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Abstract

The so-called Stein problem is addressed in the estimation of a mean vector of a multivariate normal distribution with a known covariance matrix. For general prior distributions with sphericity, the paper derives conditions on priors under which the resulting generalized Bayes estimators are minimax. It is also shown that the conditions can be expressed based on the inverse Laplace transform of the general prior. Stein's super-harmonic condition is derived from the general conditions. Finally, the priors are characterized for the admissibility.

Key words and phrases: Admissibility, decision theory, estimation, generalized Bayes estimator, inverse Laplace transform, James-Stein estimator, minimaxity, risk function, shrinkage estimation, Stein problem, uniform domination.

1 Introduction

The Stein problem is one of the most attractive topics in theoretical statistics. In the estimation of a mean vector of a multivariate normal distribution, Stein (1956) and James and Stein (1961) discovered the inadmissibility of the maximum likelihood estimator (MLE) when the dimension of the mean vector is larger than or equal to three. A considerable amount of studies have been devoted to this topic for half a century. Of these, Baranchik (1970), Brown (1971), Strawderman (1971), Alam (1973) and Berger (1976) developed classes of generalized Bayes estimators with minimaxity and/or admissibility. The classes of generalized Bayes minimax and/or admissible estimators have been extended by Faith (1978), Stein (1981), Fourdrinier, Strawderman and Wells (1998) and Maruyama (1998). These results imply a characterization of prior distributions such that the resulting generalized Bayes estimators are minimax and/or admissible. Such a characterization of prior distributions in hierarchical Bayes models has been studied by Berger and Robert (1990) and Kubokawa and Strawderman (2004) for minimaxity and by Berger and Strawderman (1996) for admissibility. Most of these studies treated the scale-mixture of normal distributions as prior distributions except for Stein (1981) who derived the super-harmonic

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condition of the general prior distributions to satisfy the minimaxity of the Bayes estimators. In this paper, we obtain a class of general prior distributions with sphericity which results in the generalized Bayes estimators with minimaxity and/or admissibility.

To explain the outlines of the paper, we describe the model and the estimation problem. Let $\boldsymbol{X} = (X_1, \dots, X_p)^t$ be a random vector distributed as $\boldsymbol{X} \sim \mathcal{N}_p(\boldsymbol{\theta}, \boldsymbol{I}_p)$ for $\boldsymbol{\theta} = (\theta_1, \dots, \theta_p)^t$ and the $p \times p$ identity matrix $\boldsymbol{I}_p$. The problem of estimating the mean vector $\boldsymbol{\theta}$ by $\widehat{\boldsymbol{\theta}}$ is considered relative to the quadratic loss $L(\boldsymbol{\theta}, \widehat{\boldsymbol{\theta}}) = \|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}\|^2$. Estimator $\widehat{\boldsymbol{\theta}}$ is evaluated in terms of the risk function $R(\boldsymbol{\theta}, \widehat{\boldsymbol{\theta}}) = E_{\boldsymbol{\theta}}[L(\boldsymbol{\theta}, \widehat{\boldsymbol{\theta}})]$. The maximum likelihood estimator of $\boldsymbol{\theta}$ is $\widehat{\boldsymbol{\theta}}_0 = \boldsymbol{X}$. Since it is minimax with a constant risk $R(\omega, \widehat{\boldsymbol{\theta}}_0) = p$, the improvement on $\widehat{\boldsymbol{\theta}}_0$ is equivalent to deriving minimax estimators but $\widehat{\boldsymbol{\theta}}_0$. To find a minimax estimator, Stein (1956) considered a class of estimators

$$\widehat{\boldsymbol{\theta}}_\psi = (1 - \psi(W)/W)\,\boldsymbol{X} \quad \text{for} \quad W = \|\boldsymbol{X}\|^2, \tag{1.1}$$

where $\psi(w)$ is a function of w. As stated in Stein (1956), this is a class of estimators equivariant under the transformation $\boldsymbol{X} \to \boldsymbol{\Gamma X}$ and $\boldsymbol{\theta} \to \boldsymbol{\Gamma\theta}$ for any $p \times p$ orthogonal matrix $\boldsymbol{\Gamma}$, namely, $\widehat{\boldsymbol{\theta}}(\boldsymbol{\Gamma X}) = \boldsymbol{\Gamma}\widehat{\boldsymbol{\theta}}(\boldsymbol{X})$. Out of the class, James and Stein (1961) found the estimator $\widehat{\boldsymbol{\theta}}^{JS} = (1 - (p-2)/W)\,\boldsymbol{X}$, and established that if $p \geq 3$, then the shrinkage estimator $\widehat{\boldsymbol{\theta}}^{JS}$ dominates $\widehat{\boldsymbol{\theta}}_0$, namely, $\widehat{\boldsymbol{\theta}}^{JS}$ is minimax. The James-Stein estimator can be further dominated by the positive-part Stein estimator, which is still inadmissible. This fact is the primary motivation to derive generalized Bayes and minimax estimators, some of which may be admissible and minimax.

In this paper, we handle the general form of prior distributions with sphericity, given by $h(\|\boldsymbol{\theta}\|^2)\mathrm{d}\boldsymbol{\theta}$. As noted in Section 2, the generalized Bayes estimator against the prior $h(\|\boldsymbol{\theta}\|^2)\mathrm{d}\boldsymbol{\theta}$ belongs to the class (1.1). In a precise sense, the generalized Bayes estimator for $h(\|\boldsymbol{\theta}\|^2)\mathrm{d}\boldsymbol{\theta}$ is identical to the generalized Bayes estimator against the prior $\pi(\lambda)\mathrm{d}\lambda = \lambda^{p/2-1}h(\lambda)\mathrm{d}\lambda$ within the equivariant class (1.1) for $\lambda = \|\boldsymbol{\theta}\|^2$. This is called the Bayes equivariant estimator in this paper and denoted by $\widehat{\boldsymbol{\theta}}^{\pi}$.

In Section 3, we obtain general conditions on $h(\lambda)$ under which $\widehat{\boldsymbol{\theta}}^{\pi}$ is minimax. For the first and second derivatives $h'(\lambda)$, $h''(\lambda)$ of $h(\lambda)$, the function $k(\lambda) \equiv \{(p-\lambda)h'(\lambda) + 2\lambda h''(\lambda)\}/h(\lambda)$ is assumed to be decomposed as $k(\lambda) = k_1(\lambda) + k_2(\lambda)$, where $k_1(\lambda)$ is a nondecreasing function of λ and $k_2(\lambda)$ is a function. Then, the general conditions are described as

(1) the first derivative of $h(\lambda)$ is not positive,

(2) $h(\lambda)$ satisfies the inequality

$$k_0 + 2\sup_{\lambda>0}\left\{\frac{(p+2-\lambda)h''(\lambda) + 2\lambda h'''(\lambda) - \{k_1(\lambda)h(\lambda)\}'}{h'(\lambda)}\right\} - \inf_{\lambda>0} k_2(\lambda) \leq p,$$

for $\{k_1(\lambda)h(\lambda)\}' = k_1'(\lambda)h(\lambda) + k_1(\lambda)h'(\lambda)$, the third derivative $h'''(\lambda)$ and a constant k_0 defined by (3.20). Under these conditions, the Bayes equivariant estimator is minimax. Especially, in the case that $k(\lambda)$ is nondecreasing, the condition (2) can be simplified as $k_0 \leq p-2$. An example using the general conditions is given in Section 3.

In Section 4, the general conditions can be expressed based on the inverse Laplace transform of $h(\lambda)$. When $h(\lambda)$ is written as $h(\lambda) = \int_0^\infty H(t)e^{-\lambda t}\mathrm{d}t$, the function $H(t)$ is called the inverse Laplace transform of $h(\lambda)$. The general conditions on $h(\lambda)$ derived in Section 3 can be rewritten by simple conditions based on the inverse Laplace transform $H(t)$. Especially, in the case that $h(\lambda)$ is completely monotone, it is known that $H(t)$ is a nonnegative function. Then, the Bayes equivariant estimator is minimax if $H(t)$ satisfies the inequality

$$K_0 + 2\sup_t K_2(t) - \inf_t K_2(t) \leq p - 3,$$

where $K(t) \equiv -(p-4)t + t(1+2t)H'(t)/H(t)$ is decomposed as $K(t) = K_1(t) + K_2(t)$ for a nonincreasing function $K_1(t)$ and a function $K_2(t)$, and K_0 is a constant defined by (4.7). This condition is similar to that of Fourdrinier *et al.* (1998), though more general conditions are provided in Section 4. When we check the conditions for the minimaxity for a given function $h(\lambda)$, the conditions given in Section 4 are not very useful, because, in general, it is hard to derive the inverse Laplace transform $H(t)$ of $h(\lambda)$. However, the conditions in Section 4 are useful for constructing prior distributions of the form $h(\lambda) = \int H(t)e^{-\lambda t}\mathrm{d}t$ such that the resulting Bayes equivariant estimators are minimax.

Section 5 explains how Stein's super-harmonic condition can be derived from the general conditions in Sections 3 and 4. Examples are given where the conditions in Section 4 do not work, but Stein's super-harmonic condition works well.

The admissibility of the Bayes equivariant estimators is studied in Section 6 based on Brown's admissibility condition. The prior distributions for the admissibility are characterized, and some examples of admissible and minimax estimators are provided.

Finally, it is remarked that the idea of using the inverse Laplace transform appeared in Kubokawa (2006) who dealt with a linear regression model with an error term having a normal distribution with an unknown variance. Since the generalized Bayes estimators are complicated in the case of the unknown variance, the estimators treated in Kubokawa (2006) were focused on a class of estimators (1.1) with monotone nondecreasing functions $\psi(\cdot)$. This paper, however, handles more general classes without assuming the monotonicity of $\psi(\cdot)$.

2 Bayes equivariant estimators

In this section, we derive the Bayes estimator within the class of equivariant estimators (1.1), called *the Bayes equivariant estimator*, and demonstrate that the Bayes equivariant estimator can be obtained as the generalized Bayes estimator against a prior distribution of $\boldsymbol{\theta}$. The minimaxity of the Bayes equivariant estimator will be discussed in the next sections.

We begin with providing the risk function of the equivariant estimator $\widehat{\boldsymbol{\theta}}_\psi$ given by (1.1). It is assumed that the function $\psi(w)$ is absolutely continuous with respect to the Lebesgue measure and satisfies that $E[\{\psi(W)\}^2/W] < \infty$. Using integration by parts called the Stein identity, Stein (1973, 1981) showed that the risk function of $\widehat{\boldsymbol{\theta}}_\psi$ is given by

$$R(\lambda, \widehat{\boldsymbol{\theta}}_\psi) = p + E\left[\frac{\psi^2(W) - 2(p-2)\psi(W)}{W} - 4\psi'(W)\right], \tag{2.1}$$

which can be expressed as

$$R(\lambda, \widehat{\boldsymbol{\theta}}_\psi) = p + \int_0^\infty \left\{\frac{\psi^2(w) - 2(p-2)\psi(w)}{w} - 4\psi'(w)\right\} g(w;\lambda)\mathrm{d}w,$$

where $g(w;\lambda)$ is a density of a noncentral chi-square distribution $\chi_p^2(\lambda)$ with p degrees of freedom and the noncentrality $\lambda = \|\boldsymbol{\theta}\|^2$, give by

$$g(w;\lambda) = 2^{-p/2}\sum_{j=0}^\infty \frac{(\lambda/2)^j}{j!\Gamma(p/2+j)2^j} e^{-\lambda/2} w^{p/2+j-1} e^{-w/2}.$$

Let $\pi(\lambda)$ be a prior distribution of λ, and the marginal density of W is given by $g_\pi(w) = \int_0^\infty g(w;\lambda)\pi(\lambda)\mathrm{d}\lambda$. Then the difference of the Bayes risks of the estimators $\widehat{\boldsymbol{\theta}}_\psi$ and $\widehat{\boldsymbol{\theta}}_0$ is written as

$$\begin{aligned}\Delta(\pi, \widehat{\boldsymbol{\theta}}_\psi) &= \int_0^\infty \left\{R(\lambda, \widehat{\boldsymbol{\theta}}_\psi) - R(\lambda, \widehat{\boldsymbol{\theta}}_0)\right\}\pi(\lambda)\mathrm{d}\lambda \\ &= \int_0^\infty \left\{\frac{\psi^2(w) - 2(p-2)\psi(w)}{w} - 4\psi'(w)\right\} g_\pi(w)\mathrm{d}w\end{aligned}$$

where this integral is assumed to be finite. By integration by parts, it is noted that

$$\begin{aligned}\int_0^\infty \psi'(w)g_\pi(w)\mathrm{d}w =& [\psi(w)g_\pi(w)]_{w=0}^\infty - \int_0^\infty \psi(w)g_\pi'(w)\mathrm{d}w \\ =& -\int_0^\infty \psi(w)g_\pi'(w)\mathrm{d}w,\end{aligned}$$

where the finiteness of $\int\{\psi^2(w)/w\}g_\pi(w)\mathrm{d}w$ implies that

$$\lim_{w\to\infty}\psi(w)g_\pi(w) = \lim_{w\to 0}\psi(w)g_\pi(w) = 0.$$

Then,

$$\Delta(\pi, \widehat{\boldsymbol{\theta}}_\psi) = \int_0^\infty \left\{\frac{\psi^2(w) - 2(p-2)\psi(w)}{w} g_\pi(w) + 4\psi(w)g_\pi'(w)\right\}\mathrm{d}w,$$

which is minimized at

$$\psi_\pi(w) = p - 2 - 2wg_\pi'(w)/g_\pi(w),$$

and we get the Bayes estimator

$$\widehat{\boldsymbol{\theta}}^\pi = (1 - \psi_\pi(W)/W)\,\boldsymbol{X}. \tag{2.2}$$

This is called the Bayes equivariant estimator, for it minimizes the Bayes risk within the class of equivariant estimators. The above expression of $\psi_\pi(w)$ was derived by Haff (1991) through the variational method. When the prior $\pi(\lambda)$ is improper, we can handle estimator (2.2) as a generalized Bayes estimator if $\psi_\pi(w)$ is finite. In this paper, we thus treat the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^{\pi}$ regardless of the finiteness of the Bayes risk and $\int \pi(\lambda)\mathrm{d}\lambda$. Another expression of $\psi_\pi(w)$ given below will be useful for deriving conditions for the minimaxity of $\widehat{\boldsymbol{\theta}}^{\pi}$. Carrying out the differentiation $g'_\pi(w)$, we may write $\psi_\pi(w)$ as

$$\begin{aligned}\psi_\pi(w) =& p-2-2w\left\{\frac{\sum_{j=0}^\infty d_j(p/2+j-1)w^{j-1}}{\sum_{j=0}^\infty d_j w^j}-\frac{1}{2}\right\}\\ =& w-2w\frac{\sum_{j=1}^\infty d_j j w^{j-1}}{\sum_{j=0}^\infty d_j w^j},\end{aligned} \tag{2.3}$$

where

$$d_j=\int_0^\infty [j!\Gamma(p/2+j)2^{2j}]^{-1}\exp\{-\lambda/2\}\lambda^j\pi(\lambda)\mathrm{d}\lambda.$$

We thus get the form

$$\psi_\pi(w)=w-2w\frac{\sum_{j=0}^\infty\{j!\Gamma(p/2+j+1)2^{2j+2}\}^{-1}w^j\int_0^\infty\lambda^{j+1}\exp\{-\lambda/2\}\pi(\lambda)\mathrm{d}\lambda}{\sum_{j=0}^\infty\{j!\Gamma(p/2+j)2^{2j}\}^{-1}w^j\int_0^\infty\lambda^{j}\exp\{-\lambda/2\}\pi(\lambda)\mathrm{d}\lambda}. \tag{2.4}$$

It may be interesting to note that the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^{\pi}$ can be derived as the generalized Bayes estimator against a spherically symmetric prior distribution of $\boldsymbol{\theta}$, given by

$$\boldsymbol{\theta}\sim\ h(\|\boldsymbol{\theta}\|^2)\mathrm{d}\boldsymbol{\theta}. \tag{2.5}$$

In fact, the generalized Bayes estimator against prior (2.5) is given by

$$\widehat{\boldsymbol{\theta}}^{GB}=\frac{\int\int\boldsymbol{\theta}\exp\{-\|\boldsymbol{X}-\boldsymbol{\theta}\|^2/2\}h(\|\boldsymbol{\theta}\|^2)\mathrm{d}\boldsymbol{\theta}}{\int\int\exp\{-\|\boldsymbol{X}-\boldsymbol{\theta}\|^2/2\}h(\|\boldsymbol{\theta}\|^2)\mathrm{d}\boldsymbol{\theta}}.$$

Using the same arguments as in Kubokawa (2006), we can show that $\widehat{\boldsymbol{\theta}}^{GB}$ is identical to the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^{\pi}$, given by (2.2), against the prior $\pi(\lambda)=\lambda^{p/2-1}h(\lambda)$. Hereafter, the prior distribution of λ is supposed to be of the form

$$\pi(\lambda)=\lambda^{p/2-1}h(\lambda), \tag{2.6}$$

namely, d_j above (2.4) is written as

$$d_j=\int_0^\infty [j!\Gamma(p/2+j)2^{2j}]^{-1}\exp\{-\lambda/2\}\lambda^{p/2+j-1}h(\lambda)\mathrm{d}\lambda. \tag{2.7}$$

and investigate the minimaxity of the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^{\pi}$.

3 General characterization of priors for minimaxity

We now address the problem of showing the minimaxity of the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^{\pi}$ against the prior $\pi(\lambda) = \lambda^{p/2-1}h(\lambda)$. In this section, we derive general sufficient conditions on $h(\lambda)$ for the minimaxity. To this end, assume the following condition.

(A.1) The function $h(\lambda)$ is three-times differentiable, and the first, second and third derivatives of $h(\lambda)$ are denoted by $h'(\lambda)$, $h''(\lambda)$ and $h'''(\lambda)$. The functions $h(\lambda)$, $h'(\lambda)$ and $h''(\lambda)$ are absolutely continuous and satisfy that $\lim_{\lambda\to 0}\lambda^{p/2}h(\lambda) = \lim_{\lambda\to 0}\lambda^{p/2}h'(\lambda) = \lim_{\lambda\to 0}\lambda^{p/2+1}h''(\lambda) = 0$ and $\lim_{\lambda\to\infty}h(\lambda)e^{-\delta\lambda} = \lim_{\lambda\to\infty}h'(\lambda)e^{-\delta\lambda} = \lim_{\lambda\to\infty}h''(\lambda)e^{-\delta\lambda} = 0$ for $0 < \delta < 1/2$.

Theorem 3.1 *Assume that $h(\lambda)$ satisfies* (A.1). *Then the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^{\pi}$ is minimax if the function $h(\lambda)$ satisfies the inequality*

$$\begin{aligned}
&2\frac{\int C_{p+2}(\lambda,w)\{(p+2-\lambda)h''(\lambda)+2\lambda h'''(\lambda)\}\mathrm{d}\lambda}{\int C_{p+2}(\lambda,w)h'(\lambda)\mathrm{d}\lambda}\\
&\qquad -\frac{\int C_{p}(\lambda,w)\{(p-\lambda)h'(\lambda)+2\lambda h''(\lambda)\}\mathrm{d}\lambda}{\int C_{p}(\lambda,w)h(\lambda)\mathrm{d}\lambda}\le p,
\end{aligned}\tag{3.1}$$

where $C_a(\lambda,w) = \sum_{j=0}^{\infty} w^j[j!\Gamma(a/2+j)2^{2j}]^{-1}\lambda^{a/2+j-1}e^{-\lambda/2}$.

Proof. From (2.3), $\psi_\pi(w)$ is written as $\psi_\pi(w) = \sum_{j=1}^{\infty} D_j w^j / \sum_{j=0}^{\infty} d_j w^j$, where $D_j = d_{j-1} - 2jd_j$. Since

$$\psi_\pi'(w) = \frac{\sum_{j=1}^{\infty} jD_jw^{j-1}}{\sum_{j=0}^{\infty} d_jw^j} - \frac{\sum_{j=1}^{\infty} D_jw^j\sum_{j=1}^{\infty} jd_jw^{j-1}}{(\sum_{j=0}^{\infty} d_jw^j)^2},$$

it is observed that

$$\begin{aligned}
\widehat{\Delta} \equiv&\{\psi_\pi(w)^2 - 2(p-2)\psi_\pi(w)\}/w - 4\psi_\pi'(w)\\
=&w\left(\frac{\sum_{j=1}^{\infty} D_jw^{j-1}}{\sum_{j=0}^{\infty} d_jw^j}\right)^2 - 2(p-2)\frac{\sum_{j=1}^{\infty} D_jw^{j-1}}{\sum_{j=0}^{\infty} d_jw^j}\\
&-4\frac{\sum_{j=1}^{\infty} jD_jw^{j-1}}{\sum_{j=0}^{\infty} d_jw^j} + 4\frac{\sum_{j=1}^{\infty} D_jw^j\sum_{j=1}^{\infty} jd_jw^{j-1}}{(\sum_{j=0}^{\infty} d_jw^j)^2}.
\end{aligned}$$

From (2.1), it follows that the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^{\pi}$ is minimax if $\widehat{\Delta}^*$ defined by

$$\begin{aligned}
\widehat{\Delta}^* \equiv&\widehat{\Delta}\sum_{j=0}^{\infty} d_jw^j / \sum_{j=1}^{\infty} D_jw^{j-1}\\
=&\frac{\sum_{j=1}^{\infty} D_jw^j}{\sum_{j=0}^{\infty} d_jw^j} - 4\frac{\sum_{j=1}^{\infty} jD_jw^{j-1}}{\sum_{j=1}^{\infty} D_jw^{j-1}} + 4\frac{\sum_{j=1}^{\infty} jd_jw^j}{\sum_{j=0}^{\infty} d_jw^j} - 2(p-2)
\end{aligned}$$

is not positive for any w. Since $2jd_j = d_{j-1} - D_j$, it is seen that $2\sum_{j=1}^{\infty} jd_jw^j / \sum_{j=0}^{\infty} d_jw^j = w - \sum_{j=1}^{\infty} D_jw^j / \sum_{j=0}^{\infty} d_jw^j$, so that $\widehat{\Delta}^*$ may be rewritten as

$$\widehat{\Delta}^* = -\frac{\sum_{j=1}^{\infty} D_jw^j}{\sum_{j=0}^{\infty} d_jw^j} + 2\frac{\sum_{j=1}^{\infty}(D_{j-1} - 2jD_j)w^j}{\sum_{j=1}^{\infty} D_jw^j} - 2(p-2), \tag{3.2}$$

where $D_0 = 0$.

We first evaluate the term D_j for $j \geq 1$. From definition (2.7) of d_j, $D_j/d_j = d_{j-1}/d_j - 2j$ is expressed as

$$D_j/d_j = 4j(p/2 + j - 1)\frac{\int_0^{\infty} \lambda^{p/2+j-2}h(\lambda)e^{-\lambda/2}\mathrm{d}\lambda}{\int_0^{\infty} \lambda^{p/2+j-1}h(\lambda)e^{-\lambda/2}\mathrm{d}\lambda} - 2j.$$

By integration by parts under assumption (A.1), it is noted that

$$\begin{aligned}
(p/2 + j - 1)&\int_0^{\infty} \lambda^{p/2+j-2}h(\lambda)e^{-\lambda/2}\mathrm{d}\lambda \\
&=[\lambda^{p/2+j-1}h(\lambda)e^{-\lambda/2}]_{\lambda=0}^{\infty} + \int_0^{\infty} \lambda^{p/2+j-1}\{h(\lambda)/2 - h'(\lambda)\}e^{-\lambda/2}\mathrm{d}\lambda \\
&=\frac{1}{2}\int_0^{\infty} \lambda^{p/2+j-1}\{h(\lambda) - 2h'(\lambda)\}e^{-\lambda/2}\mathrm{d}\lambda,
\end{aligned} \tag{3.3}$$

which yields that

$$D_j/d_j = -4j\int_0^{\infty} \lambda^{p/2+j-1}h'(\lambda)e^{-\lambda/2}\mathrm{d}\lambda / \int_0^{\infty} \lambda^{p/2+j-1}h(\lambda)e^{-\lambda/2}\mathrm{d}\lambda. \tag{3.4}$$

Using integration by parts under (A.1) again, we can demonstrate that

$$\begin{aligned}
j&\int_0^{\infty} \lambda^{j-1}\lambda^{p/2}h'(\lambda)e^{-\lambda/2}\mathrm{d}\lambda \\
&=[\lambda^{p/2+j}h'(\lambda)e^{-\lambda/2}]_{\lambda=0}^{\infty} - \frac{1}{2}\int_0^{\infty} \{(p-\lambda)h'(\lambda) + 2\lambda h''(\lambda)\}\,\lambda^{p/2+j-1}e^{-\lambda/2}\mathrm{d}\lambda \\
&= -\frac{1}{2}\int_0^{\infty} \{(p-\lambda)h'(\lambda) + 2\lambda h''(\lambda)\}\,\lambda^{p/2+j-1}e^{-\lambda/2}\mathrm{d}\lambda.
\end{aligned}$$

Hence from (3.4) and definition (2.7) of d_j, we get the expression

$$\begin{aligned}
D_j &=2\frac{\int_0^{\infty} \{(p-\lambda)h'(\lambda) + 2\lambda h''(\lambda)\}\,\lambda^{p/2+j-1}e^{-\lambda/2}\mathrm{d}\lambda}{\int_0^{\infty} \lambda^{p/2+j-1}h(\lambda)e^{-\lambda/2}\mathrm{d}\lambda}d_j \\
&=\frac{2}{j!\Gamma(p/2+j)2^{2j}}\int_0^{\infty} \{(p-\lambda)h'(\lambda) + 2\lambda h''(\lambda)\}\,\lambda^{p/2+j-1}e^{-\lambda/2}\mathrm{d}\lambda.
\end{aligned} \tag{3.5}$$

We next evaluate the term $D_{j-1} - 2jD_j$ for $j \geq 1$. From (3.4) and (2.7), the term may be written as

$$\begin{aligned}
D_{j-1} - 2jD_j &= -\left(\frac{D_{j-1}}{D_j} - 2j\right)4j\frac{\int_0^{\infty} \lambda^{p/2+j-1}h'(\lambda)e^{-\lambda/2}\mathrm{d}\lambda}{\int_0^{\infty} \lambda^{p/2+j-1}h(\lambda)e^{-\lambda/2}\mathrm{d}\lambda}d_j \\
&= -\left(\frac{D_{j-1}}{D_j} - 2j\right)\int_0^{\infty} \lambda^{p/2+j-1}h'(\lambda)e^{-\lambda/2}\mathrm{d}\lambda\frac{4j}{j!\Gamma(p/2+j)2^{2j}}.
\end{aligned} \tag{3.6}$$

Using (3.4) and (2.7) again, we observe that

$$\begin{aligned}\frac{D_{j-1}}{D_j} =& \frac{(j-1)\int_0^\infty \lambda^{p/2+j-2}h'(\lambda)e^{-\lambda/2}\mathrm{d}\lambda/\int_0^\infty \lambda^{p/2+j-2}h(\lambda)e^{-\lambda/2}\mathrm{d}\lambda}{j\int_0^\infty \lambda^{p/2+j-1}h'(\lambda)e^{-\lambda/2}\mathrm{d}\lambda/\int_0^\infty \lambda^{p/2+j-1}h(\lambda)e^{-\lambda/2}\mathrm{d}\lambda}\frac{d_{j-1}}{d_j}\\ =&4(p/2+j-1)(j-1)\frac{\int_0^\infty \lambda^{p/2+j-2}h'(\lambda)e^{-\lambda/2}\mathrm{d}\lambda}{\int_0^\infty \lambda^{p/2+j-1}h'(\lambda)e^{-\lambda/2}\mathrm{d}\lambda},\end{aligned}$$

since $d_{j-1}/d_j = 4j(p/2+j-1)\int_0^\infty \lambda^{p/2+j-2}h(\lambda)e^{-\lambda/2}\mathrm{d}\lambda/\int_0^\infty \lambda^{p/2+j-1}h(\lambda)e^{-\lambda/2}\mathrm{d}\lambda$. Then,

$$\begin{aligned}&-\left(\frac{D_{j-1}}{D_j}-2j\right)\int_0^\infty \lambda^{p/2+j-1}h'(\lambda)e^{-\lambda/2}\mathrm{d}\lambda \qquad (3.7)\\ &= -4(j-1)(p/2+j-1)\int_0^\infty \lambda^{p/2+j-2}h'(\lambda)e^{-\lambda/2}\mathrm{d}\lambda + 2j\int_0^\infty \lambda^{p/2+j-1}h'(\lambda)e^{-\lambda/2}\mathrm{d}\lambda.\end{aligned}$$

Similarly to (3.3), for $j \geq 2$, we can get the equality

$$\begin{aligned}&(p/2+j-1)\int_0^\infty \lambda^{p/2+j-2}h'(\lambda)e^{-\lambda/2}\mathrm{d}\lambda\\ &\quad = \frac{1}{2}\int_0^\infty \lambda^{p/2+j-1}\{h'(\lambda)-2h''(\lambda)\}e^{-\lambda/2}\mathrm{d}\lambda, \qquad (3.8)\end{aligned}$$

under assumption (A.1). This equality still holds for $j = 1$ since

$$\int_0^\infty \lambda^{p/2-1}e^{-\lambda/2}\{(p-\lambda)h'(\lambda)+2\lambda h''(\lambda)\}\mathrm{d}\lambda = 0, \qquad (3.9)$$

which can be derived by using integration by parts. From (3.8), the r.h.s. of equality (3.7) is rewritten as

$$\begin{aligned}&-\left(\frac{D_{j-1}}{D_j}-2j\right)\int_0^\infty \lambda^{p/2+j-1}h'(\lambda)e^{-\lambda/2}\mathrm{d}\lambda\\ &= 4(j-1)\int_0^\infty \lambda^{p/2+j-2}h''(\lambda)e^{-\lambda/2}\mathrm{d}\lambda + 2\int_0^\infty \lambda^{p/2+j-1}h'(\lambda)e^{-\lambda/2}\mathrm{d}\lambda. \qquad (3.10)\end{aligned}$$

Further using integration by parts for $j \geq 2$ gives the equality

$$\begin{aligned}&(j-1)\int_0^\infty \lambda^{p/2+j-2}h''(\lambda)e^{-\lambda/2}\mathrm{d}\lambda\\ &\quad = -\frac{1}{2}\int_0^\infty \lambda^{p/2+j-1}\left\{(p+2-\lambda)h''(\lambda)+2\lambda h'''(\lambda)\right\}e^{-\lambda/2}\mathrm{d}\lambda.\end{aligned}$$

It is noted that this equality holds for $j = 1$, because

$$\int_0^\infty \lambda^{p/2}e^{-\lambda/2}\{(p+2-\lambda)h''(\lambda)+2\lambda h'''(\lambda)\}\mathrm{d}\lambda = 0.$$

Then from (3.6) and (3.10), for $j \geq 1$, we get the expression

$$D_{j-1} - 2jD_j = \frac{8j}{j!\Gamma(p/2+j)2^{2j}} \left\{ \int_0^\infty \lambda^{p/2+j-1} h'(\lambda) e^{-\lambda/2} \mathrm{d}\lambda \right. \\ \left. - \int_0^\infty \lambda^{p/2+j-1} \left\{ (p+2-\lambda)h''(\lambda) + 2\lambda h'''(\lambda) \right\} e^{-\lambda/2} \mathrm{d}\lambda \right\}. \qquad (3.11)$$

Finally, we shall rewrite $\widehat{\Delta}^*$ given by (3.2) using expressions (3.5) and (3.11) of D_j and $D_{j-1} - 2jD_j$, respectively. Although equality (3.5) is shown for $j \geq 1$, it still holds for $j = 0$ from the equality (3.9). From this fact, (2.7) and (3.5), it follows that

$$\begin{aligned} \frac{\sum_{j=1}^\infty D_j w^j}{\sum_{j=0}^\infty d_j w^j} &= 2\frac{\sum_{j=1}^\infty c_j \int \lambda^{p/2+j-1} e^{-\lambda/2} \{(p-\lambda)h'(\lambda) + 2\lambda h''(\lambda)\} \mathrm{d}\lambda}{\sum_{j=0}^\infty c_j \int \lambda^{p/2+j-1} h(\lambda) \mathrm{d}\lambda} \\ &= 2\frac{\sum_{j=0}^\infty c_j \int \lambda^{p/2+j-1} e^{-\lambda/2} \{(p-\lambda)h'(\lambda) + 2\lambda h''(\lambda)\} \mathrm{d}\lambda}{\sum_{j=0}^\infty c_j \int \lambda^{p/2+j-1} e^{-\lambda/2} h(\lambda) \mathrm{d}\lambda} \\ &= 2\frac{\int C_p(\lambda, w) \{(p-\lambda)h'(\lambda) + 2\lambda h''(\lambda)\} \mathrm{d}\lambda}{\int C_p(\lambda, w) h(\lambda) \mathrm{d}\lambda}, \end{aligned} \qquad (3.12)$$

where $c_j = w^j/[j!\Gamma(p/2+j)2^{2j}]$. Also from (3.4) and (3.11), it is observed that

$$\begin{aligned} &\frac{\sum_{j=1}^\infty (D_{j-1} - 2jD_j) w^j}{\sum_{j=1}^\infty D_j w^j} \\ &= 2\frac{\sum_{j=1}^\infty 4jc_j \int \lambda^{p/2+j-1} \{-h'(\lambda) + (p+2-\lambda)h''(\lambda) + 2\lambda h'''(\lambda)\} e^{-\lambda/2} \mathrm{d}\lambda}{\sum_{j=1}^\infty 4jc_j \int \lambda^{p/2+j-1} h'(\lambda) e^{-\lambda/2} \mathrm{d}\lambda} \\ &= -2 + 2\frac{\int C_{p+2}(\lambda, w) \{(p+2-\lambda)h''(\lambda) + 2\lambda h'''(\lambda)\} \mathrm{d}\lambda}{\int C_{p+2}(\lambda, w) h'(\lambda) \mathrm{d}\lambda}. \end{aligned} \qquad (3.13)$$

Combining (3.12) and (3.13) yields (3.1) from (3.2). Therefore, the proof of Theorem 3.1 is complete. ■

To derive a sufficient condition on $h(\cdot)$ for the minimaxity, we use the following lemma (for the reference, see Theorem 2 in Wijsman (1985)).

Lemma 3.1 *Let X be a random variable, and let $f(x)$, $g(x)$ and $u(x)$ be functions. If both $g(x)/f(x)$ and $u(x)$ are monotone in the same direction, then the following inequality holds:*

$$E[g(X)u(X)] \cdot E[f(X)] \geq E[g(X)] \cdot E[f(X)u(X)],$$

where it is assumed that all the expectations exist and $E[f(X)] > 0$. The reversed inequality holds if $g(x)/f(x)$ and $u(x)$ are monotone in opposite directions.

Lemma 3.2 *If $b(\lambda)$ is a function of λ such that $b(\lambda)/h(\lambda)$ is nondecreasing, then the ratio of integrals*

$$\int C_p(\lambda, w) b(\lambda) \mathrm{d}\lambda \Big/ \int C_p(\lambda, w) h(\lambda) \mathrm{d}\lambda$$

is nondecreasing in w.

Proof. Differentiating $\int C_p(\lambda, w)b(\lambda)\mathrm{d}\lambda / \int C_p(\lambda, w)h(\lambda)\mathrm{d}\lambda$ with respect to w, we see that it is sufficient to show that for $c_j = w^j[j!\Gamma(p/2+j)2^{2j}]^{-1}$,

$$\sum_j jc_j \int \lambda^{p/2+j-1}b(\lambda)e^{-\lambda/2}\mathrm{d}\lambda \sum_j c_j \int \lambda^{p/2+j-1}h(\lambda)e^{-\lambda/2}\mathrm{d}\lambda$$
$$\geq \sum_j c_j \int \lambda^{p/2+j-1}b(\lambda)e^{-\lambda/2}\mathrm{d}\lambda \sum_j jc_j \int \lambda^{p/2+j-1}h(\lambda)e^{-\lambda/2}\mathrm{d}\lambda.$$

Let $f(j) = c_j \int \lambda^{p/2+j-1}h(\lambda)e^{-\lambda/2}\mathrm{d}\lambda$, $g(j) = c_j \int \lambda^{p/2+j-1}b(\lambda)e^{-\lambda/2}\mathrm{d}\lambda$ and $u(j) = j$ in Lemma 3.1. Since $u(j)$ is increasing, from Lemma 3.1, we need to show that $g(j)/f(j)$ is nondecreasing in j, namely,

$$\frac{\int \lambda^{p/2+j-1}b(\lambda)e^{-\lambda/2}\mathrm{d}\lambda}{\int \lambda^{p/2+j-1}h(\lambda)e^{-\lambda/2}\mathrm{d}\lambda} \leq \frac{\int \lambda^{p/2+j}b(\lambda)e^{-\lambda/2}\mathrm{d}\lambda}{\int \lambda^{p/2+j}h(\lambda)e^{-\lambda/2}\mathrm{d}\lambda}$$

for any j. Using Lemma 3.1 again, we can verify this inequality since both $b(\lambda)/h(\lambda)$ and λ are nondecreasing in λ. Therefore, the requested monotonicity is proved. ■

Lemma 3.3 *Assume* (A.1) *and that*

(A.2) $h(\lambda)$ *is nonincreasing.*

If $\lim_{\lambda\to 0} \lambda^{p/2}d(\lambda) = \lim_{\lambda\to\infty} e^{-\delta\lambda}d(\lambda) = 0$ *for* $0 < \delta < 1/2$*, then the following inequality holds for a differentiable function* $d(\lambda)$ *satisfying that* $d(\lambda)/h(\lambda)$ *is nondecreasing:*

$$\frac{\int C_{p+2}(\lambda, w)d'(\lambda)\mathrm{d}\lambda}{\int C_{p+2}(\lambda, w)h'(\lambda)\mathrm{d}\lambda} \leq \frac{\int C_p(\lambda, w)d(\lambda)\mathrm{d}\lambda}{\int C_p(\lambda, w)h(\lambda)\mathrm{d}\lambda}. \tag{3.14}$$

Proof. For the numerator of the r.h.s. of (3.14), integration by parts gives that

$$\int_0^\infty \lambda^{p/2+j-1}e^{-\lambda/2}d(\lambda)\mathrm{d}\lambda = \frac{1}{p/2+j}\int_0^\infty \lambda^{p/2+j}\{d(\lambda)/2 - d'(\lambda)\}\mathrm{d}\lambda. \tag{3.15}$$

Applying a similar integration by parts to the denominator, we observe that

$$\frac{\int C_p(\lambda, w)d(\lambda)\mathrm{d}\lambda}{\int C_p(\lambda, w)h(\lambda)\mathrm{d}\lambda} = \frac{\int C_{p+2}(\lambda, w)\{d(\lambda)/2 - d'(\lambda)\}\mathrm{d}\lambda}{\int C_{p+2}(\lambda, w)\{h(\lambda)/2 - h'(\lambda)\}\mathrm{d}\lambda},$$

so that inequality (3.14) can be rewritten as

$$\frac{\int C_{p+2}(\lambda, w)d'(\lambda)\mathrm{d}\lambda}{\int C_{p+2}(\lambda, w)h'(\lambda)\mathrm{d}\lambda} \leq \frac{\int C_{p+2}(\lambda, w)\{d(\lambda)/2 - d'(\lambda)\}\mathrm{d}\lambda}{\int C_{p+2}(\lambda, w)\{h(\lambda)/2 - h'(\lambda)\}\mathrm{d}\lambda},$$

which is equivalent to

$$\int C_{p+2}(\lambda, w)\{h(\lambda)/2 - h'(\lambda)\}\mathrm{d}\lambda \int C_{p+2}(\lambda, w)d'(\lambda)\mathrm{d}\lambda$$
$$\geq \int C_{p+2}(\lambda, w)\{d(\lambda)/2 - d'(\lambda)\}\mathrm{d}\lambda \int C_{p+2}(\lambda, w)h'(\lambda)\mathrm{d}\lambda,$$

since $h'(\lambda) \leq 0$. This inequality can be simplified as

$$\int C_{p+2}(\lambda, w) d'(\lambda) \mathrm{d}\lambda \int C_{p+2}(\lambda, w) h(\lambda) \mathrm{d}\lambda$$
$$\geq \int C_{p+2}(\lambda, w) d(\lambda) \mathrm{d}\lambda \int C_{p+2}(\lambda, w) h'(\lambda) \mathrm{d}\lambda. \tag{3.16}$$

By integration by parts, it can be shown that

$$\int C_{p+2}(\lambda, w) h'(\lambda) \mathrm{d}\lambda = - \int C'_{p+2}(\lambda, w) h(\lambda) \mathrm{d}\lambda,$$
$$\int C_{p+2}(\lambda, w) d'(\lambda) \mathrm{d}\lambda = - \int C'_{p+2}(\lambda, w) d(\lambda) \mathrm{d}\lambda,$$

where $C'_{p+2}(\lambda) = (\mathrm{d}/\mathrm{d}\lambda) C_{p+2}(\lambda)$. Hence, inequality (3.16) may be rewritten as

$$\int C'_{p+2}(\lambda, w) d(\lambda) \mathrm{d}\lambda \int C_{p+2}(\lambda, w) h(\lambda) \mathrm{d}\lambda$$
$$\leq \int C_{p+2}(\lambda, w) d(\lambda) \mathrm{d}\lambda \int C'_{p+2}(\lambda, w) h(\lambda) \mathrm{d}\lambda. \tag{3.17}$$

Let $f(\lambda) = C_{p+2}(\lambda, w) h(\lambda)$, $g(\lambda) = C_{p+2}(\lambda, w) d(\lambda)$ and $u(\lambda) = C'_{p+2}(\lambda, w)/C_{p+2}(\lambda, w)$ in Lemma 3.1. Since $g(\lambda)/f(\lambda) = d(\lambda)/h(\lambda)$ is nondecreasing, from Lemma 3.1, we need to check that $C'_{p+2}(\lambda, w)/C_{p+2}(\lambda, w)$ is nonincreasing in λ. Since $C_{p+2}(\lambda, w) = \sum_{j=0}^{\infty} a_j \lambda^{p/2+j} e^{-\lambda/2}$ for $a_j = w^j [j! \Gamma(p/2 + j + 1) 2^{2j}]^{-1}$, it is observed that

$$\frac{C'_{p+2}(\lambda, w)}{C_{p+2}(\lambda, w)} = \frac{\sum_j a_j \{(p/2 + j)\lambda^{p/2+j-1} - (1/2)\lambda^{p/2+j}\} e^{-\lambda/2}}{\sum_j a_j \lambda^{p/2+j} e^{-\lambda/2}}$$
$$= \frac{\sum_j a_j (p/2 + j) \lambda^{j-1}}{\sum_j a_j \lambda^j} - \frac{1}{2}. \tag{3.18}$$

Note that

$$\sum_{j=0}^{\infty} a_j (p/2 + j) \lambda^{j-1} = \frac{1}{\Gamma(p/2)\lambda} + \frac{w}{4} \sum_{j=1}^{\infty} \frac{w^{j-1} \lambda^{j-1}}{j(j-1)! \Gamma(p/2 + j) 2^{(j-1)}}$$
$$= \frac{1}{\Gamma(p/2)\lambda} + \frac{w}{4} \sum_{j=0}^{\infty} \frac{1}{j+1} a_j \lambda^j.$$

Then from (3.18), it is sufficient to show that $\sum_{j=0}^{\infty} (j+1)^{-1} a_j \lambda^j / \sum_{j=0}^{\infty} a_j \lambda^j$ is nonincreasing in λ. Since $1/(j+1)$ is decreasing in j, this monotonicity follows from the problem 4(i) in Lehmann (1986, p.428). Hence, we obtain inequality (3.17), which proves inequality (3.14) of Lemma 3.3. ■

Define $k(\lambda)$ by

$$k(\lambda) = \frac{(p - \lambda) h'(\lambda) + 2\lambda h''(\lambda)}{h(\lambda)}, \tag{3.19}$$

and assume that it is decomposed as

$$k(\lambda) = k_1(\lambda) + k_2(\lambda),$$

where $k_1(\lambda)$ is a nondecreasing and differentiable function of λ and $k_2(\lambda)$ is a function. Let k_0 be a constant such that

$$k_0 \geq \lim_{w\to\infty} \frac{\sum_j w^j [j!\Gamma(p/2+j)2^{2j}]^{-1} \int \lambda^{p/2+j-1} k_1(\lambda)h(\lambda)e^{-\lambda/2}\mathrm{d}\lambda}{\sum_j w^j [j!\Gamma(p/2+j)2^{2j}]^{-1} \int \lambda^{p/2+j-1} e^{-\lambda/2} h(\lambda)\mathrm{d}\lambda}. \tag{3.20}$$

Combining Theorem 3.1 and Lemma 3.3, we obtain sufficient conditions given by the following theorem.

Theorem 3.2 *Assume conditions* (A.1) *and* (A.2) *and that* $\lim_{\lambda\to 0} \lambda^{p/2} k_1(\lambda)h(\lambda) = \lim_{\lambda\to\infty} k_1(\lambda)h(\lambda)e^{-\cdots}$ 0 *for* $0 < \delta < 1/2$. *Then the Bayes equivariant estimator* $\widehat{\boldsymbol{\theta}}^{\pi}$ *is minimax if* $h(\lambda)$ *satisfies the inequality*

$$\begin{aligned} k_0 + 2\sup_{j\geq 0} & \frac{\int [(p+2-\lambda)h''(\lambda) + 2\lambda h'''(\lambda) - \{k_1(\lambda)h(\lambda)\}']\lambda^{p/2+j-1}e^{-\lambda/2}\mathrm{d}\lambda}{\int h'(\lambda)\lambda^{p/2+j-1}e^{-\lambda/2}\mathrm{d}\lambda} \\ & - \inf_{j\geq 0} \frac{\int k_2(\lambda)h(\lambda)\lambda^{p/2+j-1}e^{-\lambda/2}\mathrm{d}\lambda}{\int h(\lambda)\lambda^{p/2+j-1}e^{-\lambda/2}\mathrm{d}\lambda} \leq p. \end{aligned} \tag{3.21}$$

This inequality is satisfied if

$$k_0 + 2\sup_{\lambda>0}\left\{\frac{(p+2-\lambda)h''(\lambda) + 2\lambda h'''(\lambda) - \{k_1(\lambda)h(\lambda)\}'}{h'(\lambda)}\right\} - \inf_{\lambda>0} k_2(\lambda) \leq p, \tag{3.22}$$

where $\{k_1(\lambda)h(\lambda)\}' = k_1'(\lambda)h(\lambda) + k_1(\lambda)h'(\lambda)$.

Proof. From (3.1) of Theorem 3.1, it is observed that

$$\begin{aligned} & \frac{\int C_p(\lambda,w)k_1(\lambda)h(\lambda)\mathrm{d}\lambda}{\int C_p(\lambda,w)h(\lambda)\mathrm{d}\lambda} - \frac{\int C_p(\lambda,w)k_2(\lambda)h(\lambda)\mathrm{d}\lambda}{\int C_p(\lambda,w)h(\lambda)\mathrm{d}\lambda} \\ & + 2\left\{\frac{\int C_{p+2}(\lambda,w)\{(p+2-\lambda)h''(\lambda) + 2\lambda h'''(\lambda)\}\mathrm{d}\lambda}{\int C_{p+2}(\lambda,w)h'(\lambda)\mathrm{d}\lambda} - \frac{\int C_p(\lambda,w)k_1(\lambda)h(\lambda)\mathrm{d}\lambda}{\int C_p(\lambda,w)h(\lambda)\mathrm{d}\lambda}\right\} \leq p. \end{aligned} \tag{3.23}$$

For the first term in the l.h.s. of (3.23), from Lemma 3.2, it follows that

$$\sup_w \frac{\int C_p(\lambda,w)k_1(\lambda)h(\lambda)\mathrm{d}\lambda}{\int C_p(\lambda,w)h(\lambda)\mathrm{d}\lambda} = \lim_{w\to\infty} \frac{\int C_p(\lambda,w)k_1(\lambda)h(\lambda)\mathrm{d}\lambda}{\int C_p(\lambda,w)h(\lambda)\mathrm{d}\lambda}.$$

The second term is evaluated as

$$\frac{\int C_p(\lambda,w)k_2(\lambda)h(\lambda)\mathrm{d}\lambda}{\int C_p(\lambda,w)h(\lambda)\mathrm{d}\lambda} \geq \inf_\lambda k_2(\lambda).$$

Using (3.14) of Lemma 3.3 for $d(\lambda) = k_1(\lambda)h(\lambda)$, we have the inequality

$$\frac{\int C_{p+2}(\lambda,w)\{k_1(\lambda)h(\lambda)\}'\mathrm{d}\lambda}{\int C_{p+2}(\lambda,w)h'(\lambda)\mathrm{d}\lambda} \leq \frac{\int C_p(\lambda,w)k_1(\lambda)h(\lambda)\mathrm{d}\lambda}{\int C_p(\lambda,w)h(\lambda)\mathrm{d}\lambda}.$$

Using this inequality, we can see that the third term in the l.h.s. of (3.23) is less than or equal to

$$2 \sup_{j \geq 0} \frac{\int [(p+2-\lambda)h''(\lambda) + 2\lambda h'''(\lambda) - \{k_1(\lambda)h(\lambda)\}']\lambda^{p/2+j-1}e^{-\lambda/2}\mathrm{d}\lambda}{\int h'(\lambda)\lambda^{p/2+j-1}e^{-\lambda/2}\mathrm{d}\lambda},$$

which is also less than or equal to

$$2 \sup_{\lambda>0} \left\{ \frac{(p+2-\lambda)h''(\lambda) + 2\lambda h'''(\lambda) - \{k_1(\lambda)h(\lambda)\}'}{h'(\lambda)} \right\}.$$

Combining these results gives conditions (3.21) and (3.22). ■

Letting $k_1(\lambda) = 0$, we get a simple condition from Theorem 3.1.

Corollary 3.1 *Assume* (A.1) *and* (A.2)*. Then condition* (3.1) *holds if* $h(\lambda)$ *satisfies the inequality*

$$2 \sup_{\lambda} \frac{(p+2-\lambda)h''(\lambda) + 2\lambda h'''(\lambda)}{h'(\lambda)} - \inf_{\lambda} \frac{(p-\lambda)h'(\lambda) + 2\lambda h''(\lambda)}{h(\lambda)} \leq p.$$

When $k(\lambda)$ is nondecreasing, namely, in the case of $k(\lambda) = k_1(\lambda)$, we get the following proposition.

Proposition 3.1 *Assume that the function* $h(\lambda)$ *satisfies* (A.1) *and* (A.2) *for* $p \geq 3$*. Also assume that* $k(\lambda) = \{(p-\lambda)h'(\lambda) + 2\lambda h''(\lambda)\}/h(\lambda)$ *is nondecreasing in* λ*. Then* $\psi_\pi(w)$ *is nondecreasing in* w *and the Bayes equivariant estimator* $\widehat{\boldsymbol{\theta}}^\pi$ *is minimax if* $k_0 \leq p-2$ *for* k_0 *defined by* (3.20)*.*

Proof. The assumption in Proposition 3.1 corresponds to the case of $k(\lambda) = k_1(\lambda)$ or $k_2(\lambda) = 0$ in Theorem 3.2. Noting that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\{k(\lambda)h(\lambda)\} = -h'(\lambda) + (p+2-\lambda)h''(\lambda) + 2\lambda h'''(\lambda),$$

we observe that

$$2 \sup_{\lambda>0} \left\{ \frac{(p+2-\lambda)h''(\lambda) + 2\lambda h'''(\lambda) - \{k(\lambda)h(\lambda)\}'}{h'(\lambda)} \right\} = 2.$$

Hence, the condition of Proposition 3.1 is derived from Theorem 3.2. The monotonicity of $\psi_\pi(w)$ follows from Lemma 3.2. ■

Example 3.1 (Prior related to a multivariate t-distribution) Consider a prior distribution with a density proportional to $(b + \|\boldsymbol{\theta}\|^2)^{-c}\mathrm{d}\boldsymbol{\theta}$ for nonnegative constants b and c. Let $\pi(\lambda) = \lambda^{p/2-1}h(\lambda)$ for $h(\lambda) = (b+\lambda)^{-c}$. To check condition (3.22), we need the first, second and third derivatives of $h(\lambda)$, given by $h'(\lambda) = -c(b+\lambda)^{-c-1}$,

$h''(\lambda) = c(c+1)(b+\lambda)^{-c-2}$ and $h'''(\lambda) = -c(c+1)(c+2)(b+\lambda)^{-c-3}$. Then, $k(\lambda)$ defined by (3.19) has the form

$$k(\lambda) = c\left\{1 - \frac{b-2c+p-2}{b+\lambda} - \frac{2(c-1)b}{(b+\lambda)^2}\right\}. \tag{3.24}$$

When $b-2c+p-2 \geq 0$, the function $k(\lambda)$ is nondecreasing. Taking $k_0 = c$ for k_0 defined by (3.20), we see from Proposition 3.1 that the Bayes equivariant estimator is minimax if the constants b and c satisfy the condition

$$0 < c \leq \min\left\{p-2, \frac{p-2+b}{2}\right\} \quad \text{and} \quad b \geq 0. \tag{3.25}$$

We next consider the case that $b - 2c + p - 2 < 0$ and $b \geq 0$. In this case, $k(\lambda)$ is decomposed as $k(\lambda) = k_1(\lambda) + k_2(\lambda)$ for $k_1(\lambda) = c - 2c(c+1)b/(b+\lambda)^2$ and $k_2(\lambda) = c(2c-b-p+2)/(b+\lambda)$. Then we use Theorem 3.2 to derive a condition for the minimaxity. Since $\{k_1(\lambda)h(\lambda)\}' = -c^2/(b+\lambda)^{c+1} + 2c(c+1)(c+2)b/(b+\lambda)^{b+3}$, it can be seen that

$$\frac{(p+2-\lambda)h''(\lambda) + 2\lambda h'''(\lambda) - \{k_1(\lambda)h(\lambda)\}'}{h'(\lambda)} = 1 + (2c-b-p+2)\frac{c+1}{b+\lambda},$$

so that condition (3.22) is expressed by

$$k_0 + 2\sup_{\lambda>0}\left\{1 + (2c-b-p+2)\frac{c+1}{b+\lambda}\right\} - \inf_{\lambda>0}\frac{c(2c-b-p+2)}{b+\lambda} \leq p.$$

Since $k_0 = c$, it is easy to see that condition (3.22) holds if

$$c + 2(2c-b-p+2)(c+1)/b \leq p-2.$$

By solving this inequality, the condition for the minimaxity is given by

$$\begin{aligned} &\frac{b+p-2}{2} < c < \frac{1}{8}\left\{b+2p-8+\sqrt{(b+2p)^2+16(p-1)b}\right\}, \\ &0 < b < p-2. \end{aligned} \tag{3.26}$$

It can be guaranteed that there exists a c satisfying the above inequalities in (3.26) if $0 < b < p-2$.

Combining the above arguments, we conclude that the Bayes equivariant estimator is minimax if the constants b and c satisfy either (3.25) or (3.26). ■

Remark 3.1 The conditions in Theorem 3.2 may be helpful for constructing prior distributions such that the resulting Bayes equivariant estimators can be minimax. Let $k_1(\lambda)$ be a nondecreasing function and assume that there exists a constant k_0 satisfying condition (3.20). Let $k_2(\lambda)$ be an integrable function satisfying condition (3.21) or (3.22) of Theorem 3.2. Then, denote $k(\lambda) = k_1(\lambda) + k_2(\lambda)$ and solve the differential equation

$$\frac{(p-\lambda)h'(\lambda) + 2\lambda h''(\lambda)}{h(\lambda)} = k(\lambda). \tag{3.27}$$

Letting $\ell(\lambda) = (\mathrm{d}/\mathrm{d}\lambda) \log h(\lambda) = h'(\lambda)/h(\lambda)$, we can rewrite equation (3.27) as

$$\ell'(\lambda) + \ell^2(\lambda) - \frac{\lambda - p}{2\lambda}\ell(\lambda) = \frac{k(\lambda)}{2\lambda}, \tag{3.28}$$

which is the Riccati differential equation. In the case that there exists a particular solution, denoted by $\ell_0(\lambda)$, of equation (3.28), we can get the general solution of (3.28). First, let $y(\lambda) = \ell(\lambda) - \ell_0(\lambda)$ and $p(\lambda) = -2\ell_0(\lambda) + (\lambda - p)/(2\lambda)$. Then, (3.28) is rewritten by $y'(\lambda) + y^2(\lambda) - p(\lambda)y(\lambda) = 0$, or

$$\frac{y'(\lambda)}{y^2(\lambda)} - p(\lambda)\frac{1}{y(\lambda)} = -1, \tag{3.29}$$

which is the Bernoulli differential equation. By letting $z(\lambda) = 1/y(\lambda)$ again, equation (3.29) leads to the linear differential equation

$$z'(\lambda) + p(\lambda)z(\lambda) = 1,$$

which has the general solution

$$z(\lambda) = \left\{ \int_{\lambda_1}^{\lambda} \exp\left\{ \int_{\lambda_0}^{t} p(s)\mathrm{d}s \right\} \mathrm{d}t + C \right\} \exp\left\{ - \int_{\lambda_0}^{\lambda} p(s)\mathrm{d}s \right\},$$

where λ_0, λ_1 and C are constants. Since $\ell(\lambda) = \ell_0(\lambda) + 1/z(\lambda)$, this solution gives the general solution of the Riccati equation (3.28), given by

$$\ell(\lambda) = \ell_0(\lambda) + \frac{\exp\{\int_{\lambda_0}^{\lambda}[-2\ell_0(s) + (s-p)/(2s)]\mathrm{d}s\}}{\int_{\lambda_1}^{\lambda} \exp\{\int_{\lambda_0}^{t}[-2\ell_0(s) + (s-p)/(2s)]\mathrm{d}s\}\mathrm{d}t + C}.$$

In general, it is hard to find out a particular solution $\ell_0(\lambda)$. However, this idea of solving the Riccati equation possesses a possibility of extending a class of prior distributions. For example, consider the prior distribution treated in Example 3.1, namely, $h_0(\lambda) = 1/(\lambda + b)^c$, where the notation $h_0(\lambda)$ is used here instead of $h(\lambda)$. Then, $k(\lambda)$ is given by (3.24). This means that $\ell_0(\lambda) = -c/(\lambda+b)$ is a particular solution of the Riccati equation (3.28) when $k(\lambda)$ is given by (3.24). In this case, the general solution of (3.28) is

$$\ell(\lambda) = -\frac{c}{\lambda + b} + A(\lambda),$$

where

$$A(\lambda) = \frac{(\lambda + b)^{2c}\lambda^{-p/2}e^{\lambda/2}}{\int_{\lambda_1}^{\lambda}(s+b)^{2c}s^{-p/2}e^{s/2}\mathrm{d}s + C}.$$

Since $\ell(\lambda) = (\mathrm{d}/\mathrm{d}\lambda)\log h(\lambda)$, the general solution provides the solution of $h(\lambda)$ as

$$h(\lambda) = \exp\left\{ \int_{\lambda_2}^{\lambda} [-c/(t+b) + A(t)]\mathrm{d}t \right\} = \frac{C_0}{(\lambda + b)^c} \exp\left\{ \int_{\lambda_2}^{\lambda} A(t)\mathrm{d}t \right\}, \tag{3.30}$$

where C_0 and λ_2 are positive constants. As stated above, function (3.30) provides the same quantity of $k(\lambda)$ as in (3.24). Hence, the Bayes equivariant estimator for (3.30) would be minimax under the same condition (3.25) if $h(\lambda)$ given by (3.30) satisfies (A.1) and (A.2), though we need another hard work to check these conditions for function (3.30). ■

4 Expressions based on inverse Laplace transforms

The general conditions on the function $h(\lambda)$ have been derived in Section 3 for the minimaxity of the Bayes equivariant estimator. When $h(\lambda)$ has an inverse Laplace transform, denoted by $H(t)$, the general conditions can be expressed based on the inverse Laplace transform $H(t)$. This expression is not only useful for checking the minimaxity, but also helpful for constructing prior distributions which result in the generalized Bayes and minimax estimators.

For a nonnegative function $h(\lambda)$, it is assumed that

(B.1) there exists a function $H(t)$ such that

$$h(\lambda) = \int_0^\infty H(t)e^{-t\lambda}\mathrm{d}t,$$

and $H(t)$ satisfies the following conditions: $\lim_{\lambda\to 0} \lambda^{p/2-2}H(s/\lambda) = 0$ for $s > 0$ and there exists an integrable function $\Phi(s)$ such that $\lambda^{p/2-2}|H(s/\lambda)|s^n e^{-s} \le \Phi(s)$ for $n = 0, 1, 2$ and small $\lambda > 0$. Also, $\lim_{t\to 0} tH(t) = \lim_{t\to\infty} e^{-\lambda t}H(t) = 0$ for $t > 0$ and $\int t^n|H(t)|e^{-\lambda t}\mathrm{d}t < \infty$ for $n = 0, 1, 2, 3$.

The function $H(t)$ can be derived by the *inverse Laplace transformation*, defined by

$$H(t) = \lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^{R} h(\lambda + i\,\tau)e^{(\lambda+i\,\tau)t}\mathrm{d}\tau,$$

for $i = \sqrt{-1}$. The inverse Laplace transformation is guaranteed under the integrability $\int_0^\infty |H(t)e^{-t\lambda}|\mathrm{d}t < \infty$. Another derivation of $H(t)$ is given by

$$H(t) = \lim_{n\to\infty} \frac{(-1)^n}{n!}\left(\frac{n}{t}\right)^{n+1} h^{(n)}\left(\frac{n}{t}\right),$$

called *the Post's inverse formula*, where $h^{(n)}(x) = (\mathrm{d}^n/\mathrm{d}x^n)h(x)$. Since $\int_0^\infty \pi(\lambda)\mathrm{d}\lambda = \int_0^\infty H(t)\int_0^\infty \lambda^{p/2-1}e^{-t\lambda}\mathrm{d}\lambda\mathrm{d}t = \Gamma(p/2)\int_0^\infty t^{-p/2}H(t)\mathrm{d}t$ for $\pi(\lambda) = \lambda^{p/2-1}h(\lambda)$, it is seen that the prior $\pi(\lambda)$ is proper if $\int_0^\infty t^{-p/2}H(t)\mathrm{d}t < \infty$.

The inverse Laplace transform allows us to rewrite the function $\psi_\pi(w)$ based on an integral expression. The following lemma is useful for the purpose.

Lemma 4.1 *For a positive constant a and a function $f(t)$, the following equation holds:*

$$\begin{aligned}&\sum_{j=0}^\infty \frac{w^j}{j!\Gamma(a+j)2^{2j}}\int_0^\infty\int_0^\infty \lambda^{a+j-1}e^{-(1+2t)\lambda/2}f(t)\mathrm{d}\lambda\mathrm{d}t\\ =&2^a\int_0^\infty \frac{1}{(1+2t)^a}e^{w/[2(1+2t)]}f(t)\mathrm{d}t,\end{aligned} \tag{4.1}$$

where it is assumed that $\int_0^\infty\int_0^\infty \lambda^{a+j-1}e^{-(1+2t)\lambda/2}|f(t)|\mathrm{d}\lambda\mathrm{d}t < \infty$.

In fact, integrating out the l.h.s. of (4.1) with respect to λ yields

$$2^a \int_0^\infty \frac{1}{(1+2t)^a} \sum_{j=0}^\infty \left(\frac{w}{2(1+2t)} \right)^j f(t) \mathrm{d}t.$$

Noting that $\sum_j (j!)^{-1} \{w/[2(1+2t)]\}^j = \exp\{w/[2(1+2t)]\}$, we get the r.h.s. of (4.1) and Lemma 4.1 is verified.

Applying Lemma 4.1 to both the numerator and the denominator of the second term in (2.4), we can rewrite it as

$$\begin{aligned}\psi_\pi(w) =& w - w \frac{\int_0^\infty (1+2t)^{-p/2-1} \exp\{w/[2(1+2t)]\} H(t) \mathrm{d}t}{\int_0^\infty (1+2t)^{-p/2} \exp\{w/[2(1+2t)]\} H(t) \mathrm{d}t} \\ =& w \frac{\int_0^\infty 2t(1+2t)^{-p/2-1} \exp\{w/[2(1+2t)]\} H(t) \mathrm{d}t}{\int_0^\infty (1+2t)^{-p/2} \exp\{w/[2(1+2t)]\} H(t) \mathrm{d}t},\end{aligned}$$

which is equal to

$$\psi_\pi(w) = w \frac{\int_0^\infty 2t(1+2t)^{-p/2-1} \exp\{-wt/(1+2t)\} H(t) \mathrm{d}t}{\int_0^\infty (1+2t)^{-p/2} \exp\{-wt/(1+2t)\} H(t) \mathrm{d}t}.$$

Using the inverse Laplace transform, we now replace the condition in Theorem 3.1. Define the functions $K(t)$ and $q(t; w)$ by

$$\begin{aligned}K(t) =& -(p-4)t + t(1+2t)H'(t)/H(t), \\ q(t; w) =& (1+2t)^{-p/2} H(t) \exp\{w/[2(1+2t)]\}.\end{aligned} \tag{4.2}$$

Theorem 4.1 *Under assumption* (B.1)*, the Bayes equivariant estimator* $\widehat{\boldsymbol{\theta}}^\pi$ *is minimax if the following inequality is satisfied for any* $w > 0$*:*

$$2 \frac{\int K(t) t (1+2t)^{-1} q(t; w) \mathrm{d}t}{\int t(1+2t)^{-1} q(t; w) \mathrm{d}t} - \frac{\int K(t) q(t; w) \mathrm{d}t}{\int q(t; w) \mathrm{d}t} \le p - 3. \tag{4.3}$$

To prove Theorem 4.1, the following fundamental property of the Laplace transformation is useful: for positive integer n,

$$\int t^n H(t) e^{-t\lambda} \mathrm{d}t = (-\mathcal{D})^n h(\lambda) = (-1)^n h^{(n)}(\lambda),$$

where $\mathcal{D} = \mathrm{d}/\mathrm{d}t$ and $h^{(n)}(\lambda) = \mathrm{d}^n h(\lambda)/\mathrm{d}t^n$, the n-th derivative of $h(\lambda)$.

Proof. We begin with proving the following equalities:

$$\int K(t) H(t) e^{-\lambda t} \mathrm{d}t = -h(\lambda) + (p-\lambda) h'(\lambda) + 2\lambda h''(\lambda), \tag{4.4}$$

$$\int t K(t) H(t) e^{-\lambda t} \mathrm{d}t = 2h'(\lambda) - (p+2) h''(\lambda) + \lambda h''(\lambda) - 2\lambda h'''(\lambda). \tag{4.5}$$

To evaluate the term $\int K(t)H(t)e^{-\lambda t}\mathrm{d}t$, it is noted that the term is written by $-(p-4)\int tH(t)e^{-\lambda t}\mathrm{d}t + \int t(1+2t)H'(t)e^{-\lambda t}\mathrm{d}t$. By integration by parts,

$$\begin{aligned}\int t(1+2t)H'(t)e^{-\lambda t}\mathrm{d}t =& \big[t(1+2t)e^{-\lambda t}H(t)\big]_{t=0}^{\infty}\\ &- \int (1+4t)H(t)e^{-\lambda t}\mathrm{d}t + \lambda\int t(1+2t)H(t)e^{-\lambda t}\mathrm{d}t\\ =& -h(\lambda) - 4h_1(\lambda) + \lambda h_1(\lambda) + 2\lambda h_2(\lambda),\end{aligned}$$

where $h_n(\lambda) = \int t^n H(t)e^{-\lambda t}\mathrm{d}t$ for a positive integer n. Since $h_n(\lambda) = (-1)^n(\mathrm{d}^n/\mathrm{d}\lambda^n)h(\lambda)$,

$$\int t(1+2t)H'(t)e^{-\lambda t}\mathrm{d}t = -h(\lambda) + 4h'(\lambda) - \lambda h'(\lambda) + 2\lambda h''(\lambda),$$

so that equality (4.4) is obtained. For the second equality, note that

$$\int tK(t)H(t)e^{-\lambda t}\mathrm{d}t = -(p-4)\int t^2 H(t)e^{-\lambda t}\mathrm{d}t + \int t^2(1+2t)H'(t)e^{-\lambda t}\mathrm{d}t,$$

and the same arguments as discussed above can be applied. By integration by parts,

$$\begin{aligned}\int t^2(1+2t)H'(t)e^{-\lambda t}\mathrm{d}t =& \big[t^2(1+2t)e^{-\lambda t}H(t)\big]_{t=0}^{\infty}\\ &- 2\int (t+3t^2)H(t)e^{-\lambda t}\mathrm{d}t + \lambda\int t^2(1+2t)H(t)e^{-\lambda t}\mathrm{d}t,\end{aligned}$$

so that

$$\int tK(t)H(t)e^{-\lambda t}\mathrm{d}t = -2h_1(\lambda) - (p+2)h_2(\lambda) + \lambda h_2(\lambda) + 2\lambda h_3(\lambda),$$

which leads to equality (4.5).

Using Lemma 4.1, we can see that the following equalities hold:

$$\begin{aligned}&\frac{\int K(t)q(t;w)\mathrm{d}t}{\int q(t;w)\mathrm{d}t}\\ &\quad= \frac{\sum_j w^j[j!\Gamma(p/2+j)2^{2j}]^{-1}\int \lambda^{p/2+j-1}e^{-\lambda/2}\int K(t)H(t)e^{-\lambda t}\mathrm{d}t\mathrm{d}\lambda}{\sum_j w^j[j!\Gamma(p/2+j)2^{2j}]^{-1}\int \lambda^{p/2+j-1}e^{-\lambda/2}\int H(t)e^{-\lambda t}\mathrm{d}t\mathrm{d}\lambda},\end{aligned}$$

and

$$\begin{aligned}&\frac{\int K(t)t(1+2t)^{-1}q(t;w)\mathrm{d}t}{\int t(1+2t)^{-1}q(t;w)\mathrm{d}t}\\ &\quad= \frac{\sum_j w^j[j!\Gamma(p/2+j+1)2^{2j}]^{-1}\int \lambda^{p/2+j}e^{-\lambda/2}\int tK(t)H(t)e^{-\lambda t}\mathrm{d}t\mathrm{d}\lambda}{\sum_j w^j[j!\Gamma(p/2+j+1)2^{2j}]^{-1}\int \lambda^{p/2+j}e^{-\lambda/2}\int tH(t)e^{-\lambda t}\mathrm{d}t\mathrm{d}\lambda}.\end{aligned}$$

It is noted that assumption (A.1) can be satisfied by (B.1). In fact, from the condition $\lim_{\lambda\to 0}\lambda^{p/2-2}H(s/\lambda) = 0$ and the dominated convergence theorem, it follows that $\lim_{\lambda\to 0}\lambda^{p/2}h(\lambda) = \lim_{\lambda\to 0}\lambda^{p/2}h'(\lambda) = \lim_{\lambda\to 0}\lambda^{p/2+1}h''(\lambda) = 0$. Also note that for $\lambda > \lambda_1$, $\int t^n|H(t)|e^{-\lambda t}\mathrm{d}te^{-\delta\lambda} \le \int t^n|H(t)|e^{-\lambda_1 t}\mathrm{d}te^{-\delta\lambda}$, which tends to zero as $\lambda \to \infty$, because $\int t^n|H(t)|e^{-\lambda_1 t}\mathrm{d}t$ is finite from (B.1). This implies that $\lim_{\lambda\to\infty} h(\lambda)e^{-\delta\lambda} = \lim_{\lambda\to\infty} h'(\lambda)e^{-\delta\lambda} = \lim_{\lambda\to\infty} h''(\lambda)e^{-\delta\lambda} = 0$ for $0 < \delta < 1/2$. Hence, condition (4.3) can be derived from the above equalities and Theorem 3.1 by noting equalities (4.4) and (4.5). The proof of Theorem 4.1 is therefore complete. ■

Assume that $K(t)$ is decomposed as

$$K(t) = K_1(t) + K_2(t),$$

where $K_1(t)$ is a nonincreasing function of t and $K_2(t)$ is a function. Let K_0 be a constant such that

$$K_0 \ge \lim_{w\to\infty} \int K_1(t)q(t;w)\mathrm{d}t \Big/ \int q(t;w)\mathrm{d}t. \tag{4.6}$$

Making the transformation $s = 2wt/(1+2t)$, we can express inequality (4.6) as

$$K_0 \ge \lim_{w\to\infty} \frac{\int_0^w K_1(s/[2(w-s)])H(s/[2(w-s)])(1-s/w)^{p/2-2}e^{-s/2}\mathrm{d}s}{\int_0^w H(s/[2(w-s)])(1-s/w)^{p/2-2}e^{-s/2}\mathrm{d}s}, \tag{4.7}$$

which is useful for getting the limiting value.

Theorem 4.2 *Assume* (B.1) *and that*

(B.2) $\int tH(t)e^{-\lambda t}\mathrm{d}t$ *is a nonnegative function.*

Then the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^{\pi}$ *is minimax if the following inequality is satisfied:*

$$\begin{aligned} K_0 + 2\sup_w &\frac{\int K_2(t)t(1+2t)^{-p/2-1}H(t)\exp\{w/[2(1+2t)]\}\mathrm{d}t}{\int t(1+2t)^{-p/2-1}H(t)\exp\{w/[2(1+2t)]\}\mathrm{d}t} \\ &- \inf_w \frac{\int K_2(t)(1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}\mathrm{d}t}{\int (1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}\mathrm{d}t} \le p-3. \end{aligned} \tag{4.8}$$

Further, if

(B.2′) $H(t)$ *is a nonnegative function,*

then inequality (4.8) *is satisfied under the condition*

$$K_0 + 2\sup_t K_2(t) - \inf_t K_2(t) \le p-3. \tag{4.9}$$

In the case that $K(t) = K_1(t)$, $\widehat{\boldsymbol{\theta}}^{\pi}$ *is minimax when* $K_0 \le p-3$.

Proof. From the monotonicity of $K_1(t)$, it follows that

$$\frac{\int K_1(t)t(1+2t)^{-1}q(t;w)\mathrm{d}t}{\int t(1+2t)^{-1}q(t;w)\mathrm{d}t} \le \frac{\int K_1(t)q(t;w)\mathrm{d}t}{\int q(t;w)\mathrm{d}t}.$$

In fact, this inequality can be verified by putting $f(t) = q(t;w)$, $g(t) = t(1+2t)^{-1}q(t;w)$ and $u(t) = K_1(t)$ in Lemma 3.1. Thus, condition (4.3) holds if for any $w > 0$,

$$\frac{\int K_1(t)q(t;w)\mathrm{d}t}{\int q(t;w)\mathrm{d}t} + 2\frac{\int K_2(t)t(1+2t)^{-1}q(t;w)\mathrm{d}t}{\int t(1+2t)^{-1}q(t;w)\mathrm{d}t} - \frac{\int K_2(t)q(t;w)\mathrm{d}t}{\int q(t;w)\mathrm{d}t} \le p-3. \tag{4.10}$$

We here show that the ratio of integrals $\int K_1(t)q(t;w)\mathrm{d}t / \int q(t;w)\mathrm{d}t$ is nondecreasing in w. The derivative with respect to w is proportional to

$$\int \frac{K_1(t)}{1+2t}q(t;w)\mathrm{d}t \int q(t;w)\mathrm{d}t - \int K_1(t)q(t;w)\mathrm{d}t \int \frac{1}{1+2t}q(t;w)\mathrm{d}t. \tag{4.11}$$

Letting $f(t) = q(t;w)$, $g(t) = K_1(t)q(t;w)$ and $u(t) = 1/(1+2t)$ and noting that both $g(t)/f(t) = K_1(t)$ and $u(t) = 1/(1+2t)$ are nonincreasing, we can see that quantity (4.11) is nonnegative, so that the ratio of integrals $\int K_1(t)q(t;w)\mathrm{d}t / \int q(t;w)\mathrm{d}t$ is nondecreasing in w. Hence,

$$\sup_{w>0} \frac{\int K_1(t)q(t;w)\mathrm{d}t}{\int q(t;w)\mathrm{d}t} = \lim_{w\to\infty} \frac{\int K_1(t)q(t;w)\mathrm{d}t}{\int q(t;w)\mathrm{d}t}. \tag{4.12}$$

Therefore, condition (4.8) in Theorem 4.2 is obtained from (4.10) and (4.12). It can be easily verified that inequality (4.9) implies inequality (4.8). ■

It is noted that assumption (B.2′) is equivalent to the function $h(\lambda)$ being completely monotone (see Feller (1971)). Then, condition (4.9) is similar to that of Fourdrinier *et al.* (1998). When $K(t)$ is nonincreasing, namely, in the case of $K(t) = K_1(t)$, we get the following proposition from Theorem 4.2.

Proposition 4.1 *Assume that the function $H(t)$ satisfies conditions* (B.1) *and* (B.2′) *for $p \ge 3$. Also assume that $K(t) = -(p-4)t + t(1+2t)H'(t)/H(t)$ is nonincreasing in t. Then $\psi_\pi(w)$ is nondecreasing in w and the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^\pi$ is minimax if $K_0 \le p-3$ for K_0 defined by* (4.6) *or* (4.7).

Theorem 4.2 provides a class of prior distributions such that the resulting Bayes equivariant estimators can be minimax. Let $K_1(t)$ be a nonincreasing function and assume that there exists a constant K_0 such that

$$K_0 \ge \lim_{w\to\infty} \int K_1(t)q(t;w)\mathrm{d}t \Big/ \int q(t:w)\mathrm{d}t,$$

which is also described by (4.7). Let $K_2(t)$ be an integrable function satisfying condition (4.8) or (4.9). Then, denote $K(t) = K_1(t) + K_2(t)$ and solve the differential equation

$$-(p-4)t + t(1+2t)\frac{\mathrm{d}}{\mathrm{d}t}\log H(t) = K(t),$$

for a positive function $H(t)$. A solution of this equation is given by

$$H(t) = (1+2t)^{(p-4)/2} \exp\left\{ \int_{t_0}^{t} \frac{K(x)}{x(1+2x)}\mathrm{d}x \right\},$$

where t_0 is a positive constant. Then the Bayes equivariant estimator against the prior $h(\|\boldsymbol{\theta}\|^2) = \int H(t) \exp\{-\|\boldsymbol{\theta}\|^2 t\} \mathrm{d}t$ is minimax if $H(t)$ satisfies (B.1).

Applying Theorem 4.2 to the prior distribution treated in Example 3.1, we can get the same conditions as in Example 3.1. In fact, the function $h(\lambda)$ can be expressed by $h(\lambda) = (b+\lambda)^{-c} = \int_0^\infty H(t)e^{-t\lambda}\mathrm{d}t$ for $H(t) = t^{c-1}e^{-bt}$, which gives

$$K(t) = -2bt^2 - (b - 2c + p - 2)t + c - 1.$$

Using condition (4.8) in Theorem 4.2, we can derive the same conditions as in (3.25) and (3.26) for the minimaxity of the Bayes equivariant estimator. Another example is given below.

Example 4.1 (Scale mixture of a normal distribution) Consider the scale mixture of the normal distribution

$$\begin{aligned} \boldsymbol{\theta}|t \sim & \mathcal{N}_p\left(\mathbf{0}, (2t)^{-1}\boldsymbol{I}_p\right), \\ t \sim & \frac{t^{b-2}}{(1+2t)^a}\nu(t)\mathrm{d}t, \quad t > 0, \end{aligned} \tag{4.13}$$

for constants a, b and a function $\nu(t)$ satisfying the conditions

(NM-1) $a > b$ and $1 - p/2 + a < b \leq (p-2)/2$,

(NM-2) the function $\nu(t)$ is nonnegative, differentiable and bounded.

The function $h(\lambda)$ is given by $h(\lambda) = \int_0^\infty H(t)e^{-\lambda t}\mathrm{d}t$ for $H(t) = t^{p/2+b-2}(1+2t)^{-a}\nu(t)$ where the normalization constant is omitted. It can be verified that assumptions (B.1) and (B.2$'$) are satisfied under the conditions (NM-1) and (NM-2). From the arguments between (2.5) and (2.6), the generalized Bayes estimator against prior (4.13) is the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^\pi$ against the prior $\pi(\lambda) = \lambda^{p/2-1}h(\lambda)$. The function $K(t)$ defined by (4.2) may be written as

$$\begin{aligned} K(t) =& -(2a+p-4)t + (p/2+b-2)(1+2t) + t(1+2t)\nu'(t)/\nu(t) \\ =& 2(b-a)t + (p/2+b-2) + t(1+2t)\nu'(t)/\nu(t). \end{aligned}$$

When $(1+2t)t\nu'(t)/\nu(t)$ is nonincreasing in t, the function $K(t)$ is nonincreasing under the conditions (NM-1) and (NM-2). Noting that the constant K_0 defined by (4.6) or (4.7) is given by $K_0 = p/2 + b - 2$, we see from Proposition 4.1 that the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^\pi$ against prior (4.13) is minimax if $(1+2t)t\nu'(t)/\nu(t)$ is nonincreasing in t.

When $(1+2t)t\nu'(t)/\nu(t)$ does not have a monotonicity property, let $K_1(t) = 2(b-a)t + (p/2+b-2)$ and $K_2(t) = t(1+2t)\nu'(t)/\nu(t)$. From condition (4.9), it follows that the Bayes equivariant estimator is minimax if

$$b + 2\sup_t \frac{(1+2t)t\nu'(t)}{\nu(t)} - \inf_t \frac{(1+2t)t\nu'(t)}{\nu(t)} \leq (p-2)/2. \tag{4.14}$$

For a suitable function $m(t)$, solve the differential equation

$$(1+2t)t\nu'(t)/\nu(t) = m(t).$$

A solution of this equation is given by

$$\nu(t) = \exp\left\{\int_{t_0}^{t} \frac{m(s)}{(1+2s)s}\mathrm{d}s\right\},$$

for a positive constant t_0. Take the function $m(t)$ such that $\nu(t)$ is bounden for any t. Then, $\nu(t)$ satisfies the condition (NM-2) and condition (4.14) is expressed by

$$b + 2\sup_t m(t) - \inf_t m(t) \le (p-2)/2. \tag{4.15}$$

For example, consider the function $m(t) = -2ct/(1+t^2)$ for $c > 0$. Then, $\nu(t) \le 1$, namely, $\nu(t)$ is bounded, and condition (4.15) holds if $b + c \le (p-2)/2$.

In the case that $m(t) = (1+2t)t\nu'(t)/\nu(t)$ is nondecreasing in t, condition (4.14) can be used. However, we can derive a better condition from (4.8) by using the monotonicity property of $m(t)$. From the monotonicity of $K_2(t) = m(t)$, it is noted that

$$\frac{\int m(t)t(1+2t)^{-p/2-1}H(t)\exp\{w/[2(1+2t)]\}\mathrm{d}t}{\int t(1+2t)^{-p/2-1}H(t)\exp\{w/[2(1+2t)]\}\mathrm{d}t}$$

is nonincreasing in w, so that

$$\begin{aligned}&\sup_w \frac{\int m(t)t(1+2t)^{-p/2-1}H(t)\exp\{w/[2(1+2t)]\}\mathrm{d}t}{\int t(1+2t)^{-p/2-1}H(t)\exp\{w/[2(1+2t)]\}\mathrm{d}t}\\ &= \frac{\int m(t)t(1+2t)^{-p/2-1}H(t)\mathrm{d}t}{\int t(1+2t)^{-p/2-1}H(t)\mathrm{d}t}.\end{aligned}$$

Similary,

$$\begin{aligned}&\inf_w \frac{\int m(t)(1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}\mathrm{d}t}{\int (1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}\mathrm{d}t}\\ &= \lim_{w\to\infty} \frac{\int m(t)(1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}\mathrm{d}t}{\int (1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}\mathrm{d}t}.\end{aligned}$$

Since $K_0 = p/2 + b - 2$, condition (4.8) can be rewritten as

$$\begin{aligned}b &+ 2\frac{\int m(t)t(1+2t)^{-p/2-1}H(t)\mathrm{d}t}{\int t(1+2t)^{-p/2-1}H(t)\mathrm{d}t}\\ &- \lim_{w\to\infty} \frac{\int m(t)(1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}\mathrm{d}t}{\int (1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}\mathrm{d}t} \le (p-2)/2.\end{aligned} \tag{4.16}$$

For example, consider the increasing function $m(t) = 2ct/(1+2t)$ for $c > 0$. Then, the function $\nu(t)$ is written by

$$\nu(t) = \exp\left\{2c\int_{t_0}^{t}(1+2s)^{-2}\mathrm{d}s\right\} = C_0\exp\{-c/(1+2t)\},$$

which is bounded. It can be verified that

$$\lim_{w\to\infty} \frac{\int m(t)(1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}\mathrm{d}t}{\int (1+2t)^{-p/2}H(t)\exp\{w/[2(1+2t)]\}\mathrm{d}t} = 0.$$

Hence, condition (4.16) can be expressed as

$$b + 4c\frac{\int_0^\infty t^{p/2+b}(1+2t)^{-p/2-a-2}\exp\{-c/(1+2t)\}\mathrm{d}t}{\int_0^\infty t^{p/2+b-1}(1+2t)^{-p/2-a-1}\exp\{-c/(1+2t)\}\mathrm{d}t} \le (p-2)/2,$$

or

$$b + 2c\frac{\int_0^1 z^{a-b}(1-z)^{p/2+b}\exp\{-cz\}\mathrm{d}z}{\int_0^1 z^{a-b}(1-z)^{p/2+b-1}\exp\{-cz\}\mathrm{d}z} \le (p-2)/2, \tag{4.17}$$

which is derived by making the transformation $z = 1/(1+2t)$. On the other hand, (4.15) yields the condition that $b + 2c \le (p-2)/2$, which is not better than (4.17), although we need to resort to numerical computation to check condition (4.17). ■

5 Derivation of Stein's super-harmonic condition

In this section, we shall provide another expression of condition (3.1) and clarify the relationship between condition (3.1) and the super-harmonic condition of the prior density.

Theorem 5.1 *Assume condition* (A.1). *Then condition* (3.1) *is equivalent to*

$$2\frac{\int C_p(\lambda,w)s(\lambda)\mathrm{d}\lambda}{\int C_p(\lambda,w)\int_\lambda^\infty h'(t)e^{-t/2}\mathrm{d}te^{\lambda/2}\mathrm{d}\lambda} + \frac{\int C_p(\lambda,w)\{-\lambda h'(\lambda)+s(\lambda)\}\mathrm{d}\lambda}{\int C_p(\lambda,w)h(\lambda)\mathrm{d}\lambda} \ge 0, \tag{5.1}$$

where

$$s(\lambda) = ph'(\lambda) + 2\lambda h''(\lambda). \tag{5.2}$$

Proof. From (3.15), it is observed that for an absolutely continuous function $f(\lambda)$,

$$\int_0^\infty C_{p+2}(\lambda,w)\{f(\lambda)/2 - f'(\lambda)\}\mathrm{d}\lambda = \int_0^\infty C_p(\lambda,w)f(\lambda)\mathrm{d}\lambda.$$

We here consider the differential equation

$$f(\lambda)/2 - f'(\lambda) = h'(\lambda),$$

which has a solution of the form $f(\lambda) = \int_\lambda^\infty h'(t)e^{-t/2}\mathrm{d}t\, e^{\lambda/2}$. Then under condition (A.1), we get the equality

$$\int C_{p+2}(\lambda,w)h'(\lambda)\mathrm{d}\lambda = \int C_p(\lambda,w)\int_\lambda^\infty h'(t)e^{-t/2}\mathrm{d}t\, e^{\lambda/2}\mathrm{d}\lambda. \tag{5.3}$$

Since $s'(\lambda) = (p+2)h''(\lambda) + 2\lambda h'''(\lambda)$, the same argument is used to get that

$$\begin{aligned}\int & C_{p+2}(\lambda, w)\left\{(p+2-\lambda)h''(\lambda) + 2\lambda h'''(\lambda)\right\} \mathrm{d}\lambda \\ &= \int C_{p+2}(\lambda, w)\left\{-\lambda h''(\lambda) + s'(\lambda)\right\} \mathrm{d}\lambda \\ &= \int C_p(\lambda, w) \int_\lambda^\infty \left\{-t h''(t) + s'(t)\right\} e^{-t/2} \mathrm{d}t\, e^{\lambda/2} \mathrm{d}\lambda.\end{aligned} \tag{5.4}$$

By integration by parts, it is noted that

$$\int_\lambda^\infty s'(t) e^{-t/2} \mathrm{d}t = -s(\lambda)e^{-\lambda/2} + \frac{1}{2}\int_\lambda^\infty s(t)e^{-t/2}\mathrm{d}t,$$

which gives that

$$\int_\lambda^\infty \left\{-th''(t) + s'(t)\right\} e^{-t/2}\mathrm{d}t = -s(\lambda)e^{-\lambda/2} + \frac{p}{2}\int_\lambda^\infty h'(t)e^{-t/2}\mathrm{d}t.$$

Hence from (5.4),

$$\begin{aligned}\int & C_{p+2}(\lambda, w)\left\{(p+2-\lambda)h''(\lambda) + 2\lambda h'''(\lambda)\right\} \mathrm{d}\lambda \\ &= -\int C_p(\lambda, w)s(\lambda)\mathrm{d}\lambda + \frac{p}{2}\int C_p(\lambda, w)\int_\lambda^\infty h'(t)e^{-t/2}\mathrm{d}t\, e^{\lambda/2}\mathrm{d}\lambda.\end{aligned} \tag{5.5}$$

Combining (5.3) and (5.5), we obtain condition (5.1) from Theorem 3.1. ■

Assume that $s(\lambda)$ is decomposed as

$$s(\lambda) = ph'(\lambda) + 2\lambda h''(\lambda) = s_1(\lambda) + s_2(\lambda),$$

where $s_1(\lambda) \le 0$ and $s_2(\lambda) > 0$ for any $\lambda > 0$. Then from Theorem 5.1, we get the following condition.

Proposition 5.1 *Assume* (A.1) *and* (A.2). *Then, the Bayes equivariant estimator* $\widehat{\boldsymbol{\theta}}^\pi$ *is minimax if*

$$2\inf_\lambda\left\{\frac{s_2(\lambda)e^{-\lambda/2}}{\int_\lambda^\infty h'(t)e^{-t/2}\mathrm{d}t}\right\} + \inf_\lambda\left\{\frac{-\lambda h'(\lambda) - s_1(\lambda) + s_2(\lambda)}{h(\lambda)}\right\} \ge 0. \tag{5.6}$$

If $\{(\mathrm{d}/\mathrm{d}\lambda)[s_2(\lambda)e^{-\lambda/2}]\}/\{h'(\lambda)e^{-\lambda/2}\}$ *is nonincreasing, then*

$$\inf_\lambda\left\{\frac{s_2(\lambda)e^{-\lambda/2}}{\int_\lambda^\infty h'(t)e^{-t/2}\mathrm{d}t}\right\} = \lim_{\lambda\to 0}\left\{\frac{s_2(\lambda)e^{-\lambda/2}}{\int_\lambda^\infty h'(t)e^{-t/2}\mathrm{d}t}\right\}. \tag{5.7}$$

Proof. Noting that

$$\int_\lambda^\infty h'(t)e^{-t/2}\mathrm{d}t = -h(\lambda)e^{-\lambda/2} + \frac{1}{2}\int_\lambda^\infty h(t)e^{-t/2}\mathrm{d}t \geq -h(\lambda)e^{-\lambda/2},$$

we observe that

$$\frac{\int C_p(\lambda, w)s_1(\lambda)\mathrm{d}\lambda}{\int C_p(\lambda, w)\int_\lambda^\infty h'(t)e^{-t/2}\mathrm{d}te^{\lambda/2}\mathrm{d}\lambda} \geq -\frac{\int C_p(\lambda, w)s_1(\lambda)\mathrm{d}\lambda}{\int C_p(\lambda, w)h(\lambda)\mathrm{d}\lambda}.$$

Hence from Theorem 5.1, it suffices to show that

$$2\frac{\int C_p(\lambda, w)s_2(\lambda)\mathrm{d}\lambda}{\int C_p(\lambda, w)\int_\lambda^\infty h'(t)e^{-t/2}\mathrm{d}te^{\lambda/2}\mathrm{d}\lambda} + \frac{\int C_p(\lambda, w)\{-\lambda h'(\lambda) + s(\lambda) - 2s_1(\lambda)\}\mathrm{d}\lambda}{\int C_p(\lambda, w)h(\lambda)\mathrm{d}\lambda} \geq 0,$$

which yields sufficient condition (5.6).

To verify equality (5.7), we show that the ratio $g(\lambda)/\int_\lambda^\infty h'(t)e^{-t/2}\mathrm{d}t$ is nondecreasing in λ for $g(\lambda) = s_2(\lambda)e^{-\lambda/2}$. Since $g(\lambda) = -\int_\lambda^\infty g'(t)\mathrm{d}t$, the derivative of the ratio is proportional to

$$\begin{aligned} &g'(\lambda)\int_\lambda^\infty h'(t)e^{-t/2}\mathrm{d}t + g(\lambda)h'(\lambda)e^{-\lambda/2} \\ &\quad = \int_\lambda^\infty h'(t)h'(\lambda)e^{-(t+\lambda)/2}\left\{\frac{g'(\lambda)}{h'(\lambda)e^{-\lambda/2}} - \frac{g'(t)}{h'(t)e^{-t/2}}\right\}\mathrm{d}t, \end{aligned}$$

which is nonnegative if $g'(t)/\{h'(t)e^{-t/2}\}$ is nonincreasing in t. Therefore, Proposition 5.1 is established. ■

When $s(\lambda) \leq 0$ for any $\lambda > 0$, this condition implies that $h'(\lambda) \leq 0$ for any $\lambda > 0$. In fact, whenever $h'(\lambda) > 0$, the derivative of $h'(\lambda)$ is negative since $h''(\lambda) \leq -ph'(\lambda)/(2\lambda)$. This fact means that $h'(\lambda) \leq 0$, and assumption (A.2) holds. Then from Proposition 5.1, we get

Corollary 5.1 *If $s(\lambda) \leq 0$ for any $\lambda > 0$, then the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^\pi$ is minimax under assumption* (A.1)*.*

Stein (1981) showed that the Bayes equivariant estimator is minimax if the prior density $h(\|\boldsymbol{\theta}\|^2)$ is super-harmonic, namely $\sum_{i=1}^p(\partial^2/\partial\theta_i^2)h(\|\boldsymbol{\theta}\|^2) \leq 0$ for any $\boldsymbol{\theta}$. Since $\sum_{i=1}^p(\partial^2/\partial\theta_i^2)h(\|\boldsymbol{\theta}\|^2)$ is identical to $s(\lambda)$, it is seen that the condition $s(\lambda) \leq 0$ corresponds to the super-harmonic condition.

We now express the Stein super-harmonic condition based on the inverse Laplace transform $H(t)$ of $h(\lambda)$. Let

$$S(t) = -(p-4)tH(t) + 2t^2H'(t)$$

and assume that $S(t)$ is decomposed as

$$S(t) = S_1(t) + S_2(t),$$

where $S_1(t) \leq 0$ and $S_2(t) \geq 0$ for any $t > 0$.

Proposition 5.2 *Assume* (B.1) *and* (B.2). *Then the function* $s(\lambda)$ *can be expressed by*

$$s(\lambda) = \int S(t)e^{-\lambda t}\mathrm{d}t,$$

and the Bayes equivariant estimator is minimax if

$$-2\sup_{\lambda}\frac{\int S_2(t)e^{-\lambda t}\mathrm{d}t}{\int t(t+1/2)^{-1}H(t)e^{-\lambda t}\mathrm{d}t} + \inf_{\lambda}\frac{\int\{tH'(t)-S_1(t)+S_2(t)\}e^{-\lambda t}\mathrm{d}t}{\int H(t)e^{-\lambda t}\mathrm{d}t} + 1 \geq 0. \quad (5.8)$$

If $(t+1/2)S_2(t)/\{tH(t)\}$ *is nondecreasing in* t, *then*

$$\sup_{\lambda}\frac{\int S_2(t)e^{-\lambda t}\mathrm{d}t}{\int t(t+1/2)^{-1}H(t)e^{-\lambda t}\mathrm{d}t} = \lim_{\lambda\to 0}\frac{\int S_2(t)e^{-\lambda t}\mathrm{d}t}{\int t(t+1/2)^{-1}H(t)e^{-\lambda t}\mathrm{d}t}. \quad (5.9)$$

Proof. By using the same arguments as in the proof of Theorem 4.1, it is observed that

$$\begin{aligned} s(\lambda) =& ph'(\lambda) + 2\lambda h''(\lambda) \\ =& -p\int tH(t)e^{-\lambda t}\mathrm{d}t + 2\lambda\int t^2H(t)e^{-\lambda t}\mathrm{d}t \\ =& \int\left\{-ptH(t)+4tH(t)+2t^2H'(t)\right\}e^{-\lambda t}\mathrm{d}t, \end{aligned}$$

which is equal to $\int S(t)e^{-\lambda t}\mathrm{d}t$. Similarly,

$$-\lambda h'(\lambda) = \lambda\int tH(t)e^{-\lambda t}\mathrm{d}t = \int\left\{H(t)+tH'(t)\right\}e^{-\lambda t}\mathrm{d}t. \quad (5.10)$$

It is also noted that

$$\begin{aligned} -\int_{\lambda}^{\infty}h'(x)e^{-x/2}\mathrm{d}x =& \int_{\lambda}^{\infty}\int tH(t)e^{-tx}\mathrm{d}te^{-x/2}\mathrm{d}x \\ =& \int t(t+1/2)^{-1}H(t)e^{-\lambda t}\mathrm{d}te^{-\lambda/2}. \end{aligned} \quad (5.11)$$

Letting $s_1(\lambda) = \int S_1(t)e^{-\lambda t}\mathrm{d}t$ and $s_2(\lambda) = \int S_2(t)e^{-\lambda t}\mathrm{d}t$, we can see that $s_1(\lambda) \leq 0$ and $s_2(\lambda) \geq 0$ for any $\lambda > 0$ since $S_1(t) \leq 0$ and $S_2(t) \geq 0$ for any $t > 0$. From expressions (5.10) and (5.11), condition (5.6) in Proposition 5.1 is described by the condition (5.8).

To establish equality (5.9), we need to show that the ratio $\int S_2(t)e^{-\lambda t}\mathrm{d}t/\int t(t+1/2)^{-1}H(t)e^{-\lambda t}\mathrm{d}t$ is nonincreasing in λ. The derivative of the ratio is proportional to

$$\begin{aligned} &-\int tS_2(t)e^{-\lambda t}\mathrm{d}t\int t(t+1/2)^{-1}H(t)e^{-\lambda t}\mathrm{d}t \\ &+\int S_2(t)e^{-\lambda t}\mathrm{d}t\int t^2(t+1/2)^{-1}H(t)e^{-\lambda t}\mathrm{d}t, \end{aligned}$$

which can be shown to be non-positive by using Lemma 3.2 if $(t+1/2)S_2(t)/\{tH(t)\}$ is nondecreasing. ■

Corollary 5.2 *If $S(t) \leq 0$ for any $t > 0$, then the super-harmonic condition $s(\lambda) \leq 0$ holds.*

Applying the super-harmonic condition to the prior distribution treated in Example 3.1, we see that the function $s(\lambda)$ defined by (5.2) is written as

$$s(\lambda) = ph'(\lambda) + 2\lambda h''(\lambda) = -\frac{2c(c+1)b}{(b+\lambda)^{c+2}} + \frac{c(2c+2-p)}{(b+\lambda)^{c+1}},$$

which is not positive for $2c + 2 - p \leq 0$. Thus, the super-harmonic condition for the minimaxity is

$$0 < c \leq (p-2)/2.$$

It is noted that the same condition can be derived from the condition $S(t) < 0$ in Corollary 5.2. However, it is quite restrictive in comparison with conditions (3.25) and (3.26). Although the Stein super-harmonic condition is more restrictive in this example, it can provide nice and simple conditions for the minimaxity as demonstrated in the following example.

Example 5.1 (Prior based on the arctan function) Let us treat a prior distribution of the form

$$\boldsymbol{\theta} \sim \frac{1}{\alpha}\left\{\tan^{-1}\frac{\alpha}{\|\boldsymbol{\theta}\|^2}\right\} \mathrm{d}\boldsymbol{\theta}, \tag{5.12}$$

where α is a positive constant. In this case, the function $h(\lambda)$ is written by

$$h(\lambda) = \frac{1}{\alpha}\tan^{-1}\frac{\alpha}{\lambda} = \frac{1}{\alpha}\left(\frac{\pi}{2} - \tan^{-1}\frac{\lambda}{\alpha}\right) = \frac{1}{\alpha}\int_{\lambda/\alpha}^{\infty} \frac{1}{1+x^2}\mathrm{d}x,$$

and the inverse Laplace transform is given by

$$H(t) = \frac{\sin \alpha t}{\alpha t},$$

namely, $h(\lambda) = \int\{\sin \alpha t/(\alpha t)\}e^{-\lambda t}\mathrm{d}t$. It is noted that $H(t)$ goes to zero with taking positive and negative values periodically as t tends to infinity. Since $h(\lambda)$ is rewritten as $h(\lambda) = \int_\lambda^\infty (\alpha^2 + s^2)^{-1}\mathrm{d}s$ by making the transformation $s = \alpha x$, it is observed that $h'(\lambda) = -(\alpha^2 + \lambda^2)^{-1}$ and $h''(\lambda) = 2\lambda(\alpha^2 + \lambda^2)^{-2}$. Since $H(t)$ takes negative values periodically, the conditions given in Section 4 do not work well. The conditions derived in Section 3 can give a feasible but somewhat restrictive condition on α^2 and p. However, the Stein super-harmonic condition provides a nice condition for the minimaxity. That is, the function $s(\lambda)$ defined by (5.2) can be written as

$$s(\lambda) = -\left\{(p-4)\lambda^2 + p\alpha^2\right\}/(\alpha^2 + \lambda^2)^2,$$

which is not positive if $p \geq 4$. Hence from Corrolary 5.1, the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^{\pi}$ against prior (5.12) is minimax for $p \geq 4$. Although it is interesting to clarify whether $\widehat{\boldsymbol{\theta}}^{\pi}$ is minimax for $p = 3$, it is not easy to show.

The Stein super-harmonic condition can be applied to another type of prior distributions. When the prior of $\boldsymbol{\theta}$ is $(\alpha^2 + \|\boldsymbol{\theta}\|^4)^{-1}\mathrm{d}\boldsymbol{\theta}$, it corresponds to the case that $h(\lambda) = (\alpha^2 + \lambda^2)^{-1}$ and $H(t) = \sin(\alpha t)$, namely, $h(\lambda) = \int \sin(\alpha t) \exp\{-\lambda t\}\mathrm{d}t$. Then,

$$s(\lambda) = -\frac{2\lambda}{(\alpha^2 + \lambda^2)^2}\left\{p - 6 + \frac{8\alpha^2}{\alpha^2 + \lambda^2}\right\},$$

which is not positive if $p \geq 6$. Hence, the Bayes equivariant estimator is minimax for $p \geq 6$. ■

6 Admissibility of the Bayes equivariant estimators

The conditions for the minimaxity have been investigated for the Bayes equivariant estimators. Another interesting topic is to provide a characterization of prior distributions for the admissibility. Using Brown's admissibility condition, in this final section, we derive conditions on priors for the admissibility of the Bayes equivariant estimator. The results given here may be helpful for checking the admissibility for general priors, though most of them are known in the literature.

We begin with stating Brown's admissibility condition, which is known as a very useful tool for checking the admissibility in the Stein problem. As noted in Section 2, the generalized Bayes estimator against a prior distribution with the spherically symmetric density $h(\|\boldsymbol{\theta}\|^2)\mathrm{d}\boldsymbol{\theta}$ is the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^{\pi}$ against the prior $\pi(\lambda) = \lambda^{p/2-1}h(\lambda)$ for $\lambda = \|\boldsymbol{\theta}\|^2$. Define $A(h)$ by

$$A(h) = \int_1^{\infty} \left\{r^{p-1} f_h(r)\right\}^{-1} \mathrm{d}r,$$

where

$$f_h(\|\boldsymbol{x}\|) = \int (2\pi)^{-p/2} \exp\{\|\boldsymbol{x} - \boldsymbol{\theta}\|^2/2\} h(\|\boldsymbol{\theta}\|^2)\mathrm{d}\boldsymbol{\theta}.$$

Theorem 6.1 (Brown (1971)) *The Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^{\pi}$ is inadmissible if $A(h) < \infty$. When $f_h(\|\boldsymbol{x}\|)$ and $\|\widehat{\boldsymbol{\theta}}^{\pi} - \boldsymbol{x}\|$ are uniformly bounded with respect to $\boldsymbol{x}$, $\widehat{\boldsymbol{\theta}}^{\pi}$ is admissible if $A(h) = \infty$.*

It is noted that $f_h(\|\boldsymbol{x}\|)$ is the marginal density with respect to $\mathrm{d}\boldsymbol{x}$ while

$$g_{\pi}(w) = \frac{1}{2^{p/2}} \sum_{j=0}^{\infty} \frac{w^{p/2+j-1} \exp\{-w/2\}}{j!\Gamma(p/2 + j)2^{2j}} \int \lambda^{p/2+j-1} e^{-\lambda/2} h(\lambda)\mathrm{d}\lambda$$

is the marginal density with respect to $\mathrm{d}w$ for $w = \|\boldsymbol{x}\|^2$. It is thus seen that $g_{\pi}(w) = w^{p/2-1} f_h(\sqrt{w})$, or $f_h(r) = r^{2-p} g_{\pi}(r^2)$ for $r = \sqrt{w}$, and $A(h)$ is written as

$$A(h) = \int_1^{\infty} \frac{1}{r g_{\pi}(r^2)} \mathrm{d}r.$$

Since $\|\widehat{\boldsymbol{\theta}}^{\pi} - \boldsymbol{x}\| = \psi_{\pi}(w)/\sqrt{w}$, Theorem 6.1 is rewritten in the following.

Lemma 6.1 *Assume that there exists a constant δ such that $g_\pi(r^2) \sim C_0 r^\delta$ for some generic constant C_0 as $r \to \infty$. If $\delta > 0$, the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^\pi$ is inadmissible. When $g_\pi(w)/w^{p/2-1}$ and $\psi_\pi(w)/\sqrt{w}$ given by* (2.4) *are uniformly bounded, $\widehat{\boldsymbol{\theta}}^\pi$ is admissible if $\delta \leq 0$.*

Though this section, we use the notations C, C', C_0, C_1 and C_2 as as generic positive constants, namely, for example we use the same notation C for different constants without anything confusing.

It may be hard to check the conditions in Lemma 6.1. However, the use of the inverse Laplace transform of $h(\lambda)$ can make them tractable. Since $h(\lambda) = \int H(t)e^{-\lambda t}\mathrm{d}t$, it is observed that

$$\int \lambda^{p/2+j-1}e^{-\lambda/2}h(\lambda)\mathrm{d}\lambda = \Gamma(p/2+j)\int H(t)\frac{1}{(t+1/2)^{p/2+j}}\mathrm{d}t,$$

so that $g_\pi(w)$ is expressed as

$$\begin{aligned} g_\pi(w) =& w^{p/2-1}e^{-w/2}\int \sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{w}{2(1+2t)}\right)^j \frac{H(t)}{(1+2t)^{p/2}}\mathrm{d}t \\ =& w^{p/2-1}\int \frac{H(t)}{(1+2t)^{p/2}}\exp\left\{-\frac{2t}{2(1+2t)}w\right\}\mathrm{d}t. \end{aligned}$$

Making the transformation $z = 2t/(1+2t)$ gives the expression $g_\pi(w) = w^{p/2-1}G_\pi(\sqrt{w})/2$ where

$$G_\pi(r) = \int_0^1 (1-z)^{p/2-2}H\left(\frac{z}{2(1-z)}\right)e^{-r^2z/2}\mathrm{d}z. \tag{6.1}$$

Also, the function $\psi_\pi(w)$ is written by

$$\psi_\pi(w) = w\frac{\int_0^1 z(1-z)^{p/2-2}H(z/[2(1-z)])\exp\{-wz/2\}\mathrm{d}z}{\int_0^1 (1-z)^{p/2-2}H(z/[2(1-z)])\exp\{-wz/2\}\mathrm{d}z}. \tag{6.2}$$

Then Lemma 6.1 is expressed in the following.

Lemma 6.2 *Assume that there exists a constant δ such that $G_\pi(r) \sim C_0 r^{2-p+\delta}$ for some generic constant C_0 as $r \to \infty$. If $\delta > 0$, the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^\pi$ is inadmissible. When $G_\pi(r)$ and $\psi_\pi(r^2)/r$ are uniformly bounded, $\widehat{\boldsymbol{\theta}}^\pi$ is admissible if $\delta \leq 0$.*

When $H(t)$ is a positive function, it follows from (6.2) that $\psi_\pi(w) \leq w$. If $\psi_\pi(w)$ is bounded, namely,

$$\psi_\pi(w) \leq C,$$

for a constant C, then it is observed that $\psi_\pi(r^2)/r \leq \min\{r, C/r\} \leq \sqrt{C}$, so that $\psi_\pi(r^2)/r$ is uniformly bounded. Hence, the boundedness of $\psi_\pi(r^2)$ is sufficient for the boundedness of $\psi_\pi(r^2)/r$. Lemma 6.3 is also useful for checking the conditions in Lemma 6.2.

Lemma 6.3 *For positive constants a, b and d, there are constants C_1 and C_2 such that*

$$C_1 \int_0^1 z^{a-1} e^{-dz} \mathrm{d}z \le \int_0^1 z^{a-1}(1-z)^{b-1} e^{-dz} \mathrm{d}z \le C_2 \int_0^1 z^{a-1} e^{-dz} \mathrm{d}z.$$

This implies that $\int_0^1 z^{a-1}(1-z)^{b-1}e^{-dz}\mathrm{d}z \sim C_0/d^a$ for a constant C_0 as $d \to \infty$.

Proof. In the case of $0 < b < 1$, $(1-z)^{b-1}$ is increasing and $1 \le (1-z)^{b-1}$, so that the left inequality holds. Since $(1-z)^{b-1}$ and e^{-dz} are monotone in opposite directions, Lemma 3.1 implies that

$$E[(1-Z)^{b-1}e^{-dZ}] \le E[(1-Z)^{b-1}]E[e^{-dZ}], \tag{6.3}$$

where Z is a random variable having the density az^{a-1}. Inequality (6.3) is expressed as

$$\int_0^1 z^{a-1}(1-z)^{b-1} e^{-dz} \mathrm{d}z \le a \int_0^1 z^{a-1}(1-z)^{b-1} \mathrm{d}z \int_0^1 z^{a-1} e^{-dz} \mathrm{d}z,$$

which yields the right inequality. In the case of $b \ge 1$, $(1-z)^{b-1}$ is deccreasing and $(1-z)^{b-1} \le 1$. The same arguments can be used to get the inequalities in Lemma 6.3.

Since $\int_0^1 z^{a-1}e^{-dz}\mathrm{d}z = d^{-a}\int_0^d x^{a-1}e^{-x}\mathrm{d}x$, the inequalities in Lemma 6.3 implies that $C_1 \le \lim_{d\to\infty} d^a \int_0^1 z^{a-1}(1-z)^{b-1}e^{-dz}\mathrm{d}z \le C_2$, which means that $\int_0^1 z^{a-1}(1-z)^{b-1}e^{-dz}\mathrm{d}z \sim C_0/d^a$ for $C_1 \le C_0 \le C_2$ as $d \to \infty$. ■

We conclude this section with checking the conditions in Lemma 6.2 for the admissibilty of the Bayes equivariant estimators treated in examples in the previous sections.

Example 6.1 (Prior related to a multivariate t-distribution. Continued) In this example, we treat the prior distribution discussed in Example 3.1, namely, $H(t)$ is given by $H(t) = t^{c-1}e^{-bt}$. For some generic constant C_0, $G_\pi(r)$ may be written as

$$G_\pi(r) = C_0 \int_0^1 z^{c-1}(1-z)^{p/2-c-1} e^{-bz/(1-z)-(r^2/2)z} \mathrm{d}z,$$

which can be seen to be uniformly bounded with respect to r for $0 < c < p/2$ and $b \ge 0$.

We here prove the inequalities

$$\begin{aligned} C_1 &\int_0^1 z^{c-1}(1-z)^{p/2-c-1} e^{-(r^2/2)z} \mathrm{d}z \\ &\le G_\pi(r) \le C_0 \int_0^1 z^{c-1}(1-z)^{p/2-c-1} e^{-(r^2/2)z} \mathrm{d}z. \end{aligned} \tag{6.4}$$

In fact, $G_\pi(r)$ is expressed by $G_\pi(r) = CE[e^{-bZ/(1-Z)}e^{-(r^2/2)Z}]$ for a random variable Z having the density $z^{c-1}(1-z)^{p/2-c-1}/B(c, p/2-c)$. Since both $e^{-bZ/(1-Z)}$ and $e^{-(r^2/2)Z}$ are decreasing in Z, it follows from Lemma 3.1 that

$$\begin{aligned} G_\pi(r) &\ge CE[e^{-bZ/(1-Z)}]E[e^{-(r^2/2)Z}] \\ &= C_1 \int_0^1 z^{c-1}(1-z)^{p/2-c-1} e^{-(r^2/2)z} \mathrm{d}z, \end{aligned}$$

which shows the left inequality in (6.4). The right inequality is trivial.

Combining inequalities (6.4) and Lemma 6.3 gives that

$$G_\pi(r) \sim C \int_0^1 z^{c-1}(1-z)^{p/2-c-1} e^{-(r^2/2)z} \mathrm{d}z \sim C' r^{2-p+(p-2-2c)},$$

as $r \to \infty$. Inequality (6.4) and Lemma 6.3 are again used to evaluate $\psi_\pi(w)$ as

$$\begin{aligned}
\psi_\pi(w) =& w \frac{\int_0^1 z^c (1-z)^{p/2-c-1} e^{-bz/(1-z)-(w/2)z} \mathrm{d}z}{\int_0^1 z^{c-1}(1-z)^{p/2-c-1} e^{-bz/(1-z)-(w/2)z} \mathrm{d}z} \\
\leq & Cw \frac{\int_0^1 z^c e^{-(w/2)z} \mathrm{d}z}{\int_0^1 z^{c-1} e^{-(w/2)z} \mathrm{d}z} \\
=& C \frac{\int_0^w x^c e^{-x/2} \mathrm{d}x}{\int_0^w x^{c-1} e^{-x/2} \mathrm{d}x} \leq C \frac{\int_0^\infty x^c e^{-x/2} \mathrm{d}x}{\int_0^\infty x^{c-1} e^{-x/2} \mathrm{d}x},
\end{aligned} \tag{6.5}$$

which is bounded. Since $G_\pi(r)$ is bounded, we can use Lemma 6.2 for $\delta = p - 2 - 2c$ and $0 < c < p/2$. Hence, the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^\pi$ is inadmissible for $0 < c < (p-2)/2$ and $b \geq 0$, and admissible for $(p-2)/2 \leq c < p/2$ and $b \geq 0$. When $(p-2)/2 \leq c \leq \min\{p-2, (p-2+b)/2\}$, $c < p/2$ and $b \geq 0$, $\widehat{\boldsymbol{\theta}}^\pi$ is admissible and minimax. This result suggests to take $c = (p-2)/2$, because the resulting Bayes equivariant estimator is admissible and minimax for any $b \geq 0$. ■

Example 6.2 (Scale mixture of a normal distribution. Continued) We next treat the prior distribution in Example 4.1, $H(t)$ is given by $H(t) = t^{p/2+b-2}(1+2t)^{-a}\nu(t)$. Then,

$$G_\pi(r) = C_0 \int_0^1 z^{p/2+b-2}(1-z)^{a-b} \nu\left(\frac{z}{2(1-z)}\right) e^{-(r^2/2)z} \mathrm{d}z,$$

which is bounded when $1 - p/2 < b < a + 1$ and $\nu(t)$ is bounded as $0 < \nu_1 \leq \nu(t) \leq \nu_2$ for some positive constants ν_1 and ν_2. Since

$$\begin{aligned}
& C_0\nu_1 \int_0^1 z^{p/2+b-2}(1-z)^{a-b} e^{-(r^2/2)z} \mathrm{d}z \\
& \qquad \leq G_\pi(r) \leq C_0\nu_2 \int_0^1 z^{p/2+b-2}(1-z)^{a-b} e^{-(r^2/2)z} \mathrm{d}z,
\end{aligned}$$

Lemma 6.3 implies that $G_\pi(r) \sim C r^{2-p-2b}$ as $r \to \infty$. The boundedness of $\psi_\pi(w)$ can be verified by the same arguments as in (6.5). Hence from Lemma 6.2, it is concluded that $\widehat{\boldsymbol{\theta}}^\pi$ is inadmissible for $1 - p/2 < b < \min(0, a+1)$, and admissible for $0 \leq b < a + 1$. When $0 \leq b < a$ and $1 - p/2 + a < b \leq (p-2)/2$, the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^\pi$ is admissible and minimax, where $\nu(t)$ is a nonnegative and differentiable function such that $0 < \nu_1 \leq \nu(t) \leq \nu_2$ and $(1+2t)t\nu'(t)/\nu(t)$ is nonincreasing in t. ■

Example 6.3 (Prior based on the arctan function. Continued) For the prior distribution treated in Example 5.1, $H(t)$ is given by $H(t) = (\sin \alpha t)/(\alpha t)$. Since $(\sin t)/t \leq$

1, it is observed that

$$
\begin{aligned}
G_\pi(r) &= \int_0^1 (1-z)^{p/2-2} \frac{\sin(\alpha z/[2(1-z)])}{\alpha z/[2(1-z)]} e^{-(r^2/2)z} \mathrm{d}z \\
&\leq \int_0^1 (1-z)^{p/2-2} e^{-(r^2/2)z} \mathrm{d}z,
\end{aligned}
$$

which is bounded for $p \geq 3$. Also from Lemma 6.3, it can be seen that

$$
\int_0^1 (1-z)^{p/2-2} e^{-(r^2/2)z} \mathrm{d}z \sim C_0 r^{2-p+(p-4)}.
$$

On the other hand, $h(\lambda)$ is evaluated as

$$
\begin{aligned}
h(\lambda) &= \int H(t) e^{-\lambda t} \mathrm{d}t = \int_\lambda^\infty (\alpha^2 + s^2)^{-1} \mathrm{d}s \\
&\geq \int_\lambda^\infty (\alpha + s)^{-2} \mathrm{d}s = \frac{1}{\alpha + \lambda} = \int H_0(t) e^{-\lambda t} \mathrm{d}t,
\end{aligned}
$$

for $H_0(t) = e^{-\alpha t}$. This inequality implies that

$$
\begin{aligned}
G_\pi(r) \geq & \int_0^1 (1-z)^{p/2-2} e^{-(\alpha/2)z/(1-z)} e^{-(r^2/2)z} \mathrm{d}z \\
\geq & C \int_0^1 e^{-(r^2/2)z} \mathrm{d}z \sim C' r^{2-p+(p-4)},
\end{aligned}
$$

where the second inequality follows from inequality (6.4) and Lemma 6.3. Combining these observations gives that $G_\pi(r) \sim C r^{2-p+(p-4)}$ as $r \to \infty$. Since $\psi_\pi(w)$ can be verified to be bounded, it is concluded from Lemma 6.2 that the Bayes equivariant estimator $\widehat{\boldsymbol{\theta}}^\pi$ is inadmissible for $p > 4$, and admissible for $p = 3$ and $p = 4$. The estimator $\widehat{\boldsymbol{\theta}}^\pi$ is admissible and minimax for $p = 4$, though it is minimax, but inadmissible for $p \geq 5$. ■

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