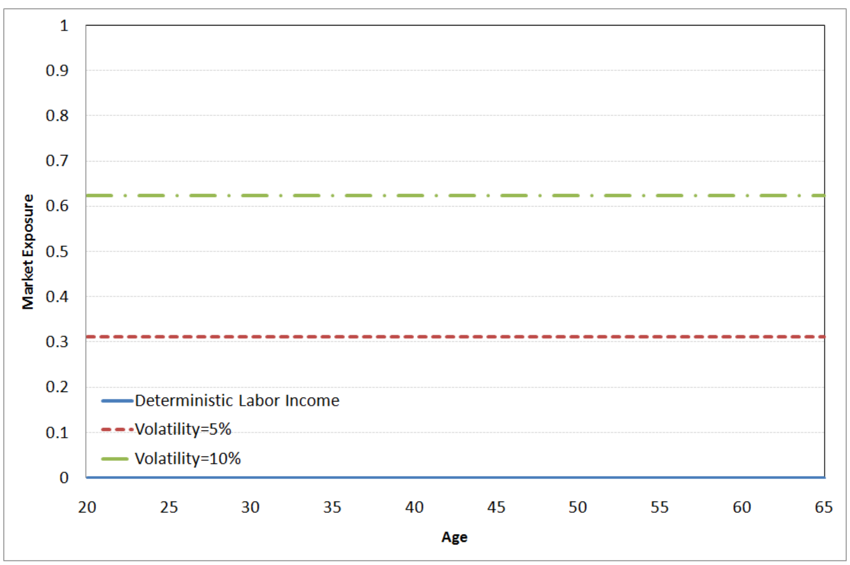

Figure 4. Market Exposure of Human Capital.

In Figure 5, we simulated the optimal stock holdings for agents with different σ_y.

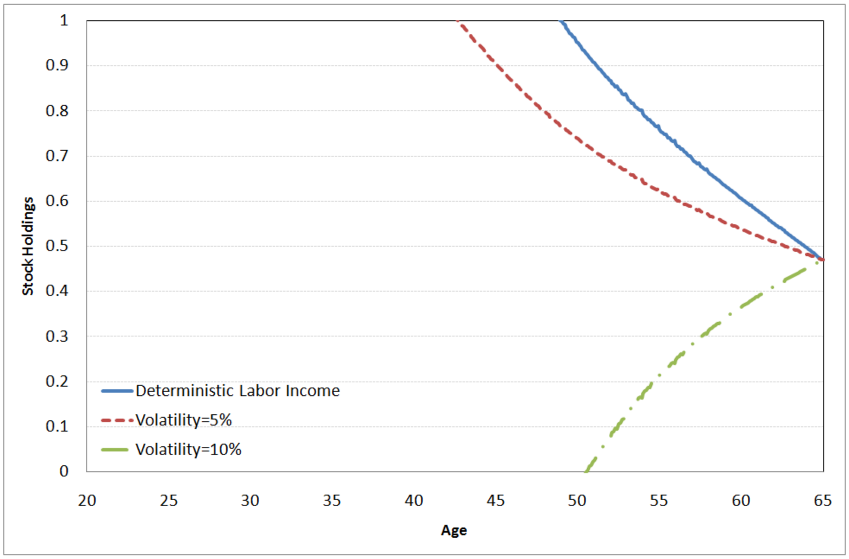

Figure 5. Optimal Stock Holdings.

When wages have equity exposure, the human asset is given some stock element. Thus, the optimal stock weight falls below 100% earlier than the case of riskless wages. When the value of σ_y is as large as 10%, the optimal portfolio strategy becomes an increasing function of age. The intuition for this result is that, when the future flow of labor income is adequately "stock-like", the young agent will find himself implicitly overexposed to market risk. Thus, it is optimal for him to take a short position in the risky asset. As he ages and accumulates his financial wealth, however, the human capital declines in relative importance. Then it becomes optimal for him to place a larger fraction of his financial wealth in the risky asset to maintain the optimal market exposure of his total portfolio. We note that the optimal stock holdings is a strictly decreasing or a strickly increasing function of age in this case. In other words, we cannot observe an hump-shaped stock holding profile because human capital is equally exposed to the market risk throughout lifetime.

Case 3: Labor Income with No Short-run Risk

For an individual, a wage is like a "money market mutual fund": the amount he is going to receive in one or two years is almost certain but the wages after many years develops a wide dispersion reflecting fluctuations in economic conditions. For the baseline case, we assume that the drift term of labor income growth is perfectly correlated with stock returns, while labor income growth itself is deterministic contemporaneously. In addition, $\mu_y(t)$ is linked to its average level $\alpha_\mu(t)$ by the parameter θ where, to generate the same income process presented in Figure 1, we set $\alpha_\mu(t) = 0.0443 - 0.0024t$, $\sigma_\mu = 0.0283$ and $\theta = 0.15$. We also consider the case of imperfect correlation. For that case, we use the dynamic programming approach to solve it numerically. The details are explained in the next section.

Using Ito's lemma, we can show that human capital is perfectly correlated with the stock market in this case, namely,

$$Corr\left(d\log Y(t), d\log S(t)\right) = 0 \tag{3.3}$$

and

$$Corr\left(d\log L(t), d\log S(t)\right) = 1.0 \tag{3.4}$$

The equity weight in human asset is given by $\Lambda\left(t, T, \mu_y(t)\right)\sigma_\mu\sigma^{-1}$. Figure 6 illustrates the value of human capital at each age, whereas Figure 7 depicts the components of human capital over the life-cycle.

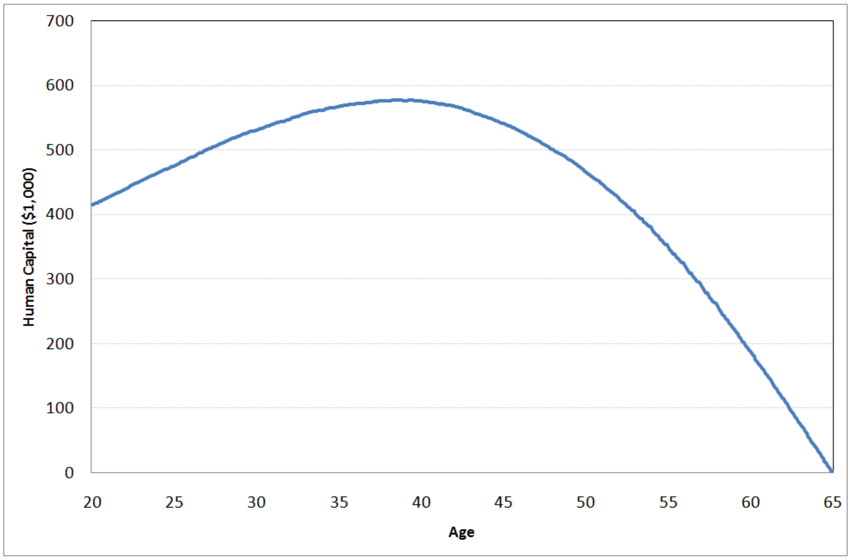

Figure 6. Human Capital.

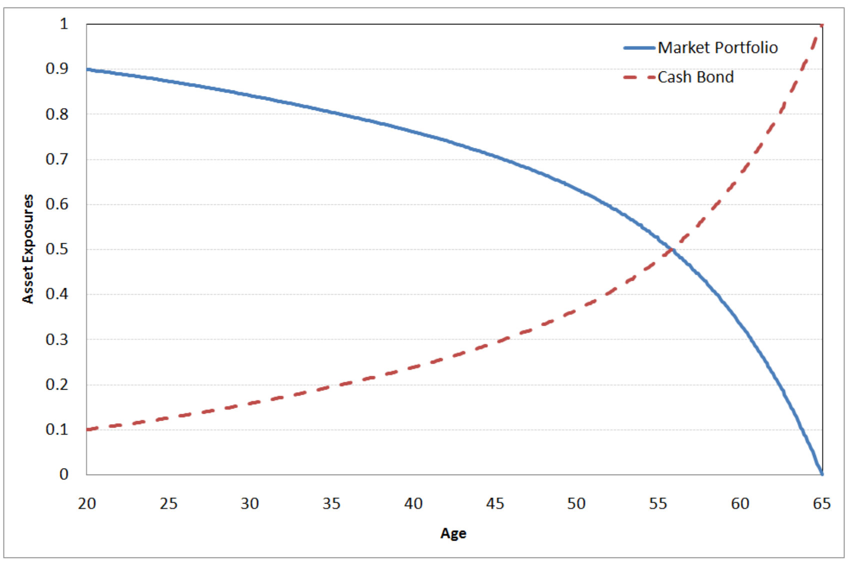

Figure 7. Asset Exposures of Human Capital.

The human capital is about 26 times the initial annual labor income at age-20, corresponding to the report of Bodie and Treussard (2006), and has a hump-shaped profile. The most distinct difference between our model and the one in Case 2 is found in the transition of the nature of human capital. We find that the fraction of the agent's human capital tied up in the market portfolio is more than 90% at $T = 0$ and then gradually decreases toward 0. Aside from the above, the fraction of cash-bond increases toward 100% as the agent approaches retirement. In other words, human capital becomes more "bond-like" as the investor ages while the correlation of stock and human-capital returns are 1.0 throughout his life. This is another reason which makes the hump of human capital larger than Case 2. That implies that when the agent is young, his human capital is mainly dominated by risky assets. Thus, early labor income flows have to be discounted at a higher rate reflecting the APT. As he ages, however, his human capital becomes bond-like, thus the income flows occuring at older ages have to be discounted at a lower rate. Due to this effect, human capital has a larger hump than the one in Case 2 and peaks at a later time point.

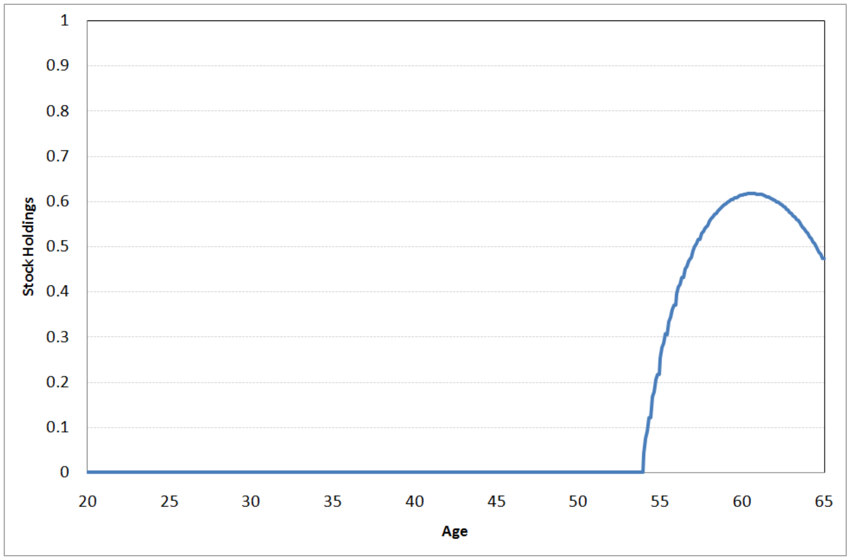

Figure 8. Optimal Stock Holdings.

Figure 8 illustrates the optimal stock holdings of this agent. We observe a hump-shaped optimal stock holding profile which insists that the young agent should not invest in risky assets. The reason for this phenomenon is that, for a young agent, most of his wealth is tied up in his future labor income. Thus, when he finds that his human capital is occupied mainly by risky assets (as shown in Figure 7), he will short the market portfolio to achieve optimal risk exposure. As he ages, however, his human capital becomes more "bond-like." Then, he has to invest more in stocks than the young agents. As Dufresne noted in their paper, when the investor approaches retirement, he will face two offsetting effect. First, as his human capital becomes more and more "bond-like," it becomes optimal for him to invest more in risky assets. Secondly, human capital itself decreases to 0, and therefore his implicit bond holdings through his human capital also decreases to 0. This effect drives him to invest more in risk-free assets. These two types of effect conflict with each other creating the hump-shaped stock holdings.

Asset Allocation vs. Mean-Reversion For Figure 9 and 10 we changed the mean-reversion level, θ, to 0.13 and 0.18.

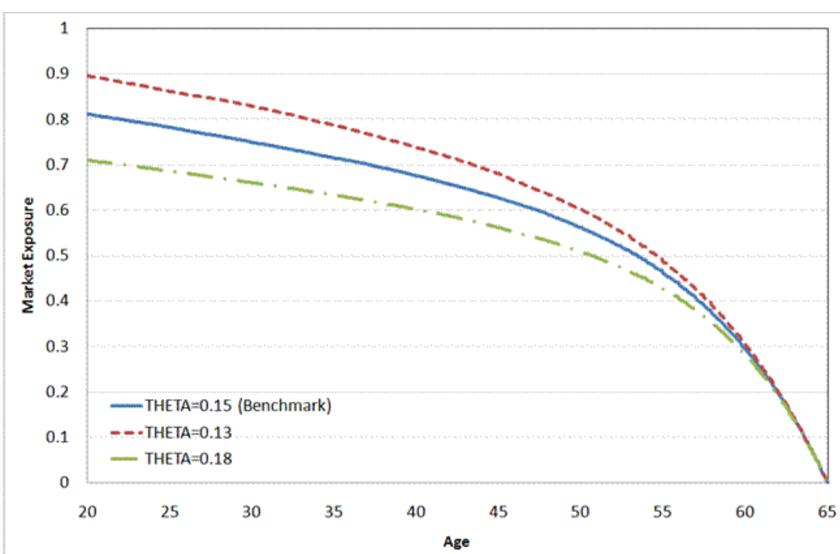

Figure 9. Market Exposure of Human Capital by θ.

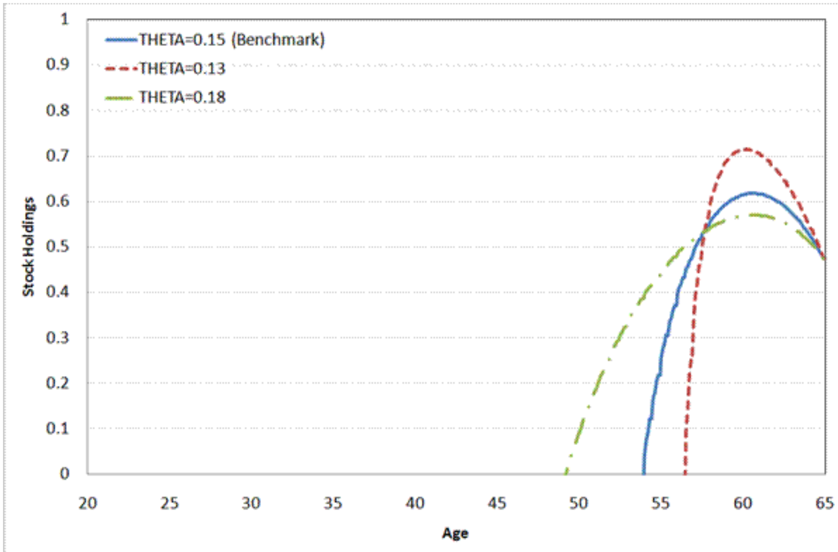

Figure 10. Optimal Stock Holdings by θ.

With higher speed of mean-reversion ($\theta = 0.18$), human capital shows more "bond-like" character because the labor income process is strongly pulled back to its average level. Thus, it is optimal for the agent to take a long position in stocks earlier than the baseline case. This result is actually obeservable in a mature economy where labor income is enough stable to drive young agents to participate in the stock market. On the contrary, people in a growing economy (like Japan in 1980's) will find their human capital is more attractive than stocks so that they refrain from holding stocks for a longer period. Interestingly, a larger value of θ relaxes the two offsetting effects noted above to make the hump smaller.

Asset Allocation vs. Risk Premium For Figure 11 and 12 we reduced the equity return, μ, to 5%.

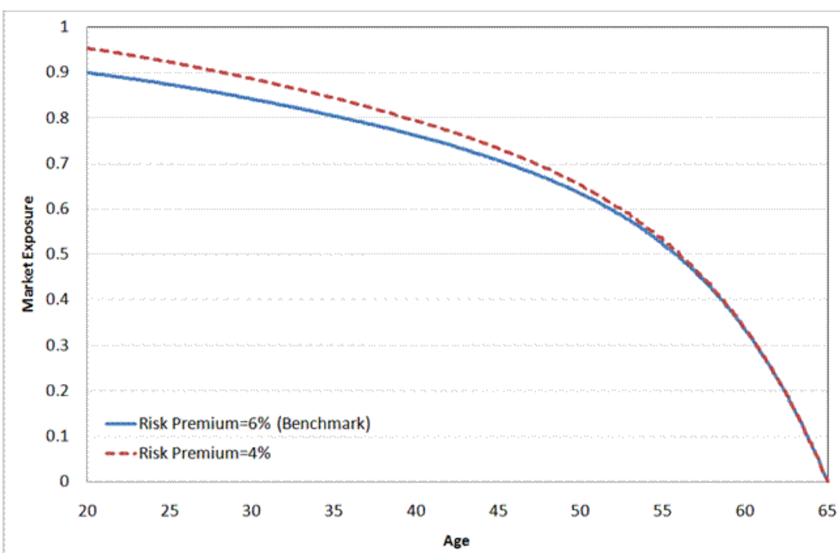

Figure 11. Market Exposure of Human Capital by μ.

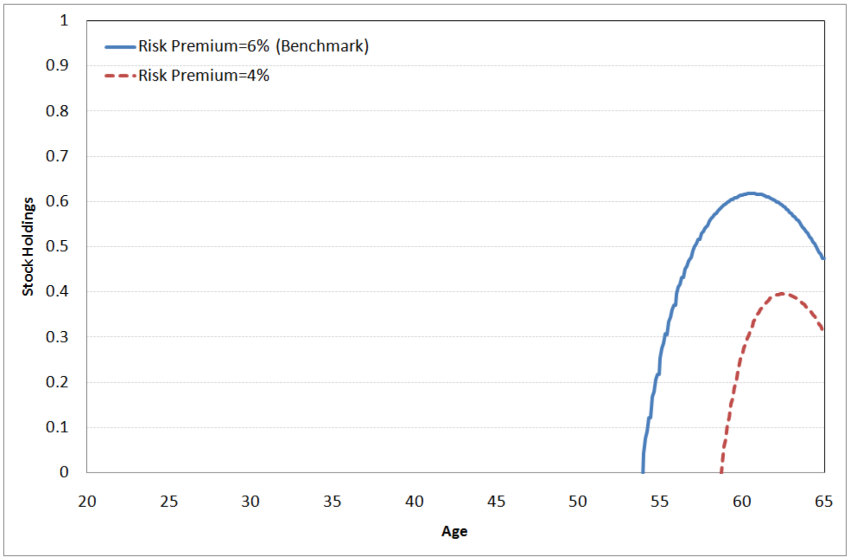

Figure 12. Optimal Stock Holdings by μ.

As can be seen, the value of risk premium affects the market exposure of human capital slightly. However, it heavily reduces the stock holdings at every age. Intuitively, an agent with worse investment opportunity will reduce the market exposure of his total porfolio because the risky asset becomes less attractive. Therefore, when the equity premium decreases to 4%, most agents find themselves implicitly overinvested in stocks through their human capital and therefore they will start holding stocks only when they are close to retirement and their human capital becomes sufficiently "bond-like."

Asset Allocation vs. Risk Aversion For Figure 13 we changed the level of relative risk aversion, γ, to 4 and 6.

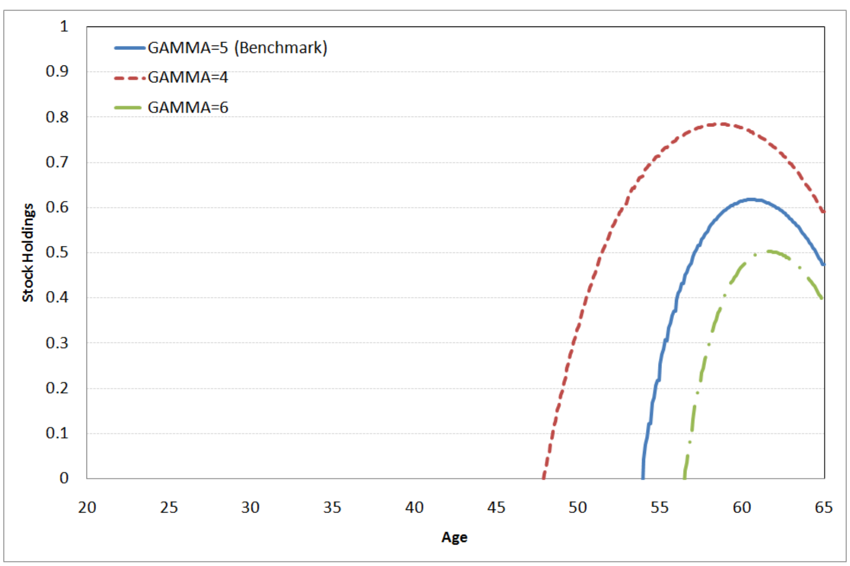

Figure 13. Optimal Stock Holdings by γ.

Changes in the risk aversion coefficient exercise no effect on the market exposure of human capital. Even though, a less risk-averse agent perceives that his human capital is not "stock-like" and thus he will start investment in stocks earlier. Contrarily, a more risk-averse agent finds that his human capital is too "stock-like" even as he approaches retirement, and will thus refrain from participating in the stock market for a longer period.

Asset Allocation vs. Time Preference For Figure 14 and 15 we increased the time preference coefficient, β, to 7%.

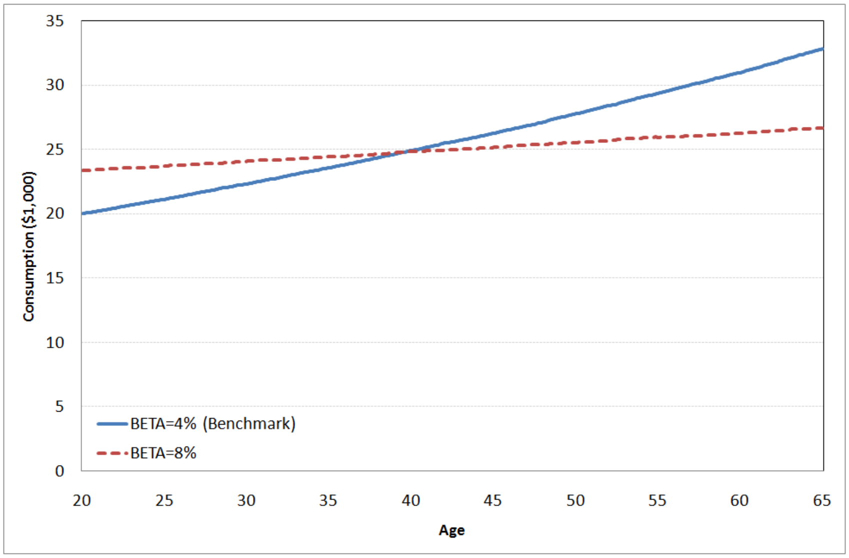

Figure 14. Optimal Consumption by β.

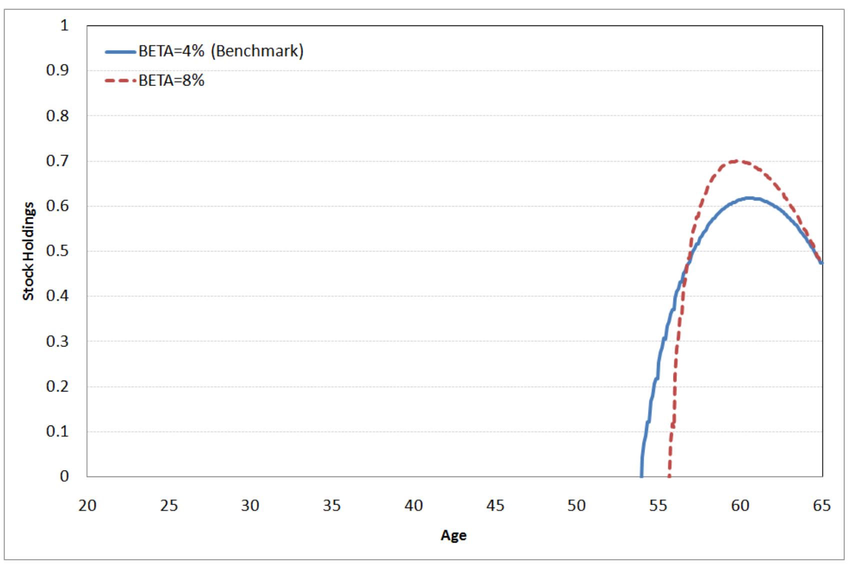

Figure 15. Optimal Stock Holdings by β.

It is generally agreed that the time preference parameter has negligible effects on the consumption profile. Figure 14 illustrates that the "grasshopper-type" agent consumes more in young days than the "ant-type" agent. That is, the agent with stronger time preference has to take a short position heavily in market portfolio to afford his consumption. Hence, his financial wealth grows slowly to reduce his consumption after middle age and postpone his participation in the stock market.

Asset Allocation vs. Bequest Motive Finally, for Figure 16 and 17 we changed the bequest motive coefficient, ε, to 12^5 and 1^5.

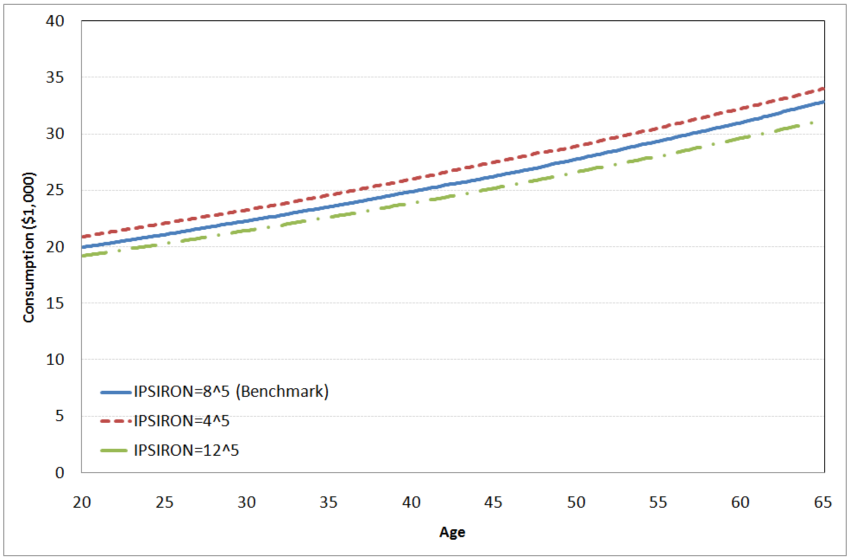

Figure 16. Optimal Consumption by ε.

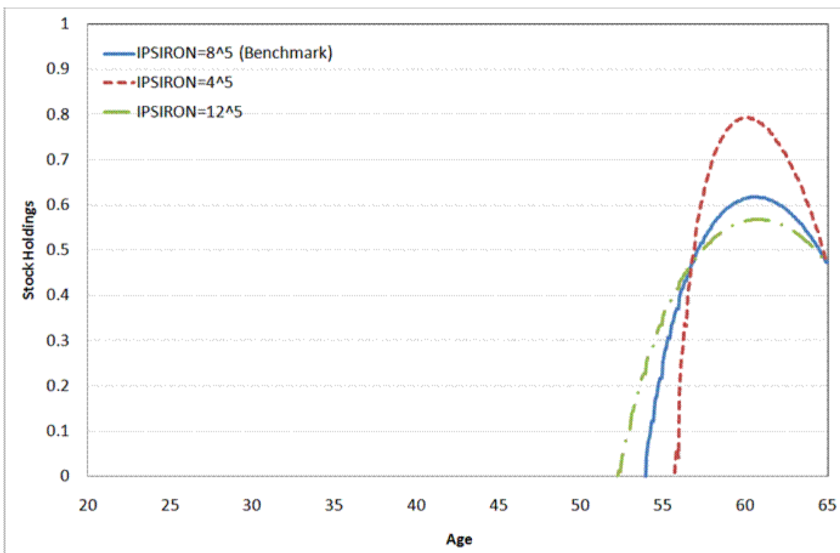

Figure 17. Optimal Stock Holdings by ε.

Contrary to the illustration in Figure 13, an agent with a weaker bequest motive will consume more throughout his working lifetime even if this disturbs the growth of his financial wealth because it is not important for him to accumulate wealth for his after-retirement life.

4 Optimal Consumption and Portfolio Choice in the Incomplete Market

In previous sections, we uses the Martingale method to derive the optimal consumption and portfolio policy. When the market is complete, we can find a unique pricing kernel which enables us to change the dynamic problem into a static one. Thus we can solve for consumption and investment policy directly. When the market is not complete, however, there exist infinitely many pricing kernels that are consistent both with no arbitrage conditions and with a family of static budget constraints that has to be considered in order to ensure the feasibility of solutions. Hence, as long as no judgement can be made to explain which pricing kernel is the "right" one to employ, it seems likely that using Martingale method makes it difficult to obtain the exact solution.

In this section, we add another idiosyncratic risk to our model of labor income processes. Since labor income is non-tradable, we cannot observe the market price of its idiosyncratic risk. Hence, we use the dynamic programming approach to solve the problem numerically. The details are presented below.

Consistent with the empirical observation, we assume that, besides market risk, there is another idyosyncratic risk in an individual's labor income. Namely,

$$dy\left(t\right)=\mu_{y}\left(t\right)dt \tag{4.1}$$

and

$$d\mu_{y}\left(t\right)=\theta\left(\alpha_{\mu}\left(t\right)-\mu_{y}\left(t\right)\right)dt+\sigma_{\mu}\left(\rho dw\left(t\right)+\sqrt{1-\rho^{2}}dw_{i}\left(t\right)\right) \tag{4.2}$$

where $w_i\left(t\right)$ is a standard Brownian motion independent of $w\left(t\right)$ representing the idiosyncratic shocks.

If the investor places a proportion $\pi\left(t\right)$ of his current financial wealth in the risky asset and $\left(1-\pi\left(t\right)\right)$ in the risky-free asset, then his wealth dynamics follow

$$dW\left(t\right)=-C\left(t\right)dt+\left(1-\pi\left(t\right)\right)W\left(t\right)\frac{dB\left(t\right)}{B\left(t\right)}+\pi\left(t\right)W\left(t\right)\frac{dS\left(t\right)}{S\left(t\right)}+Y\left(t\right)dt, \tag{4.3}$$

or

$$\frac{dW\left(t\right)}{W\left(t\right)}=\left(r+\pi\left(t\right)\left(\mu-r\right)+\frac{Y\left(t\right)}{W\left(t\right)}-\frac{C\left(t\right)}{W\left(t\right)}\right)dt+\pi\left(t\right)\sigma dz_{3}\left(t\right). \tag{4.4}$$

From (2.1) and (2.2), the objective function of this investor can be written as

$$J\left(t,Y\left(t\right),W\left(t\right),\mu_{y}\left(t\right)\right)\triangleq\max_{\{C,\pi\}}E_{t}\left[\int_{t}^{T}\exp\left(-\beta u\right)\frac{\left(C\left(u\right)\right)^{1-\gamma}}{1-\gamma}du+\varepsilon\exp\left(-\beta T\right)\frac{\left(W_{T}\right)^{1-\gamma}}{1-\gamma}\right]. \tag{4.5}$$

Then the HJB equation is

$$\begin{aligned}0 &= \exp\left(-\beta t\right)\frac{C^{1-\gamma}}{1-\gamma}+J_{t}+WJ_{W}\left(r+\pi\left(\mu-r\right)+\frac{Y}{W}-\frac{C}{W}\right)+J_{Y}\mu_{y}Y+J_{\mu_{y}}\theta\left(\alpha_{\mu}-\mu_{y}\right)\\ &\quad+\frac{1}{2}\left\{W^{2}J_{WW}\pi^{2}\sigma^{2}+J_{\mu_{y}\mu_{y}}\sigma_{\mu}^{2}+2WJ_{W\mu_{y}}\sigma_{\mu}\rho\pi\sigma\right\}.\end{aligned} \tag{4.6}$$

And the two first-order conditions are

$$C=\left(\exp\left(\beta t\right)J_{W}\right)^{-\frac{1}{\gamma}}, \tag{4.7}$$

$$\pi=-\frac{J_{W}}{WJ_{WW}}\frac{\left(\mu-r\right)}{\sigma^{2}}-\frac{J_{W\mu_{y}}}{WJ_{WW}}\frac{\sigma_{\mu}\rho}{\sigma}. \tag{4.8}$$

Like Dufresne , we use the scaling feature of CRRA utility functions to eliminate $W\left(t\right)$ from the state variables, namely

$$C\left(1,\frac{Y}{W},\mu_{y},t\right)=\frac{1}{W}C\left(W,Y,\mu_{y},t\right). \tag{4.9}$$

Defining $X\triangleq\frac{Y}{W}$, $c\left(X,\mu_{y},t\right)\triangleq\frac{C(W,L,y,t)}{W}$ and using (4.7), the optimal portfolio decision can be written in terms of $c\left(X,\mu_{y},t\right)$ as

$$\pi=\frac{c}{\left(c-Xc_{X}\right)}\frac{\mu-r}{\gamma\sigma^{2}}+\frac{c_{\mu_{y}}}{\left(c-Xc_{X}\right)}\frac{\sigma_{\mu}\rho}{\sigma}. \tag{4.10}$$

Further we first differentiate the HJB equation with respect to W and use $c(X, y, t)$ to rewrite it again, then we get the following equation;

$$\begin{aligned} 0 \;=\; & -c_t + \frac{c}{\gamma}\left(r + \pi(\mu - r) - \beta\right) - \left(r + \pi(\mu - r) + X - c + \sigma^2\pi^2\right)(c - Xc_X) - \mu_y X c_X \\ & + \sigma_\mu \rho \pi \sigma \left((\gamma - 1) c_{\mu_y} - (\gamma + 1) c^{-1} X c_X c_{\mu_y} + X c_{X\mu_y}\right) - \theta\left(\alpha_\mu - \mu_y\right) c_{\mu_y} \\ & + \frac{1}{2}\sigma^2\pi^2 \left\{(\gamma + 1) c^{-1} (c - Xc_X)^2 - X^2 c_{XX}\right\} \\ & + \frac{1}{2}\sigma_\mu^2 \left\{(\gamma + 1) c^{-1} c_{\mu_y}^2 - c_{\mu_y\mu_y}\right\}, \end{aligned} \tag{4.11}$$

and the terminal condition is

$$c\left(X, \mu_y, T\right) = \varepsilon^{-\frac{1}{\gamma}} \quad \forall \left(X, \mu_y\right) \tag{4.12}$$

We use the alternative direction implicit (ADI) finite-difference method to solve this problem backward from its terminal condition. The details are almost the same with Dufresne et al.(2007), but we set the upper bound of $\mu_y(t)$ at $\mu_{y\,\max} = \mu_y(0) + 2\sigma_\mu$ and the lower bound at $\mu_{y\,\min} = \mu_y(0) - 4\sigma_\mu$ where $\mu_y(0) = 0.0473$. We construct the μ_y-grid with a $\triangle\mu_y = 0.005$ mesh. We simulate $100,000$ paths for W, Y, X and μ_y and take the average. The results are presented in Figure 18.

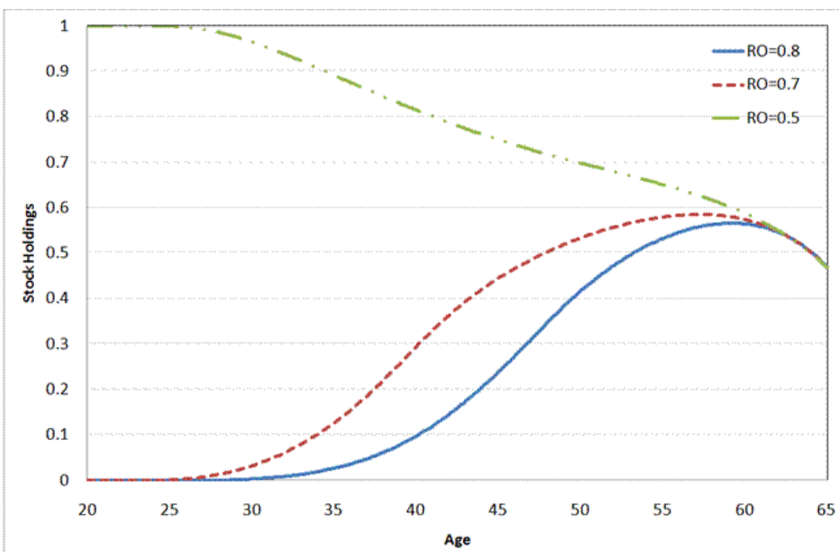

Figure 18. Optimal Stock Holdings in Incomplete Market.

Imposition of the short-sale constraints induces gradual change of the stock holdings. When the labor income growth is adequately correlated with market returns, we can still observe a hump-shaped life-cycle investment strategy. However, when the value of ρ becomes as low as 0.5, the agent will find that his human capital is too "bond-like." Hence, for a young agent whose human capital dominates his total wealth, investing more in the stock market is optimal.

5 The Model of Cointegrated Labor Income (The Model of Benzoni, Collin-Dufresne, Goldstein(2007))

Benzoni, Collin-Dufresne, Goldstein (2007) studied the effect of cointegration between labor income and dividends on the optimal portfolio choice. They specified the aggregate labor income to be cointegrated with dividends and showed that this cointegration can explain why younger wage earners do not hold risky assets. It is widely-known that closed-form solutions for the life-cycle portfolio choice problem are generally unavailable. However we emphaticaly state that even if the market is incomplete, using Martingale method still enables us to achieve meaningful economic implications. Here, we derive the analytical solution for the model of Dufresne and simulate $10,000$ paths again. The details of calculations are available in Appendix 2. When we assume that the two portfolios which enabling us to perfectly hedge against the idiosyncratic risks $dz_1(t)$ and $dz_{2,i}(t)$ have the same market price of risk with the market portfolio, namely, $\kappa_1 = \kappa_2 = \kappa_3$, the analytical solution is approximated to the numerical one.[1] The results are illustrated in Figure 19.

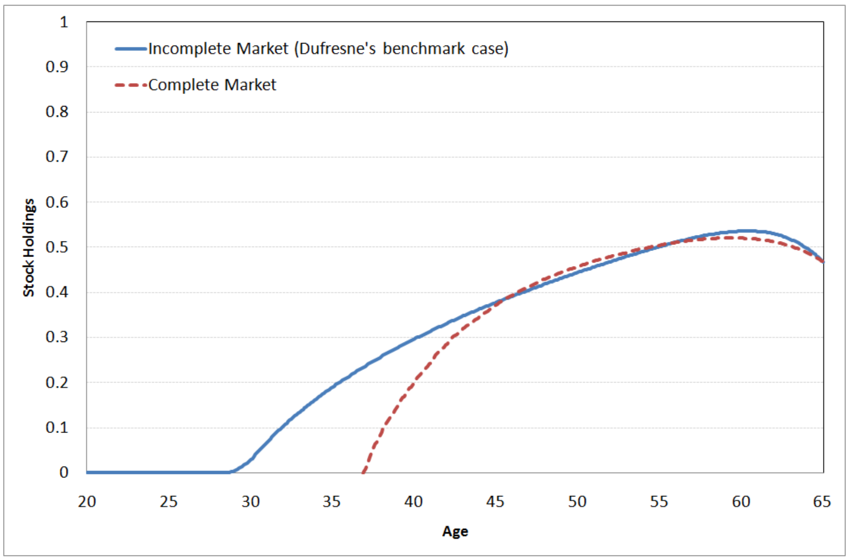

Figure 19. Optimal Stock Holdings.

It is possibles for these two results to be more similar to each other if we relax the short-sale constraint imposed in simulation. Furthermore, using the Martingale method makes it possible for the components of human capital over the life-cycle to be observable. Those are presented in Figure 20.

[1] We found that the original paper had some errors in the HJB equation: lack of $\pi(\mu - r)$ in the second term, $\sigma^2\pi^2$ in the third term of first line, it is not $(1+\gamma)$ but γ in line 3, and we have to double c_y in line 5. The correct version should read:

$$
\begin{aligned}
0 \;=\; & -c_t + \frac{1}{\gamma}\left(r + \pi(\mu - r) - \psi\right) - \left(r + \pi(\mu - r) + X - c + \sigma^2\pi^2 + \beta^2\right)(c - Xc_X) \\
& -\frac{1}{2}\left(\sigma^2\pi^2 + \beta^2\right)\left\{X^2 c_{XX} - (\gamma+1)c^{-1}(c - Xc_X)^2\right\} + \pi\sigma(\sigma - \nu_3)\gamma X c_X \\
& -Xc_X\left(-\kappa y + g_D - \frac{\sigma^2}{2} + \alpha(t) + \frac{\nu_1^2}{2} + \frac{(\sigma - \nu_3)^2}{2}\right) - \frac{1}{2}\left(\nu_1^2 + \nu_3^2\right)\left\{c_{yy} - (\gamma+1)c^{-1}c_y^2\right\} \\
& -\frac{1}{2}\left\{\nu_1^2 + \nu_2^2 + (\sigma - \nu_3)^2 - 2\pi\sigma(\sigma - \nu_3)\right\}\left\{X^2 c_{XX} - (\gamma+1)c^{-1}X^2 c_X^2\right\} + \varkappa\xi c_\xi \\
& +\nu_3\sigma\pi\left\{2c_y - Xc_{Xy} - (\gamma+1)c^{-1}c_y(c - Xc_X)\right\} \\
& -\left(\nu_1^2 - \nu_3(\sigma - \nu_3)\right)\left\{Xc_{Xy} - (\gamma+1)c^{-1}Xc_Xc_y\right\}
\end{aligned}
$$

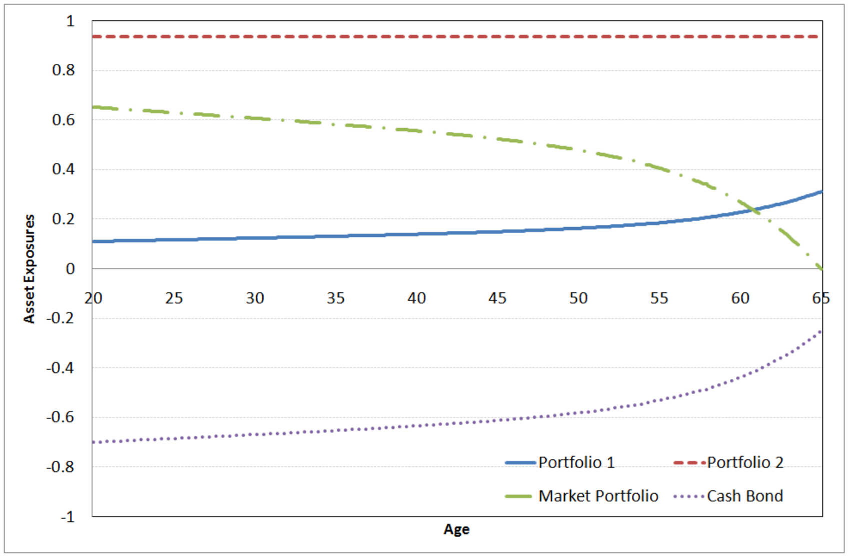

Figure 20. Asset Exposures of Human Capital.

As can be deduced from the figure, the human capital in this model is almost equivalent to a long position in the market portfolio and portfolio 2 that is hedged with $dz_{2,i}(t)$. Meanwhile the portion of market portfolio decreases as the investor ages and portfolio 2 keeps a 93.75% fraction throughout the life-cycle. Hence, we conclude that the position in the market portfolio implicit in the agent's human capital is the main factor that explains his hump-shaped stock holdings. More interestingly, unlike our model, the agent still shorts bond implicitly even as he approaches retirement.

6 Conclusion

In 1969, Merton established a model which implied that investors should optimally maintain the "constant-mix" strategy throughout their lifetime, pointing out simultaneously the importance of human capital. From his pioneering work onward, the optimal asset allocation problem has received considerable attention in Economics literature. In 2007, Benzoni, Collin-Dufresne and Goldstein showed that when labor income and dividends are cointegrated, young agents should not invest in stocks – a result that is consistent with empirical observations. In this paper, we solved the dynamic asset allocation problem analytically with the Martingale method, which freed us from the difficult task of solving the complex HJB equation. We find Merton's constant-mix strategy is still useful from the perspective of "total portfolio" management. We also find that we can observe hump-shaped stock holdings, similar to the result of Dufresne et al. (2007) when we consider the labor income growth follows a mean-reverting process and is adequately correlated with market returns.

This study offers several possible implications for future research. One would be to extend our model to the case of an incomplete market. In such a case, the pricing kernel is no more unique and the Martingale method is no longer useful as in the complete market case. Thus, relaxing the assumption of the market's completeness compromises our ability to obtain analytical solutions. Cuoco (1997) examined the intertemporal optimal consumption and investment problem in the presence of a stochastic endowment and constraints on the portfolio choices including incomplete markets. He proved the existence of an optimal consumption and investment policy in incomplete market and showed that even when the market is incomplete, we still can find a "right" pricing kernel that allows us to solve our optimizing problem. That is to say, once the kernel is found, we can fictitiously assume the market to be complete and solve the problem using the same method as in the complete market case. Although finding the "right" one among infinitely many candidates is problematic, we believe that this method might offer some promoting advances.

Another direction for research might be a calibration of the model using real data. Within the academic debate about whether human capital is stock-like or bond-like in behavior, varying research has offered different conclusions. Our theoretical results suggest that human capital is stock-like when the agent is young, and then becomes bond-like as he ages. Thus, one possible suggestion is to test this theory using empirical data.

Obviously, our last theme is to include life insurance and housing. Japan, in particular, offers an interesting case whereby life insurance represents more than 30% of households' financial wealth (and is the second largest asset class after savings) and housing is a necessary investment for many families. Examining the data from these cases could make our model more suitable for practical applications.

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Appendix1

Here we want to show that if (c, W) is financed by an adimissible pair of (α, θ), then the cost of $\{c(s); s \in [t, T]\}$ and W at time t is

$$\alpha(t) B(t) + \theta(t)^T S(t) = B(t) E^Q \left[\int_t^T \frac{c(s)}{B(s)} ds - \int_t^T \frac{Y(s)}{B(s)} ds + \frac{W}{B(T)} \mid \mathcal{F}_t \right]$$

Proof. If (c, W) is financed by an admissible pair of (α, θ), letting $W(t) \triangleq \alpha(t) B(t) + \theta(t)^T S(t)$, we have from (2.4)

$$dW(t) + c(t)\,dt - Y(t)\,dt = \alpha(t) B(t) r(t)\,dt + \theta(t)^T \mu(t)\,dt + \theta(t)^T \sigma(t)\,dw(t) \tag{A1.1}$$

Using Ito's lemma, we have

$$\begin{aligned} d\left(\frac{W(t)}{B(t)}\right) &= \frac{dW(t)}{B(t)} - \frac{W(t)\,dB(t)}{(B(t))^2} \\ &= \frac{dW(t) - r(t) W(t)\,dt}{B(t)} \\ &= \frac{-c(t)\,dt + Y(t)\,dt + \alpha(t) B(t) r(t)\,dt + \theta(t)^T \mu(t)\,dt + \theta(t)^T \sigma(t)\,dw(t)}{B(t)} \\ &\quad \frac{-r(t)\left[\alpha(t) B(t) + \theta(t)^T S(t)\right] dt}{B(t)} \\ &= \frac{-c(t)\,dt + Y(t)\,dt + \theta(t)^T \mu(t)\,dt + \theta(t)^T \sigma(t)\left[dw^*(t) - \kappa(t)\,dt\right] - r(t)\theta(t)^T S(t)\,dt}{B(t)} \\ &= \frac{-c(t)\,dt + Y(t)\,dt + \theta(t)^T \sigma(t)\,dw^*(t)}{B(t)} \end{aligned} \tag{A1.2}$$

Integration yeilds

$$\frac{W(T)}{B(T)} = \frac{W(t)}{B(t)} - \int_t^T \frac{c(s)}{B(s)}ds + \int_t^T \frac{Y(s)}{B(s)}ds + \int_t^T \frac{\theta(s)^T \sigma(s)\,dw^*(s)}{B(s)}ds \tag{A1.3}$$

Since $W(T) = W$ and $W(t) = \alpha(t) B(t) + \theta(t)^T S(t)$, we can rewrite this relation as

$$\frac{\alpha(t) B(t) + \theta(t)^T S(t)}{B(t)} = \int_t^T \frac{c(s)}{B(s)}ds - \int_t^T \frac{Y(s)}{B(s)}ds + \frac{W}{B(T)} - \int_t^T \frac{\theta(s)^T \sigma(s)\,dw^*(s)}{B(s)}ds, \forall t \in [0, T], a.s. \tag{A1.4}$$

Taking expectation conditional on $\mathcal{F}_t$ under Q, we get

$$\frac{\alpha(t) B(t) + \theta(t)^T S(t)}{B(t)} = E^Q\left[\int_t^T \frac{c(s)}{B(s)}ds - \int_t^T \frac{Y(s)}{B(s)}ds + \frac{W}{B(T)} \mid \mathcal{F}_t\right] \tag{A1.5}$$

or

$$\alpha(t) B(t) + \theta(t)^T S(t) = B(t) E^Q\left[\int_t^T \frac{c(s)}{B(s)}ds - \int_t^T \frac{Y(s)}{B(s)}ds + \frac{W}{B(T)} \mid \mathcal{F}_t\right] \tag{A1.6}$$

■

Appendix 2

Here, we describe the details of the model of Dufresne et al. (2007) and then derive the analytical solution for their model by using the martingale method and by assuming the market is complete.[2]

Let $y_1(t) \triangleq \log Y_1(t)$ fbe denoted for the aggregate income associated with the investor's career choice and $y_2(t) \triangleq \log Y_2(t)$ for the idiosyncratic income shocks. Let $\widehat{d}(t) \triangleq \log D(t)$ denotes the log dividend where

$$d\widehat{d}(t) = \left(g_D - \frac{\sigma^2}{2}\right)dt + \sigma dz_3 \tag{A2.1}$$

[2] To prevent overlapping, we changed some notations. We use $\{\beta, y, \xi, \varkappa, \varphi, \varepsilon\}$ for $\{\psi, l, y, \kappa, \beta, \epsilon^\gamma\}$ in their original paper, respectively.

The log stock price (cum dividend) process $s(t) \triangleq \log S(t)$ follows

$$ds(t) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz_3 \tag{A2.2}$$

Thus we have $d\widehat{d}(t) - ds(t) = (g_D - \mu)\, dt$. Assume that

$$\xi(t) \triangleq y_1(t) - \widehat{d}(t) - \overline{y}_d \tag{A2.3}$$

is a stationary mean-reverting process,

$$d\xi(t) = -\varkappa\xi(t)\, dt + \nu_1 dz_1(t) - \nu_3 dz_3(t) \tag{A2.4}$$

where $\overline{y}_d$ is the mean ratio of $\log Y_1(t)$ to $\log D(t)$. Thus, $y_1(t)$ follows

$$\begin{aligned} dy_1(t) &= d\xi(t) + d\widehat{d}(t) \\ &= \left(-\varkappa\xi(t) + g_D - \frac{\sigma^2}{2}\right) dt + \nu_1 dz_1(t) + (\sigma - \nu_3)\, dz_3(t) \end{aligned} \tag{A2.5}$$

They also specify that

$$dy_2(t) = \left(\alpha(t) - \frac{\nu_2^2}{2}\right) dt + \nu_2 dz_{2,i}(t) \tag{A2.6}$$

Combining the above, the log income $y(t) \triangleq y_1(t) + y_2(t)$ follows

$$dy(t) = \left(-\varkappa\xi(t) + g_D - \frac{\sigma^2}{2} + \alpha(t) - \frac{\nu_2^2}{2}\right) dt + \nu_1 dz_1(t) + \nu_2 dz_{2,i}(t) + (\sigma - \nu_3)\, dz_3(t) \tag{A2.7}$$

Note that $z_1(t)$, $z_{2,i}(t)$ and $z_3(t)$ are independent of each other.

If we assume the market is complete, we can also find the analytical solution to this model. As we did in Case 3, firstly we derive the process of $y(t)$. From (A2.4) and (A2.7), we obtain

$$\begin{aligned} y(s) &= y(t) - \xi(t)\{1 - \exp(-\varkappa(s-t))\} - \nu_1 \int_t^s \{1 - \exp(-\varkappa(s-v))\}\, dz_1(v) \\ &\quad + \nu_3 \int_t^s \{1 - \exp(-\varkappa(s-v))\}\, dz_3(v) + \left(g_D - \frac{\sigma^2}{2} - \frac{\nu_2^2}{2}\right)(s-t) + \int_t^s \alpha(u)\, du \\ &\quad + \nu_1 \int_t^s dz_1(v) + \nu_2 \int_t^s dz_{2,i}(v) + (\sigma - \nu_3) \int_t^s dz_3(v) \end{aligned} \tag{A2.8}$$

This implies

$$\begin{aligned} Y(s) &= Y(t) \exp\Bigg[-\xi(t)\{1 - \exp(-\varkappa(s-t))\} - \nu_1 \int_t^s \{1 - \exp(-\varkappa(s-v))\}\, dz_1(v) \\ &\quad + \nu_3 \int_t^s \{1 - \exp(-\varkappa(s-v))\}\, dz_3(v) + \left(g_D - \frac{\sigma^2}{2} - \frac{\nu_2^2}{2}\right)(s-t) + \int_t^s \alpha(u)\, du \\ &\quad + \nu_1 \int_t^s dz_1(v) + \nu_2 \int_t^s dz_{2,i}(v) + (\sigma - \nu_3) \int_t^s dz_3(v)\Bigg] \end{aligned} \tag{A2.9}$$

Suppose the pricing kernel of this case is given by

$$\begin{aligned} \frac{m(s)}{m(t)} &= \exp\left\{-\left(r + \frac{1}{2}\left(\kappa_1^2 + \kappa_2^2 + \kappa_3^2\right)\right)(s-t)\right\} \\ &\quad \times \exp\left\{-\kappa_1 \int_t^s dz_1(v) - \kappa_2 \int_t^s dz_{2,i}(v) - \kappa_3 \int_t^s dz_3(v)\right\} \end{aligned} \tag{A2.10}$$

the human capital at time t is

$$\begin{aligned} L(t) &= Y(t)\int_t^T \exp\left\{-\left(r+\frac{1}{2}\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)\right)(s-t)\right\} \\ &\times\exp\left\{-\xi(t)\{1-\exp(-\varkappa(s-t))\}+\left(g_D-\frac{\sigma^2}{2}-\frac{\nu_2^2}{2}\right)(s-t)+\int_t^s \alpha(u)\,du\right\} \\ &\times E_t\left[\int_t^s \exp\{-\kappa_1+\nu_1\exp(-\varkappa(s-v))\}\,dz_1(v)\right] \\ &\times E_t\left[\int_t^s \exp\{-\kappa_2+\nu_2\}\,dz_{2,i}(v)\right] \\ &\times E_t\left[\int_t^s \exp\{\sigma-\kappa_3+\nu_3\exp(-\varkappa(s-v))\}\,dz_3(v)\right]ds. \end{aligned} \tag{A2.11}$$

By rewriting the last three expectation terms, we finally obtain

$$\begin{aligned} L(t) &= Y(t)\int_t^T \exp\left\{-\left(r+\frac{1}{2}\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)\right)(s-t)\right\} \\ &\times\exp\left\{-\xi(t)\{1-\exp(-\varkappa(s-t))\}+\left(g_D-\frac{\sigma^2}{2}-\frac{\nu_2^2}{2}\right)(s-t)+\int_t^s \alpha(u)\,du\right\} \\ &\times\exp\left\{\frac{1}{2}\int_t^s \{-\kappa_1+\nu_1\exp(-\varkappa(s-v))\}^2\,dv\right\} \\ &\times\exp\left\{\frac{1}{2}\int_t^s \{-\kappa_2+\nu_2\}^2\,dv\right\} \\ &\times\exp\left\{\frac{1}{2}\int_t^s \{\sigma-\kappa_3+\nu_3\exp(-\varkappa(s-v))\}^2\,dv\right\}ds. \end{aligned} \tag{A2.12}$$

Define

$$R(\xi(t),t) \triangleq \int_t^T a(t,s)\exp\left[-\xi(t)\{1-\exp(-\varkappa(s-t))\}\right]ds \tag{A2.13}$$

where

$$\begin{aligned} a(t,s) &\triangleq \exp\left\{-\left(r+\frac{1}{2}\left(\kappa_1^2+\kappa_2^2+\kappa_3^2\right)-g_D+\frac{\sigma^2}{2}+\frac{\nu_2^2}{2}\right)(s-t)+\int_t^s \alpha(u)\,du\right\} \\ &\times\exp\left\{\frac{1}{2}\int_t^s \{-\kappa_1+\nu_1\exp(-\varkappa(s-v))\}^2\,dv+\frac{1}{2}\int_t^s \{-\kappa_2+\nu_2\}^2\,dv\right\} \\ &\times\exp\left\{\frac{1}{2}\int_t^s \{\sigma-\kappa_3+\nu_3\exp(-\varkappa(s-v))\}^2\,dv\right\}ds, \end{aligned} \tag{A2.14}$$

$L(t)$ can be rewritten as

$$L(t) = Y(t)\,R(\xi(t),t). \tag{A2.15}$$

To specify the optimal investment strategy, we have to know the volatility of human capital $\sigma_L(t)$ again. As we did above, we devide the change of $\log L(t)$ into two parts,

$$d\log L(t) = d\log Y(t) + d\log R(\xi(t),t). \tag{A2.16}$$

From (A2.7) we can deduce that the log-volatility of $Y(t)$ is given by $\nu_1 dz_1(t)+\nu_2 dz_{2,i}(t)+(\sigma-\nu_3)\,dz_3(t)$. Now we have to calculate the log-volatility of $R(\xi(t),t)$. Appling Ito's lemma,

$$dR(\xi(t),t) = \left[\frac{\partial R(\xi(t),t)}{\partial t}\right]dt + \left[\frac{\partial R(\xi(t),t)}{\partial \xi(t)}\right]d\xi(t). \tag{A2.17}$$

From (A2.4) and (A2.13),

$$dR(\xi(t),t) = (\cdot)\,dt - \left[\int_t^T a(t,s)\{1-\exp(-\varkappa(s-t))\}\exp[-\xi(t)\{1-\exp(-\varkappa(s-t))\}]\,ds\right](\nu_1 dz_1(t) - \nu_3 dz_3(t)). \tag{A2.18}$$

Thus we obtain

$$dl(t) = (\cdot)\,dt + \nu_1 dz_1(t) + \nu_2 dz_{2,i}(t) + (\sigma-\nu_3)\,dz_3(t) - \Lambda(t,T,\xi(t))(\nu_1 dz_1(t) - \nu_3 dz_3(t)) \tag{A2.19}$$

where

$$\Lambda(t,T,\xi(t)) \triangleq \frac{\int_t^T a(t,s)\{1-\exp(-\varkappa(s-t))\}\exp[-\xi(t)\{1-\exp(-\varkappa(s-t))\}]\,ds}{\int_t^T a(t,s)\exp[-\xi(t)\{1-\exp(-\varkappa(s-t))\}]\,ds} \tag{A2.20}$$

and $l(t) \triangleq \log L(t)$. And thus the contemporaneous correlation between human capital and stock return is

$$\frac{cov(ds(t),dl(t))}{\sigma(ds(t))\,\sigma(dl(t))} = \frac{[(\sigma-\nu_3)+\nu_3\Lambda]}{\sqrt{\nu_1^2(1-\Lambda)^2+\nu_2^2+[(\sigma-\nu_3)+\nu_3\Lambda]^2}} \tag{A2.21}$$

Suppose that we can find another two portfolio of stocks, as well as the market portfolio, each of the two enables us to perfectly hedge against $dz_1(t)$ and $dz_{2,i}(t)$ respectively. Namely,

$$\begin{cases} dS_1(t) = (\mu_1 dt + \sigma_1 dz_1(t))\,S_1(t) \\ dS_2(t) = (\mu_2 dt + \sigma_2 dz_{2,i}(t))\,S_2(t) \\ dS(t) = (\mu dt + \sigma dz_3(t))\,S(t) \end{cases}. \tag{A2.22}$$

Besides the fictitiousness of these stock portfolios, if we accept the restrictive assumption that returns of these portfolios are mutually independent, that is,

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu \end{pmatrix} \quad \sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma \end{pmatrix}, \tag{A2.23}$$

then the "total" optimal dollar investment of this investor is

$$\begin{pmatrix} \frac{(\mu_1-r)}{\gamma\sigma_1^2} \\ \frac{(\mu_2-r)}{\gamma\sigma_2^2} \\ \frac{(\mu-r)}{\gamma\sigma^2} \end{pmatrix}(W^*(t)+L(t)). \tag{A2.24}$$

If we assume that the stock portfolios which hedged against the labor income shocks $dz_1(t)$, $dz_{2,i}(t)$ have no risk premium, investment to S_1 and S_2 must be zero. In this case the optimal total dollar investment to each stock portfolio is

$$\begin{pmatrix} 0 \\ 0 \\ \frac{(\mu-r)}{\gamma\sigma^2}(W^*(t)+L(t)) \end{pmatrix} \tag{A2.25}$$

Note that this is not the optimal amount of financial wealth to invest practically because it is also including the "implicit" part of investment embedded in human capital, $\sigma_L(t)\,\sigma^{-1}$, which in this case is

$$\begin{pmatrix} \nu_1(1-\Lambda(t))\,L(t)/\sigma_1 \\ \nu_2 L(t)/\sigma_2 \\ (\sigma-\nu_3+\nu_3\Lambda(t))\,L(t)/\sigma \end{pmatrix}. \tag{A2.26}$$

Thus, the time-t optimal dollar investment from financial wealth is tha balance between (A2.24) and (A2.26).