CIRJE-F-390

Real Indeterminacy of Stationary Equilibria in Matching Models with Divisible Money

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November 2005

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Abstract

In this paper, it is shown that real indeterminacy of stationary equilibria generically arises in most matching models with perfectly divisible fiat money. In other words, the real indeterminacy follows from the condition for stationarity of money holdings, and surprisingly it has nothing to do with the other specifications, e.g., the bargaining procedures, of the models. Thus if we assume the divisibility of money in money search models, it becomes quite difficult to make accurate predictions of the effects of some policies.

Keywords: Real Indeterminacy, Matching Model, Divisible Money.

Journal of Economic Literature Classification Number: C78, D51, D83, E40.

1 Introduction

Recently, real indeterminacy of stationary equilibria has been found in matching models with fiat money. (See, for example, Green and Zhou [5] [6], Matsui and Shimizu [15], and Zhou [22].[1]) In this paper, it is shown that real indeterminacy generically arises in most matching models with perfectly divisible money. In other words, the real indeterminacy follows from the condition for stationarity of money holdings, and surprisingly it has nothing to do with the other specifications, e.g., the bargaining procedures, of the models.

It is well known that some general equilibrium models have intrinsic multiplicity of equilibria. (See, for example, Gale [3], Geanakoplos and Mas-Colell [4], Herings [8],

*This paper is a version of the manuscript originally circulated under the title "Real Indeterminacy of Stationary Equilibria in Matching Models with Media of Exchange." The authors would like to acknowledge seminar participants at Hitotsubashi University, Kansai University, Osaka University, University of Tokyo, and GETA, as well as Nobu Kiyotaki, Michihiro Kandori, Noritaka Kudoh, Akihiko Matsui, Koji Shimomura, Dolf Talman, and Akira Yamazaki for helpful comments and references. This research is financially supported by Grant-in-Aid for Scientific Research from JSPS and MEXT. The second author also acknowledges the financial support by Zengin Foundation for studies on Economics and Finance. Of course, any remaining error is our own.

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[1] Green and Zhou [6] have found real indeterminacy of dynamic equilibria from an initial state as well.

Samuelson [17], and van der Laan [21].) Overlapping generations models, for example, have such real indeterminacy of equilibria; in the so-called Samuelson's case, there is a continuum of equilibria parameterized by the first period consumption. Nevertheless, stationary equilibria are typically determinate in intertemporal general equilibrium models; for example, overlapping generations models have generically a finite number of stationary equilibria.[2] (See Kehoe and Levine [14].)

However, it is recently shown that even stationary equilibria are indeterminate in a few matching models with divisible money referred above. Though some authors intuitively argued that specific bargaining procedures lead to the intrinsic multiplicity of equilibria, the logic behind the real indeterminacy has not been found so far. In this paper, it is shown that the real indeterminacy results from the condition for stationarity of money holdings, independently of the details of the models.

A sketch of our idea is as follows. Suppose the nominal stock of money is given. When the price level is lower, there is more liquidity in the economy, the trade is more frequent, and therefore the welfare level is higher. When the price level is higher, there is less liquidity in the economy, the trade is less frequent, and therefore the welfare level is lower. If we can find the corresponding equilibrium values of the other variables, such as the money holdings distribution and the value function, as the price level continuously varies, then the real indeterminacy follows. More precisely, if the number of variables is larger than that of equations, then by applying the implicit function theorem this property holds. In this paper, we show that the stationary condition of money holdings, common to all random matching models of money, has at least one more variable than the number of equations. Thus the stationary equilibria in such models are indeterminate.

More specifically, we consider the case of one fiat money. Suppose it is perfectly divisible and there is an upper bound of its holdings. We confine our attention to stationary equilibria in which, for some positive number p, all trades occur with its integer multiple amounts of money.[3] We focus on stationary distributions on $\{0, \ldots, N\}$ expressed by $h = (h(0), \ldots, h(N))$, where $h(n)$ is the measure of the set of agents with np amount of money, and $N < \infty$ is the upper bound. In the condition for stationarity of money holdings, there are $(N+1)$ variables, $h(n)$, $n = 0, \ldots, N$. On the other hand, since $\sum_{n=0}^{N} O_n = \sum_{n=0}^{N} I_n$ always holds, where O_n (I_n) is the outflow (inflow resp.) at

[2] Recently, Nishimura and Shimomura [16] find a continuum of stationary equilibria in a dynamic trade model. However, the logic behind the indeterminacy is quite different from ours.

[3] Any type of trades with integer multiple of p, e.g., equilibrium price dispersion as in Kamiya and Sato [11], is allowed.

n, then, at first glance, there seem to be $(N+1)$ independent equations, $O_n = I_n$, $n = 1, \ldots, N$, and $\sum_{n=0}^{N} h(n) = 1$. Thus it seems that the numbers of independent equations and variables, $h(n), n = 0, \ldots, N$, are the same. However, surprisingly it can be shown that one more equation is always redundant and that the system of equations has always at least one degree of freedom; namely, $\sum_{n=1}^{N} nO_n = \sum_{n=1}^{N} nI_n$ always holds. This fact is the key to the real indeterminacy of stationary equilibria.

We present the concept of a stationary quasi-equilibrium which is weaker than a stationary equilibrium. It enables us to analyze matching models in a general way. Note that, in most of the specific models, it can easily be shown that a stationary quasi-equilibrium is indeed a stationary equilibrium. Let $V = (V(0), \ldots, V(N))$ and β be a value function and a vector of proportions of agents who take a certain pure strategy, respectively, and (V^*, h^*, β^*) be a stationary quasi-equilibrium. Then, due to the indeterminacy of stationary distributions, it seems that there exists another stationary quasi-equilibrium (V, h, β) in a small neighborhood of (V^*, h^*, β^*). Indeed, using differential topology, we can show that the existence of a stationary quasi-equilibrium generically leads to the existence of a continuum of them. It can also be shown that real allocations are generically not constant in a connected set of the stationary quasi-equilibria.

We also present a sufficient condition that the indeterminacy of stationary quasi-equilibria implies that of stationary equilibria. That is any model satisfying this condition has a continuum of stationary equilibria as well as a continuum of stationary quasi-equilibria. In some matching models with indivisible money, such as Camera and Corbae [2], Shi [18], and Trejos and Wright [20], the stationary equilibria are determinate. However, if once they are extended to the models with perfectly divisible money, then real indeterminacy generically arises. Indeed, these models satisfy the sufficient condition. Moreover, we directly show that the Camera and Corbae's model with divisible money has real indeterminacy of stationary equilibria.

Even if we maintain the assumption of indivisible money, the above arguments suggest that the greater the divisibility of money, the larger the number of equilibria. In other words, for a fixed money supply and a fixed upper bound of money holdings, there are much larger number of equilibria in the case of one unit of money being one cent than in the case of one hundred dollars.

We believe that the general results found in the present paper are worthy by themselves, but they also shed a new light on other aspects of monetary economics. In the literature, the welfare effect of monetary policy has often been discussed in matching models with money, and in most of these models money is indivisible and the stationary equilibria are determinate. Thus the effects of the policies are determinate as well. However, if we assume the divisibility of money in these models, the stationary equilibria become indeterminate. Thus it is quite difficult to make accurate predictions of the effects of simple policies in such models. Instead, in the accompanying paper [12], we investigate a sophisticated policy which selects a determinate efficient equilibrium. It is also worthwhile noting that by using the results on indeterminacy we can relatively easily prove the existence of stationary equilibria in matching models with divisible money. (See Kamiya and Shimizu [13] and Kamiya et al. [10].)

The plan of this paper is as follows. In Section 2, we first present our basic model and examples. In Section 3, the key feature of stationary distributions is proved, and then in Section 4, the real indeterminacy is informally discussed; the rigorous discussion and the proofs are given in Appendix B. Some models with a continuum of stationary equilibria are also given. Moreover, we present a sufficient condition that the indeterminacy of stationary quasi-equilibria implies that of stationary equilibria, and discuss the case of indivisible money. In Section 5, we relax some assumptions given in Section 2; such as possibility of multiple money, possibility of the matchings not being pairwise, possibility of money holdings giving some utility, and possibility of discarding the upper bound of money holdings.

2 The Basic Model and Examples

In this section, we present the basic model. Since our concern is mainly on the stationarity of money holdings, the other aspects of the model are described in a quite general way. For concrete examples of the basic model, see Zhou [22]'s model in Section 2.2 and a divisible money version of Camera and Corbae [2]'s model in Section 2.3.

2.1 The Basic Model

We make the following assumptions in most parts of this paper for simplicity: there is only one kind of money, the matching is pairwise, money holdings give no utility, and there is an upper bound of money holdings. All of these assumptions will be relaxed in Section 5.

Throughout this paper, we assume that there are infinitely lived agents with a nonatomic mass of measure one. Our model can be considered both as a continuous-time model and as a discrete-time model depending on the interpretations of the matching technology presented below. There is one fiat money which is perfectly durable and divisible. Although money is traded for perishable goods, we do not explicitly specify them; all the results in what follows can be obtained no matter what the specification is.

We confine our attention to stationary equilibria in which, for some positive number p, all trades occur with its integer multiple amounts of fiat money.[4] In what follows, we focus on stationary distributions on $\{0, \ldots, N\}$ expressed by $h = (h(0), \ldots, h(N))$, where $h(n)$ is a measure of the set of agents with np amount of money, and the upper bound $N < \infty$ can be either exogenous or endogenous. Of course, $h(n) \geq 0$ and $\sum_{n=0}^{N} h(n) = 1$ hold. Let $M > 0$ be a given supply of the medium of exchange. Since p is uniquely determined by $\sum_{n=0}^{N} pnh(n) = M$ for a given h (unless $h(0) = 1$), then, deleting p from $\{0, p, 2p, \ldots, Np\}$, the set $\{0, \ldots, N\}$ can be considered as the state space.

An agent with n chooses an action in $A_n = \{a_1, \ldots, a_{k_n}\}$. Note that we restrict our attention to a finite action space. Let $\beta_{nj} \geq 0$ be the proportion of the agents choosing an action a_j among the agents with n, and $\beta = (\beta_{01}, \ldots, \beta_{nj}, \ldots, \beta_{Nk_N})$. Thus $\sum_{j=1}^{k_n} \beta_{nj} = 1$ holds. Define $h(n, j)$ as $h(n, j) = \beta_{nj} h(n)$. Let $\gamma \in \mathbb{R}^L$ be the parameter of the model.

The technology of pairwise matching is described by random matching process and the following function f. When an agent with (n, j) meets an agent with (n', j'), the former's and the latter's states will be $n + f((n, j), (n', j'))$ and $n' - f((n, j), (n', j'))$, respectively.[5] That is f maps an ordered pair $((n, j), (n', j'))$ to a non-negative integer $f((n, j), (n', j'))$. Here "ordered" means, for example, that the former is a seller and the latter is a buyer. When N is exogenously determined, we assume

$$N \geq n + f((n, j), (n', j')) \quad \text{and} \quad n' - f((n, j), (n', j')) \geq 0.$$

When N is endogenously determined, we assume the latter condition while the former one should be satisfied on the equilibrium path.

By random matching process, the rate of matching between agents with (n, j) and (n', j') is written as $\alpha h(n, j) h(n', j')$ for some $\alpha > 0$. Note that α is a parameter and is

[4] Note that we do not exclude the case in which one good is traded for multiple prices, i.e., the case of price dispersion.

[5] In this formulation it is implicitly assumed that the bargaining immediately ends on the equilibrium path. However, it is not a substantial drawback of our model, because we could analyze a situation in which the bargaining delays by extending the state space.

included in γ. Needless to say, in discrete time cases, the proportion $\alpha h(n,j)h(n',j')$ of agents move from n to $n+f((n,j),(n',j')$ and n' to $n'-f((n,j),(n',j'))$ in each period. In continuous time cases, $\alpha h(n,j)h(n',j')$ is the time derivative of the proportion of such movements.

We adopt a Bellman equation approach. Let $V(n)$ be the value of state n, $n = 0,\ldots,N$. The variables in the model are denoted by $x = (V,h,\beta)$. Let $W_{nj}(x,\gamma)$ be the value of action j at state n. Thus, in equilibria, $W_{nj}(x,\gamma) = V(n)$ holds for j such that $\beta_{nj} > 0$. Note that $W_{nj}(x,\gamma)$ includes the utility and/or the production cost of perishable goods.

Remark 1 One may think that our model is too restrictive in two points: confining our attention to stationary equilibria in which all trades occur with some integer multiple of p and to a finite action space. Moreover, some might think that such an equilibrium does not exist in Camera and Corbae [2]'s model or in Trejos and Wright [20]'s model if money is divisible. However, we will later show that it really exists. Also, as for finite action space, we will later show that many matching models with divisible money can be converted into the models with finite action spaces. More specifically, see Section 2.2 and 4.1 for Zhou [22]'s model, and see Section 2.3 and 4.2 for the divisible money version of Camera and Corbae [2]'s model. For Trejos and Wright [20]'s model and a general discussion, see Section 4.4.

Remark 2 The rate of matching can be much more general. That is even if it is some function of $(n,j),(n',j'),h$, and β, the arguments in what follows do not change. The general model, for example, includes so-called "directed search" models such as Matsui and Shimizu [15].

Remark 3 It is worthwhile noting that, in the case of time-additive expected utility in discrete time, W_{nj} can be for example written as:

$$W_{nj}(x,\gamma) = U(h,\beta,j,\gamma) + \kappa E(V(n')|h,\beta,j,n),$$

where κ is a discount factor, U is the temporal utility, and $E(V(n')|h,\beta,j,n)$ is the expectation of $V(n')$ conditional on h,β,j,n. We can also analyze the case that U depends on the amount of money. For the details, see Section 5.

Remark 4 We can easily extend our model to the case that prices are not necessarily nonnegative integer multiples of p. For example, suppose the state space is

$\{0, p, \sqrt{2}p, \ldots, (n_1 + n_2\sqrt{2})p, \ldots, (N_1 + N_2\sqrt{2})p\}$, and p and $\sqrt{2}p$ are the equilibrium prices. Then we can obtain the same results as in the case of nonnegative integer multiples of p. In fact, our argument is applicable to any finite state space.

Remark 5 Our model includes the case that both barter and monetary exchange are possible, such as Shi [18]. It is worthwhile noting that, as far as some monetary exchange exists in equilibria, our argument in the following sections are applicable.

2.2 Zhou Model

In Zhou [22], time is continuous, and pairwise random matchings take place according to Poisson process with a parameter μ. There are k types of agents with equal fractions and the same number of types of goods. Only one unit of good i can be produced and held by a type $i-1 \pmod{k}$ agent. The production cost is c. A type i agent obtains utility $u > 0$ only when she consumes one unit of good i. Fiat money is divisible and there is no inventory constraint on fiat money. For every matched pair, the seller posts a take-it-or-leave-it price offer, ignorant of the buyer's money holdings.

Zhou shows the existence of "single price equilibria" in which all trades occur with a price p^*. In the equilibria, the support of money holdings distribution is endogenously bounded. Let the support be $\{0, p^*, 2p^*, \ldots, Np^*\}$, where N is endogenously determined. Note that, as long as symmetric Markov equilibria concerned, the value function depends only upon current money holdings. It is verified that this type of equilibria is included in our model as follows:

- $h(n)$ is the fraction of agents with np^* amount of fiat money.
- $A_n = \{a_j\}_{j \in K_n}$, where $K_n = \{(o, r) \,|\, o = 0, 1, \ldots, \hat{N}, r = 0, 1, \ldots, n\}$ for some finite $\hat{N}$, i.e., $k_n = \#K_n$ (here, we have slightly abused the notations; j denotes an action instead of an integer). An action $a_j = a_{(o,r)}$ means that an agent offers op^* when she is a seller, and she accepts the partner's offer if and only if the offer price is less than or equal to rp^* when she is a buyer.
- $f((n, j), (n', j'))$ is the monetary transfer between a seller (n, j) and a buyer (n', j'). Thus

$$f((n, (o, r)), (n', (o', r'))) = \begin{cases} o & \text{if } o \leq r' \\ 0 & \text{otherwise.} \end{cases}$$

- The time derivative of the matching between a seller (n, j) and a buyer (n', j') is $(\mu/k)h(n, j)h(n', j')$.
- $V(n)$ is the value of np^*.

In order to discuss stationary equilibrium, we also need to consider actions excluded from our action space, and the strategy and the value at $\eta \notin \{0, p^*, \ldots, Np^*\}$. In Section 4.1, we will show that the above specifications are sufficient.

2.3 Divisible Money Version of Camera and Corbae Model

Camera and Corbae [2] (referred to below as CC) analyze a model in which fiat money is indivisible, there is an exogenously given upper bound of money holdings, and goods are perfectly divisible. In this subsection, we extend the model to the case of perfectly divisible fiat money. Later we show that there is a continuum of stationary equilibria of which strategies are similar to the strategy in CC.

CC's model is similar to Zhou's. The differences are the divisibility of goods, the bargaining procedure, and the specifications of fiat money. By consuming q unit of goods, an agent obtains utility $U(q) = q^{1-\lambda}/(1-\lambda)$, where $\lambda \in (0, 1)$ is a parameter. The cost function is $C(q) = q$. After observing the seller's money holdings, the buyer posts a take-it-or-leave-it offer (d, q), where d and q are quantities of money and goods, respectively.

In the original version of CC model, one unit is no longer divisible. Agents are under a money holding constraint; N is the maximum unit they can hold. Let M^S be the total units of money supply. Note that N and M^S are exogenously given in CC.

Let us turn to the case of divisible money. For a given $p^* > 0$, we will later analyze an equilibrium in which all trades occur with p^* amount of fiat money. Agents behave as if p^* were the minimum unit of divisibility. Let $\bar{N}$ and M be the upper bound of money holdings and the total quantity of fiat money, respectively. Then let

$$N = \lfloor \bar{N}/p^* \rfloor \quad \text{and} \quad M^s = M/p^*,$$

where $\lfloor x \rfloor$ denotes the integer part of x, then the divisible money version looks similar to the original version. The only difference is that N and M^S are endogenously given in the divisible version.

One might think that the divisible money version of CC model is not a special case of our model, since the action space includes the choice of quantity offer in R_+. However, since a buyer can exploit all gains from trade, then the equilibrium quantity is uniquely determined as the function of the offer price, the partner's money holdings, and the value function. Thus we can confine our attention to a simpler action space as in CC. Now, we can check that our model includes that of the divisible money version of CC model as follows:

- $h(n)$ is the measure of the set of agents with np^* amount of fiat money.
- $A_n = \{a_j\}_{j \in K_n}$, where $K_n = \{(o_0, o_1, \ldots, o_N) \mid o_{\hat{n}} = 0, \ldots, \max\{n, N - \hat{n}\}, \hat{n} = 0, \ldots, N\}$, i.e., $k_n = \# K_n$ (here, we have slightly abused the notations; j denotes an action instead of an integer). An action $a_j = a_{(o_0, \ldots, o_N)}$ means that the agent offers $o_{\hat{n}} p^*$ amount of money when she is a buyer and the partner's money holdings are $\hat{n} p^*$. Note that the set of a seller's actions is a singleton, since all of his gain from trade is extracted on the equilibrium path.
- $f\left((n, (o_0, \ldots, o_N)), (n', (o'_0, \ldots, o'_N))\right) = o_{n'}$.
- The time derivative of the matching between a seller (n, j) and a buyer (n', j') is $(\mu/k) h(n, j) h(n', j')$.
- $V(n)$ is the value of np^*.

Similarly as in Zhou model, although we also need to consider actions excluded from our action space, and the strategy and the value at $\eta \notin \{0, p^*, \ldots, Np^*\}$, we will show that the above specifications are sufficient. For the details, see Section 4.2.

3 Stationarity

By the definition of f, the outflow O_n and the inflow I_n at state n, functions of h, β, and α, are defined as follows:

$$O_n(h, \beta, \alpha) = \sum_{j, i', j'} \alpha h(n, j) h(i', j') + \sum_{i, j, j'} \alpha h(i, j) h(n, j'),$$
$$I_n(h, \beta, \alpha) = \sum_{(i, j, i', j') \in B_n} \alpha h(i, j) h(i', j') + \sum_{(i, j, i', j') \in B'_n} \alpha h(i, j) h(i', j'),$$

where

$$B_n = \{(i, j, i', j') \mid i + f((i, j), (i', j')) = n\},$$
$$B'_n = \{(i, j, i', j') \mid i' - f((i, j), (i', j')) = n\}.$$

The condition for stationarity is $O_n = I_n, n = 0, \ldots, N$, and $\sum_{n=0}^{N} h(n) = 1$. In fact, O_n (I_n) is the "gross" outflow (inflow resp.), since it includes the fraction of agents who are matched with others but make no trade using the medium of exchange, i.e., the case of $f((i,j),(i',j')) = 0$, where $i = n$ or $i' = n$. Since such fractions are included both in O_n and in I_n, then they are clearly canceled out. Thus even if we replace them with the "net" outflow and inflow, the results in what follows do not change.

Clearly, $\sum_{n=0}^{N}(O_n - I_n) = 0$ holds and thus at least one equation is redundant. At first glance, both the numbers of linearly independent equations and of variables seem to be $N+1$. However, the following theorem shows that one more equation is always redundant.

Theorem 1

$$\sum_{n=0}^{N} nO_n = \sum_{n=0}^{N} nI_n. \tag{1}$$

(Note that the terms O_0 and I_0 are multiplied by 0.)

Below, we present a simple proof. A direct proof is also given in Appendix A.[6]

Proof: Consider a pair of pairs (n,j) and (n',j'). By the matchings between them, the proportion $\alpha h(n,j)h(n',j')$ of agents move from n to $n + f((n,j),(n',j'))$, and the same proportion of agents move from n' to $n' - f((n,j),(n',j'))$. Corresponding to the moves, the following terms appear in the RHS and in the LHS of (1):

$$\begin{array}{cc} \text{the LHS} & \text{the RHS} \\ n\alpha h(n,j)h(n',j') & (n + f((n,j),(n',j')))\alpha h(n,j)h(n',j') \\ n'\alpha h(n,j)h(n',j') & (n' - f((n,j),(n',j')))\alpha h(n,j)h(n',j') \end{array}$$

Clearly, the sum of the terms in the LHS is equal to that in the RHS. Since this holds for any pair of pairs (n,j) and (n',j'), (1) holds. ∎

The interpretation of the theorem is simple. The LHS of (1) is the total amount of money held by the agents involved in trading *before* the trade, while the RHS of (1) is the total amount of money held by them *after* the trade. Clearly, they must coincide. Note that (1) holds even in non-stationary and/or non-equilibrium situations.

[6] The direct proof is suggested by Prof. Koji Shimomura.

By the above theorem,

$$0 = \sum_{n=0}^{N} nO_n - \sum_{n=0}^{N} nI_n = \sum_{n=1}^{N} n(O_n - I_n)$$

holds, i.e., $O_n - I_n, n = 1, \ldots, N$, are linearly dependent. By $\sum_{n=0}^{N}(O_n - I_n) = 0$, without loss of generality, we can first delete $O_0 - I_0 = 0$ and then, by the above theorem, can delete $O_1 - I_1 = 0$. Thus the distribution is stationary if and only if $O_n - I_n = 0, n = 2, \ldots, N$, and $\sum_{n=0}^{N} h(n) - 1 = 0$ hold. That is, for a given β, the number of linearly independent equations is less than that of variables. Namely, the condition for stationarity has at least one degree of freedom. In the next section, it is shown that this is the main cause of the real indeterminacy.

4 Real Indeterminacy of Stationary Equilibria

In this section, we show that Theorem 1 implies the real indeterminacy of stationary equilibria. First, without using Theorem 1, we directly show that there exits a continuum of stationary equilibria in Zhou [22]'s model and the divisible money version of Camera and Corbae [2]'s model. Next, using Theorem 1, we present a general theory of the real indeterminacy. More precisely, we define a stationary quasi-equilibrium which is easy to deal with and show its real indeterminacy. Then we present a sufficient condition that a stationary quasi-equilibrium is a stationary equilibrium. The rigorous discussion about real indeterminacy, based on differential topology, is given in Appendix B. Moreover, the case of indivisible money is discussed.

4.1 Zhou Model

Zhou [22] shows the existence of "single price equilibria" having the following feature: the stationary distribution has masses only at 0 and p^*, i.e., the endogenously determined upper bound of money holdings N is 1, sellers without money always offer p^*, sellers with p^* always offer ∞, and thus trades occur only between sellers without money and buyers with p^*.

In order to show that this can happen as an equilibrium phenomenon, we first convert Zhou model into our framework as in Section 2.2. In this type of the equilibria, sellers with p^* cannot sell their production goods on the equilibrium path, since there are no agents who afford to accept their offers. Then, even if we modify the equilibrium strategy such that agents with p^* offer $\hat{N}p^*$, where $\hat{N} \geq 2$, the value on the equilibrium path does not change.

Since $N = 1$ is endogenously determined, we should check incentives at $np^*, n \geq 2$. As is the case of agents with p^*, we consider equilibrium strategy such that agents with np^* $(n \geq 2)$ offer $\hat{N}p^*$, where $\hat{N} \geq 2$. Thus this type of the equilibrium can be expressed in our model as follows:

$$\beta^*_{0j} = \begin{cases} 1 & \text{if } j = (1,0) \\ 0 & \text{otherwise} \end{cases} \qquad \beta^*_{nj} = \begin{cases} 1 & \text{if } j = (\hat{N},1) \\ 0 & \text{otherwise} \end{cases} \quad \text{for} \quad n \geq 1.$$

Let $\phi = kr/\mu$, where r is a discount rate. Then the Bellman equation is as follows:

$$V(0) = \frac{1}{\phi+2}\left[(1-h(0))(-c+V(1)) + h(0)V(0) + V(0)\right],$$
$$V(n) = \frac{1}{\phi+2}\left[V(n) + (1-h(1))(u+V(n-1)) + h(1)V(n)\right], \quad n \geq 1.$$

Let $h(0) = 1-m$ and $h(1) = m$ for some $m > 0$. Then we obtain

$$V(n) = \frac{1-m}{\phi}u - A^n\frac{\phi+1-m}{\phi(1+\phi)}\left[(1-m)u + mc\right],$$

where $A = \frac{1-m}{\phi+1-m}$.

If an agent without money offers a price larger than p^*, she cannot trade. Thus she prefers to offer p^* if and only if $V(0) \geq 0$. It is verified that this is equivalent to the condition

$$\frac{u}{c} \geq 1 + \frac{\phi}{1-m}.$$

Next, we check an incentive for agents with p^* to offer $\hat{N}p^*$, where $\hat{N} \geq 2$. This is equivalent to the condition that offering p^* makes a loss, i.e., $V(2) - c \leq V(1)$. This holds if and only if

$$\frac{u}{c} \leq \frac{\phi(1+\phi)(\phi+1-m) - \phi m(1-m)}{\phi(1-m)^2}.$$

And thus, if

$$1+\phi < \frac{u}{c} < (1+\phi)^2 \tag{2}$$

holds, then, for any sufficiently small m, any agent with np^* has no incentive to deviate by an action included in A_n.

Based on this result, we extend the distribution, the value, and the strategy to state space $[0, \infty)$. More precisely, define the distribution $\tilde{h}$ as a natural extension of h. Next, define the value function on $[0, \infty)$ such that $\tilde{V}(\eta) = V(\lfloor \eta / p^* \rfloor)$. Lastly, define the equilibrium strategy as follows: (i) when a seller's money holding is less than $2p^*$, then she offers p^*, (ii) otherwise she offers $\hat{N}p^*$, where $\hat{N}$ is defined as the above, (iii) when a buyer's money holding is less than p^*, then he always rejects the seller's offer, and (iv) otherwise he has some reservation price larger than or equal to p^*.[7]

Let us consider a sufficient condition that the profile above indeed forms a stationary equilibrium. First, it is clear that $\tilde{h}$ is a stationary distribution. Of course, p^* is determined by $p^* \tilde{h}(p^*) = p^* m = M$. Note that $p^* > M$ holds. Next, it is easily verified that $\tilde{V}$ is consistent with the equilibrium strategy. Lastly, we need to consider a condition that an agent has no incentive to deviate from the equilibrium strategy. However, we can show that (2) is sufficient. For, since $\tilde{V}$ is a step function with the steps of length p^*, a seller has no strict incentive to offer a price other than an integer multiple of p^*.

In summary, if (2) holds, then, for any sufficiently small $m > 0$, there is no incentive to deviate from the actions specified above. It follows that there is a continuum of stationary equilibria with different m. Note that p^* should also be different in equilibria, since $p^* m = M$.

Remark 6 It may seem strange that our condition is different from that in Corollary 2.1 in Zhou [22]. The difference arises from the fact that we modify the equilibrium strategy and thus the "weak undominatedness" in Zhou [22] is not necessarily satisfied. Consider the case that parameters satisfy only our condition. If there were agents who accept the offer, then there should be some offer prices more profitable than ∞; that is, the offer ∞ is weakly dominated. However, there do not exist such agents on the equilibrium path. Therefore the offer price ∞ can also be an equilibrium offer.

4.2 Divisible Money Version of Camera and Corbae Model

Some might think that the real indeterminacy result in Zhou [22] crucially depends upon the assumptions held in the model: money holdings of a matched partner are unobservable and a bargaining proceeds in a way like double auction. We try to refute this. For this purpose, we show that there is also a continuum of stationary equilibria

[7] Strictly speaking, the strategy that (iv) specifies at $\eta \neq p^*$ is not a direct extension of β^*. But, this makes no problem, since no price other than p^* is ever offered on the equilibrium path.

in the divisible money version of CC model. Recall that CC assume that the money holding of a partner is observable and a bargaining proceeds by buyer's take-it-or-leave-it offer. It suggests that the real indeterminacy is independent of the informational setting and the bargaining procedure assumed in Zhou [22].

Let us start with considering the original version of CC in which fiat money is indivisible. CC show that there exists a stationary equilibrium in which all trades occur with one unit of fiat money in some region of parameters. More precisely, they construct the strategy in which buyers with positive money holdings always offer one unit of fiat money and the quantity of goods such that the seller is indifferent between accept and reject. Let $\phi = kr/\mu$.[8] Then their result (Proposition 2, [2]) is as follows:

Proposition 1 Suppose that fiat money is indivisible. There exists $\Phi(\lambda, N, M^S) > 0$ such that, for any $N \geq 1$, any M^S, any $\lambda > 1 - (1/(N-1))$, and any $\phi < \Phi(\lambda, N, M^S)$, the strategy stated above, together with some distribution of money holdings, forms a stationary equilibrium.

Let us turn to the case of divisible money. Recall that $\bar{N}$ is the upper bound of money holdings. Then, by Proposition 1, we can construct the stationary equilibrium in which all agents behave as if p^* were the minimum unit for any $p^* \in (0, \bar{N}]$.

Consider the discrete distribution with masses only at $\eta = 0, p^*, 2p^*, \ldots, \lfloor \bar{N}/p^* \rfloor p^*$, which is a natural extension of the distribution in Proposition 1. Define the equilibrium strategy as follows: (i) when a seller has an integer multiple of p^*, then the partner offers p^* quantity of money and the corresponding quantity of goods if his money holding is more than or equal to p^*, and he does not trade if his money holding is less than p^*, and (ii) the other cases, which happen with probability zero, are given below.

Below, we present conditions for the above profile of distribution and strategy to form a stationary equilibrium. Clearly, the distribution is stationary. Similarly, the incentive compatibility conditions for deviating from offering p^* to another integer multiple of p^* are equivalent to those of the case of indivisible money. Below, we check the other conditions, i.e., the incentive compatibility conditions for deviating to offering non-integer multiple of p^*

First of all, by the above distribution and strategy, the (candidate for) value function defined on $[0, \bar{N}]$, denoted by $\tilde{V}$, must be a step function with the steps of length p^*.[9]

[8]Then ϕ in this paper is the reciprocal of ϕ in CC [2].

[9]In this paper, a value function defined on a certain interval in R is denoted by $\tilde{V}$ whereas the one defined on $\{0, 1, \ldots, N\}$ is denoted by V.

Next, check the incentive of buyers. First, consider the case that a seller's money holding is an integer multiple of p^*. Let η (possibly non-integer multiple of p^*) be the buyer's money holdings, η' be the quantity of money she offers, and np^* be the seller's money holdings. Let $q_{(np^*,\eta')}$ be the quantity of goods which the seller is indifferent between accepting and rejecting, then

$$q_{(np^*,\eta')} = \tilde{V}(np^* + \eta') - \tilde{V}(np^*).$$

Next, let $n' = \lfloor \eta'/p^* \rfloor$, then

$$\tilde{V}(np^* + \eta') = \tilde{V}((n + n')p^*),$$

since $\tilde{V}$ is a step function with the steps of length p^*. Thus $q_{(np^*,\eta')} = q_{(np^*,n'p^*)}$. That is offering $n'p^*$ is not less profitable than offering η'. Thus it suffices to check the incentive compatibility conditions only for offering some integer multiple of p^*. Next, consider the case that a seller's money holding is not an integer multiple of p^*. In this case, choose any buyer's strategy which exploits all gains from trade. Note that this behavior does not affect the value of buyers since the above matching occurs with probability zero.

Also, given the strategy defined above, the seller's value after trade is always the same as the one before trade.

Thus the Bellman equation at η is the same as the one at $\lfloor \eta/p^* \rfloor$. Thus the Bellman equation is satisfied for all $\eta \in [0, \bar{N}]$.

For an intuitive illustration of the arguments, consider a buyer with money holding of $1.5p^*$. One might think that she has a strict incentive to offer $.5p^*$ instead of p^*. However, the money holdings of her future partners will be some integer multiple of p^* with probability 1, so that she does not appreciate smaller portion of money than p^*. Thus the quantity in compensation for $.5p^*$ is the same as for 0, i.e., $q_{(np^*,.5p^*)} = q_{(np^*,0p^*)}$.

Thus we obtain the following result:

Proposition 2 Suppose that fiat money is perfectly divisible. Then for $p^* \in (0, \bar{N}]$, any M, any $\lambda > 1 - (1/(\lfloor \bar{N}/p^* \rfloor - 1))$, and any $\phi < \Phi(\lambda, \lfloor \bar{N}/p^* \rfloor, M/p^*)$ where Φ appears in Proposition 1, the profile of distribution and strategy stated above forms a stationary equilibrium.

Since p^* is an endogenous variable in the divisible money model, there is a continuum of stationary equilibria with different p^*.

4.3 General Theory of the Real Indeterminacy

In this subsection, we present an informal discussion of the general theory of the real indeterminacy by using Theorem 1. See Appendix B for the detailed presentation.

First, we present the definition of a stationary quasi-equilibrium.

Definition 1 A triple $x^* = (V^*, h^*, \beta^*) \in \mathbb{R}^{N+1} \times \mathbb{R}_+^{N+1} \times \mathbb{R}_+^{\sum_{n=0}^N k_n}$ is said to be *a stationary quasi-equilibrium* for a given γ if

$$O_n(h^*, \beta^*, \alpha) - I_n(h^*, \beta^*, \alpha) = 0, \qquad n = 2, \dots, N, \tag{3}$$

$$\sum_{n=0}^{N} h^*(n) - 1 = 0, \tag{4}$$

$$V^*(n) - W_{nj}(x^*, \gamma) = 0 \qquad \text{if } \beta^*_{nj} > 0, \tag{5}$$

$$\sum_{j=1}^{k_n} \beta^*_{nj} - 1 = 0, \qquad n = 0, \dots, N, \tag{6}$$

$$V^*(n) - W_{nj}(x^*, \gamma) \geq 0 \qquad \text{if } \beta^*_{nj} = 0. \tag{7}$$

Recall that α is included in γ.

Note that we do not require (3) for $n = 0, 1$; it suffices to define a stationary distribution due to Theorem 1. We call x^* a "quasi-equilibrium" because we need some additional conditions in order for x^* to be a real stationary equilibrium: (i) the existence of $p > 0$ satisfying $\sum_{n=0}^N pnh(n) = M$, which is equivalent to the condition $h(0) < 1$, (ii) the incentive not to choose an action out of our action space, and (iii) the existence of strategies at state $\eta \notin \{0, p, \dots, Np\}$ consistent with the given stationary quasi-equilibrium. For the details, see Section 4.4.

Remark 7 In order for V^* to optimize the real objective function, the transversality condition in dynamic programming should be satisfied. In the case of time additive expected utility with a discount factor $\kappa \in (0, 1)$, it is clearly satisfied.

First, we fix the set of equilibrium actions,[10] denoted by b, i.e., b is a set of (n, j), and we confine the domain of β to

$$\Omega^b = \{(\beta_{nj})_{(n,j)\in b} \mid \beta_{nj} > 0 \quad \text{for} \quad (n, j) \in b\}. \tag{8}$$

[10] Without fixing equilibrium actions, we can formulate the problem as a kind of nonlinear complementarity problems. However, the special structure of the problem prevents us to use the standard technique. That is, without fixing it, the dimension of equilibria may not be determinate. (See Appendix B.)

In the previous section, we showed that two of $O_n(h,\beta,\alpha) - I_n(h,\beta,\alpha), n = 0,\ldots,N$, are redundant. Thus, for given b and β, h is determined up to at least one degree of freedom. Suppose, for a given stationary quasi-equilibrium (V^*, h^*, β^*), (7) is satisfied with strict inequality for all (n,j) such that $\beta_{nj} = 0$. Then all stationary quasi-equilibria in a small neighborhood of (V^*, h^*, β^*) are determined by (3)-(6), and, by the above argument on stationary distributions, the number of equations and variables are $2N+\#b+1$ and $2N+\#b+2$, respectively. Thus the set of equilibria is generically at least one-dimensional. This means that the main cause of indeterminacy is the feature of stationary distributions shown in Theorem 1.

To be more precise, consider a stationary quasi-equilibrium (V^*, h^*, β^*) and the corresponding b^*. Then

$$V^*(n) - W_{nj}(x^*,\gamma) \geq 0 \quad \text{for} \quad (n,j) \notin b^* \tag{9}$$

holds. Suppose in (9) all of inequalities are strict. Then, besides degenerate cases, it follows from the implicit function theorem that the dimension of the set of stationary quasi-equilibria around (V^*, h^*, β^*) is at least one. Of course, $V^*(n) = W_{nj}(x^*,\gamma)$ may hold for some $(n,j) \notin b^*$. However, under mild conditions, we can show that generically only one of inequalities in (9) can be equal. If there is just one equality in (9), then it is on the boundary of a connected set of stationary quasi-equilibria of which dimension is more than or equal to one. Thus the dimension of the set of stationary quasi-equilibria is generically more than or equal to one.

The rigorous discussions of the above and several indeterminacy theorems will be given in Appendix B. Below, we only present the most important theorems. For a given γ, let $E_\gamma^{b^*}$ be the set of stationary quasi-equilibria such that β_{nj} can be positive only if $(n,j) \in b^*$. Let g^{b^*} be the function expressed by the LHS of (3)-(6) and (9) replacing "if $\beta^*_{nj} > 0$" in (5) by "if $\beta_{nj} \in b^*$". Let C^{b^*}, $C^{b^*(n,j)}$, and $C^{b^*(n,j)(n',j')}$ be the subsets in the final set[11] corresponding to the set of stationary quasi-equilibria in which all inequalities in (9) are strict, only the (n,j)th one is equal, and only the (n,j)th and the (n',j')th ones are equal, respectively. (For the precise definitions, see Appendix B.)

Theorem 2 Let $\Gamma \subset \mathbb{R}^L$ be a C^2 manifold without boundary.[12] For a given b^*, suppose that $E_\gamma^{b^*} \neq \emptyset$ holds for all $\gamma \in \Gamma$, and that g^{b^*} is C^2 and is transversal to C^{b^*}, $C^{b^*(n,j)}$, and $C^{b^*(n,j)(n',j')}$ for all $(n,j),(n',j') \notin b^*$. Then, for almost every $\gamma \in \Gamma$, $E_\gamma^{b^*}$ is a

[11] For a function $F : X \to Y$, X, Y, and $\{y \in Y | \exists x \in X, F(x) = y\}$ are called the domain, the final set, and the range, respectively.

[12] Γ can be considered as the set of γ such that some strategies using actions in b^* can be a stationary quasi-equilibrium.

one-dimensional manifold with boundary. Moreover, at any endpoint of the manifold, only one $V^*(n) - W_{nj}(x^*, \gamma) \geq 0, (n,j) \notin b^*$, can be binding.

For simplicity, we assume that Γ is an open set in $\mathbb{R}^L$. For example, "g^{b^*} is transversal to C^{b^*}" means if $g^{b^*}(\hat{x}, \hat{\gamma}) \in C^{b^*}$ holds for some $(\hat{x}, \hat{\gamma})$, then, together with the tangent space of C^{b^*} at $g^{b^*}(\hat{x}, \hat{\gamma})$, the space $\{Dg^{b^*}_{(\hat{x},\hat{\gamma})}(x^T, \gamma)^T \mid (x, \gamma) \text{ is in the domain}\}$ spans the final set,[13] where $Dg^{b^*}_{(\hat{x},\hat{\gamma})}$ is the Jacobian matrix at $(\hat{x}, \hat{\gamma})$ and T denotes transpose. As shown in the examples in the following subsection, the conditions in the theorem are quite mild and can often be easily verified at least locally. Note that, by verifying the condition locally, we can show that there is a continuum of equilibria. We should verify the condition globally in order to find some features of the set of equilibria. Using the features, we can numerically compute a connected component of equilibria. (See [13].)

Although we have shown that there is a kind of indeterminacy, it might not be a real one. That is, in a connected component of the set of equilibria, the real variables h and V might be the same. For real indeterminacy, it suffices to show that the welfare $\sum_{n=0}^N h(n)V(n)$ can be the same only in a set of measure zero in the set of stationary quasi-equilibria. To see this, for a given b^*, we analyze $\sum_{n=0}^N h(n)V(n) = a$ together with (3)-(6) and (9) replacing "if $\beta^*_{nj} > 0$" in (5) by "if $\beta_{nj} \in b^*$". Fix $a \in R$. Then the numbers of equations and variables are the same and thus the dimension of the set of stationary quasi-equilibria with welfare a is generically one dimension less than that of the set of stationary quasi-equilibrium. The theorem can be stated as follows. First, we modify g^{b^*}, denoted by $g^{b^*}_w$, adding one equation $\sum_{n=0}^N h(n)V(n) - a = 0$ and one variable $a \in R$. We should also modify C^{b^*}, $C^{b^*(n,j)}$, and $C^{b^*(n,j)(n',j')}$ denoted by $C^{b^*}_a$, $C^{b^*(n,j)}_a$, and $C^{b^*(n,j)(n',j')}_a$, respectively; for example, $C^{b^*}_a$ is the subsets in the final set corresponding to the cases that all inequalities in (9) are strict and the welfare is a.

Theorem 3 Let $\Gamma \subset \mathbb{R}^L$ be a manifold without boundary. For b^*, suppose that $E^{b^*}_\gamma \neq \emptyset$ holds for all $\gamma \in \Gamma$, and that, for any given a, $g^{b^*}_w(\cdot, a)$ is C^1 and is transversal to $C^{b^*}_a$, $C^{b^*(n,j)}_a$, and $C^{b^*(n,j)(n',j')}_a$ for all $(n,j), (n',j') \notin b^*$. Then, for almost every $\gamma \in \Gamma$, $E^{b^*}_\gamma \cap \{x | \sum_{n=0}^N h(n)V(n) = a\}$ is a zero-dimensional manifold.

4.4 Stationary Quasi-Equilibrium and Stationary Equilibrium

In the previous subsection, we focused on stationary quasi-equilibria instead of stationary equilibria, because the former is easier to deal with than the latter. In this

[13] In general, the tangent space at $g^{b^*}(\hat{x}, \hat{\gamma})$ should be spanned.

subsection, we discuss a sufficient condition that the indeterminacy of stationary quasi-equilibria implies that of stationary equilibria. In general, the following three conditions are the additional conditions for the definition of stationary equilibria: (i) the existence of $p > 0$ such that $\sum_{n=0}^{N} pnh(n) = M$, i.e., $h(0) < 1$, (ii) the incentive not to choose an action out of our action space, and (iii) the existence of strategies at state $\eta \notin \{0, p, \ldots, Np\}$ consistent with the given stationary quasi-equilibrium. (i) can be easily checked. We need to check (ii) and (iii) carefully. One might think that an agent, for example, has strict incentive to offer $.5p$, since she may obtain (pay) $.5p$ later. However, under some reasonable assumptions, this is not true, because such a trade cannot occur later on the equilibrium path. Below, we discuss this point rigorously.

We focus on economies in which each matched agents observe the partners' money holdings. We consider a model with the state space $[0, \bar{N}]$ for some positive real number $\bar{N}$, the set of money holdings, and the action space $\tilde{A} = \Pi_{\eta \in [0,\bar{N}]} \tilde{A}_\eta$, where each element in $\tilde{A}_\eta$ is represented by a finite dimensional vector of amounts of money $r = (r_i)$ transferred along with trade[14] and the other factor $t \in T$ related to the bargaining, where T is a finite set.[15] In other words, we focus on the case that the other factors in trade, e.g., the amount of goods, can be considered to be determined by (r, t) and V. (See examples below.) We first suppose that

> $(*)$ (a) a bargaining game between a matched pair[16] has a pure strategy Markov perfect equilibrium, and (b) the bargaining game immediately ends on the equilibrium path.

For a matched pair, let $V_{s0} = V(\eta_s)$ and $V_{b0} = V(\eta_b)$ be the seller's and the buyer's values of their money holdings, respectively, and $F \subset \mathbb{R}^2$ be the feasible set of values of their money holdings after trade. We further suppose that

> $(**)$ (a) F does not directly depend on r but on $(V(\eta_s + r_i), V(\eta_b - r_i))$, where $(r, t) = ((r_i), t)$ is in $\tilde{A}_{\eta_s}$ or in $\tilde{A}_{\eta_b}$ for some $t \in T$, [17] and (b) an outcome of bargaining game (V_{s1}, V_{b1}, y), where (V_{s1}, V_{b1}) are the values of the seller's and buyer's money holdings after trade, and y is a vector of the other elements of outcome,[18] depends only on (V_{s0}, V_{b0}, F).

[14]For example, $r = (r_1, r_2)$ is the amounts of money offer when an agent is a seller and when she is a buyer.

[15]For example, T includes "replies" such as "accept" or "reject".

[16]Here a bargaining game between a matched pair is defined as a sequential game in which the initial node is an instant when they are matched and every terminal node is an instant when the pair dissolves.

[17]F does not necessarily consist of $\{(V(\eta_s + r_i), V(\eta_b - r_i)) | (r, t) = ((r_i), t)$ is in $\tilde{A}_{\eta_s}$ or in $\tilde{A}_{\eta_b}$ for some $t \in T\}$. That is some element in this set might be rejected in the bargaining. $(**)$(a) simply means that F does not directly depend on amounts of money but on the values after trades.

[18]For example, y includes the amount of the commodity goods.

By $(*)$, the bargaining game has an outcome, and thus $(**)$(b) has meaning.

For a given N, we focus on a set of $p > 0$ such that $N = \lfloor \frac{\bar{N}}{p} \rfloor$. We restrict our attention to the state space $\{0, 1, \ldots, N\}$ and the action space $\Pi_{n=0}^{N} A_n$, where A_n is a finite set such that, for each $p > 0$, each element $a \in A_n$ corresponds to $(r, t) \in \tilde{A}_{np}$ and all elements of r are integer multiples of p. For example, $a = (n_1, n_2)$ corresponds to $r = (n_1 p, n_2 p)$, where $n_1 p$ and $n_2 p$ are the amounts of money offer when an agent is a seller and when she is a buyer, respectively. Then a stationary quasi-equilibrium (V^*, h^*, β^*) with $p^* = \frac{M}{\sum_n n h^*(n)}$ is defined. Below, we show that it corresponds to a stationary equilibrium if $h^*(0) < 1$. More precisely, define $(\tilde{V}^*, \tilde{h}^*, (\tilde{\beta}^*_{np^*})_{n=0}^{N})$ as $\tilde{V}^*(\eta) = V^*\left(\left\lfloor \frac{\eta}{p^*} \right\rfloor\right)$, $\tilde{h}^*$ is the natural extension of h^* to $[0, \bar{N}]$, and $\tilde{\beta}^*_{np^*}$ is naturally defined from β^*_n. Then it will be shown that $(\tilde{V}^*, \tilde{h}^*, (\tilde{\beta}^*_{np^*})_{n=0}^{N})$ satisfies the following two conditions: (ii') each agent at np^* has no strict incentive to choose actions out of the support of $\tilde{\beta}^*_{np}$, and (iii') for each $\eta \in [0, \bar{N}]$, which is not an integer multiple of p^*, there exists some $(\tilde{\beta}^*_{\eta})$ consistent with $\tilde{V}^*$. Note that (ii'), (iii') corresponds to (ii), (iii) stated above, respectively. Thus, (ii') and (iii') imply that $(\tilde{V}^*, \tilde{h}^*, \tilde{\beta}^*)$ forms a stationary equilibrium.

First, since $\tilde{V}^*$ is a step function with the steps of length p^*, we obtain the same feasible set as V^* and thus, by the assumption $(**)$, (ii') is satisfied.

Next, we check (iii'). Consider an agent with η such that η/p^* is not an integer. Clearly, her partner holds np^* for some n with probability one. Since $\tilde{V}^*(\eta) = V^*(\lfloor \frac{\eta}{p^*} \rfloor)$ and the feasible set for this pair is the same as that of the pair of agents with $\lfloor \frac{\eta}{p^*} \rfloor p^*$ and np^*, and thus, by the assumption $(**)$, the outcome of the bargaining game is the same as that of the pair of agents with $\lfloor \frac{\eta}{p^*} \rfloor p^*$ and np^*, and the value of η is indeed $V^*(\lfloor \frac{\eta}{p^*} \rfloor)$, i.e., the outcome is consistent with $\tilde{V}^*$. For a pair of agents with η and η' such that both η/p^* and η'/p^* are not integers, we can choose any Markov perfect equilibria of the bargaining game. In other words, the choice does not affect the value function since η's partner is np^* for some n with probability one.

It is verified that the divisible money version of Camera and Corbae's model satisfies $(*)$ and $(**)$ as follows. In Camera and Corbae's model, if the seller and the buyer have η and η', respectively, then the buyer's offer η'' maximizes $U(q_{(\eta,\eta'')}) + V(\eta' - \eta'')$, where $q_{(\eta,\eta'')}$ is a solution to

$$\tilde{V}(\eta + \eta'') - q = \tilde{V}(\eta). \tag{10}$$

$(*)$ and $(**)$ are clearly satisfied. Note that the amount of goods in trade is determined by (10), i.e., it is an element of the other outcome stated in $(**)$.

Similarly, we can deal with a divisible money version of Trejos and Wright [20]'s model. Let us extend their model to the one with divisible fiat money in the same way as we did in Section 2.3, and consider the distribution and the strategy in which agents behave as if p^* were the minimum unit of divisibility. Then the (candidate for) value function is a step function with the steps of length p^*. Thus, if the seller and the buyer have η and η', respectively, and the amount of money d maximizes

$$\max_{d} \; (U(q_d) + \tilde{V}(\eta' - d))(-C(q_d) + \tilde{V}(\eta + d)), \tag{11}$$

where q_d is a solution to

$$\max_{q} \; (U(q) + \tilde{V}(\eta' - d))(-C(q) + \tilde{V}(\eta + d)), \tag{12}$$

(see Trejos and Wright [20], p. 134.), then $(*)$ and $(**)$ are clearly satisfied. In other words, some integer multiple of p^* is necessarily one maximizer of the above, so, if such a distribution and a strategy form a stationary quasi-equilibrium, they are also a stationary equilibrium.[19] A similar argument applies to Shi [19].

Although the discussion above depends upon the assumption that each matched agent observes the partners' money holdings, this is not a crucial assumption. For example, Zhou's model assumes that each matched agents cannot observe the partner's money holdings. It follows that a bargaining outcome depends upon a distribution of $(V_{s0},\ V_{b0}, F)$ determined by h and β. However, this would not virtually change the analysis above.

4.5 Indivisible Money

In the case of indivisible money, our results suggest that the greater the divisibility of money , the larger the number of equilibria. Suppose that p should be in a finite set $P = \{p_1, p_2, \ldots, p_L\}$, where $p_\ell < p_{\ell+1}$. For example, $p_\ell = \ell$ *dollars*. Suppose p^* in P is a solution to $M = \sum_{n=0}^{N} pnh^*(n)$, where h^* is an equilibrium distribution. By the above arguments, all h in a neighborhood of h^* can be equilibrium distributions in the perfectly divisible case. If some $p_{\bar{\ell}}$ in P is a solution to $\sum_{n=0}^{N} pnh(n) = M$, where h is in the neighborhood, then h is also an equilibrium distribution in the indivisible case.

[19] Consider the following equilibrium candidate: Let $\bar{N}$ be the upper bound of money holdings. Choose any p^* such that $\bar{N}/2 < p^* \leq \bar{N}$, i.e., $\lfloor \bar{N}/p^* \rfloor = 1$. Let the money holdings distribution have masses only at 0 and p^*, and the strategy specify that any agent offers a money quantity p^* (if possible). Then, it is easy to show that this profile of the distribution and the strategy form a stationary equilibrium since p^* is necessarily a maximizer of (11).

Therefore, if $p_{\ell+1} - p_\ell$ is small, then there must be a lot of equilibria. In other words, for a fixed money supply and a fixed upper bound of money holdings, there are much larger number of equilibria in the case that one unit of money is one cent compared with the case which the minimum unit is a hundred dollar.

Formally, for a given b^*, we analyze $\sum_{n=0}^{N} nh(n) = t$ together with (3)-(6) and (9) replacing "if $\beta^*_{nj} > 0$" in (5) by "if $\beta_{nj} \in b^*$". The theorem can be stated as follows. First, we modify g^{b^*}, denoted by $g_d^{b^*}$, adding one equation $\sum_{n=0}^{N} nh(n) - t = 0$ and one variable $t \in R$. We should also modify C^{b^*}, $C^{b^*(n,j)}$, and $C^{b^*(n,j)(n',j')}$ denoted by $C_d^{b^*}$, $C_d^{b^*(n,j)}$, and $C_d^{b^*(n,j)(n',j')}$, respectively; for example, $C_d^{b^*}$ is the subsets in the final set corresponding to the cases that all inequalities in (9) are strict and $\sum_{n=0}^{N} nh(n) - t = 0$.

Theorem 4 Let M and N be given and $\Gamma \subset \mathbb{R}^L$ be a manifold without boundary. For b^*, suppose that $E_\gamma^{b^*} \neq \emptyset$ holds for all $\gamma \in \Gamma$, and that $g_d^{b^*}$ is C^2 and is transversal to $C_d^{b^*}$, $C_d^{b^*(n,j)}$, and $C_d^{b^*(n,j)(n',j')}$ for all $(n,j),(n',j') \notin b^*$. Then, for almost every $\gamma \in \Gamma$, and for any positive integer I, there exists a positive integer L such that the number of stationary quasi-equilibria with

$$p \in P_L = \{\ell_1 + \frac{\ell_2}{\ell_3} \mid \ell_1 \text{ is a nonnegative integer}, \ell_2 = 0, 1, \ldots, L, \ell_3 = 1, 2, \ldots, L, \ell_2 \leq \ell_3\}$$

is larger than I.

Proof: Let $x^* = (V^*, h^*, \beta^{b^*})$ be a stationary quasi-equilibrium. Let $t^* = \sum_{n=0}^{N} nh^*(n)$. By the assumptions and implicit function theorem, there exists an $\varepsilon > 0$ and a function $\varphi : (t^* - \varepsilon, t^* + \varepsilon) \to E_\gamma^{b^*}$. Thus for a large enough L, the number of $p \in P_L$ such that $\frac{M}{p} \in (t^* - \varepsilon, t^* + \varepsilon)$ is larger than I. ∎

5 Extensions of the Basic Model

In the previous sections, we assumed that there is only one kind of money, the matching is pairwise, money holdings give no utility, and there is an upper bound of money holdings. All of these assumptions can be easily relaxed. However, in order to avoid complicated notations, we mainly show them by examples.

First, we can allow for multiple fiat money. For simplicity, we assume that there are two money. Let the state space be $\{0, \ldots, N^1\} \times \{0, \ldots, N^2\}$. Suppose that an agent with (n^1, n^2), a seller, meets an agent with $(\bar{n}^1, \bar{n}^2)$, a buyer, and that a trade occurs. The seller pays (m^1, m^2) to the buyer. Suppose, at each period, the proportion of the above type of matching is ξ. Thus, by the trade, the agents in the match move from (n^1, n^2) to $(n^1 + m^1, n^2 + m^2)$, and $(\bar{n}^1, \bar{n}^2)$ to $(\bar{n}^1 - m^1, \bar{n}^2 - m^2)$, respectively, and the proportion of each move is of course ξ. Thus the same argument as in the proof of Theorem 1 applies and

$$\sum_{n^1=0}^{N^1} \sum_{n^2=0}^{N^2} (n^1 + n^2) O_{(n^1,n^2)} = \sum_{n^1=0}^{N^1} \sum_{n^2=0}^{N^2} (n^1 + n^2) I_{(n^1,n^2)}$$

holds, where $O_{(n^1,n^2)}$ and $I_{(n^1,n^2)}$ are the outflow and the inflow at (n^1, n^2), respectively. That is, by the condition for stationarity, the stationary distribution is determined up to at least one degree of freedom and the dimension of the set of stationary equilibria is typically more than or equal to one. Of course, this argument can be applied to much more general cases.

Second, matchings need not be pairwise. We consider the following model. There are $k + 1$ goods, where $k \geq 4$. The first k goods are indivisible and immediately perishable, and good i is consumed by type i agents. The remaining good is a perfectly divisible and durable fiat-money object. A type i agent and a type $i + 1$ agent can cooperate to produce one unit of good $i + 2$ (mod. 3). A type i agent consumes only good i and derives instantaneous utility. Each agent is characterized by her type and the amount of money she holds. Suppose there is a matching technology that always chooses 3 agents. If their types are i, $i+1$, and $i+2$ (mod. 3), then a trade potentially occurs. Let their money holdings be pn_1, pn_2, and pn_3, respectively. Suppose, by a bargaining procedure, a trade occurs and the type $i + 2$ agent pays the type i agent pm_i and the type $i + 1$ agent pm_{i+1}. Suppose the proportion of the above type of matching is ξ. Thus, by the trade, the agents in the match move from n_i to $n_i + m_i$, n_{i+1} to $n_{i+1} + m_{i+1}$, and n_{i+2} to $n_{i+2} - m_i - m_{i+1}$, respectively, and the proportion of each move is of course ξ. Thus the same argument as in the proof of Theorem 1 applies and $\sum_{n=0}^{N} nO_n = \sum_{n=0}^{N} nI_n$ holds. That is, by the condition for stationarity, h is determined up to at least one degree of freedom and the dimension of the set of stationary equilibria is typically more than or equal to one. Of course, this argument can be applied to much more general cases.

Third, as we mentioned in Section 2, we can deal with models in which money holdings give some utility to the holder. In those models, W_{nj} have the following form:

$$W_{nj}(x, p, \gamma) = U(np, h, \beta, j, \gamma) + \kappa E(V(n')|h, \beta, n, j),$$

where κ is the discount factor, U is the temporal utility, and $E(V(n')|h, \beta, n, j)$ is the expectation of $V(n')$ conditional on h, β, n, j. Here U depends on np, the quantity of the medium of exchange. In this case, we cannot deal with $\sum_{n=0}^{N} pnh(n) = M$ separately, since W_{nj} depends on p. However, analyzing $\sum_{n=0}^{N} pnh(n) = M$ and (3)-(7) in the definition of stationary quasi-equilibrium simultaneously, we can obtain the same results as in Section 4. Intuitively, even in this case, the number of equations is less than that of variables. Of course, we should notice that the sufficient conditions that a stationary quasi-equilibrium is a stationary equilibrium, stated in Section 4.4, may not be satisfied in some models. However, Zhou [23] shows that there also exists a continuum of stationary equilibria in Green and Zhou model even if money holdings give a dividend in the form of utility.

Finally, we discuss the case of $N = \infty$. In Appendix C, we show that $\sum_{n=0}^{\infty} n(O_n - I_n) = 0$ holds under a mild condition which is satisfied in Green and Zhou [5]. As in Section 3, $O_0 - I_0$ and $O_1 - I_1$ are redundant in the condition of stationary distribution. Thus, together with $\sum_{n=0}^{\infty} h(n) = 1$, the stationary distribution could be determined with at least one degree of freedom. Of course, this argument is very rough. For the rigorous arguments, we should use the implicit function theorem or the transversality theorem in infinite dimensional spaces. (See, for example, Abraham and Robbin [1].) However, it seems that the conditions for these theorems are typically satisfied in our environment. Indeed, in Green and Zhou [5], the conditions are satisfied and the stationary distribution has (at least) one degree of freedom.

Appendix

A A Direct Proof of the Identity

Since at each time period the total amount of money before the trades is equal the one after the trades,

$$\sum_{i,j,i',j'} i\alpha h(i,j)h(i',j') + \sum_{i,j,i',j'} i'\alpha h(i,j)h(i',j')$$

$$= \sum_{i,j,i',j'} ((i + f((i,j),(i',j')))\alpha h(i,j)h(i',j') + \sum_{i,j,i',j'} (i' - f((i,j),(i',j')))\alpha h(i,j)h(i',j') \tag{13}$$

clearly holds. Below, we show that the LHS of (13) is equal to $\sum_n nO_n$ and the RHS of (13) is equal to $\sum_n nI_n$.

$$\begin{aligned}
\text{The LHS of (13)} &= \sum_{i=0}^{N}\sum_{j,i',j'} i\alpha h(i,j)h(i',j') + \sum_{i'=0}^{N}\sum_{j,i',j'} i'\alpha h(i,j)h(i',j') \\
&= \sum_{n=0}^{N}\sum_{j,i',j'} n\alpha h(n,j)h(i',j') + \sum_{n=0}^{N}\sum_{j,i',j'} n\alpha h(i,j)h(n,j') \\
&= \sum_{n=0}^{N} nO_n.
\end{aligned}$$

$$\begin{aligned}
\text{The RHS of (13)} &= \sum_{n=0}^{N}\sum_{(i,j,i',j')\in B_n} n\alpha h(i,j)h(i',j') + \sum_{n=0}^{N}\sum_{(i,j,i',j')\in B'_n} n\alpha h(i,j)h(i',j') \\
&= \sum_{n=0}^{N}\left[\sum_{(i,j,i',j')\in B_n} n\alpha h(i,j)h(i',j') + \sum_{(i,j,i',j')\in B'_n} n\alpha h(i,j)h(i',j')\right] \\
&= \sum_{n=0}^{N} nI_n.
\end{aligned}$$

B Real Indeterminacy Theorems

We denote $O_n(h,\beta,\alpha) - I_n(h,\beta,\alpha)$ by $D_n(h,\beta,\alpha)$. Let B be the power set of $\{(n,j) \mid j = 1,\ldots,k_n, n = 0,\ldots,N\}$ and $\hat{B}$ be $\{b \in B \mid \forall n, \exists j, (n,j) \in b\}$. $b \in \hat{B}$ can be considered as a set of actions used in an equilibrium. For a given $b \in \hat{B}$, let

$$\Omega^b = \{(\beta_{nj})_{(n,j)\in b} \mid \beta_{nj} > 0 \quad \text{for} \quad (n,j) \in b\}.$$

Let $x^b = (V, h, \beta^b)$, where $\beta^b \in \Omega^b$. For a given $b \in \hat{B}$ and all $(n,j) \in b$, $W^b_{nj}(x^b,\gamma)$ is defined from $W_{nj}(x,\gamma)$ by setting $\beta_{n'j'} = 0$ for all $(n',j') \notin b$. In parallel with this, $D^b_n(h,\beta^b,\alpha)$ is defined for $n = 2,\ldots,N$.

Below, we show that the dimension of the set of equilibria is at least one. However, in fact, there are many types of stationary equilibria depending on which $b \in \hat{B}$ is used in equilibria. We first consider the simplest case; namely, the case that $h(n) > 0, n = 0,\ldots,N$, hold and $D^b_n = 0, n = 2,\ldots,N$, can be linearly independent, i.e., $\exists(h,\beta^b)$ such that $D^b_n(h,\beta^b,\alpha) = 0, n = 2,\ldots,N$, are linearly independent for any γ.[20]

[20] The result would not change if this holds only for almost every γ.

Let $K = \sum_{n=0}^{N} k_n$. Recall that, for a given $b \in \hat{B}$, the condition for a stationary quasi-equilibrium is as follows:

$$\begin{aligned}
D_n^b(h, \beta^b, \alpha) &= 0, && n = 2, \dots, N \\
\sum_{n=0}^{N} h(n) - 1 &= 0, && \\
V(n) - W_{nj}^b(x^b, \gamma) &= 0, && (n,j) \in b^* \\
\sum_{j \in \{j' | (j', n) \in b\}} \beta_{nj} - 1 &= 0, && n = 0, \dots, N \\
V(n) - W_{nj}^b(x^b, \gamma) &\geq 0, && (n,j) \notin b^*.
\end{aligned}$$

Let $g^b : \mathbb{R}^{N+1} \times \mathbb{R}_{++}^{N+1} \times \Omega^b \times \mathbb{R}^L \to \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R}^{\#b} \times \mathbb{R}^{N+1} \times \mathbb{R}^{K-\#b}$ be the LHS of the above condition.

Let

$$C^b = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+\#b+1} \times \underbrace{\mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++}}_{K-\#b},$$

and, for $(n,j) \notin b$,

$$C^{b(n,j)} = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+\#b+1} \times \underbrace{\mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++} \times \{0\} \times \mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++}}_{K-\#b},$$

where the last $\{0\}$ corresponds to $V(n) - W_{nj}^b(x^b, \gamma), (n,j) \notin b$. Moreover, for $(n,j), (n',j') \notin b$ such that $(n,j) \neq (n',j')$,

$$C^{b(n,j)(n',j')} = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+\#b+1} \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{K-\#b},$$

where the last two $\{0\}$s correspond to $V(n) - W_{nj}^b(x^b, \gamma), (n,j) \notin b$, and $V(n) - W_{n'j'}^b(x^b, \gamma), (n',j') \notin b$, respectively. For γ, let E_γ^b be the set of stationary quasi-equilibria for $b \in \hat{B}$. Then clearly

$$\begin{aligned}
&E_\gamma^b \subset (g^b(\cdot, \gamma))^{-1} \left(C^b \cup (\cup_{(n,j)} C^{b(n,j)}) \cup (\cup_{(n,j),(n',j')} C^{b(n,j)(n',j')}) \right) \quad \text{and} \\
&E_\gamma^b \supset (g^b(\cdot, \gamma))^{-1} \left(C^b \cup (\cup_{(n,j)} C^{b(n,j)}) \right)
\end{aligned}$$

hold.

Theorem 2 Let $\Gamma \subset \mathbb{R}^L$ be a C^2 manifold without boundary. For $b \in \hat{B}$, suppose that $E^b_\gamma \neq \emptyset$ holds for all $\gamma \in \Gamma$, and that g^b is C^2 and is transversal to C^b, $C^{b(n,j)}$, and $C^{b(n,j)(n',j')}$ for all $(n,j),(n',j') \notin b$. Then, for almost every $\gamma \in \Gamma$, E^b_γ is a one-dimensional manifold with boundary.

Proof: (i) By the parametric transversality theorem (see, for example, Guillemin and Pollack ([7], Chapter 2) and Hirsch ([9], Chapter 3)), for almost every $\gamma \in \Gamma$, $g^b(\cdot,\gamma)$ is transversal to all of C^b, $C^{b(n,j)}$ and $C^{b(n,j)(n',j')}$, $(n,j),(n',j') \notin b$,. Let Γ' be the set of such γs.

(ii) Let $\gamma \in \Gamma'$. Suppose $(g^b(\cdot,\gamma))^{-1}(C^{b(n,j)(n',j')}) \neq \emptyset$ for some $(n,j),(n',j') \notin b$. Then it is a submanifold in the domain and the codimension of the manifold is equal to the codimension of $C^{b(n,j)(n',j')}$. Since codim $C^{b(n,j)(n',j')} = 2N + \#b + 3$ and the dimension of the domain of $g^b(\cdot,\gamma)$ is $2N + \#b + 2$, then the dimension of $(g^b(\cdot,\gamma))^{-1}(C^{b(n,j)(n',j')})$ is minus one, i.e., it is empty. This is a contradiction. Thus

$$E^b_\gamma = (g^b(\cdot,\gamma))^{-1}\left(C^b \cup (\cup_{(n,j)} C^{b(n,j)})\right) \neq \emptyset$$

holds.

(iii) Suppose $(g^b(\cdot,\gamma))^{-1}(C^b) \neq \emptyset$. Then it is a submanifold in the domain. Moreover, the codimension of the manifold is equal to the codimension of C^b. (See, for example, Guillemin and Pollack ([7], Chapter 1).) Since codim $C^b = 2N + \#b + 1$ and the dimension of the domain of $g^b(\cdot,\gamma)$ is $2N + \#b + 2$, then $(g^b(\cdot,\gamma))^{-1}(C^b)$ is a one-dimensional manifold; more precisely, each connected component is diffeomorphic either to an open interval or to a circle.

(iv) Suppose $(g^b(\cdot,\gamma))^{-1}(C^{b(n,j)}) \neq \emptyset$ for some $(n,j) \notin b$. Let $x^b \in (g^b(\cdot,\gamma))^{-1}(C^{b(n,j)})$. Below, we show that x^b is an endpoint of some one-dimensional manifold in $(g^b(\cdot,\gamma))^{-1}(C^b)$, i.e., the connected component containing x^b is homeomorphic to an interval. Let $\varphi^b : \mathbb{R}^{2N+K+1} \to \mathbb{R}^{2N+\#b+2}$ be the projection map from the range of g^b to the space of elements which correspond to $\{0\}$s in $C^{b(n,j)}$. Then, by the assumption, $\varphi^b \circ g^b(\cdot,\gamma)$ is a submersion at x^b, i.e., the linear map defined by the Jacobian matrix at x^b, denoted by $d_{x^b}(\varphi^b \circ g^b(\cdot,\gamma))$, is surjective. Since the domain and the range of $\varphi^b \circ g^b(\cdot,\gamma)$ are the same, the inverse function theorem can be applied. Thus there exist an open neighborhood of x^b denoted by D, and an open neighborhood of $(0,\ldots,0) \in R^{2N+\#b+2}$, denoted by D', such that the restriction of $\varphi^b \circ g^b(\cdot,\gamma)$ to D is a diffeomorphism from D to D'. Since, for sufficiently small $\varepsilon > 0$, $D'_\varepsilon = \{(0,\ldots,0,t) | -\varepsilon < t < \varepsilon\}$ is a subset of D', then $(\varphi^b \circ g^b(\cdot,\gamma))^{-1}(D'_\varepsilon)$ is diffeomorphic to an open interval. Note that $(\varphi^b \circ g^b(\cdot,\gamma))^{-1}(\{(0,\ldots,0,t)|0 \le t < \varepsilon\})$ is a subset of E^b_γ and diffeomorphic to $[0,1)$. Since $(\varphi^b \circ g^b(\cdot,\gamma))^{-1}((0,\ldots,0,\frac{\varepsilon}{2})) \in (g^b(\cdot,\gamma))^{-1}(C^b)$, it belongs to a connected component obtained in (iii). Thus the component should be diffeomorphic to an open interval in $(g^b(\cdot,\gamma))^{-1}(C^b)$. Of course, one of its endpoints is x^b. ∎

Next, we consider the case that (i) $D^b_n = 0, n = 2,\ldots,N$, are not linearly independent for all β^b and h, and/or that (ii) $h(n) = 0$ for some n in equilibria.

Example 1 For some $b \in \hat{B}$, suppose $f((n,j),(n',j'))$ is equal to 0 or 2 for all (n,j) and (n',j'). Then if n is even (odd), then $n + f((n,j),(n',j'))$ and $n - f((n,j),(n',j'))$ are even (odd). Thus the stationary distribution can be divided into two distribution so that $D^b_n = 0, n = 0,\ldots,N$, has more than one degree of freedom.

Example 2 Under the assumption in Example 1, there exists a stationary distribution such that $h(n) = 0$, for all $n = 2m, m = 1,2,\ldots$.

We first consider the case that, for any γ, $D^b_n = 0, n = 2,\ldots,N$, are not linearly independent for all β^b and h, and that $h(n) > 0$ for all n. Let $M(b)$ be the maximal number of (potentially) independent equations in $D^b_n = 0, n = 0,\ldots,N$. For simplicity, we assume $D^b_n = 0, n = N - M(b)+1,\ldots,N$, can be linearly independent. We define $g^b : \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}_{++} \times \Omega^b \times \mathbb{R}^L \to \mathbb{R}^{M(b)} \times \mathbb{R} \times \mathbb{R}^{\#b} \times \mathbb{R}^{N+1} \times \mathbb{R}^{K-\#b}$ by replacing

$$D^b_n(h,\beta^b,\alpha), \quad n = 2,\ldots,N,$$

in the definition of g^b in Theorem 2 by

$$D^b_n(h,\beta^b,\alpha), \quad n = N - M(b)+1,\ldots,N.$$

By the same argument as in Theorem 2, we obtain the following theorem.

Theorem 5 Let $\Gamma \subset \mathbb{R}^L$ be a $C^{2-(N-1-M(b))}$ manifold without boundary. For $b \in \hat{B}$, suppose that $E^b_\gamma \neq \emptyset$ holds for all $\gamma \in \Gamma$, and that g^b is $C^{2-(N-1-M(b))}$ and is transversal to C^b, $C^{b(n,j)}$, and $C^{b(n,j)(n',j')}$ for all $(n,j),(n',j') \notin b$, where the definitions of them should be slightly modified. Then, for almost every $\gamma \in \Gamma$, E^b_γ is a $(N - M(b))$-dimensional manifold with boundary.

We next consider the case that, for some $b \in \hat{B}$ and for any γ, $h(n) = 0$ holds for some n in equilibria. That is we assume that there exists a set $N(b) \subset \{0,1,\ldots,N\}$ such that if the players take actions in b and $h(n) = 0, n \notin N(b)$, and $h(n) > 0, n \in N(b)$, hold, then $I_n = 0$ and $O_n = 0$ hold for $n \notin N(b)$. For simplicity, we assume that $\#N(b) - 2$ of $D_n^b = 0, n \in N(b)$, can be linearly independent in equilibria. (We will discuss the general case later.) Let $N'(b)$ be the subset of $N(b)$ such that $D_n^b = 0, n \in N'(b)$, can be linearly independent. We define $g^b : \mathbb{R}^{N+1} \times \mathbb{R}_{++}^{\#N(b)} \times \Omega^b \times [\mathbb{R}]^L \to \mathbb{R}^{\#N'(b)} \times \mathbb{R} \times \mathbb{R}^{\#b} \times \mathbb{R}^{N+1} \times \mathbb{R}^{K-\#b}$ by replacing

$$D_n^b(h, \beta^b, \alpha), \quad n = 2, \ldots, N,$$

in the definition of g^b in Theorem 2 by

$$D_n^b(h, \beta^b, \alpha), \quad n \in N'(b).$$

By the same argument as in Theorem 2, we obtain the following theorem.

Theorem 6 Let $\Gamma \subset \mathbb{R}^L$ be a C^2 manifold without boundary. For $b \in \hat{B}$, suppose that $E_\gamma^b \neq \emptyset$ holds for all $\gamma \in \Gamma$, and that g^b is C^2 and is transversal to C^b, $C^{b(n,j)}$, and $C^{b(n,j)(n',j')}$ for all $(n,j), (n',j') \notin b$, where the definitions of them should be slightly modified. Then, for almost every $\gamma \in \Gamma$, E_γ^b is a one-dimensional manifold with boundary.

In general, $\#N'(b)$ can be less than $\#N(b) - 2$. Applying the same argument as in Theorems 6 and 5, the dimension of the set of equilibrium for b is more than one.

In order to show real indeterminacy of equilibria, it suffices to prove that the welfare $\sum_{n=0}^N h(n)V(n)$ is not constant on each connected set of equilibria. For a given $a \in R$, the condition for a stationary quasi-equilibrium with welfare a is as follows:

$$\begin{aligned}
D_n^b(h, \beta^b, h, \alpha) &= 0, && n = 2, \ldots, N \\
\sum_{n=0}^N h(n) - 1 &= 0, && \\
V(n) - W_{nj}^b(x^b, \gamma) &= 0, && (n,j) \in b^* \\
\sum_{j \in \{j' | (j',n) \in b\}} \beta_{nj} - 1 &= 0, && n = 0, \ldots, N \\
V(n) - W_{nj}^b(x^b, \gamma) &\geq 0, && (n,j) \notin b^* \\
\sum_{n=0}^N h(n)V(n) - a &= 0.
\end{aligned}$$

Let $g_w^b : \mathbb{R}^{N+1} \times \mathbb{R}^{N+1}_{++} \times \Omega^b \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R}^{\#b} \times \mathbb{R}^{N+1} \times \mathbb{R}^{K-\#b} \times \mathbb{R}$ be the LHS of the above condition, where the last R in the domain is the set of a in the last equation.

Let

$$C_a^b = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+\#b+1} \times \underbrace{\mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++}}_{K-\#b} \times \{0\},$$

and, for $(n,j) \notin b$,

$$C_a^{b(n,j)} = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+\#b+1} \times \underbrace{\mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++} \times \{0\} \times \mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++}}_{K-\#b} \times \{0\}.$$

Moreover, for $(n,j),(n',j') \notin b$ such that $(n,j) \neq (n',j')$,

$$C_a^{b(n,j)(n',j')} = \underbrace{\{0\} \times \cdots \times \{0\}}_{2N+\#b+1} \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{K-\#b} \times \{0\}.$$

By the same argument as in the proof of Theorem 2, the following theorem holds.

Theorem 3 Let $\Gamma \subset \mathbb{R}^L$ be a C^1 manifold without boundary. For $b \in \hat{B}$, suppose that $E_\gamma^b \neq \emptyset$ holds for all $\gamma \in \Gamma$, and that, for any given a, $g_w^b(\cdot, a)$ is C^1 and is transversal to C_a^b, $C_a^{b(n,j)}$, and $C_a^{b(n,j)(n',j')}$ for all $(n,j),(n',j') \notin b$. Then, for almost every $\gamma \in \Gamma$, $E_\gamma^b \cap \{x^b | \sum_{n=0}^N h(n)V(n) = a\}$ is a zero-dimensional manifold.

Together with Theorem 2, the above theorem implies real indeterminacy of E_γ^b. That is, for any given welfare level a, the dimension of the set of equilibria with welfare level a is one dimension less than that of the set of equilibria. The same argument applies to the cases in Theorems 5 and 6; the dimension of the set of equilibria that have the same welfare level is one dimension less than that of the set of equilibria.

C The Case of $N = \infty$

Theorem 7 Suppose, for some integer $\delta > 0$, $f((i,j),(i',j')) \leq \delta$ holds for all (i,j) and (i',j'). Then

$$\sum_{n=0}^{\infty} n(O_n - I_n) = 0.$$

Proof: It suffices to show that $\sum_{n=0}^{\infty} n\left(O_n - I_n\right)$ absolutely converges. Let

$$\hat{O}_n = \sum_{i=n}^{\infty} O_i \quad \text{and} \quad \hat{I}_n = \sum_{i=n}^{\infty} I_i.$$

(Note that we can define $\hat{O}_n$ and $\hat{I}_n$ since $\sum_{i=0}^{\infty} h\left(i\right)$ converges.) Clearly, it suffices to show that $\sum_{n=1}^{\infty}\left(\hat{O}_n - \hat{I}_n\right)$ absolutely converges.

The presumption implies that

$$\begin{aligned} O_i = &\sum_{k=0}^{\delta} \sum_{\{j,i',j' \mid f((i,j),(i',j'))=k\}} \alpha h(i,j)h(i',j') \\ &+ \sum_{k=0}^{\delta} \sum_{\{j,i',j' \mid f((i',j'),(i,j))=k\}} \alpha h(i',j')h(i,j), \end{aligned}$$

and

$$\begin{aligned} I_i = &\sum_{k=0}^{\delta} \sum_{\{j,i',j' \mid f((i-k,j),(i',j'))=k\}} \alpha h(i-k,j)h(i',j') \\ &+ \sum_{k=0}^{\delta} \sum_{\{j,i',j' \mid f((i',j'),(i+k,j))=k\}} \alpha h(i',j')h(i+k,j) \end{aligned}$$

hold for all $i \geq \delta$. Thus

$$\begin{aligned} \left|\hat{O}_n - \hat{I}_n\right| = &\left|\sum_{d=1}^{\delta} \sum_{k=0}^{d-1} \sum_{\{j,i',j' \mid f((i',j'),(n+k,j))=d\}} \alpha h(i',j')h(n+k,j) \right. \\ &\left. - \sum_{d=1}^{\delta} \sum_{k=1}^{d} \sum_{\{j,i',j' \mid f((n-k,j),(i',j'))=d\}} \alpha h(n-k,j)h(i',j') \right| \\ \leq &\ \alpha \sum_{i=n-\delta}^{n+\delta-1} h\left(i\right) \end{aligned}$$

holds for all $n \geq \delta$. So, for all $N_2 > N_1 > \delta$, we obtain

$$\sum_{n=1}^{N_2} \left|\hat{O}_n - \hat{I}_n\right| - \sum_{n=1}^{N_1} \left|\hat{O}_n - \hat{I}_n\right| \leq 2\alpha \sum_{i=N_1-\delta+1}^{N_2+\delta-1} h\left(i\right).$$

Since $\sum_{n=0}^{\infty} h\left(n\right)$ converges, so does $\sum_{n=1}^{\infty} \left|\hat{O}_n - \hat{I}_n\right|$. ∎

References

[1] Ralph Abraham and Joel Robbin. *Transversal Mappings and Flows*. W. A. Benjamin, 1967.

[2] Gabriele Camera and Dean Corbae. Money and price dispersion. *International Economic Review*, 40(4):985–1008, 1999.

[3] David Gale. Pure exchange equilibrium of dynamic economic models. *Jouranl of Eocnomic Theory*, 6:12–36, 1973.

[4] John Geanakoplos and Andreu Mas-Colell. Real indeterminacy with financial assets. *Journal of Economic Theory*, 47(1):22–38, 1989.

[5] Edward J. Green and Ruilin Zhou. A rudimentary random-matching model with divisible money and prices. *Journal of Economic Theory*, 81(2):252–271, 1998.

[6] Edward J. Green and Ruilin Zhou. Dynamic monetary equilibrium in a random matching economy. *Econometrica*, 70(3):929–969, 2002.

[7] Victor Guillemin and Alan Pollack. *Differential Topology*. Prentice-Hall, 1974.

[8] Jean-Jacques Herings. On the existence of a continuum of constrained equilibria. *Journal of Mathematical Economics*, 8(1):257–273, 1998.

[9] Morris W. Hirsch. *Differential Topology*. Springer-Verlag, 1976.

[10] Kazuya Kamiya, Noritsugu Morishita, and Takashi Shimizu. The existence of single-price equilibria in a matching model with divisible money and production cost. CIRJE Discussion Paper CIRJE-F-283, University of Tokyo, http://www.e.u-tokyo.ac.jp/cirje/research/dp/2004/2004cf283.pdf, 2004.

[11] Kazuya Kamiya and Takashi Sato. Equilibrium price dispersion in a matching model with divisible money. *International Economic Review*, 45:413–430, 2004.

[12] Kazuya Kamiya and Takashi Shimizu. On the role of the tax-subsidy scheme in money search models. CIRJE Discussion Paper CIRJE-F-323, University of Tokyo, http://www.e.u-tokyo.ac.jp/cirje/research/dp/2002/2005cf323.pdf.

[13] Kazuya Kamiya and Takashi Shimizu. A new technique for proving the existence of stationary equilibrium in matching models with divisible money. 2005.

[14] Timothy J. Kehoe and David K. Levine. Comparative statics and perfect foresight in infinite horizon economies. *Econometrica*, 53(2):433–53, 1985.

[15] Akihiko Matsui and Takashi Shimizu. A theory of money and marketplaces. *International Economic Review*, 46:35–59, 2005.

[16] Kazuo Nishimura and Kouji Shimomura. Trade and indeterminacy in a dynamic general equilibrium model. *Journal of Economic Theory*, 105:244–260, 2002.

[17] Paul A. Samuelson. An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy*, 66(6):467–482, 1958.

[18] Shouyong Shi. Money and prices: A model of search and bargaining. *Journal of Economic Theory*, 67:467–496, 1995.

[19] Shouyong Shi. A divisible search model of fiat money. *Econometrica*, 65(1):75–102, 1997.

[20] Alberto Trejos and Randall Wright. Search, bargaining, money, and prices. *Journal of Political Economy*, 103(1):118–141, 1995.

[21] Gerard van der Laan. Simplicial approximation of unemployment equilibria. *Journal of Mathematical Economics*, 9(1):83–97, 1982.

[22] Ruilin Zhou. Individual and aggregate real balances in a random-matching model. *International Economic Review*, 40(4):1009–1038, 1999.

[23] Ruilin Zhou. Does commodity money eliminate the indeterminacy of equilibrium. *Journal of Economic Theory*, 110:176–190, 2003.