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Pricing Currency Options with a Market Model of Interest Rates under Jump-Diffusion Stochastic Volatility Processes of Spot Exchange Rates

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Abstract

This paper proposes a pricing method of currency options with a market model of interest rates. Using a simple approximation and a Fourier transform method, we derive a formula of the option pricing under jump-diffusion stochastic volatility processes of spot exchange rates. As an application, we apply the formula to the calibration of volatility smiles in the JPY/USD currency option market. Moreover, using the approximate prices as a control variate, we achieve substantial variance reduction in Monte Carlo simulation.

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1 Introduction

In this paper we propose a new approximation formula for the valuation of currency options under jump-diffusion stochastic volatility processes of spot exchange rates in stochastic interest rates environment. In particular, we apply *market models* developed by Brace, Gatarek and Musiela[1998], Jamshidian[1997] and Miltersen, Sandmann and Sondermann[1997] to modeling term structures of interest rates.

Recently, currency options with maturities up to ten years become common in G7 currencies' markets and *smiles* (*skews*) for those maturities are usually observed. Even longer maturities options such as fifteen and twenty years become gradually liquid especially for at-the-money (ATM) options. Moreover, popular currency derivatives such as Power-Reverse-Dual-Currency (PRDC) swaps have long maturities.

Thus, because it is well known that the effect of interest rates become more important in long-term maturities, we have to take term structure models into account for the currency options. Further, we need a stochastic volatility model of foreign exchange rates for calibration of smiles (skews); for shorter maturities, we sometimes need a jump component as well. As for term structure models, market models become popular in matured interest rates markets since calibrations of caps, floors and swap options are necessary and market models are regarded as most useful.

Hence, our objective is to develop a model with stochastic volatilities of exchange rates and with a market model of interest rates. Moreover, a closed-form formula is desirable in practice especially for calibration because it is very time consuming by numerical methods such as Monte Carlo simulation.

Because it is impossible to obtain an exact closed-form formula, we derive an explicit formula by Fourier transform method through a simple approximation of the characteristic function of a forward foreign exchange rate distribution under some independence assumption; we assume independence between currency components and interest rates components while allowing the correlation between a spot exchange rate and its volatility, and the correlation between domestic and foreign interest rates.

Garman and Kohlhagen[1983] and Grabbe[1983] started research for currency options based on contingent claim analysis; they directly applied framework of Black and Scholes[1973], Merton[1973] and Black[1976] to pricing currency options. Grabbe[1983]'s formula also included the case of stochastic interest rates following Gaussian processes though he did not specify the processes explicitly. Under deterministic interest rates, Melino and Turnbull[1991] examined the foreign exchange rate process underlying the observed option prices and Rumsey[1991] considered cross-currency options.

Amin and Jarrow[1991] and Hilliard, Madura and Tucker[1991] derived formulas of currency options with Gaussian stochastic interest rates; in particular, Amin and Jarrow[1991] combined term structure models under the framework of Heath, Jarrow and Morton [1992](HJM[1992]) with currency options.

Amin and Bodurtha[1995] and Takahashi and Tokioka[1999] gave numerical solutions to pricing currency American options with stochastic interest rates by lattice methods; Amin and Bodurtha[1995] used HJM[1992] models and Takahashi and Tokioka[1999] applied Hull and White[1990,1994] term structure models. Dempster and Hutton[1997] considered terminable (Bermudan) differential swaps with Gaussian interest rates models following partial differential equations(PDE) approach.

Bates[1996] developed a jump-diffusion stochastic volatility model with constant interest rates for currency options based on Heston[1993]'s Fourier transform method. Bakshi, Cao and Chen [1997] combined stochastic interest rates of Cox Ingersoll and Ross[1985] with a jump-diffusion stochastic volatility model under independence assumption between the domestic interest rate and the other components though they did not consider currency options, but equity options. Carr and Wu[2005] proposed new models, *stochastic skew models* for pricing currency options based on time-changed Levy processes with deterministic interest rates, and investigated skew structures of options with short maturities such as one, three and twelve months in detail.

Schlogl[2002] extended market models to a cross-currency framework. He did not take stochastic volatilities into account and focus on cross currency derivatives such as differential swaps and options on the swaps as applications; he did not consider currency options. Mikkelsen[2001] considered cross-currency options with market models of interest rates and deterministic volatilities of spot exchange rates by simulation. Piterbarg[2005] developed a model for cross-currency derivatives such as

PRDC swaps with calibration to currency options; he used neither market models nor stochastic volatility models.

This paper develops a jump-diffusion stochastic volatility model with a market model of interest rates for pricing and calibration of currency options with long maturities more than one year.

The organization of the paper is as follows: After the next section describes basic structure of our model, section 3 derives an approximation formula. Section 4 shows numerical examples and the final section states conclusion.

2 Models

We first define domestic and foreign forward interest rates as:

$$f_n^d(t) = \left(\frac{B_n^d(t)}{B_{n+1}^d(t)} - 1 \right) \frac{1}{\delta_n},$$

$$f_n^f(t) = \left(\frac{B_n^f(t)}{B_{n+1}^f(t)} - 1 \right) \frac{1}{\delta_n},$$

respectively, where $B_n^d(t)$ and $B_n^f(t)$ denote the prices of zero coupon bonds with maturity T_n at time $t\,(\le T_n)$, and $n = m(t), m(t)+1, \cdots, N$, $m(t) = \min\{k : t \le T_k\}$, $\delta_n = T_{n+1} - T_n$. In this paper, we specify the forward interest rate processes as follows:

$$\frac{df_n^d(t)}{f_n^d(t)} = \vec{\sigma}_n^d(t) \cdot d\vec{W}_{n+1}^d(t), \tag{1}$$

$$\frac{df_n^f(t)}{f_n^f(t)} = \vec{\sigma}_n^f(t) \cdot d\vec{W}_{n+1}^f(t), \tag{2}$$

where $\vec{W}_{n+1}^d(t)$ and $\vec{W}_{n+1}^f(t)$ are D dimensional Brownian motions under the forward martingale measures of numeraire $B_{n+1}^d(t)$ and $B_{n+1}^f(t)$ respectively, and $\vec{\sigma}_n^d(t)$ and $\vec{\sigma}_n^f(t)$ denote some R^D-valued deterministic functions. Here, "$\vec{x} \cdot \vec{y}$" denotes the inner product of vectors $\vec{x}$ and $\vec{y}$. We also define spot interest rates to an initial fixing date as:

$$f^d_{m(t)-1}(t) = \left(\frac{1}{B^d_{m(t)}(t)} - 1\right)\frac{1}{T_{m(t)} - t},$$

and

$$f^f_{m(t)-1}(t) = \left(\frac{1}{B^f_{m(t)}(t)} - 1\right)\frac{1}{T_{m(t)} - t}.$$

We assume that a variance process of a spot exchange rate denoted by $V(t)$ follows under the domestic risk neutral measure:

$$dV(t) = \xi(\eta - V(t))dt + \theta\vec{\vartheta}\cdot\sqrt{V(t)}d\vec{W}^*(t),$$

where $\vec{W}^*(t)$ is a D dimensional Brownian motion under the domestic risk neutral measure, ξ, η and θ are positive scalar parameters and $\vec{\vartheta}$ is a D dimensional constant vector with $\left\|\vec{\vartheta}\right\| = 1$ to represent the correlations between the variance and other factors. The condition $2\xi\eta > \theta^2$ ensures that the variance process $V(t)$ remains positive starting from a positive initial variance $V(0)$. This stochastic volatility model is introduced by Heston [1993] and its closed-form formula of option price with deterministic interest rates or some spot rate models is well known. Therefore, we focus on its approximate solution with a market model of interest rates.

Next, we unify numeraires of above models into the domestic discount bond $B^d_{N+1}(t)$ with maturity T_{N+1}, the currency option maturity. We first note the following relations among Brownian motions under different martingale measures:

$$d\vec{W}^d_n(t) = d\vec{W}^d_{n+1}(t) - \frac{\delta_n f^d_n(t)}{1 + \delta_n f^d_n(t)}\vec{\sigma}^d_n(t)dt,$$

$$d\vec{W}^f_n(t) = d\vec{W}^f_{n+1}(t) - \frac{\delta_n f^f_n(t)}{1 + \delta_n f^f_n(t)}\vec{\sigma}^f_n(t)dt,$$

$$d\vec{W}^f_{n+1}(t) = d\vec{W}^d_{n+1}(t) - \left\{\sum_{m\in M_{N+1}(t)}\left(\frac{\delta_m f^d_m(t)}{1 + \delta_m f^d_m(t)}\vec{\sigma}^d_m(t) - \frac{\delta_m f^f_m(t)}{1 + \delta_m f^f_m(t)}\vec{\sigma}^f_m(t)\right) + \sqrt{V(t)}\vec{\omega}\right\}dt,$$

$$d\vec{W}^*(t) = d\vec{W}^d_{N+1}(t) - \left\{ \sum_{m \in M_{N+1}(t)} \frac{\delta_m f^d_m(t)}{1+\delta_m f^d_m(t)} \vec{\sigma}^d_m(t) \right\} dt.$$

Then, we can obtain forward interest rates and variance processes under the domestic forward measure of numeraire $B^d_{N+1}(t)$:

$$\frac{df^d_n(t)}{f^d_n(t)} = -\sum_{k=n+1}^{N} \frac{\delta_k f^d_k(t)}{1+\delta_k f^d_k(t)} \vec{\sigma}^d_k(t) \cdot \vec{\sigma}^d_n(t) dt + \vec{\sigma}^d_n(t) \cdot d\vec{W}^d_{N+1}(t),$$

$$\begin{aligned} \frac{df^f_n(t)}{f^f_n(t)} = & \left\{ \begin{aligned} & \sum_{m \in M_{N+1}(t)} \left(\frac{\delta_m f^f_m(t)}{1+\delta_m f^f_m(t)} \vec{\sigma}^f_m(t) - \frac{\delta_m f^d_m(t)}{1+\delta_m f^d_m(t)} \vec{\sigma}^d_m(t) \right) \\ & - \sum_{k=n+1}^{N} \frac{\delta_k f^f_k(t)}{1+\delta_k f^f_k(t)} \vec{\sigma}^f_k(t) - \sqrt{V(t)}\vec{\omega} \end{aligned} \right\} \cdot \vec{\sigma}^f_n(t) dt \\ & + \vec{\sigma}^f_n(t) \cdot d\vec{W}^d_{N+1}(t), \end{aligned}$$

$$\begin{aligned} dV(t) = & \left\{ \xi(\eta - V(t)) - \theta\vec{\vartheta} \cdot \sqrt{V(t)} \sum_{m \in M_{N+1}(t)} \frac{\delta_m f^d_m(t)}{1+\delta_m f^d_m(t)} \vec{\sigma}^d_m(t) \right\} dt \\ & + \theta\vec{\vartheta} \cdot \sqrt{V(t)} d\vec{W}^d_{N+1}(t), \end{aligned} \tag{3}$$

where $M_{N+1}(t) = \{m(t)-1, m(t), m(t)+1, \cdots, N\}$ and $\vec{\omega}$ is a D dimensional constant vector with $\|\vec{\omega}\| = 1$ to represent the correlations between a spot exchange rate and the other factors.

Let $F_{N+1}(t)$ denote a forward exchange rate with maturity T_{N+1} at time t. The process of $F_{N+1}(t)$ under the domestic forward measure of numeraire $B^d_{N+1}(t)$ can be expressed as:

$$\frac{dF_{N+1}(t)}{F_{N+1}(t)} = \vec{\sigma}^F_{N+1}(t) \cdot d\vec{W}^d_{N+1}(t), \tag{4}$$

where

$$\begin{aligned} \vec{\sigma}^F_{N+1}(t) & = -\sum_{m \in M_{N+1}(t)} \left(\frac{\delta_m f^f_m(t)}{1+\delta_m f^f_m(t)} \vec{\sigma}^f_m(t) - \frac{\delta_m f^d_m(t)}{1+\delta_m f^d_m(t)} \vec{\sigma}^d_m(t) \right) + \sqrt{V(t)}\vec{\omega} \\ & = \vec{b}^f_{N+1}(t) - \vec{b}^d_{N+1}(t) + \sqrt{V(t)}\vec{\omega}. \end{aligned} \tag{5}$$

Here we use the notations:

$$\vec{b}^{f}_{N+1}(t) \equiv -\sum_{m\in M_{N+1}(t)} \frac{\delta_m f^{f}_m(t)}{1+\delta_m f^{f}_m(t)} \vec{\sigma}^{f}_m(t),$$
$$\vec{b}^{d}_{N+1}(t) \equiv -\sum_{m\in M_{N+1}(t)} \frac{\delta_m f^{d}_m(t)}{1+\delta_m f^{d}_m(t)} \vec{\sigma}^{d}_m(t). \qquad (6)$$

Moreover, we can add a jump component to the equation (4). For example, in the case of Merton's jump-diffusion [1976] it can be rewritten as follows:

$$\frac{dF_{N+1}(t)}{F_{N+1}(t)} = \vec{\sigma}^{F}_{N+1}(t)\cdot d\vec{W}^{d}_{N+1}(t) + dJ(t) - \lambda\kappa dt, \qquad (7)$$

where $J(t)$ denotes a compound Poisson process with intensity λ and a log-normal distribution of jump size, if K is a random variable representing jump size then $\ln(1+\mathrm{K}) \sim N(\ln(1+\kappa) - \frac{1}{2}\delta^2, \delta^2)$. Note that the third term on the right-hand side of the equation (7) is the compensator of $dJ(t)$. This model can also be viewed as the Bates model [1996] with a market model of interest rates.

3 Pricing Options

Let $C(S,K,T_{N+1})$ be the value at time 0 of a currency call option written on a spot exchange rate S with expiry date T_{N+1} and strike rate K. In order to evaluate a currency option in our model, we apply the forward measure pricing approach. Then, its discounted value is expressed as:

$$\frac{C(S,K,T_{N+1})}{B^{d}_{N+1}(0)} = E^{d}_{N+1}[(F_{N+1}(T_{N+1}) - K)^{+}],$$

where $E^{d}_{N+1}[\cdot]$ denotes the expectation operator under the domestic forward measure of numeraire $B^{d}_{N+1}(t)$.

We use a Fourier transform method for option pricing introduced by Carr and Madan [1998]. In order to apply the method to our models, we first set log-price $X(t) = \ln F_{N+1}(t)/F$ and log-strike $k = \ln K/F$ where $F = F_{N+1}(0)$. Then, the call option value is given by:

$$\frac{C(S,K,T_{N+1})}{B_{N+1}^{d}(0)} = \frac{1}{2\pi} F \int_{-\infty}^{+\infty} e^{-iku} \frac{\Phi_X(u-i)-1}{iu(1+iu)} du + (F-K)^{+}, \tag{8}$$

where $i=\sqrt{-1}$ and $\Phi_X(u)$ is the characteristic function of log-price $X(T_{N+1})$. Therefore, we concentrate on the characteristic function $\Phi_X(u)$ to use the pricing formula (8).

Applying Ito's formula to the equation (7), we can obtain the equation for log-price $X(t)$:

$$dX(t) = -\frac{1}{2}\left\| \vec{\sigma}_{N+1}^{F}(t) \right\|^{2} dt + \vec{\sigma}_{N+1}^{F}(t)\cdot d\vec{W}_{N+1}^{d}(t) + dJ^{*}(t) - \lambda\kappa dt, \tag{9}$$

where $J^{*}(t)$ follows a compound Poisson process with intensity λ and a Gaussian distribution of jump size. Unfortunately, we cannot obtain the characteristic function $\Phi_X(u)$ explicitly, because $X(t)$ is complicated process under the equation (9). Thus, we assume independence of interest rates and the foreign exchange rate, and then we apply a simple approximation to the equation (9).

First, assume that domestic and foreign forward interest rates are independent of the spot exchange rate and its variance. That is, we suppose that

$$\vec{\sigma}_{j}^{d}(t)\cdot\vec{\omega} = 0, \quad \vec{\sigma}_{j}^{f}(t)\cdot\vec{\omega} = 0,$$

and

$$\vec{\sigma}_{j}^{d}(t)\cdot\vec{\vartheta} = 0, \quad \vec{\sigma}_{j}^{f}(t)\cdot\vec{\vartheta} = 0,$$

for all $j = 1,2,\cdots N$. Under the assumptions, we can decompose the equation (6) as follows:

$$X(t) = Y(t) + Z(t) + J^{*}(t) - \lambda\kappa t,$$

where

$$dY(t) = -\frac{1}{2}\left\| \vec{b}^{f}_{N+1}(t) - \vec{b}^{d}_{N+1}(t) \right\|^2 dt + (\vec{b}^{f}_{N+1}(t) - \vec{b}^{d}_{N+1}(t)) \cdot d\vec{W}^{d}_{N+1}(t), \quad Y(0) = 0,$$
$$dZ(t) = -\frac{1}{2}V(t)dt + \sqrt{V(t)}\vec{\omega} \cdot d\vec{W}^{d}_{N+1}(t), \quad Z(0) = 0.$$

Note that $Y(t)$ and $Z(t)$ are independent. Moreover, because of the independence, the equation (3) can be simplified as:

$$dV(t) = \xi(\eta - V(t))dt + \theta\vec{\vartheta} \cdot \sqrt{V(t)}d\vec{W}^{d}_{N+1}(t).$$

Therefore, the characteristic function of $X(T_{N+1})$ is given by:

$$\Phi_X(u) = \Phi_Y(u)\Phi_Z(u)\Phi_{J^*}(u)\exp\{-iu\lambda\kappa T_{N+1}\}, \tag{10}$$

where $\Phi_Y(u)$, $\Phi_Z(u)$ and $\Phi_{J^*}(u)$ are characteristic functions of $Y(T_{N+1})$, $Z(T_{N+1})$ and $J^*(T_{N+1})$ respectively. Since $J^*(t)$ is a compound Poisson process and $Z(t)$ can be classified into affine state processes (See Duffie, Pan and Singleton [1999]), we can derive the concrete expression of $\Phi_Z(u)$ and $\Phi_{J^*}(u)$ as follows:

$$\Phi_Z(u) = \left(\cosh\frac{\gamma T_{N+1}}{2} + \frac{\xi - i\rho\theta u}{\gamma}\sinh\frac{\gamma T_{N+1}}{2} \right)^{-\frac{2\xi\eta}{\theta^2}} \exp\left\{ \frac{\xi\eta(\xi - i\rho\theta u)T_{N+1}}{\theta^2} \right\}$$
$$\times \exp\left\{ -\frac{(u^2 + iu)V(0)}{\gamma\coth\frac{\gamma T_{N+1}}{2} + \xi - i\rho\theta u} \right\},$$

where $\rho = \vec{\omega} \cdot \vec{\vartheta}$ and $\gamma = \sqrt{\theta^2(u^2 + iu) + (\xi - i\rho\theta u)^2}$,

$$\Phi_{J^*}(u) = \exp\left\{ \lambda T_{N+1}\left(\exp\left\{ -\frac{1}{2}\delta^2 u^2 + \left(\ln(1+\kappa) - \frac{1}{2}\delta^2 \right) iu \right\} - 1 \right) \right\}.$$

On the other hand, it is impossible to derive the closed-form expression of $\Phi_Y(u)$.

Next, we consider a simple approximation of $\Phi_Y(u)$. We replace processes $\delta_m f^d_m(t)/(1+\delta_m f^d_m(t))$ and $\delta_m f^f_m(t)/(1+\delta_m f^f_m(t))$ with $\delta_m f^d_m(0)/(1+\delta_m f^d_m(0))$ and $\delta_m f^f_m(0)/(1+\delta_m f^f_m(0))$ respectively. This argument often appears in some literatures to derive approximate swaption formula in a lognormal forward LIBOR model. Then, we can obtain the approximation of the equation (6) as:

$$\vec{b}^{f}_{N+1}(t) \approx \vec{b}^{f,0}_{N+1}(t) \equiv -\sum_{m\in M_{N+1}(t)} \frac{\delta_m f^f_m(0)}{1+\delta_m f^f_m(0)} \vec{\sigma}^f_m(t),$$
$$\vec{b}^{d}_{N+1}(t) \approx \vec{b}^{d,0}_{N+1}(t) \equiv -\sum_{m\in M_{N+1}(t)} \frac{\delta_m f^d_m(0)}{1+\delta_m f^d_m(0)} \vec{\sigma}^d_m(t), \tag{11}$$

and the approximate process of $Y(t)$ as:

$$dY(t) \approx dY^0(t) \equiv -\frac{1}{2}\left\| \vec{b}^{f,0}_{N+1}(t) - \vec{b}^{d,0}_{N+1}(t) \right\|^2 dt + (\vec{b}^{f,0}_{N+1}(t) - \vec{b}^{d,0}_{N+1}(t)) \cdot d\vec{W}^d_{N+1}(t). \tag{13}$$

Note that $Y^0(t)$ is a Gaussian process, because $\vec{b}^{f,0}_{N+1}(t)$ and $\vec{b}^{d,0}_{N+1}(t)$ are deterministic functions. Hence, the characteristic function $\Phi_{Y^0}(u)$ is given by:

$$\Phi_{Y^0}(u) = \exp\left\{ -\frac{1}{2}(u^2 + iu) \int_0^{T_{N+1}} \left\| \vec{b}^{f,0}_{N+1}(t) - \vec{b}^{d,0}_{N+1}(t) \right\|^2 dt \right\}.$$

Therefore, using the following approximation of the characteristic function of $X(t)$, we obtain an approximate formula of currency options with a market model of interest rates.

$$X(t) \approx X^0(t) \equiv Y^0(t) + Z(t) + J^*(t) - \lambda\kappa t, \tag{14}$$

$$\Phi_X(u) \approx \Phi_{X^0}(u) \equiv \Phi_{Y^0}(u)\Phi_Z(u)\Phi_{J^*}(u)\exp\{-iu\lambda\kappa T_{N+1}\}, \tag{15}$$

The option pricing formula (8) consists of the intrinsic value part and the Fourier transform part in terms of the characteristic function of the log-price, which can be viewed as its time value. Computing the exact value, it is important to consider the convergence problem about the integrand on the Fourier transform at infinity. In many cases, the convergence can be dramatically improved by replacing the intrinsic value part with the Black-Scholes call price with a suitable volatility in the equation (8). Thus, the approximate formula of the option price is given by:

$$\frac{C(S,K,T_{N+1})}{B^d_{N+1}(0)} \approx \frac{1}{2\pi} F \int_{-\infty}^{+\infty} e^{-iku} \frac{\Phi_{X^0}(u-i) - \Phi_0(u-i)}{iu(1+iu)} du + \frac{C_0(S,K,T_{N+1})}{B^d_{N+1}(0)}. \tag{16}$$

Here we use the notations:

$$\Phi_0(u) = \exp\left\{-\frac{\sigma_0^2 T_{N+1}}{2}(u^2 + iu)\right\},$$

$$C_0(S, K, T_{N+1}) = B_{N+1}^d(0)(FN(d_+) - KN(d_-)),$$

$$d_\pm = \frac{-k \pm \frac{1}{2}\sigma_0^2 T_{N+1}}{\sigma_0 \sqrt{T_{N+1}}},$$

$$\sigma_0 = \sqrt{\frac{1}{T_{N+1}} \int_0^{T_{N+1}} \left\| -\sum_{m \in M_{N+1}(t)} \left(\frac{\delta_m f_m^f(0)}{1 + \delta_m f_m^f(0)} \vec{\sigma}_m^f(t) - \frac{\delta_m f_m^d(0)}{1 + \delta_m f_m^d(0)} \vec{\sigma}_m^d(t) + \sqrt{V(0)}\vec{\omega} \right) \right\|^2 dt},$$

where $N(x)$ stands for the standard normal cumulative distribution function. Since σ_0 is a constant, we can interpret the first term on the right-hand side of the formula (8) as difference between the call price of our model and the Black-Scholes call price.

4 Numerical Examples

In this section we show three numerical examples using the option pricing formula. First, comparing our approximate option prices with exact prices, we verify whether we obtain sufficient accuracy of approximate solutions. Second, in order to improve the efficiency of Monte Carlo simulation, we apply the formula to control variates, which is one of variance reduction techniques. Third, we apply the formula to the calibration of volatility smiles in the JPY/USD currency option market.

4.1 Testing Accuracy of Approximate Solutions

Note that our approximation depends on replacing the processes of forward interest rates with initial forward interest rates. Because of that, we need to compare the approximate solutions with the exact option prices under different settings of interest rates. In subsection 4.1 and 4.2 we suppose that the exchange rate process follows a stochastic volatility model without a jump component. The parameters used for the model are $S = 100$, $V(0) = 0.015$, $\xi = 0.5$, $\eta = 0.015$, $\theta = 0.1$ and $\rho = -0.5$. We also assume that domestic and foreign forward interest rates are expressed as one-factor market models with constant volatilities $\sigma_n^d(t) \equiv \sigma_n^d$ and $\sigma_n^f(t) \equiv \sigma_n^f$ respectively, and $\delta_n = 0.5$ for all n. We examine the option prices in two cases where domestic and foreign interest rates are independent or their correlation is equal to 0.5. And we consider three cases for initial forward interest rates and their volatilities as follows:

Table 1: Initial domestic and foreign forward interest rates and volatilities for all n

	$f_n^d(0)$	σ_n^d	$f_n^f(0)$	σ_n^d
Case 1	0.05	0.20	0.05	0.20
Case 2	0.02	0.50	0.05	0.20
Case 3	0.05	0.20	0.02	0.50

Here we regard call prices calculated by Monte Carlo simulation with 1,000,000 sample paths as exact solutions.

Tables 2~5 show the exact and approximate call prices with 3, 5 and 10-year maturities and their errors. Figures 1~3 plot the errors against the moneyness, which shows that the longer call maturity and the wider difference of domestic and foreign interest rates, the larger errors the approximate prices suffer. When the call maturity is 3 or 5-year, the absolute errors between the exact and approximate prices are less than 0.06. However, when domestic and foreign interest rates are different and the maturity is 10-year, the errors increase. On one hand we can obtain the sufficient accuracy in terms of medium-term currency option prices, on the other hand we might need to improve the approximation method to price long-term options. It seems that there is no impact of the correlation between domestic and foreign interest rates on our approximation.

Table 2: Call prices with 3-year maturity

		interest rates' correlation = 0.0			interest rates' correlation = 0.5		
moneyness(K/F)		0.4	1	1.6	0.4	1	1.6
Case 1	exact	51.745	7.084	0.035	51.744	7.028	0.033
	approximation	51.745	7.083	0.037	51.744	7.028	0.034
	error	0.000	-0.001	0.001	0.000	0.000	0.001
	error ratio	0.000%	-0.010%	4.160%	0.000%	-0.006%	4.261%
Case 2	exact	51.745	7.088	0.035	51.744	7.032	0.032
	approximation	51.745	7.085	0.037	51.744	7.029	0.034
	error	0.000	-0.003	0.002	0.000	-0.003	0.002
	error ratio	0.000%	-0.043%	5.416%	0.000%	-0.045%	7.225%
Case 3	exact	56.530	7.746	0.037	56.530	7.684	0.035
	approximation	56.530	7.740	0.040	56.530	7.679	0.037
	error	0.000	-0.006	0.003	0.000	-0.005	0.002
	error ratio	0.000%	-0.072%	7.833%	0.000%	-0.063%	7.028%

error = approximation – exact

error ratio = error / exact

Table 3: Call prices with 5-year maturity

		interest rates' correlation = 0.0			interest rates' correlation = 0.5		
moneyness(K/F)		0.4	1	1.6	0.4	1	1.6
Case 1	exact	46.915	8.528	0.270	46.912	8.325	0.221
	approximation	46.914	8.519	0.268	46.912	8.317	0.221
	error	0.000	-0.010	-0.002	0.000	-0.008	0.000
	error ratio	0.000%	-0.112%	-0.634%	0.000%	-0.093%	0.026%
Case 2	exact	46.915	8.556	0.308	46.912	8.354	0.243
	approximation	46.914	8.524	0.270	46.912	8.320	0.222
	error	0.000	-0.032	-0.039	0.000	-0.034	-0.021
	error ratio	-0.001%	-0.370%	-12.629%	-0.001%	-0.405%	-8.709%
Case 3	exact	54.370	9.932	0.308	54.366	9.690	0.258
	approximation	54.366	9.879	0.312	54.364	9.642	0.257
	error	-0.003	-0.054	0.005	-0.002	-0.048	-0.001
	error ratio	-0.006%	-0.539%	1.518%	-0.004%	-0.495%	-0.202%

Table 4: Call prices with 10-year maturity

		interest rates' correlation = 0.0			interest rates' correlation = 0.5		
moneyness(K/F)		0.4	1	1.6	0.4	1	1.6
Case 1	exact	36.912	10.798	2.250	36.843	9.955	1.603
	approximation	36.891	10.732	2.164	36.833	9.890	1.540
	error	-0.021	-0.066	-0.086	-0.010	-0.066	-0.063
	error ratio	-0.057%	-0.612%	-3.824%	-0.026%	-0.658%	-3.902%
Case 2	exact	36.912	10.939	2.771	36.851	10.152	2.031
	approximation	36.893	10.756	2.183	36.834	9.903	1.550
	error	-0.019	-0.183	-0.588	-0.017	-0.249	-0.481
	error ratio	-0.050%	-1.672%	-21.222%	-0.047%	-2.454%	-23.687%
Case 3	exact	49.859	14.763	2.989	49.672	13.693	2.245
	approximation	49.544	14.445	2.931	49.465	13.299	2.081
	error	-0.315	-0.318	-0.058	-0.207	-0.394	-0.164
	error ratio	-0.631%	-2.154%	-1.925%	-0.416%	-2.879%	-7.311%

Figure 1: Error of call prices with 3-year maturity

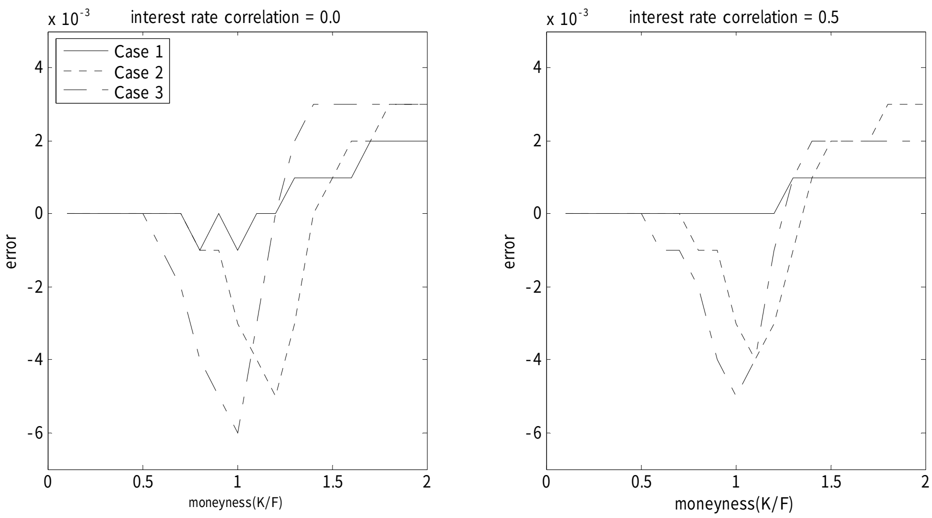


Figure 2: Error of call prices with 5-year maturity

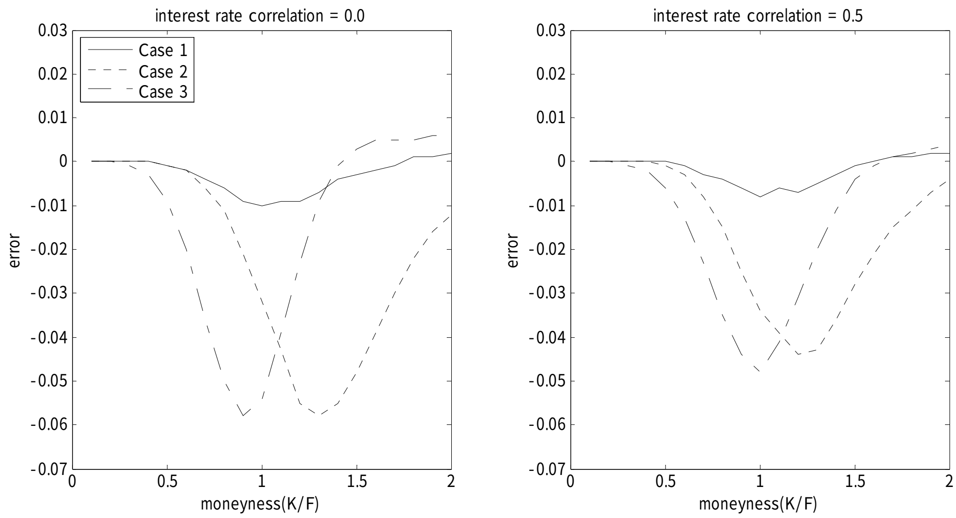
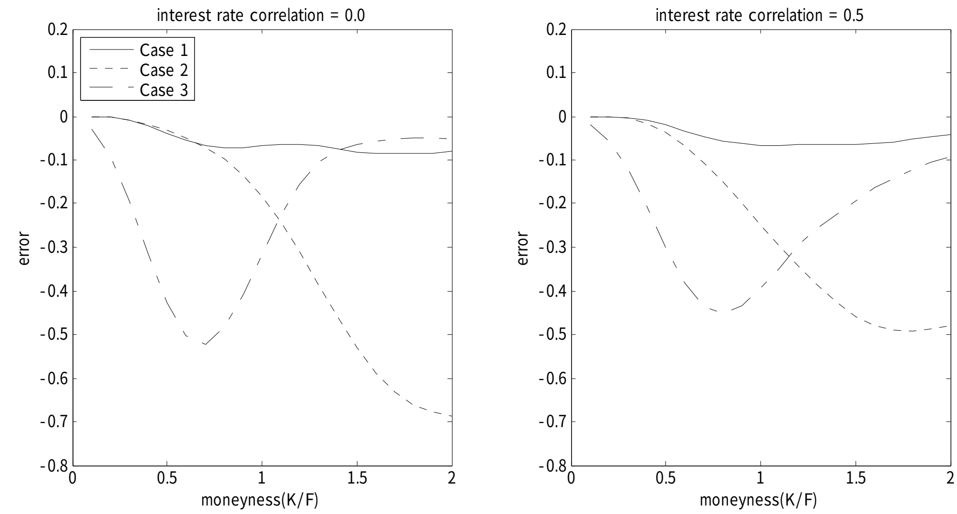


Figure 3: Error of call prices with 10-year maturity



4.2 A Control Variate Method

The method of control variates is well-known technique for improving the efficiency of Monte Carlo simulation. If we have to obtain more accurate prices in Monte Carlo simulation, we can utilize our approximate price as a control variate estimator. Since the approximate process $X^0(t)$ is close to the exact process $X(t)$, we expect that using an approximate price for the estimator is effective. To investigate the effect, we compare the convergence of our method with that of a crude Monte Carlo method; we also apply antithetic variable technique to both methods.

In the following examples, we calculate forward ATM call prices (strike rate $K = F$) with maturities T = 3, 5 and 10, and the interest rates' correlation is 0.5 under Case2 in the previous subsection. And the parameters used for the stochastic volatility model of the spot exchange rate are $S = 100$, $V(0) = 0.04$, $\xi = 0.5$, $\eta = 0.04$, $\theta = 0.18$ and $\rho = -0.5$. The procedure of the examples is as follows: (I) We generate 10,000 sample paths (trials), compute the price and take the average of prices for 10,000 trials. (II) We repeat algorithm (I) by 100 times (cases) and take the average of 100 cases. (III) We extract the result of simulation.

Tables 5~7 expresses the performance of both methods, where we show the averages, the standard deviations, the maximum and minimum values of each case. The standard deviations in our method are much smaller than those in the crude Monte Carlo method. Moreover, the maximum and minimum values in our method are very closer to the averages than those in the crude Monte Carlo method. These results show that our control variate method seems useful in Monte Carlo simulation. Figures 4~6 show the convergence of the call prices. The convergence in our method is much faster than in the crude Monte Carlo method. In particular, for the call prices with 3 and 5-year maturities, our method is extremely effective and it improves the convergence dramatically.

Table 5: Performance of control variate for ATM call price with 3-year maturity

	(A) Crude MC	(B) Control Variate	(A) / (B)
average	11.210	11.211	
standard deviation	0.269	0.008	32.825
max	11.768	11.237	
difference ratio	4.984%	0.238%	20.909
min	10.410	11.190	
difference ratio	-7.134%	-0.182%	39.297

difference ratio = (max or min – average) / average

Table 6: Performance of control variate for ATM call price with 5-year maturity

	(A) Crude MC	(B) Control Variate	(A) / (B)
average	13.137	13.101	
standard deviation	0.289	0.026	11.301
max	13.788	13.164	
difference ratio	4.951%	0.482%	10.262
min	12.348	13.034	
difference ratio	-6.008%	-0.507%	11.852

Table 7: Performance of control variate for ATM call price with 10-year maturity

	(A) Crude MC	(B) Control Variate	(A) / (B)
average	15.001	15.015	
standard deviation	0.260	0.065	3.978
max	15.688	15.157	
difference ratio	4.582%	0.941%	4.871
min	14.222	14.867	
difference ratio	-5.192%	-0.986%	5.266

Figure 4: Convergence of ATM call price with 3-year maturity

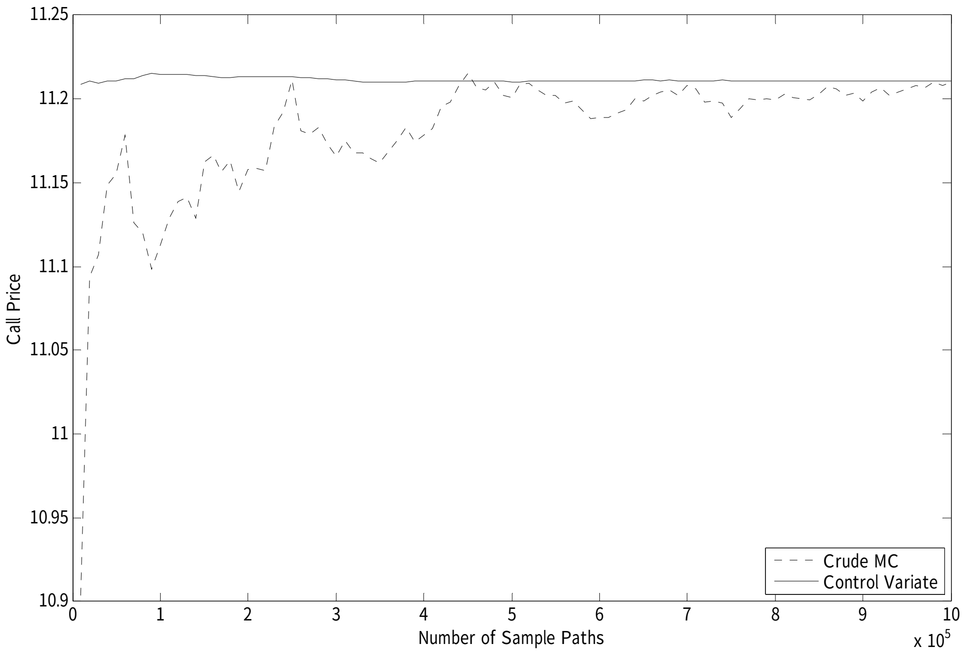


Figure 5: Convergence of ATM call price with 5-year maturity

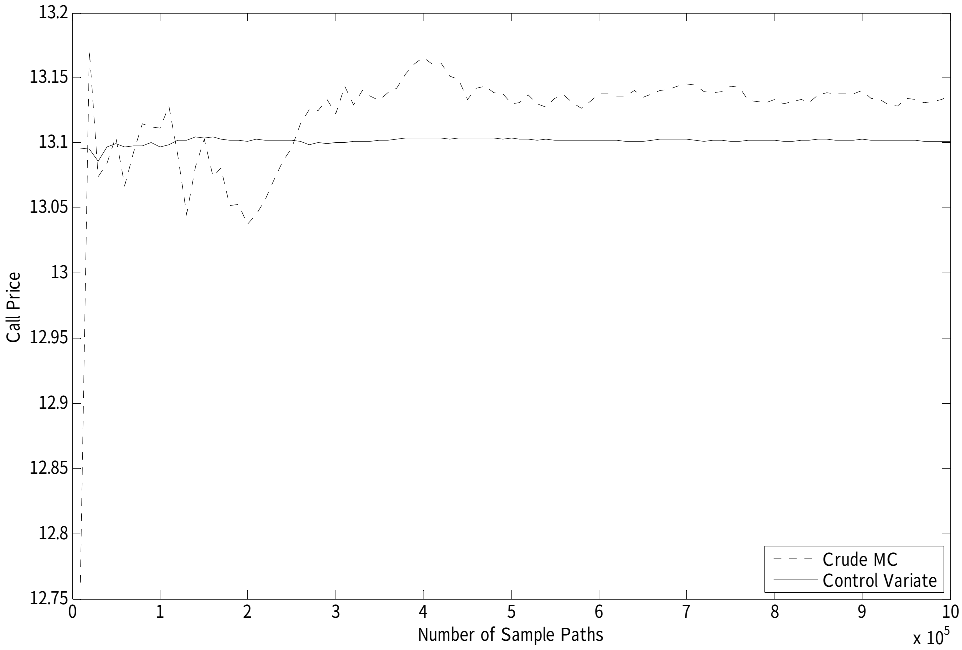
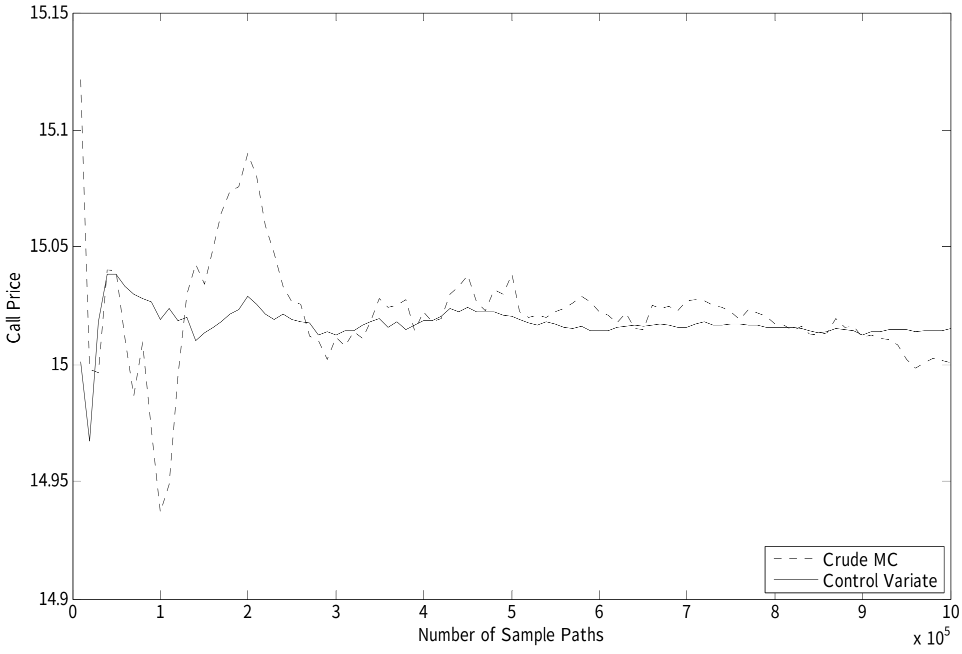


Figure 6: Convergence of ATM call price with 10-year maturity



4.3 Calibration

We calibrate our model to observed volatilities in the JPY/USD currency option market and draw volatility surfaces. Market makers in OTC currency option markets routinely provide quotes based on Black-Scholes implied volatilities and the moneyness of an option is expressed in terms of Black-Scholes delta, rather than its strike price (See Carr and Wu [2005] for the detail). Here we use 10c, 25c, 10p, 25p and ATM to denote 10-delta call, 25-delta call, 10-delta put, 25-delta put and at-the-money respectively. Therefore the observed data consist of implied volatilities against the delta of the options (10c, 25c, ATM, 25p and 10p) at five maturities (1, 2, 3, 4 and 5-year) as of April 28, 2006, May 25, 2006 and Jun 26, 2006. The data are provided by Forex Division of Mizuho Corporate Bank, Ltd. We also need information on domestic and foreign interest rates and their volatilities. We construct our forward interest rates and their volatilities using swap rates and cap volatilities data respectively downloaded from Bloomberg.

Let us suppose that the spot exchange rate follows a jump-diffusion stochastic volatility process given by the equation (7), and that domestic and foreign interest rates follow one-factor market models with constant volatilities. Under the assumptions, we calibrate the model to volatility surfaces.

Table 8 reports the estimated parameters under the restriction of $2\xi\eta > \theta^2$. Note that because observed volatility smiles are asymmetry so-called volatility skew, the parameters ρ denoting the correlation between the spot exchange rate and its variance are strongly negative. Furthermore the correlations between interest rates are strongly positive, which offsets the impact of domestic and foreign interest rates' volatilities. Figures 7~9 plot observed and model-based implied volatility surfaces and Table 9 shows the difference between model-based and observed volatilities. Although the maximum difference is 1.20%, most of the differences are less than 0.50%. Consequently, we can conclude that the model-based volatility surfaces calibrate the observed surfaces very well.

Table 8: Estimated parameters

		April 28, 2006	May 25, 2006	Jun 26, 2006
Stochastic volatility	$V(0)$	0.0074	0.0086	0.0071
	ξ	0.1660	0.3379	0.4044
	η	0.0039	0.0119	0.0098
	θ	0.0358	0.0896	0.0888
	ρ	-0.6899	-0.9990	-0.9461
Jump-diffusion	λ	0.0060	0.1218	0.0718
	κ	-0.2932	0.1096	0.1176
	δ	1.2382	0.0586	0.0531
Interest rate correlation		0.9990	0.9990	0.9990

Figure 7: Volatility surface as of April 28, 2006

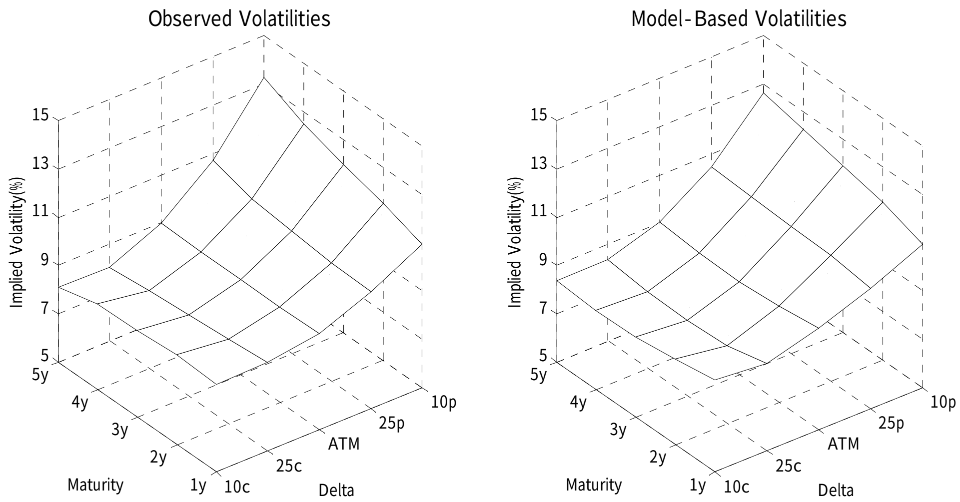


Figure 8: Volatility surface as of May 25, 2006

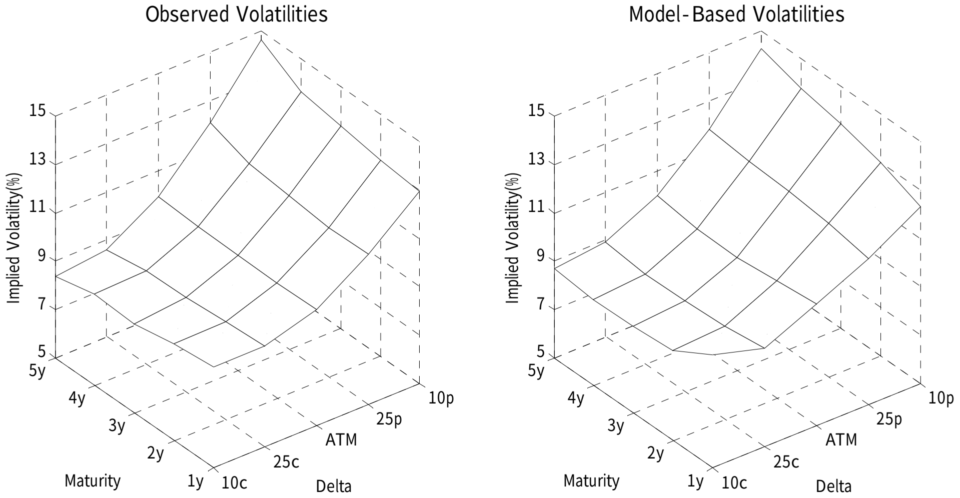


Figure 9: Volatility surface as of Jun 26, 2006

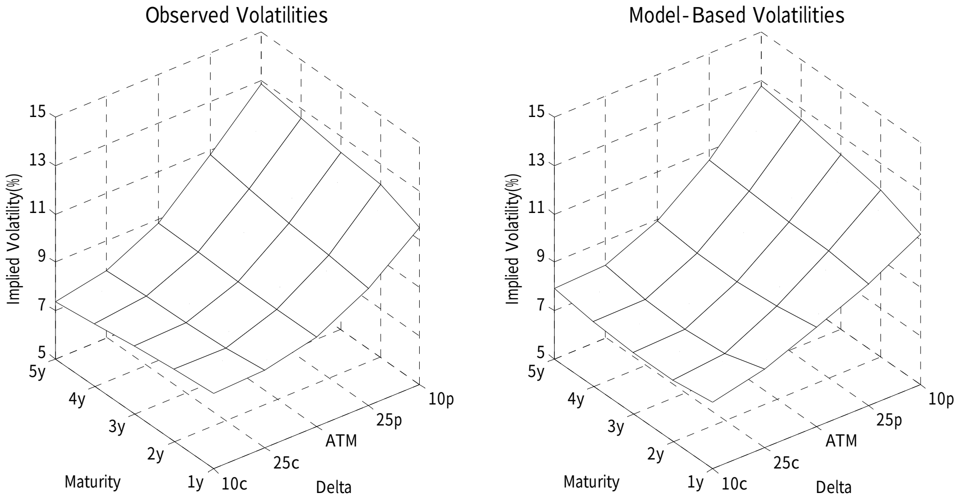


Table 9: Differences between model-based and observed implied volatilities

		10c	25c	ATM	25p	10p
1-year	28-Apr-06	-0.09%	-0.12%	-0.01%	-0.21%	0.03%
	25-May-06	-0.14%	-0.23%	0.13%	-0.25%	-0.78%
	26-Jun-06	-0.35%	-0.29%	0.14%	-0.02%	-0.52%
2-year	28-Apr-06	-0.20%	-0.07%	0.08%	0.07%	0.29%
	25-May-06	-0.08%	-0.39%	0.21%	0.10%	-0.34%
	26-Jun-06	-0.29%	-0.31%	0.19%	0.08%	-0.54%
3-year	28-Apr-06	-0.11%	-0.07%	0.09%	0.06%	0.29%
	25-May-06	0.02%	-0.26%	0.27%	0.07%	-0.31%
	26-Jun-06	-0.21%	-0.18%	0.25%	0.03%	-0.47%
4-year	28-Apr-06	-0.07%	-0.05%	0.08%	-0.02%	0.13%
	25-May-06	-0.16%	-0.13%	0.27%	-0.03%	-0.43%
	26-Jun-06	0.00%	-0.02%	0.25%	-0.17%	-0.61%
5-year	28-Apr-06	0.27%	0.04%	0.02%	-0.32%	-0.21%
	25-May-06	0.20%	0.20%	0.25%	-0.47%	-1.20%
	26-Jun-06	0.22%	0.13%	0.26%	-0.36%	-0.78%

difference = model-based volatility(%) − observed volatility(%)

5 Conclusion

In this paper, we propose an approximate solution to evaluate currency options with a market model of interest rates under jump-diffusion stochastic volatility processes of spot exchange rates. We find that our approximation could give accurate option prices except for a few cases. Moreover, using the approximate price as a control variate estimator, we improve the efficiency of Monte Carlo simulation. Finally, we calibrate the models to observed volatility surfaces in the JPY/USD currency option market.

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