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# Effects of Reputation in Bubbles and Crashes

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## Abstract

We analyze the stock market by modeling it as a timing game among arbitrageurs for beating the gun. We assume that (1) arbitrageurs are behavioral with a small probability, (2) the bubble soft-lands, and (3) the postcrash price increases as the X-day is postponed. Due to these assumptions, the effect of reputation assumes importance because any rational arbitrageur is willing to build a reputation in order to ride the bubble. It is demonstrated that the bubble persists for a long period as an outcome of a unique symmetric Nash equilibrium, even if all arbitrageurs are almost certainly rational.



**Keywords:** Bubbles and Crashes, Timing Games, Soft-Landing, Behavioral Finance, Reputation, Self-Control.

**JEL Classification Numbers:** C72, C73, D84, G12, G14.

## 1. Introduction

This paper demonstrates the theoretical foundation underlying the willingness of rational arbitrageurs to develop *reputations* to ride the *bubble* in the stock market, instead of competing with each other to beat the gun at the earliest. We modeled the stock market as a *timing game* among arbitrageurs. Our model is inspired by Abreu and Brunnermeier (2003); however, it does not require their key assumptions such as sequential awareness and coordinated attacks.[3] Instead, we assume that arbitrageurs are *behavioral* with a small probability, in that they are subject to momentum trading just like amateur behavioral traders. We also assume that the bubble *soft-lands* after it *crashes*, and that the postcrash price increases as the X-Day is postponed. Based on these assumptions, we can apply the basic concept of the reputation theory explored by Kreps, Milgrom, Roberts, and Wilson (1982) in the finitely repeated prisoners' dilemma[4] to the stock market. According to the effect of the arbitrageurs' reputation, the bubble can persist for a long period, even if arbitrageurs are *almost certainly* rational.

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[3] See also Abreu and Brunnermeier (2002) and Brunnermeier and Morgan (2006).
[4] For general surveys on the reputation theory, see Fudenberg and Tirole (1991, Chapter 9) and Mailath and Samuelson (2006, Part IV).

The efficient market hypothesis asserts that by reflecting all relevant information, the stock price is always adjusted to its fundamental value (for instance, see Fama (1970, 1991)). However, even though it is a cornerstone of modern financial theory, the efficient market hypothesis is highly controversial. There are considerable empirical evidences that contradict this hypothesis: the stock price sometimes increases beyond the fundamental value, and continues to increase until it goes into a free fall. In other words, the bubble sometimes develops, persists, and then suddenly crashes. Advocates of behavioral finance, such as Shleifer (2000) and Shiller (2000), argued that the bubble is driven by behavioral traders; they incorrectly believe that the stock price will increase at a high growth rate in perpetuity. The efficient market hypothesis, on the other hand, claims that rational arbitrageurs quickly undo this mispricing; their selling pressure dampens the enthusiasm of these traders, immediately bursting the bubble.

In contrast to this ideal of rational arbitrageurs, actual professional arbitrageurs, who are mostly considered to be rational, generally do not think that the best strategy is to undo mispricing quickly. Instead, they would like to ride the bubble and sell out just prior to the X-Day on which they anticipate the bubble to crash. On the basis of historical facts and experiences, several authors such as Kindleberger (1978) and Soros (1994) emphasized the notion of *self-feeding* bubbles. In their view, speculative price movements involve multiple professional arbitrageurs who continuously drive the stock price up and then sell out at the top to the behavioral traders, who, in turn, sell out at the bottom.

However, we disagree with this view because arbitrageurs may compete with each other to beat the gun at the earliest; this phenomenon, along with the backward induction method, prevents a bubble from persisting. Hence, in order for the notion of self-feeding bubbles to be convincing, we need to demonstrate a theoretical foundation based on which each rational arbitrageur is willing to terminate this chain reaction of competition and develop a reputation in order to ride the bubble for a long period.

Based on these arguments, the present paper models the stock market as a timing game among arbitrageurs. The stock market operates during the continuous time interval $[0, \tau_0]$, and each arbitrageur selects a timing at or before the terminal time $\tau_0 > 0$ to exit the market by selling her/his share. As long as no arbitrageur has sold, the bubble continues to be driven by the behavioral traders. Once any arbitrageur sells, the bubble crashes and the stock price decreases drastically. Any arbitrageur who fails to keep up with this crash cannot enjoy the benefit of bubbles as well as the winner can.

We assume that every arbitrageur is not necessarily rational; there is a small probability that she/he is behavioral like the behavioral traders, and never bursts the bubble of her/his own accord. Moreover, we assume that the bubble soft-lands after it crashes, i.e., the stock price does not immediately decline to its fundamental value. Hence, even a rational arbitrageur who fails to keep up with this crash can sell at the postcrash price that is still greater than the fundamental value. We also assume that the postcrash price increases exponentially as the X-Day is postponed.

Based on these assumptions, we observe that the reputation of the arbitrageur facilitates the persistence of bubbles. On witnessing the persistence of a bubble, each rational arbitrageur is increasingly convinced that the other arbitrageurs are behavioral, which incentivizes her/him to further postpone timing the market. With a minor restriction on timing games, it is shown that there exists a *unique* symmetric mixed strategy Nash equilibrium. This equilibrium describes a pattern of bubbles and crashes where a particular point of time referred to as the *hazard time* exists such that:

(i) the bubble never crashes until the hazard time, and

(ii) the bubble crashes at a constant hazard rate after the hazard time.

Given that the terminal time $\tau_0$ is sufficiently late, we can make the probability of each arbitrageur being behavioral close to zero. It is important to note that any rational arbitrageur expects to earn hardly any profit by exploiting behavioral arbitrageurs; instead, she/he expects to make a considerable profit from the increase in the postcrash price by postponing the X-Day.

The present paper is closely related to Abreu and Brunnermeier (2003), who also modeled the stock market as a timing game, and were the first to present a theoretical ground which explained that the resilience of the bubble stems from the inability of arbitrageurs to coordinate their selling strategies. Abreu and Brunnermeier assumed sequential awareness and coordinated attacks, i.e., the arbitrageurs become sequentially aware that the bubble has developed, and selling pressure bursts the bubble only when a sufficient number of arbitrageurs have sold out. The present paper does not require these assumptions; instead, we assume that (1) each arbitrageur is not necessarily rational, (2) the bubble soft-lands, and (3) the delay of the X-Day increases the postcrash price. Using these assumptions along with mixed strategies, we can show an alternative ground with regard to the inability of arbitrageurs to coordinate their selling strategies.

In Abreu and Brunnermeier (2003), the endogenous determination of the X-Day is not greatly influenced by the delay of the terminal time $\tau_0$, at which the bubble inevitably crashes for exogenous reasons. In contrast, in the model proposed by the present paper, the determination of the X-Day is sensitive to the delay of the terminal time $\tau_0$ in a substantial manner. This delay strengthens the effect of reputation, which can prompt the bubble to persist much longer.

Keynes (1936, Chapter 12) likened the competition among professional arbitrageurs to a *beauty contest*, in which people guess which faces others will find to be the most beautiful. Experimental economists such as Nagel (1995) designed a simple beauty contest game that captures the basic concept of reasoning that Keynes had in mind. They reported that subjects in laboratories are mostly irrational in terms of reasoning ability; they tend to enforce only two or three rounds of iteration in order to eliminate dominated strategies.[5] The experimental economists explained that the bubble can persist, since arbitrageurs are almost certainly irrational.

The main body of the present paper, however, assumes that arbitrageurs are almost certainly rational; there is only a very small probability of each arbitrageur being irrational—not in terms of reasoning ability but of behavioral bias. A huge volume of empirical and experimental researches in behavioral finance reported that amateur traders are tend to be greatly influenced by behavioral cultures that lead them to engage in momentum trading.[6] Even professional arbitrageurs are not immune to the effect of these cultures; as Shiller (2000) has explained, there may be no clear distinction between professionals and amateurs, because the professionals advise the amateurs.

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[5] See Camerer (2003, Chapters 1 and 5) for experimental researches on beauty contest games.

[6] See Barberis and Thaler (2003) for a general survey of behavioral finance. See also DeLong, Shleifer, Summers, and Waldmann (1990a, 1990b).

In the final section, like Strotz (1955) and Rabin (1998), we shall replace the reputation theory with the *self-control* problem in the following manner. We assume that each arbitrageur is *quasi-rational*, in that at the early stage of the timing game, she/he does not know whether she/he will become behavioral in the future. Hence, any quasi-rational arbitrageur may control her/himself by advancing the timing to exit the market out of the fear that she/he becomes behavioral and fails to sell out before the stock price reaches bottom. We, however, show that as long as the terminal time $\tau_0$ is sufficiently late, any quasi-rational arbitrageur will be willing to take this risk. The persistence of bubbles is robust to the self-control exercised by arbitrageurs, because we can make the probability of each arbitrageur becoming behavioral close to zero.

The rest of the paper is organized as follows. Section 2 defines timing games. Section 3 indicates a necessary and sufficient condition under which the quick crash of bubbles at the initial time can be supported by a symmetric Nash equilibrium. Section 4 presents a specification of symmetric Nash equilibria, according to which the bubble persists for a while. Section 5 focuses on the case when the terminal time is sufficiently late, and illustrates that the bubble can persist for a long period even if every arbitrageur is almost certainly rational. Section 6 characterizes the set of symmetric Nash equilibria and considers the uniqueness issue. Section 7 is devoted to the self-control problem.

## 2. The Model

This paper considers the trade in a company's shares on a stock market during the time interval $[0, \tau_0]$. The fundamental value of this company's shares is considered to be $y \in [0,1)$ at the initial time 0, and it grows exponentially at the market interest rate $\delta \in [0, \infty)$. Further, we assume that no dividends are paid. There exist $n \geq 2$ arbitrageurs, each of whom decides the timing to exit the stock market by selling out her/his stockholding, which is normalized to a single share. At the initial time 0, they recognize that the bubble has occurred at or before this time. This bubble persists as long as no arbitrageur sells her/his stockholding.

Figure 1 illustrates the process that leads to bubbles and crashes. If no arbitrageur sells before time $\tau \in [0, \tau_0]$, the stock price per share grows at a rate higher than the interest rate. It is considered to be the *precrash* price $e^{\rho\tau}$, where $\rho > \delta$ is referred to as the *growth rate of bubbles*. The precrash price at the initial time 0 is normalized to 1. If there exists any arbitrageur who sells at time $\tau$, we regard this time as the X-Day. This selling pressure bursts the bubble at the moment immediately after X-Day $\tau$, and the stock price declines to the *postcrash* price $\lambda e^{\rho\tau}$. We assume that $y < \lambda < 1$, i.e., the postcrash price $\lambda e^{\rho\tau}$ is less than precrash price $e^{\rho\tau}$ but greater than the fundamental value $ye^{\delta\tau}$ at time $\tau$. This arbitrageur can sell at the precrash price $e^{\rho\tau}$, and the bubble crashes just after she/he has completed selling out her/his stockholding. If no arbitrageur has sold during the time interval $[0,\tau_0]$, the bubble inevitably crashes at the moment immediately after the terminal time $\tau_0$ for exogenous reasons.[7]

**[Figure 1]**

Against the abovementioned background, we implicitly assume the presence of behavioral traders who have psychological biases that lead them to engage in momentum trading. They incorrectly believe that the stock price will grow at the growth rate $\rho > \delta$ in perpetuity, and attempt to maintain the stock price at the precrash price. The moment any arbitrageur sells her/his share, the resulting selling pressure immediately dampens their enthusiasm and leads to the crash of the bubble. Even if no arbitrageur has sold out, their enthusiasm is automatically dampened for exogenous reasons at the moment immediately after the terminal time $\tau_0$.

The selling pressure does not instantaneously cause a drastic dampening in the enthusiasm of behavioral arbitrageurs, since $\lambda e^{\rho\tau} > ye^{\delta\tau}$ for all $\tau \in [0,\tau_0]$. Hence, the

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[7] We assume that each arbitrageur's position is restricted to be either 0 or +1.

stock price soft-lands, i.e., it does not immediately decline to the fundamental value $ye^{\delta\tau}$. Let us refer to *the soft-landing index* as $\lambda$. The postcrash price increases at the same rate that the precrash price does, namely, $\rho$.

We arbitrarily set $\varepsilon \in (0,1)$ and $r \in (0,1)$. By regarding the $n$ arbitrageurs as players, we define a *timing game* $G = (n, \tau_0, \delta, \rho, \lambda, r, \varepsilon)$ as follows. A strategy for each arbitrageur $i \in \{1,...,n\}$ is defined as a cumulative distribution $q_i : [0, \tau_0] \to [0,1]$ that is nondecreasing, right continuous, and satisfies $q_i(\tau_0) = 1$.[8] Let $Q_i$ denote the set of strategies for arbitrageur $i$.

We assume that each arbitrageur $i \in \{1,...,n\}$ is not necessarily rational. She/he is either rational or behavioral in the following manner. With the probability $1-\varepsilon$, arbitrageur $i$ is rational; further, according to strategy $q_i$, she/he plans to sell at or before each time $a_i \in [0, \tau_0]$ with the probability $q_i(a_i) \in [0, \tau_0]$. With regard to the remaining probability $\varepsilon$, arbitrageur $i$ is behavioral like the behavioral traders; she/he does not follow strategy $q_i$ and never bursts the bubble of her/his own accord. Whether or not each arbitrageur is behavioral is independently determined. Each arbitrageur does not know if the other arbitrageurs are rational.

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[8] We will consider $q_i = a_i$ to be a *pure* strategy if $q_i(\tau) = 0$ for all $\tau \in [0, a_i)$, and $q_i(\tau) = 1$ for all $\tau \in [a_i, \tau_0]$.

Assuming that arbitrageur $i$ is rational, if arbitrageur $i's$ time choice $a_i$ is earlier than that of any other rational arbitrageur, she/he sells just at her/his planned time $a_i$ and earns the present value of the precrash price $e^{(\rho-\delta)a_i}$ as *the winner's gain*. If there exists any other arbitrageur $j \neq i$ who is rational and whose time choice $a_j$ is the earliest (and earlier than $a_i$), arbitrageur $i$ loses. She/he earns the present value of the postcrash price $\lambda e^{(\rho-\delta)a_j}$ as *the loser's gain* by selling out at the next moment. If arbitrageur $i's$ time choice is the earliest but there exists another arbitrageur whose time choice is the same as that of arbitrageur $i$, arbitrageur $i$ wins only with the probability $r \in (0,1)$ .[9] In this case, her/his expected earning is given by $\{r+(1-r)\lambda\}e^{(\rho-\delta)a_i}$ .

Let $Q = Q_1 \times \cdots \times Q_n$ . Let $q = (q_1,...,q_n) \in Q$ denote a strategy profile. We define the payoff function $u_i : Q \to R$ for each arbitrageur $i \in \{1,...,n\}$ as follows. We arbitrarily set $a = (a_j)_{j\in N} \in [0,\tau^0]^n$ and assume that any rational arbitrageur $i \in \{1,...,n\}$ selects $a_i$ . We then arbitrarily set any subset $H \subset \{1,...,n\}$ . We assume that any arbitrageur in subset $H$ could be behavioral, while any arbitrageur in its complementary set $\{1,...,n\} \setminus H$ could be rational. Then, any rational arbitrageur

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[9] At the expense of complexity, we could allow $r$ to be dependent on the number of arbitrageurs whose time choices are the earliest. This dependence is irrelevant to this paper since tie-breaking hardly occurs, except at the initial time 0.

$i \in \{1,...,n\} \setminus H$ can obtain the payoff $v_i(H,a) \in R$ as follows:

$$v_i(H,a) = e^{(\rho-\delta)a_i} \quad \text{if } a_i < a_j \text{ for all } j \notin H \cup \{i\},$$

$$v_i(H,a) = \{r+(1-r)\lambda\}e^{(\rho-\delta)a_i} \quad \text{if } a_i \le a_j \text{ for all } j \notin H \cup \{i\}, \text{ and}$$

$$a_j = a_i \text{ for some } j \notin H \cup \{i\}.$$

For every $j \notin H \cup \{i\}$,

$$v_i(H,a) = \lambda e^{(\rho-\delta)a_j} \quad \text{if } a_j < a_i, \text{ and } a_j \le a_{j'} \text{ for all } j' \notin H \cup \{j\}.$$

We define $u_i(q)$ as the expected value of $v_i(H,a)$:

$$u_i(q) \equiv E[\sum_{H \subset \{1,...,n\} \setminus \{i\}} v_i(H,a)(1-\varepsilon)^{n-1-|H|}\varepsilon^{|H|} \mid q].$$

The probability that there exists any arbitrageur $j \neq i$ other than arbitrageur $i$ who sells at or before time $\tau$ is given by

(1) $$D_i(\tau;q_{-i}) \equiv 1 - \prod_{j \neq i}\{1-(1-\varepsilon)q_j(\tau)\}.$$

From (1), it follows that

(2) $$u_i(0,q_{-i}) = 1 - D_i(0;q_{-i}) + \{r+(1-r)\lambda\}D_i(0;q_{-i}),$$

and

(3) $$u_i(a_i,q_{-i}) = e^{(\rho-\delta)a_i}\{1-D_i(a_i;q_{-i})\}$$

$$+\{r+(1-r)\lambda\}e^{(\rho-\delta)a_i}\{D_i(a_i;q_{-i}) - \lim_{\tau \uparrow a_i} D_i(\tau;q_{-i})\}$$

$$+\int_{\tau=0}^{a_i} \lambda e^{(\rho-\delta)\tau} dD_i(\tau;q_{-i}) + \lambda D_i(0;q_{-i}) \quad \text{for all } a_i \in (0,\tau_0].$$

Moreover, the probability that the bubble crashes at or before time $\tau$ is given by

$$D(\tau;q) \equiv 1 - \prod_{i=1}^{n} [\varepsilon + (1-\varepsilon)\{1 - q_i(\tau)\}] .$$

We define the *hazard rate* at time $\tau \in [0, \tau_0]$ for strategy profile $q \in Q$ by

$$\theta(\tau;q) \equiv \frac{\frac{dD(\tau;q)}{d\tau}}{1 - D(\tau,q)} .$$

The hazard rate multiplied by $d\tau$ indicates the conditional probability that the bubble crashes over the interval $[\tau, \tau + d\tau]$, given that the bubble has persisted up to time $\tau$.

A strategy profile $q \in Q$ is said to be *symmetric* in $G$ if $q_i = q_1$ for all $i \in \{1,...,n\}$. A strategy profile $q \in Q$ is said to be a Nash equilibrium in $G$ if

$$u_i(q) \geq u_i(q_i', q_{-i}) \text{ for all } i \in \{1,...,n\} \text{ and all } q_i' \in Q_i .$$

## 3. Quick Crashes

We denote the symmetric pure strategy profile by $\tilde{q} \equiv (0,...,0)$, according to which any rational arbitrageur sells at the initial time 0. The bubble quickly crashes at the initial time 0, except in the case where all arbitrageurs are behavioral. Note that for every $i \in \{1,...,n\}$,

$$u_i(\tilde{q}) = \varepsilon^{n-1} + (1-\varepsilon^{n-1})\{r + (1-r)\lambda\},$$

and

$$u_i(a_i, \tilde{q}_{-i}) = \varepsilon^{n-1} e^{(\rho-\delta)a_i} + (1-\varepsilon^{n-1})\lambda \quad \text{for all} \quad a_i \in (0, \tau_0].$$

Hence, $\tilde{q}$ is a Nash equilibrium in $G$ if and only if

$$\varepsilon^{n-1} + (1-\varepsilon^{n-1})\{r + (1-r)\lambda\} \geq \varepsilon^{n-1} e^{(\rho-\delta)\tau_0} + (1-\varepsilon^{n-1})\lambda,$$

that is,

$$\text{(4)} \qquad \varepsilon^{n-1} e^{(\rho-\delta)\tau_0} \leq (1-\varepsilon^{n-1})(1-\lambda)r + \varepsilon^{n-1}.$$

This implies that *if the terminal time* $\tau_0$ *is not greatly delayed in relation to the smallness of* $\varepsilon$*, timing game* $G$ *can describe the case where any rational arbitrageur never rides the bubble as a symmetric Nash equilibrium.* Note that $\varepsilon = 0$ automatically guarantees (4), i.e., it guarantees the Nash equilibrium property of $\tilde{q}$.

The main body of this paper will assume that $\varepsilon > 0$. We shall also focus on other symmetric Nash equilibria, according to which rational arbitrageurs *do* ride the bubble. The assumption of $\varepsilon > 0$ is very crucial, without which the backward induction method eliminates all strategy profiles except $\tilde{q}$, i.e., $\tilde{q}$ becomes the unique Nash equilibrium.

## 4. Bubbles and Crashes

We arbitrarily set $\hat{\tau} \in [0, \tau_0]$, which we refer to as the *hazard time*. Let us specify a symmetric strategy profile $q^{\hat{\tau}} = (q_i^{\hat{\tau}})_{i=1}^n \in Q$ as follows: for every $i \in \{1,...,n\}$,

$$q_i^{\hat{\tau}}(a_i) = 0 \quad \text{for all} \quad a_i \in [0, \hat{\tau}),$$

and

$$(5) \qquad q_i^{\hat{\tau}}(a_i) = \frac{1 - \varepsilon \exp[\dfrac{(\rho - \delta)(\tau_0 - a_i)}{(1-\lambda)(n-1)}]}{1-\varepsilon} \quad \text{for all} \quad a_i \in [\hat{\tau}, \tau_0].$$

In order for $q^{\hat{\tau}}$ to be well specified, we need to assume that

$$(6) \qquad \varepsilon \leq \exp[\frac{-(\rho - \delta)(\tau_0 - \hat{\tau})}{(n-1)(1-\lambda)}],$$

because (6) guarantees $q_i^{\hat{\tau}}(\hat{\tau}) \geq 0$.

According to $q^{\hat{\tau}}$, no arbitrageur ever bursts the bubble until the hazard time $\hat{\tau}$. The X-Day is randomly selected over the interval $[\hat{\tau}, \tau_0]$; the hazard rate $\theta(\tau; q^{\hat{\tau}})$ at time $\tau$ for strategy profile $q^{\hat{\tau}}$ is equal to

$$\theta(\tau; q^{\hat{\tau}}) = 0 \qquad \text{if} \quad 0 < \tau < \hat{\tau},$$

and

$$(7) \qquad \theta(\tau; q^{\hat{\tau}}) = \frac{(n-1)(\rho - \delta)}{n(1-\lambda)} \qquad \text{if} \quad \hat{\tau} < \tau < \tau_0.$$

Note that the hazard rate after the hazard time $\hat{\tau}$ is constant with respect to time $\tau$, and increases with respect to the growth rate $\rho$ and the soft-landing index $\lambda$.

**Theorem 1:** *Based on the assumption in (6), strategy profile* $q^{\hat{\tau}}$ *is a Nash equilibrium in* $G$ *if and only if (6) holds with equality, i.e.,*

(8) $$\varepsilon = \exp[\frac{-(\rho-\delta)(\tau_0-\hat{\tau})}{(n-1)(1-\lambda)}],$$

*where for every* $i \in \{1,...,n\}$,

(9) $$u_i(q^{\hat{\tau}}) = e^{(\rho-\delta)\hat{\tau}} = \exp[(\rho-\delta)\tau_0 + (n-1)(1-\lambda)\log\varepsilon].$$

**Proof:** Note that (8) is necessary for $q^{\hat{\tau}}$ to be a Nash equilibrium. Without (8), $q_i^{\hat{\tau}}$ is discontinuous at the hazard time, i.e., $q_i^{\hat{\tau}}(\hat{\tau}) > \lim_{\tau \uparrow \hat{\tau}} q_i^{\hat{\tau}}(\tau) = 0$; each arbitrageur prefers to sell slightly earlier than $\hat{\tau}$ instead of $\hat{\tau}$, which contradicts the Nash equilibrium property.

Suppose that (8) holds. From (1),

$$D_i(\tau; q_{-i}^{\hat{\tau}}) = 0 \quad \text{for all} \quad \tau \in [0, \hat{\tau}),$$

and

$$D_i(\tau; q_{-i}^{\hat{\tau}}) = 1 - \varepsilon^{n-1} e^{\frac{(\rho-\delta)(\tau_0-\tau)}{1-\lambda}} \quad \text{for all} \quad \tau \in [\hat{\tau}, \tau_0].$$

Hence, from (2), (3), and (8),

$$u_i(a_i, q_{-i}^{\hat{\tau}}) = e^{(\rho-\delta)a_i} \quad \text{for all} \quad a_i \in [0, \hat{\tau}],$$

and the following first-order conditions hold. For every $a_i \in [\hat{\tau}, \tau_0]$,

$$\frac{\partial u_i(a_i, q_{-i}^{\hat{\tau}})}{\partial a_i} = \frac{\partial}{\partial a_i}[e^{(\rho-\delta)a_i}\{1 - D_i(a_i; q_{-i}^{\hat{\tau}})\}$$

$$+ \int_{\tau=0}^{a_i} \lambda e^{(\rho-\delta)\tau} dD_i(\tau; q_{-i}^{\hat{\tau}}) + \lambda D_i(0; q_{-i}^{\hat{\tau}})]$$

$$= \rho e^{(\rho-\delta)a_i}\{1 - D_i(a_i; q_{-i}^{\hat{\tau}})\} - e^{(\rho-\delta)a_i}(1-\lambda)\frac{dD_i(a_i; q_{-i}^{\hat{\tau}})}{da_i}$$

$$= (\rho-\delta)e^{(\rho-\delta)a_i}\varepsilon^{n-1}e^{\frac{(\rho-\delta)(\tau_0-a_i)}{1-\lambda}} - e^{(\rho-\delta)a_i}(1-\lambda)(\frac{-(\rho-\delta)}{1-\lambda}\varepsilon^{n-1}e^{\frac{(\rho-\delta)(\tau_0-a_i)}{1-\lambda}})$$

$$= 0 .$$

From the continuity of $q^{\hat{\tau}}$ and (8),

$$u_i(\hat{\tau}, q_{-i}^{\hat{\tau}}) = e^{(\rho-\delta)\hat{\tau}} = \exp[(\rho-\delta)\tau_0 + (n-1)(1-\lambda)\log\varepsilon],$$

which implies (9). Since $e^{(\rho-\delta)\hat{\tau}} \geq e^{(\rho-\delta)a_i}$ for all $a_i \in [0, \hat{\tau})$, it follows that

$$u_i(q^{\hat{\tau}}) \geq u_i(a_i, q_{-i}^{\hat{\tau}}) \text{ for all } a_i \in [0, \tau_0].$$

**Q.E.D.**

We show that for every $\hat{\tau} \in (0, \tau_0)$, there exists $\varepsilon \in (0,1)$ such that *timing game* $G = (n, \tau_0, \delta, \rho, \lambda, r, \varepsilon)$ *can describe the pattern of bubbles and crashes where the bubble persists until terminal time* $\hat{\tau}$ *, and later crashes at the hazard rate* $\frac{(n-1)(\rho-\delta)}{n(1-\lambda)}$*, as is characteristic of a symmetric Nash equilibrium.*

**Theorem 2:** *For every* $\hat{\tau} \in (0, \tau_0)$ *, there exist* $\varepsilon \in (0,1)$ *and a symmetric Nash equilibrium* $q$ *in* $G = (n, \tau_0, \delta, \rho, \lambda, r, \varepsilon)$ *such that*

$$q_i(0) = 0 \quad \textit{for all} \quad i \in \{1,...,n\},$$

$$\theta(\tau; q) = 0 \quad \textit{for all} \quad \tau \in (0, \hat{\tau}),$$

*and*

$$\theta(\tau; q) = \frac{(n-1)(\rho - \delta)}{n(1-\lambda)} \quad \textit{for all} \quad \tau \in (\hat{\tau}, \tau_0).$$

**Proof:** In this case, we only have to show that there exists $\varepsilon \in (0,1)$ such that $q^{\hat{\tau}}$ is a Nash equilibrium in $G = (n, \tau^0, \delta, \rho, \lambda, r, \varepsilon)$. From Theorem 1, (8) is sufficient for $q^{\hat{\tau}}$ to be a Nash equilibrium. Since $0 < \exp[\frac{-(\rho - \delta)(\tau_0 - \hat{\tau})}{(n-1)(1-\lambda)}] < 1$, it follows that $\varepsilon \in (0,1)$.

**Q.E.D.**

## 5. Late Terminal Time

From (8), the distance between the hazard time $\hat{\tau}$ and terminal time $\tau_0$ is given by

$$\tau_0 - \hat{\tau} = -\frac{(n-1)(1-\lambda)\log\varepsilon}{\rho-\delta}.$$

This distance does not depend on the terminal time $\tau_0$. *The hazard time is postponed at the same rate as the terminal time.*

Let us confine our attention to timing games where the terminal time $\tau_0$ is sufficiently late. We will show that even a very small probability $\varepsilon$ of each arbitrageur's being behavioral facilitates the persistence of bubbles. We arbitrarily set any infinite sequence of vectors $(\tau_0^{[m]}, \hat{\tau}^{[m]})_{m=1}^{\infty}$ such that

$$\tau_0^{[m]} > \hat{\tau}^{[m]} > 0 \text{ for all } m = 1, 2, \dots,$$

$$\lim_{m\to\infty} \hat{\tau}^{[m]} = \infty, \quad \lim_{m\to\infty} \tau_0^{[m]} = \infty, \text{ and } \lim_{m\to\infty}(\tau_0^{[m]} - \hat{\tau}^{[m]}) = \infty.$$

For every $m = 1, 2, \dots$, we specify that

(10) $$\varepsilon^{[m]} \equiv \exp\left[\frac{-(\rho-\delta)(\tau_0^{[m]} - \hat{\tau}^{[m]})}{(n-1)(1-\lambda)}\right].$$

From (10), note that $\varepsilon^{[m]} \in (0,1)$, i.e., $\varepsilon^{[m]}$ is well specified. From (8) and (10), for every $m = 1, 2, \dots$, strategy profile $q^{\hat{\tau}^{[m]}}$ is a Nash equilibrium in $G$, where $(\tau_0, \varepsilon) = (\tau_0^{[m]}, \varepsilon^{[m]})$ was assumed. From (10) and $\lim_{m\to\infty}(\tau_0^{[m]} - \hat{\tau}^{[m]}) = \infty$, it follows that

$$\lim_{m\to\infty}\varepsilon^{[m]} = \lim_{m\to\infty}\exp[\frac{-(\rho-\delta)(\tau_0^{[m]}-\hat{\tau}^{[m]})}{(n-1)(1-\lambda)}] = 0 .$$

This along with $\lim_{m\to\infty}\hat{\tau}^{[m]} = \infty$ implies that *even with a very small probability* $\varepsilon$ *that each arbitrageur is behavioral, we can describe the infinitely long persistence of bubbles as a symmetric Nash equilibrium.*

According to the Nash equilibrium property of $q^{\hat{\tau}}$, any time choice in the interval $[\hat{\tau},\tau_0]$ is a best response to $q^{\hat{\tau}}$. This implies that each rational arbitrageur $i$ has an incentive to postpone the market timing until the terminal time $\tau_0$. Hence,

$$u_i(q^{\hat{\tau}}) = u_i(\tau_0, q_{-i}^{\hat{\tau}}) = \varepsilon^{n-1}e^{(\rho-\delta)\tau_0} + \lim_{\tau\uparrow\tau_0}\int_{\tau'=0}^{\tau}\lambda e^{(\rho-\delta)\tau}dD_i(\tau';q_{-i}) .$$

Let us denote

$$w = w(\tau_0,\varepsilon) \equiv \varepsilon^{n-1}e^{(\rho-\delta)\tau_0} ,$$

which we refer to as *the gain from behavioral arbitrageurs*; in other words, she/he can earn $w$ as the winner's gain when all the other arbitrageurs are behavioral. The following theorem states that *the gain from behavioral arbitrageurs* $w$ *is negligible in relation to* $u_i(q^{\hat{\tau}}) = e^{(\rho-\delta)\hat{\tau}}$.

**Theorem 3:** *The following property holds:*

$$\lim_{m\to\infty}\frac{w(\tau_0^{[m]},\varepsilon^{[m]})}{e^{(\rho-\delta)\hat{\tau}^{[m]}}} = 0 .$$

**Proof:** From (10) and $\lim_{m\to\infty}(\tau_0^{[m]} - \hat{\tau}^{[m]}) = \infty$, it follows that

$$\lim_{m\to\infty} \frac{w(\tau_0^{[m]}, \varepsilon^{[m]})}{e^{(\rho-\delta)\hat{\tau}^{[m]}}} = \lim_{m\to\infty} (\varepsilon^{[m]})^{n-1} e^{(\rho-\delta)(\tau_0^{[m]}-\hat{\tau}^{[m]})}$$

$$= \lim_{m\to\infty} \exp[\frac{-(\tau_0^{[m]} - \hat{\tau}^{[m]})(\rho-\delta)}{(1-\lambda)} + (\rho-\delta)(\tau_0^{[m]} - \hat{\tau}^{[m]})]$$

$$= \lim_{m\to\infty} \exp[(\tau_0^{[m]} - \hat{\tau}^{[m]})(\rho-\delta)(1-\frac{1}{1-\lambda})] = 0 .$$

**Q.E.D.**

The remaining payoff $u_i(q^{\hat{\tau}}) - w$ is equal to $\lim_{\tau\uparrow\tau_0} \int_{\tau'=0}^{\tau} \lambda e^{(\rho-\delta)\tau} dD_i(\tau'; q_{-i})$, which corresponds to the loser's gain when some of the other arbitrageurs are rational. Theorem 3 implies that any rational arbitrageur expects to earn hardly any profit by exploiting behavioral arbitrageurs; instead, she/he expects a considerable increase in the loser's gain by postponing the market timing. The following theorem shows that *the gain from behavioral arbitrageurs* $w$ *can be small even in absolute terms.*

**Theorem 4:** *The following properties hold:*

(i) $[0 < \lim_{m\to\infty} w(\tau_0^{[m]}, \varepsilon^{[m]}) < \infty] \Rightarrow [\lim_{m\to\infty} \frac{\hat{\tau}^{[m]}}{\tau_0^{[m]}} = \lambda]$;

(ii) $[\lim_{m\to\infty} \frac{\hat{\tau}^{[m]}}{\tau_0^{[m]}} < \lambda] \Rightarrow [\lim_{m\to\infty} w(\tau_0^{[m]}, \varepsilon^{[m]}) = 0]$;

(iii) $[\lim_{m\to\infty} \frac{\hat{\tau}^{[m]}}{\tau_0^{[m]}} > \lambda] \Rightarrow [\lim_{m\to\infty} w(\tau_0^{[m]}, \varepsilon^{[m]}) = \infty]$.

**Proof:** From (10),

(11) $$\lim_{m\to\infty} w(\tau_0^{[m]}, \varepsilon^{[m]}) = \lim_{m\to\infty} (\varepsilon^{[m]})^{n-1} e^{(\rho-\delta)\tau_0^{[m]}}$$

$$= \lim_{m\to\infty} \exp[\frac{-(\tau_0^{[m]} - \hat{\tau}^{[m]})(\rho-\delta)}{1-\lambda} + (\rho-\delta)\tau_0^{[m]}]$$

$$= \lim_{m\to\infty} \exp[\frac{(\rho-\delta)\tau_0^{[m]}}{1-\lambda}(\frac{\hat{\tau}^{[m]}}{\tau_0^{[m]}} - \lambda)].$$

If $\lim_{m\to\infty} \frac{\hat{\tau}^{[m]}}{\tau_0^{[m]}} < \lambda$, then from $\lim_{m\to\infty} \tau_0^{[m]} = \infty$,

$$\lim_{m\to\infty} \exp[\frac{(\rho-\delta)\tau_0^{[m]}}{1-\lambda}(\frac{\hat{\tau}^{[m]}}{\tau_0^{[m]}} - \lambda)] = 0 .$$

This along with (11) implies property (ii). If $\lim_{m\to\infty} \frac{\hat{\tau}^{[m]}}{\tau_0^{[m]}} > \lambda$, then from $\lim_{m\to\infty} \tau_0^{[m]} = \infty$,

$$\lim_{m\to\infty} \exp[\frac{(\rho-\delta)\tau_0^{[m]}}{1-\lambda}(\frac{\hat{\tau}^{[m]}}{\tau_0^{[m]}} - \lambda)] = \infty .$$

This along with (11) implies property (iii). From properties (ii) and (iii), it follows that if $0 < \lim_{m\to\infty} (\varepsilon^{[m]})^n e^{(\rho-\delta)\tau_0^{[m]}} < \infty$, then $\lim_{m\to\infty} \frac{\hat{\tau}^{[m]}}{\tau_0^{[m]}} = \lambda$ must hold. This implies property (i).

**Q.E.D.**

Property (i) of Theorem 4 is particularly important because *whenever the gain from behavioral arbitrageurs is non-negligible, i.e.,* $\lim_{m\to\infty} w(\tau_0^{[m]}, \varepsilon^{[m]}) > 0$, *and moderate, i.e.,* $\lim_{m\to\infty} w(\tau_0^{[m]}, \varepsilon^{[m]}) < \infty$, *then the ratio* $\frac{\hat{\tau}}{\tau_0} = \frac{\hat{\tau}^{[m]}}{\tau_0^{[m]}}$ *of the hazard time to the terminal time is approximated by the soft-landing index* $\lambda$. Note that given that the terminal time $\tau_0$ is sufficiently late, it is almost certain that the ratio of the X-Day to the terminal time is approximated by $\frac{\hat{\tau}}{\tau_0}$. Hence, it is almost certain that *the ratio of the X-Day to the* *terminal time is approximated by the soft-landing index* $\lambda$. Figure 2 illustrates these observations.

**[Figure 2]**

By approximating the ratio of the X-Day to $\lambda\tau_0$, we can regard $(1-\lambda)e^{\lambda\rho\tau_0}$ as the range of the stock price drop on the X-Day. Note that $(1-\lambda)e^{\lambda\rho\tau_0}$ increases with respect to $\lambda \in [0, 1-\frac{1}{(\rho-\delta)\tau_0}]$, where $1-\frac{1}{(\rho-\delta)\tau_0}$ is close to 1. Hence, *the more the bubble soft-lands, the greater is the range of the stock price drop on the X-Day.*

Property (ii) implies that whenever $\frac{\hat{\tau}}{\tau_0} = \frac{\hat{\tau}^{[m]}}{\tau_0^{[m]}}$ is less than $\lambda$, the gain from behavioral arbitrageurs must be negligible, i.e., $\lim_{m\to\infty} w(\tau_0^{[m]}, \varepsilon^{[m]}) = 0$. Hence, *even if any rational arbitrageur hardly expects to have an advantage over the other behavioral arbitrageurs in absolute terms, the bubble persists for a significantly long time.*

## 6. Uniqueness

This section characterizes the set of all symmetric Nash equilibria, and then shows a sufficient condition for strategy profile $q^{\hat{\tau}}$ to be the *unique* symmetric Nash equilibrium. We arbitrarily set $\hat{\tau} \in (0, \tau_0)$ and $k \in (0,1)$. Let us specify another symmetric strategy profile $q^{(\hat{\tau},k)} = (q_i^{(\hat{\tau},k)})_{i=1}^n \in Q$ as follows: for every $i \in \{1,...,n\}$,

$$q_i^{(\hat{\tau},k)}(a_i) = k \quad \text{for all} \quad a_i \in [0, \hat{\tau}),$$

and

$$q_i^{(\hat{\tau},k)}(a_i) = \frac{(1-\varepsilon)(1-k) + \varepsilon - \varepsilon \exp[\dfrac{(\rho-\delta)(\tau_0 - a_i)}{(1-\lambda)(n-1)}]}{(1-\varepsilon)(1-k)} \quad \text{for all} \quad a_i \in [\hat{\tau}, \tau_0].$$

In order for $q^{(\hat{\tau},k)}$ to be well specified, we need to assume

(12) $$\varepsilon \leq \{1-(1-\varepsilon)k\} \exp[\frac{-(\rho-\delta)(\tau_0 - \hat{\tau})}{(n-1)(1-\lambda)}],$$

because (12) guarantees $q_i^{(\hat{\tau},k)}(\hat{\tau}) \geq 0$.

According to $q^{(\hat{\tau},k)}$, with probability $k > 0$, any rational arbitrageur bursts the bubble at the *initial* time 0. With probability $1-k > 0$, she/he never bursts the bubble until the hazard time $\hat{\tau}$. The hazard rate $\theta(\tau; q^{(\hat{\tau},k)})$ for $q^{(\hat{\tau},k)}$ is the same as that for $q^{\hat{\tau}}$:

$$\theta(\tau; q^{(\hat{\tau},k)}) = 0 \qquad \text{if} \quad 0 < \tau < \tilde{\tau},$$

and

$$\theta(\tau; q^{(\hat{\tau},k)}) = \frac{(n-1)(\rho-\delta)}{n(1-\lambda)} \quad \text{if} \quad \hat{\tau} < \tau < \tau_0 .$$

**Proposition 5:** *Based on the assumption in (12), strategy profile* $q^{(\hat{\tau},k)}$ *is a Nash equilibrium in* $G$ *if and only if*

(13) $$e^{(\rho-\delta)\hat{\tau}} = \frac{(1-\lambda)r - \{(1-\lambda)r - 1\}\{1-(1-\varepsilon)k\}^{n-1}}{\{1-(1-\varepsilon)k\}^{n-1}},$$

*and (12) holds with equality, i.e.,*

(14) $$\varepsilon = \{1-(1-\varepsilon)k\}\exp[\frac{-(\rho-\delta)(\tau_0-\hat{\tau})}{(n-1)(1-\lambda)}],$$

*where for every* $i \in \{1,...,n\}$ *,*

(15) $$u_i(q^{(\hat{\tau},k)}) = 1-(1-r)(1-\lambda)[1-\{1-(1-\varepsilon)k\}^{n-1}].$$

**Proof:** See the Appendix.

From (15), it follows that $u_i(q^{(\hat{\tau},k)}) < 1$, i.e., the payoff induced by $q^{(\hat{\tau},k)}$ is less than the precrash price at the initial time. Hence, no rational arbitrageur takes advantage of riding the bubble. The following theorem indicates a characteristic of the set of symmetric Nash equilibria, which states that there exists no symmetric Nash equilibrium other than $\tilde{q}$, $q^{\hat{\tau}}$, and $q^{(\hat{\tau},k)}$.

**Theorem 6:** *If any strategy profile* $q \in Q$ *is a symmetric Nash equilibrium in* $G$ *, then any one of the following three properties hold:*

(iv) $q = \tilde{q}$.

(v) $q = q^{\hat{\tau}}$ *, where* $\hat{\tau} \in [0, \tau_0]$ *is given by (7).*

(vi) $q = q^{(\hat{\tau},k)}$ *, where* $\hat{\tau} \in [0, \tau_0]$ *and* $k \in [0,1]$ *satisfy (13) and (14).*

**Proof:** We set any symmetric Nash equilibrium $q \in Q$ arbitrarily. Let

$$\tilde{\tau} = \inf\{\tau \in (0, \tau_0] : q_1(\tau) > q_1(0)\} \text{ and } \overline{\tau} = \min\{\tau \in [0, \tau_0] : q_1(\tau) = 1\}.$$

First, we show that $q_1(\tau)$ is continuous in $[0, \tau_0]$. Suppose that $q_1(\tau)$ is not continuous in $[0, \tau_0]$. Then, there exists $\tau' \in [\tilde{\tau}, \overline{\tau}]$ such that $\tau' > 0$ and $\lim_{\tau \uparrow \tau'} q_1(\tau) < q_1(\tau')$. From $0 < \lambda < 1$, $0 < r < 1$, and the symmetry of $q$, it follows that any arbitrageur can increase her/his probability of becoming the winner by selecting any time that is slightly earlier than time $\tau'$ instead of time $\tau'$. This implies that no arbitrageur ever selects time $\tau'$, which is a contradiction.

Second, we show that $q_1(\tau)$ is increasing in $(\tilde{\tau}, \overline{\tau})$. Suppose that $q_1(\tau)$ is not increasing in $(\tilde{\tau}, \overline{\tau})$. From the continuity of $q_1$ and the definition of $\tilde{\tau}$, we can select $\tau', \tau'' \in [\tilde{\tau}, \overline{\tau})$ such that $\tau' < \tau''$, $q_1(\tau') = q_1(\tau'')$, and time choice $\tau'$ is a best response. Since no arbitrageur selects any time $\tau$ in $(\tau', \tau'')$, by selecting time $\tau''$ instead of

$\tau'$, any arbitrageur can increase the winner's gain without decreasing the probability of winning this gain. This is a contradiction.

Third, we show that either $\bar{\tau} = 0$ or $\bar{\tau} = \tau_0$. Suppose that $0 < \bar{\tau} < \tau_0$. Since $q_1(\tau)$ is continuous at time $\bar{\tau}$, by selecting time $\tau_0$ instead of $\bar{\tau}$, any arbitrageur can increase the winner's gain from $e^{(\rho-\delta)\bar{\tau}}$ to $e^{(\rho-\delta)\tau_0}$ without decreasing the probability of winning this gain. This contradicts the fact that time choice $\bar{\tau}$ is a best response.

Suppose that $q \neq (0,...,0)$, i.e., property (v) does not hold. Since either $\bar{\tau} = 0$ or $\bar{\tau} = \tau_0$, it follows that $\bar{\tau} = \tau_0$. Since $q_1(\tau)$ is increasing in $(\tilde{\tau}, \bar{\tau})$, any time choice $\tau \in [\tilde{\tau}, \tau_0]$ is a best response to $q$. Hence, the following first-order conditions must hold: for every $\tau \in [\tilde{\tau}, \tau_0]$,

$$\frac{\partial u_1(\tau, q_{-1})}{\partial \tau} = (\rho - \delta) e^{(\rho-\delta)\tau} \{1 - D_1(\tau; q_{-1})\} - e^{(\rho-\delta)\tau} (1 - \lambda) \frac{dD_1(\tau; q_{-1})}{d\tau} = 0,$$

that is,

$$D_1(\tau; q_{-1}) = 1 - Ce^{\frac{-(\rho-\delta)\tau}{1-\lambda}} \quad \text{for all } \tau \in [\tilde{\tau}, \tau_0],$$

where $C$ is a positive real number. The continuity of $q_1$ along with the symmetry of $q$ implies that

(16) $$D_1(\tau_0; q_{-1}) = 1 - Ce^{\frac{-(\rho-\delta)\tau_0}{1-\lambda}} = 1 - \varepsilon^{n-1},$$

and

(17) $$D_1(\tilde{\tau};q_{-1}) = 1 - Ce^{\frac{-(\rho-\delta)\tilde{\tau}}{1-\lambda}} = D_1(0;q_{-1}).$$

From (16), it follows that $C = \varepsilon^{n-1} e^{\frac{(\rho-\delta)\tau_0}{1-\lambda}}$, and therefore,

(18) $$D_1(\tau;q_{-1}) = 1 - \varepsilon^{n-1} e^{\frac{(\rho-\delta)(\tau_0-\tau)}{1-\lambda}} \quad \text{for all } \tau \in [\tilde{\tau}, \tau_0].$$

Suppose that $q_1(0) = 0$. Note that the symmetry of $q$ implies that $D_1(0;q_{-1}) = 0$, which along with (17) and (18) implies that

$$\tilde{\tau} = \tau_0 + \frac{(n-1)(1-\lambda)\log\varepsilon}{\rho-\delta}.$$

These observations imply that $q = q^{\hat{\tau}}$, where $\hat{\tau}$ is given by (7). Hence, property (vi) holds.

Suppose that $q_1(0) > 0$. Then, there is a positive real number $k \in (0,1)$ such that $q_1(0) = k$ and $D_1(0;q_{-1}) = 1 - \{1-(1-\varepsilon)k\}^{n-1}$. This along with (17) and (18) implies that

$$\tilde{\tau} = \tau_0 + \frac{(n-1)(1-\lambda)\log\left(\frac{\varepsilon}{1-(1-\varepsilon)k}\right)}{\rho-\delta}.$$

These observations imply that $q = q^{(\hat{\tau},k)}$, where $(\hat{\tau},k)$ satisfies (14). Since $q^{(\hat{\tau},k)}$ is a Nash equilibrium, $(\hat{\tau},k)$ also satisfies (13). Hence, property (vii) holds.

**Q.E.D.**

The proof of Theorem 6 shows that in order for any symmetric strategy profile other than $\tilde{q} = (0,...,0)$ to be a Nash equilibrium, there must exist the hazard time $\hat{\tau}$

such that any time choice between this time and the terminal time is a best response. This implies that the first-order conditions addressed in the proof of Theorem 6 play a crucial role in eliminating all strategy profiles, except for $q^{\hat{\tau}}$ and $q^{(\hat{\tau},k)}$.

The Nash equilibrium properties of $\tilde{q}=(0,...,0)$ and $q^{(\hat{\tau},k)}$ require that time choice 0 be a best response. This along with Theorem 6 implies that whenever time choice 0 is a dominated strategy, strategy profile $q^{\hat{\tau}}$ must be the *unique* symmetric Nash equilibrium. Hence, we can show a sufficient condition for $q^{\hat{\tau}}$ to be the unique symmetric Nash equilibrium as follows.

**Theorem 7:** *With the assumption in (7), strategy profile* $q^{\hat{\tau}}$ *is the unique symmetric Nash equilibrium in* $G$ *if the gain from behavioral arbitrageurs is greater than* $1-(1-\varepsilon^{n-1})\lambda$, *i.e.,*

(19) $$w(\tau_0,\varepsilon)>1-(1-\varepsilon^{n-1})\lambda.$$

**Proof:** For every $q\in Q$,

$$u_1(0,q_{-1})\leq 1 \text{ and } u_1(\tau_0,q_{-1})\geq \varepsilon^{n-1}e^{(\rho-\delta)\tau_0}+(1-\varepsilon^{n-1})\lambda,$$

which along with (19) implies that time choice 0 is dominated by time choice $\tau_0$, i.e.,

$$u_1(\tau_0,q_{-1})>u_1(0,q_{-1}) \text{ for all } q_{-1}\in Q_{-1}.$$

Hence, any symmetric Nash equilibrium $q \in Q$ must satisfy $q_1(0) = 0$, which along with Theorem 6 implies that $q = q^{\hat{\tau}}$.

**Q.E.D.**

We assume that the terminal time $\tau_0$ is sufficiently late. Then, $\varepsilon$ must be close to zero, and therefore, the right-hand side of (19) is approximated by $1 - \lambda$. Hence, from Theorem 7, it follows that strategy profile $q^{\hat{\tau}}$ is the unique symmetric Nash equilibrium if the gain from behavioral arbitrageurs $w(\tau_0, \varepsilon)$ is greater than $1 - \lambda$. From Theorems 5 and 7, we conclude that *the high soft-landing index* $\lambda$ *not only plays a decisive role in prolonging the equilibrium persistence of bubbles but also in eliminating irrelevant equilibria.*

## 7. Quasi-Rationality

We have assumed that each arbitrageur knows whether she/he is rational or behavioral from the beginning of the timing game. This section, however, eliminates this assumption. Instead, we assume that all arbitrageurs are *quasi-rational* as follows. We arbitrarily set $\tau^* \in (0, \tau_0)$ and $\varepsilon \in (0,1)$. We shall refer to $\tau^*$ as the *critical* time. We assume (8) and $\tau^* < \hat{\tau}$. Until the critical time $\tau^*$, each arbitrageur $i$ behaves rationally and then follows $q^{\hat{\tau}}$. Once the bubble has persisted beyond the critical time $\tau^*$, she/he becomes behavioral with probability $\varepsilon$; in other words, she/he stops following $q^{\hat{\tau}}$ and is committed to not bursting the bubble of her/his own accord. With probability $1-\varepsilon$, she/he remains rational and continues to follow $q^{\hat{\tau}}$ even after the critical time $\tau^*$.

This section will provide an affirmative answer to the question of whether even quasi-rational arbitrageurs are willing to follow $q^{\hat{\tau}}$. If arbitrageur $i$ is rational, she/he obtains $u_i(q^{\hat{\tau}}) = e^{(\rho-\delta)\hat{\tau}}$. We assume that no behavioral arbitrageur can sell out before the stock price declines to the fundamental value; in this case, she/he obtains $y$ as the payoff evaluated at the initial time. Hence, according to $q^{\hat{\tau}}$, any quasi-rational arbitrageur $i$ obtains the expected payoff given by

$$(1-\varepsilon)e^{(\rho-\delta)\hat{\tau}}+\varepsilon y\ .$$

Quasi-rational arbitrageur $i$ is faced with the following *self-control* problem. By selecting the critical time $\tau^*$ as the timing to sell out instead of following $q_i^{\hat{\tau}}$, she/he can avoid the loss from her/his becoming behavioral, and obtains the winner's gain $e^{(\rho-\delta)\tau^*}$ with certainty. Hence, in order for any quasi-rational arbitrageur to follow $q^{\hat{\tau}}$, it is necessary and sufficient to satisfy not only (8) but also the inequality

(20) $$(1-\varepsilon)e^{(\rho-\delta)\hat{\tau}}+\varepsilon y \geq e^{(\rho-\delta)\tau^*}\ .$$

Note that inequality (20) does not hold if the critical time $\tau^*$ is close to the hazard time $\hat{\tau}$; in this case, $\tilde{q}=(0,...,0)$ is the unique Nash equilibrium. If the terminal time $\tau_0$ is sufficiently late, however, the requirement of (20) is not restrictive at all; in this case, $\varepsilon$ can be close to zero. Hence, the right-hand side of (20) is close to $e^{(\rho-\delta)\hat{\tau}}$, which is greater than $e^{(\rho-\delta)\tau^*}$ since $\tau^*<\hat{\tau}$. Given that the terminal time $\tau_0$ is sufficiently late, we can conclude that *the Nash equilibrium property of* $q^{\hat{\tau}}$ *is robust to the quasi-rational arbitrageurs' self-control.*

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# Appendix: Proof of Proposition 5

Note that (14) is necessary for $q^{(\hat{\tau},k)}$ to be a Nash equilibrium. Without (14), $q_i^{(\hat{\tau},k)}$ is discontinuous at the hazard time $\hat{\tau}$, i.e., $q_i^{(\hat{\tau},k)}(\hat{\tau}) > \lim_{\tau\uparrow\hat{\tau}} q_i^{(\hat{\tau},k)}(\tau) = 0$; hence, each arbitrageur prefers any time slightly earlier than $\hat{\tau}$ instead of $\hat{\tau}$, and this contradicts the Nash equilibrium property of $q_i^{(\hat{\tau},k)}$.

We suppose that (14) holds. From (1),

$$D_i(\tau; q_{-i}^{(\hat{\tau},k)}) = 1 - \{1-(1-\varepsilon)k\}^{n-1} \quad \text{for all} \quad \tau \in [0,\hat{\tau}),$$

and

$$D_i(\tau; q_{-i}^{(\hat{\tau},k)}) = 1 - \varepsilon^{n-1} e^{\frac{(\rho-\delta)(\tau_0-\tau)}{1-\lambda}} \quad \text{for all} \quad \tau \in [\hat{\tau},\tau_0].$$

From (2), (3), and (14), it follows in the same manner as in the proof of Theorem 1 that

(A1) $$u_i(0, q_{-i}^{(\hat{\tau},k)}) = 1-(1-r)(1-\lambda)[1-\{1-(1-\varepsilon)k\}^{n-1}],$$

(A2) $$u_i(a_i, q_{-i}^{(\hat{\tau},k)}) = \lambda[1-\{1-(1-\varepsilon)k\}^{n-1}] + \{1-(1-\varepsilon)k\}^{n-1} e^{(\rho-\delta)a_i}$$

$$\text{for all} \quad a_i \in (0,\hat{\tau}),$$

and

(A3) $$u_i(a_i, q_{-i}^{(\hat{\tau},k)}) = [\lambda + (1-\lambda) r \{1-(1-\varepsilon)k\}^{n-1}$$

$$+ (1-r)(1-\lambda)\varepsilon^{n-1} e^{\frac{(\rho-\delta)(\tau_0-\hat{\tau})}{1-\lambda}}] e^{(\rho-\delta)\hat{\tau}} \quad \text{for all} \quad a_i \in [\hat{\tau},\tau_0].$$

Note that (A1) implies (15). From (14),

$$[\lambda+(1-\lambda)r\{1-(1-\varepsilon)k\}^{n-1}+(1-r)(1-\lambda)\varepsilon^{n-1}e^{\frac{(\rho-\delta)(\tau_0-\hat{\tau})}{1-\lambda}}]e^{(\rho-\delta)\hat{\tau}}$$

$$=\lambda[1-\{1-(1-\varepsilon)k\}^{n-1}]+\{1-(1-\varepsilon)k\}^{n-1}e^{(\rho-\delta)\hat{\tau}},$$

which along with (A2) and (A3) imply that

$$u_i(q^{(\hat{\tau},k)})\geq u_i(a_i,q_{-i}^{(\hat{\tau},k)}) \quad \text{for all} \quad a_i\in(0,\tau_0].$$

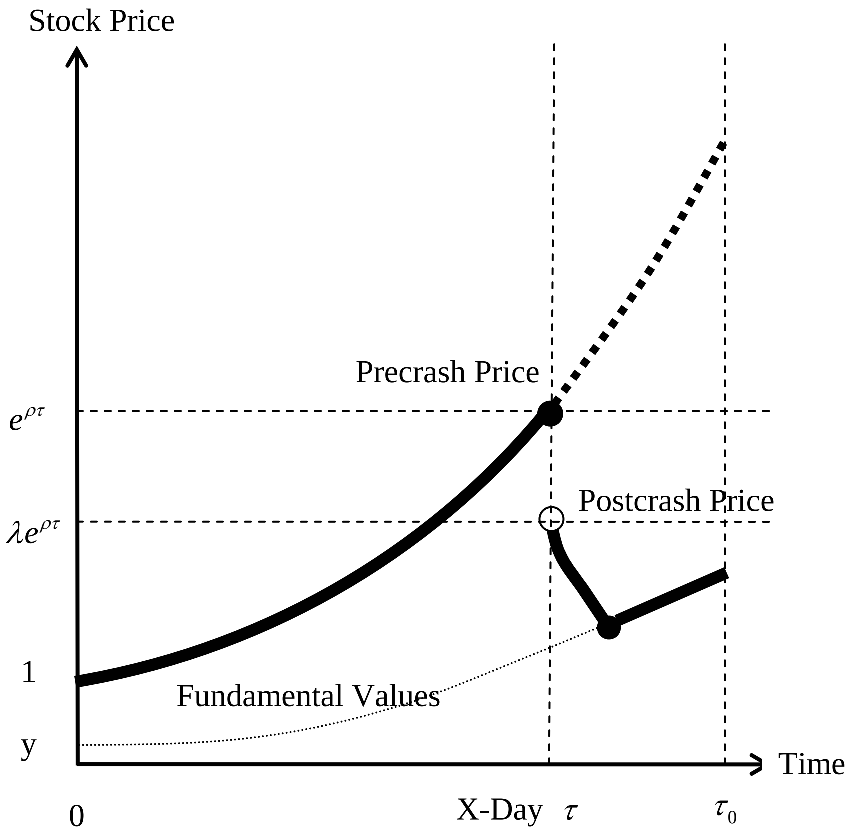
Hence, from (A1) and (A3), $q^{(\hat{\tau},k)}$ is a Nash equilibrium in $G$ if and only if $u_i(q^{(\hat{\tau},k)})=u_i(0,q_{-i}^{(\hat{\tau},k)})$, that is,

$$[\lambda+(1-\lambda)r\{1-(1-\varepsilon)k\}^{n-1}+(1-r)(1-\lambda)\varepsilon^{n-1}e^{\frac{(\rho-\delta)(\tau_0-\hat{\tau})}{1-\lambda}}]e^{(\rho-\delta)\hat{\tau}}$$

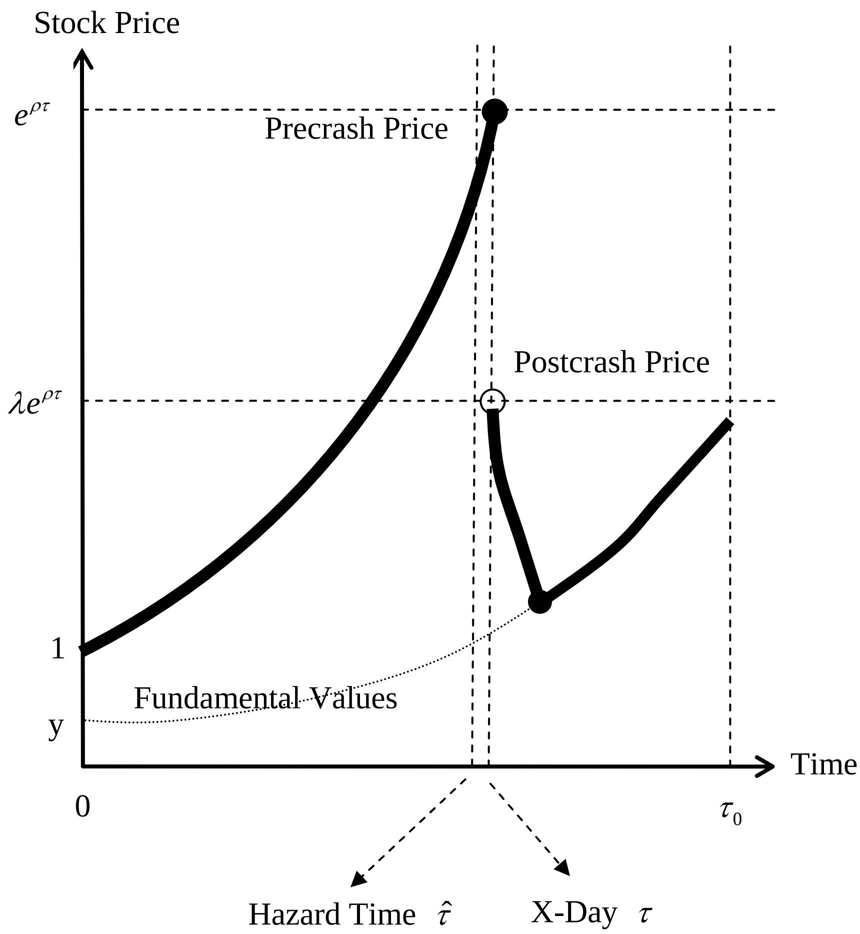
$$=1-(1-r)(1-\lambda)[1-\{1-(1-\varepsilon)k\}^{n-1}],$$

which is equivalent to (13).

**Figure 1: Bubbles and Crashes**



# Figure 2: Late Terminal Time




$$\frac{\hat{\tau}}{\tau_0} \approx \frac{\tau}{\tau_0} \approx \lambda$$