CIRJE-F-963

Optimal Position Management for a Market Maker with Stochastic Price Impacts

Masaaki Fujii
The University of Tokyo

March 2015

Optimal Position Management for a Market Maker with Stochastic Price Impacts *

Masaaki Fujii†

The first version: March 24, 2015.

Abstract

This paper provides the optimal position management strategy for a market maker who has to face uncertain customer orders in an *"illiquid"* market, where the market maker's continuous trading through a traditional exchange incurs stochastic linear price impacts. In addition, it is supposed that the market participants can *partially* infer the position size held by the market maker and their aggregate reactions affect the security prices. Although the market maker can ask its OTC counterparties to transact a block trade without causing a direct price impact in the exchange, its timing is assumed to be uncertain. Another important way for the market maker to reduce its position is to match an incoming customer order to the outstanding position being warehoused in its balance sheet. The solution of the problem is represented by a stochastic Hamilton-Jacobi-Bellman equation, which can be decomposed into three (one non-linear and two linear) backward stochastic differential equations (BSDEs). We provide the verification using the standard BSDE techniques for a single security case. For a multiple-security case, we use an interesting connection of the non-linear BSDE to a special type of backward stochastic Riccati differential equation (BSRDE) whose properties have been studied by Bismut (1976).

Keywords : stochastic control, BSDE, BSPDE, stochastic HJB, portfolio, inventory illiquidity, transaction costs, liquidation, execution

1 Introduction

The financial market currently being formed in the aftermath of the great financial crisis looks completely different from the previous one. Mandatory clearing for the standardized financial products and much higher regulatory costs for the rest of over-the-counter (OTC) contracts made many investors withdraw from the long-dated exotic derivative business and pay more attention to the trading of listed products in exchanges or standard contracts with central counterparties.

*All the contents expressed in this research are solely those of the author and do not represent any views or opinions of any institutions. The author is not responsible or liable in any manner for any losses and/or damages caused by the use of any contents in this research.

†Graduate School of Economics, The University of Tokyo. e-mail: mfujii@e.u-tokyo.ac.jp

In the new market, it is clear that the exchanges and central counterparties are the most important trading venues and have started to play a much bigger role than before. However, these new developments have not completely diminished the importance of the traditional key players in the market, that is a *market maker*. A market maker is a firm that quotes buy and sell prices for financial securities and derivatives, and stands ready to perform these deals on a regular and/or continuous basis. They are crucially important to maintain liquidity for equities, (government/corporate) bonds, currencies, commodities, and many structured products and derivatives. Even for products tradable at an exchange, market makers are playing an important role for intermediating non-financial corporates and other investors since it is not always possible for them to satisfy many regulatory conditions required to get a direct access to the exchange. There exist many other benefits such as those related to accounting, anonymity and flexibility that may be obtained with the help of market makers.

Especially due to the proposed regulation on the leverage ratio and the higher capital amount required for the open positions, the market makers have to deal with formidable tasks. Due to the smaller warehousing capability of their balance sheets, they need more active position management. At the same time, they have to optimize execution strategies in order to avoid unnecessarily big market impacts and the associated transaction costs.

In this paper, we consider the optimal position management strategy for a market maker who is facing uncertain customer orders. We are interested in a *good* market maker who accepts every customer order with a predefined bid/offer spread. The spread can be stochastic but we do not allow the market maker to control its size dynamically based on its proprietary reasons in order to give a bias to the customer flows. Otherwise the firm will not be considered as a trustful market maker [1]. We suppose that there exists a relatively liquid market for security borrowing and lending (i.e., so called repo transactions), which can be used by the market maker to answer the incoming customer orders. In addition to matching an incoming order directly to the security being warehoused in its balance sheet, the market maker can access two external trading venues. One is a traditional exchange where the market maker carries out absolutely continuous trading, which is assumed to incur stochastic linear price impact. It is also supposed that the participants of the exchange can *partially* infer the position size held by the market maker and their aggregate reactions, as a preparation for the market maker's future trading, affect the security prices linearly with respect to the current position size held by the market maker. Another venue is the aggregate of the market maker's OTC counterparties with which the firm can execute a block trade without directly affecting the price in the exchange. In this case, however, the execution timing is uncertain. The modeling of the latter venue (we call it the dark pool) is closely related to the one introduced by Kratz & Schöneborn (2013) [26] except that we allow stochasticity in its execution intensity.

There now exist the vast literature on the optimal execution problems. Our model is closely related to the line of developments made by Bertsimas & Lo (1998) [8], Almgren & Chriss (1999, 2000) [3, 4], Schied & Schöneborn (2009) [33] and to more recent works Ankirchner & Kruse (2013) [6], Ankirchner, Jeanblanc & Kruse (2014) [5] and Kratz & Schöneborn (2013) [26]. There exist many other interesting approaches, such as models of

[1] In fact, this was the common pride I observed among the fellow traders I worked together while I was in the industry.

supply curves (See, for example, Bank & Baum (2004) [7], Cetin, Jarrow & Protter (2004) [13] and Roch (2011) [32].) and those directly modeling the dynamics of Limit Order Books (See, for example, Obizhaeva & Wang (2013) [29], Alfonsi, Fruth & Schied (2008, 2010) [1, 2] and Fruth & Schöneborn (2014) [15].). We refer to the review articles Gatheral & Schied (2013) [21] and Gökay, Roch & Soner (2011) [22] for the recent developments, various other aspects and references.

The main difference of the current work is the focus on the market maker's position management problem with uncertain customer orders rather than a very short-term liquidation problem with a given initial position. For example, in the presence of the customer orders, the market maker has an incentive to keep an adequate amount of inventory (which can be short position) of the securities since matching an incoming order directly to its own position does not cause a market impact. This introduces a term not proportional to the current position size into the optimal execution strategy. Since many of the literature make use of this proportionality to find a candidate solution for the corresponding Hamilton-Jacobi-Bellman (HJB) equation, we need a more general systematic approach. We also use quite general stochastic processes, which include the price and position impact factors, the compensator of incoming customer orders, the execution intensity of the dark pool, the repo rates relevant for the security borrowing/lending, and penalties for the outstanding position size etc. In particular, we allow the unaffected security price to follow a general stochastic process adapted to a Brownian filtration and do not assume it is a martingale. This may be important since the time horizon relevant for our problem, such as Quarter to one year, is longer than the typical time horizon used for the standard execution problems, which is ranging from a few hours to a few trading days. The difference of the relevant time horizon is an also important factor for us to choose a framework of exogenous security price modeling. Although it is a simplified reduced-form approach and impossible to model market depth and resilience relevant for the dynamics of Limit Order Books, it allows more flexible modeling of the underlying processes including multiple securities and also their mutual dependence, which is expected to be more relevant for our medium-term position management problem.

We follow the technique proposed by Mania & Tevzadze (2003) [28] to derive the relevant stochastic HJB equation and its decomposition into three (one non-linear and two linear) backward stochastic differential equations (BSDEs). For a single security case, we use the standard results on BSDEs (See, for example, Pardoux & Rascanu (2014) [30].) and the comparison theorem for verifying the solution. For a multiple-security case, however, we need to handle a matrix-valued non-linear BSDE for which we cannot apply the standard results. Interestingly, we find that the relevant BSDE is actually a special type of backward stochastic Riccati differential equation (BSRDE) associated with a stochastic linear quadratic control (SLQC) problem, which has a seemingly totally different setup from our original one. Because of this relation, we can guarantee the existence of a uniformly bounded solution by theorems proved in Bismut (1976) [11].

The organization of the paper is as follows: Section 2 gives some preliminaries and Section 3 provides the detailed market description and the market maker's problem. Section 4 gives the derivation of the candidate solution and its verification. The behavior of the terminal position size with respect to the penalty size is discussed in Section 5. An extension to the multiple-security case is given in Section 6 and 7. Section 8 gives a possible approximation technique for the relevant non-linear BSDE and also a simple special case which is straightforward to evaluate.

2 Preliminaries

We consider a complete filtered probability space, in which all the stochastic processes are defined, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the filtration satisfying the usual conditions. W is the d-dimensional standard Brownian motion and the $\mathbb{P}$-augmented filtration generated by W is denoted by $\mathbb{F}^W = (\mathcal{F}_t^W)_{t\geq 0}$. We assume that $\mathbb{F}^W$ is a subset of the full filtration; $\mathbb{F}^W \subset \mathbb{F}$.

For the ease of discussion, let us define the following spaces of the stochastic processes ($p \geq 1$):
• $\mathbb{S}_r^p(t,T)$ is the set of progressively measurable process X taking values in $\mathbb{R}^r$ and satisfying

$$\mathbb{E}\Big[||X||_{[t,T]}^p\Big] := \mathbb{E}\Big[\sup_{s\in[t,T]} |X_s(\omega)|^p\Big] < \infty \tag{2.1}$$

where we use the notation

$$||x||_{[a,b]} := \sup\{|x_t|, t \in [a,b]\} \tag{2.2}$$

for $x : [0,T] \to \mathbb{R}^r$. We write $||x||_{[0,t]} = ||x||_t$. Its norm is defined by

$$||X||_{\mathbb{S}_r^p(t,T)} := \Big\{\mathbb{E}\Big[||X||_{[t,T]}^p\Big]\Big\}^{1/p}. \tag{2.3}$$

• $\mathbb{H}_r^p(t,T)$ is the set of progressively measurable process X taking values in $\mathbb{R}^r$ and satisfying

$$\mathbb{E}\left[\left(\int_t^T |X_t|^2 dt\right)^{p/2}\right] < \infty, \tag{2.4}$$

and its norm is defined by

$$||X||_{\mathbb{H}_r^p(t,T)} := \left\{\mathbb{E}\Big[\left(\int_t^T |X_s|^2 ds\right)^{p/2}\Big]\right\}^{1/p}. \tag{2.5}$$

In every space, the subscript r may be omitted if the associated dimension is clearly seen from the context.

3 Setup: Single Security Case

Firstly, let us summarize the standing assumptions. They are obviously not the weakest ones but allow simple analysis and also do not make the model unrealistic in the practical setup. The definition of each variable will appear in the following sections.

Assumption A
Firstly, all the stochastic processes which do not jump by $\mathcal{N}$ and H are assumed to be $\mathbb{F}^W$-adapted and hence continuous. This $\mathbb{F}^W$ adaptedness includes all the stochastic processes defined below.

(a_1) $S : \Omega \times [0,T] \to \mathbb{R}$ *is non-negative and* $S \in \mathbb{S}^4(0,T)$.
(a_2) $b, l : \Omega \times [0,T] \to \mathbb{R}$ *and* $b, l \in \mathbb{S}^4(0,T)$.
(a_3) $\Lambda(\cdot,\cdot) : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ *is such that* $\Lambda(t,\cdot)(\omega)$ *is a non-negative measurable function with bounded support* $K \subset \mathbb{R}$ *for every* $t \in [0,T]$ *and* $\omega \in \Omega$*, and that* $\Lambda(\cdot,z)$ *is a uniformly bounded* $\mathbb{F}^W$*-adapted process for every* $z \in K$.
(a_4) $\widetilde{\gamma} : \Omega \times [0,T] \to \mathbb{R}$ *are uniformly bounded and non-negative.*
(a_5) $M, \widetilde{\eta}, \lambda : \Omega \times [0,T] \to \mathbb{R}$ *are uniformly bounded and strictly positive.*
(a_6) $\widetilde{\xi} : \Omega \to \mathbb{R}$ *is strictly positive, bounded and* $\mathcal{F}^W_T$*-measurable.*
(a_7) $\beta : \Omega \times [0,T] \to \mathbb{R}$ *is uniformly bounded.*
(a_8) *There is no simultaneous jump between* $\mathcal{N}$ *and* H.

Notation : *For a bounded variable* x*, we express its upper bound by* $\bar{x}$.

3.1 The market description

We are interested in a market maker, who has to face uncertain customer orders regarding the single specified security. An extension to the portfolio management including multiple securities will be discussed in later sections. As a *good* market maker, the firm accepts every customer order with a predefined bid-offer spread. Although the spread can be dynamic depending on the external market variables such as the security's volatility, it is supposed that the firm does not adjust it in order to control the customer flow based on the firm's proprietary reasons. Otherwise, the firm will be nothing but an opportunistic investor and will not be treated as a trustful market maker.

The market maker is assumed to buy and sell the security through two major trading venues. The first venue is a standard exchange. The market maker adopts absolutely continuous trading strategies to access the exchange but they are assumed to incur linear stochastic price impacts. In addition, the participants of the exchange (partially) observe the current position size of the market maker. They expect future buy/sell orders from it and adjust their positioning accordingly. We assume that the aggregated effects of the participants change the market price by the amount proportional to the size held by the market maker. Although we are interested in creating the adverse price impacts, we can allow the stochastic proportional factor to take the both signs so that it may sometime represent a favorable price change. The second venue is the aggregate of the OTC block trades with the firm's customers or the dark pools. The market maker can buy/sell as a block trade without directly affecting the market price in the exchange (or the first venue), but its timing is uncertain. The modeling of this part is basically the same as the one proposed by Kratz & Schöneborn (2013) [26] except that we allow a general stochastic execution intensity. Although we call this venue *the dark pool* as in their work, it actually means the aggregate of OTC block trades with the firm's counterparties as well as potentially multiple dark pools to which the market maker can access. It is a simplistic model for which we do not consider order-size dependent intensity process nor possibility of partial executions. This is necessary for making the problem tractable.

In addition to the above two trading venues, the market maker can match an incoming customer order to its outstanding position being warehoused in its balance sheet. This is the most distinguishing feature of the market maker. Because this is the most profitable way to reduce the position, the market maker needs to adjust buy/sell orders based on the expected future customer flow. If the market maker cannot answer the customer order within its inventory, it needs to borrow the security through the corresponding repo market by paying the stochastic repo rate. On the other hand, it also earns money by lending the security through the repo market.

We model the the market-maker's position at time $s > t$ starting from the position size $x \in \mathbb{R}$ at time t as

$$X_s^{\pi,\delta}(t,x) = x + \int_t^s \int_K z\mathcal{N}(du,dz) + \int_t^s \pi_u du + \int_t^s \delta_u dH_u \tag{3.1}$$

where the second term describes the customer flow, which is represented by the marked point process expressed by the counting measure $\mathcal{N}$. $K \subset \mathbb{R}$ is a bounded support for the mark z which gives the size and direction ($+$ or $-$) of the order. (π, δ) denotes an $\mathbb{F}$-predictable trading strategy of the market maker through the exchange and the dark pool, respectively. H is the counting process, whose jump signals the happening of an execution event in the dark pool. For simplicity, we assume no simultaneous jump between $\mathcal{N}$ and H. The negative position size $X^{\pi,\delta} < 0$ is always interpreted as a short position taken by the security borrowing through the repo market. Let us assume the existence of the compensators Λ for $\mathcal{N}$ (λ for H) so that

$$\begin{aligned}\int_0^t \int_K \widetilde{\mathcal{N}}(ds,dz) &= \int_0^t \int_K \Big(\mathcal{N}(ds,dz) - \Lambda(s,z)dzds\Big) \\ \int_0^t d\widetilde{H}_s &= \int_0^t \Big(dH_s - \lambda_s ds\Big)\end{aligned} \tag{3.2}$$

for $t \in [0,T]$ are $\mathbb{F}$-martingales. This also implies that the occurrences of customer orders and the executions in the dark pool are totally inaccessible. For later convenience, let us define

$$\Phi_t := \int_K z\Lambda(t,z)dz, \quad \Psi_t := \int_K |z|\Lambda(t,z)dz, \quad \Phi_{2,t} := \int_K z^2\Lambda(t,z)dz \tag{3.3}$$

for $t \in [0,T]$, which are the moments of the size of the customer orders. By Assumption A, the above processes are uniformly bounded.

The price observed in the exchange $\widetilde{S}^{\pi,\delta}(t,x)$ i.e., the market price under the impact of the market maker's strategy (π,δ) starting from the position size x at time t, is assumed to be given by

$$\widetilde{S}_s^{\pi,\delta}(t,x) = S_s + M_s\pi_s - \beta_s X_s^{\pi,\delta}(t,x) \tag{3.4}$$

for $s \in [t,T]$. The second term denotes the stochastic linear price impact, where M is the F^W-adapted impact factor. The last term denotes the aggregate impact from the market participants' reactions to the market maker's position size.

Notice that we are not assuming the perfect observability of the market maker's position X to other investors. It is likely that they can only observe X with a big noise and so is their aggregate reaction. However, this noise part can easily be absorbed into the definition of S, *the* *unaffected price* of the security. For the market maker's point of view, X is directly observable and β is simply its coefficient to be obtained by the linear regression of the security price. We model β as a uniformly bounded process possibly being correlated with other market variables, such as volatility of the security. Although we intend to model β as a positive process to introduce an adverse price impact for the future unwinding of the market maker, one may allow it to take the both signs. Due to the presence of the customer orders, the last term is not directly determined by the trading volume in the past and hence different from the standard model of the permanent price impact. It is more closely related to the works on the *large trader's problem* studied by Jarrow (1992) [24] and Bank & Baum (2004) [7].

We model the cash flow in the interval $]t,T]$ to the market maker with strategy (π,δ) in the following way:

$$-\int_t^T \widetilde{S}_s^{\pi,\delta}(t,x)\pi_s ds - \int_t^T\int_K \widetilde{S}_{s-}^{\pi,\delta}(t,x)\big(1-\mathrm{sgn}(z)b_s\big)z\mathcal{N}(ds,dz)$$
$$-\int_t^T \Big(\big(S_s-\beta_s X_{s-}^{\pi,\delta}(t,x)\big)\delta_s+\widetilde{\eta}_s|\delta_s|^2\Big)dH_s+\int_t^T l_s X_s^{\pi,\delta}(t,x)ds \ . \tag{3.5}$$

Let us explain the economic meaning of each term below:
• (1st term) The cash flow from the trades through the exchange.
• (2nd term) The cash flow from accepting the customer orders with a (proportional) bid/offer spread b.
• (3rd term) The cash flow from the trades through the dark pool.
• (4th term) The cash flow from the security borrowing/lending with a repo rate l.

We need additional comments for the third term describing the trades with the dark pool. Firstly, the basic transaction price is given by

$$S_s-\beta_s X_{s-}^{\pi,\delta}(t,x) \tag{3.6}$$

which does not include the price impact from the continuous trading of the market maker. Inclusion of $M_s\pi_s$ to the price could induce price manipulation, and more importantly, the trading counterparties will not accept expensive price caused by the market maker's *temporal* trading activity. We also add the spread $\widetilde{\eta}|\delta|$ to the above price [2] as a *premium* that the market maker pays to the counterparty who has accepted a block trade.

We consider $T \lesssim 1$ (year) as a relevant span for the control of the market maker. More realistically, it can be a Quarter or a half year, and we neglect the net proceeds from a money market account for this time interval. This can be understood as a zero interest rate, or equivalently, we can interpret that the cost function (see below) is given in the discounted basis. We are also interested in relatively liquid market in a sense that the borrowing and lending of the security is always possible once the stochastic repo rate is being paid, and continuous trading in the exchange is possible. In a highly illiquid market, neither seamless execution of market orders in the exchange nor a functioning repo market can be expected.

[2] Thus the additional cost is given by $\widetilde{\eta}|\delta|^2$.

Even if we treat a major equity, bond or index under the normal market condition, it is still important for a global market maker to handle the adverse price impacts from its own trading activities as the fate of a *large trader*.

3.2 The market maker's problem

Definition 3.1. [3] *We define the admissible strategies $\mathcal{U}$ by the set of $\mathbb{F}$-predictable processes (π, δ) that belong to $\mathbb{H}^2(0,T) \times \mathbb{H}^2(0,T)$ and also Markovian with respect to the position size, i.e., they are expressed with some measurable functions (f^π, f^δ) by*

$$\pi_s = f^\pi(s, X^{\pi,\delta}_{s-}(t,x)), \quad \delta_s = f^\delta(s, X^{\pi,\delta}_{s-}(t,x)) \tag{3.7}$$

where, for $a \in \{\pi, \delta\}$, $f^a : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ and $f^a(\cdot, x)$ is an $\mathbb{F}^W$-adapted process for all $x \in \mathbb{R}$.

We suppose that the market maker tries to solve the following optimization problem:

$$\begin{aligned}
\widetilde{V}(t,x) &= \operatorname*{ess\,inf}_{(\pi,\delta)\in\mathcal{U}} \mathbb{E}\Bigg[\ \widetilde{\xi}|X^{\pi,\delta}_T(t,x)|^2 + \int_t^T \widetilde{\gamma}_s |X^{\pi,\delta}_s(t,x)|^2 ds \\
&+ \int_t^T \big(\widetilde{S}^{\pi,\delta}_s(t,x)\pi_s - l_s X^{\pi,\delta}_s(t,x)\big)ds + \int_t^T \int_K \widetilde{S}^{\pi,\delta}_{s-}(t,x)(1-\operatorname{sgn}(z)b_s)z\mathcal{N}(ds,dz) \\
&+ \int_t^T \Big(\big[S_s - \beta_s X^{\pi,\delta}_{s-}(t,x)\big]\delta_s + \widetilde{\eta}_s|\delta_s|^2\Big)dH_s\ \Big|\mathcal{F}_t\Bigg]\ .
\end{aligned} \tag{3.8}$$

The first two terms are introduced to give penalties for the outstanding position size. It is natural to consider that $\widetilde{\xi}$ and $\widetilde{\gamma}$ are proportional to the variance of the price process of the security. One may also want to take into account the regulatory costs arising from the outstanding position in the balance sheet. It is possible by an appropriate modification of $\widetilde{\gamma}$ and l as long as the relevant costs can be reasonably approximated by a quadratic function with respect to the position size X. Note that the coefficients of the quadratic function can be stochastic.

We can observe that the expectation in (3.8) is finite for all $(\pi, \delta) \in \mathcal{U}$. This can be easily checked by the fact $X^{\pi,\delta}(t,x) \in \mathbb{S}^2(t,T)$ and $\widetilde{S}^{\pi,\delta}(t,x) \in \mathbb{H}^2(t,T)$. However, due to the 2nd order terms of (π,δ) arising from $(-\beta X^{\pi,\delta}\pi)$ and $(-\beta X^{\pi,\delta}\delta)$, the cost function could be made infinitely small, and then the problem would be ill-defined. In order to guarantee the well-posedness of the problem, we need additional assumptions.

Firstly, let us write the dynamics of the $\mathbb{F}^W$-adapted process β as

$$d\beta_t = \mu^\beta_t dt + \sigma^\beta_t dW_t\ . \tag{3.9}$$

[3]It is not necessary to constrain the admissible strategies as Markovian with respect to X, since we know a posteriori that the global solution has this form. However, limiting the strategy space at this stage makes the following analysis much clearer.

Furthermore, we denote

$$\xi := \widetilde{\xi} - \frac{\beta_T}{2}, \quad \eta := \widetilde{\eta} + \frac{\beta}{2}, \quad \gamma := \widetilde{\gamma} + \frac{\mu^\beta}{2} \ . \tag{3.10}$$

Assumption B
(b_1) *$\mu^\beta : \Omega \times [0,T] \to \mathbb{R},\ \sigma^\beta : \Omega \times [0,T] \to \mathbb{R}^d$ are uniformly bounded.*
(b_2) *γ is non-negative $d\mathbb{P} \otimes dt$-a.e..*
(b_3) *There exists a constant $c > 0$ such that $\xi \geq c$ a.s. and $M, \eta \ \geq c\ d\mathbb{P} \otimes dt$-a.e..*

Definition 3.2. *The cost function for the market maker with a given position size $x \in \mathbb{R}$ at t is*

$$\begin{aligned} J^{t,x}(\pi,\delta) = \mathbb{E}\Bigg[& \ \xi |X_T^{\pi,\delta}(t,x)|^2 + \int_t^T \Big(\gamma_s |X_s^{\pi,\delta}(t,x)|^2 + X_s^{\pi,\delta}(t,x)(\beta_s b_s \Psi_s - l_s)\Big) ds \\ & + \int_t^T \Big(M_s \pi_s^2 + \lambda_s \eta_s \delta_s^2 + \big[(S_s + M_s\Theta_s)\pi_s + S_s \lambda_s \delta_s\big] + \big(S_s \Theta_s + \frac{\beta_s}{2}\Phi_{2,s}\big)\Big) ds \ \Big| \mathcal{F}_t^W \Bigg] \end{aligned} \tag{3.11}$$

where $\Theta : \Omega \times [0,T] \to \mathbb{R}$ is defined by $\Theta_s := \Phi_s - b_s \Psi_s$.

Proposition 3.1. *Under Assumptions A and B, the market maker's problem (3.8) is equivalent to*

$$V(t,x) = \operatorname*{ess\,inf}_{(\pi,\delta)\in\mathcal{U}} J^{t,x}(\pi,\delta) \tag{3.12}$$

and it has a unique optimal solution $(\pi^*, \delta^*) \in \mathcal{U}$.

Proof. Applying Itô-formula, one obtains

$$\begin{aligned} -\int_t^T \beta_s X_{s-}^{\pi,\delta}(t,x) dX_s^{\pi,\delta}(t,x) &= -\frac{\beta_T}{2}|X_T^{\pi,\delta}(t,x)|^2 + \frac{\beta_t}{2}x^2 + \int_t^T \frac{\beta_s}{2}\delta_s^2 dH_s \\ & + \int_t^T \int_K \frac{\beta_s}{2} z^2 \mathcal{N}(ds,dz) + \int_t^T \frac{1}{2}|X_s^{\pi,\delta}(t,x)|^2 (\mu_s^\beta ds + \sigma_s^\beta dW_s) \ . \end{aligned}$$

Then, replacing the β-proportional terms in (3.8) by the above relation yields

$$\begin{aligned} \widetilde{V}(t,x) - \frac{\beta_t}{2}x^2 = \operatorname*{ess\,inf}_{(\pi,\delta)\in\mathcal{U}} \mathbb{E}\Bigg[& \ \xi |X_T^{\pi,\delta}(t,x)|^2 + \int_t^T \Big(\gamma_s |X_s^{\pi,\delta}(t,x)|^2 + X_s^{\pi,\delta}(t,x)(\beta_s b_s \Psi_s - l_s)\Big) ds \\ & + \int_t^T \Big\{ M_s \pi_s^2 + \lambda_s \eta_s \delta_s^2 + \big[(S_s + M_s\Theta_s)\pi_s + S_s \lambda_s \delta_s\big] + \big(S_s \Theta_s + \frac{\beta_s}{2}\Phi_{2,s}\big)\Big\} \ ds \ \Big| \mathcal{F}_t \Bigg] \end{aligned}$$

where one can easily check that the local martingale parts are true martingales under the assumptions. In particular, one can use the Burkholder-Davis-Gundy (BDG) inequality, the fact that $X^{\pi,\delta} \in \mathbb{S}^2(t,T)$ and the boundedness of σ_β for the dW integration term. For the jump part, it suffices to check that the integration by the corresponding compensator is in

$\mathbb{L}^1(\Omega)$ (See, for example, Corollary C4, Chapter VIII in [12].), which can be easily confirmed by the boundedness of the compensators, λ and Λ.

Because all the processes except $(X^{\pi,\delta}, \pi, \delta)$ are $\mathbb{F}^W$-adapted and $(\pi, \delta) \in \mathcal{U}$ satisfies (3.7), the expectation conditioned on $\mathcal{F}_t$ can be replaced by $\mathcal{F}_t^W \vee \sigma\{X_t^{\pi,\delta}\}$. However, $X_t^{\pi,\delta}(t,x) = x$ has been already fixed. Thus, redefining the value function by $V(t,x) := \widetilde{V}(t,x) - \frac{\beta_t}{2}x^2$, one obtains the result (3.12).

The remaining claims easily follow from the standard arguments (See, for example, Theorem 3.1 in Bismut (1976) [11].) since now all the quadratic terms have positive coefficients. For simplicity, let us consider the case where the initial time is zero, $t = 0$. The cost function $J^{0,x}$ is a continuous map from $\mathcal{U}$ to $\mathbb{R}$ and obviously strictly convex. It is also proper since, for example, $J^{0,x}(0) < \infty$. We also have the so-called coerciveness since

$$J^{0,x}(u) \nearrow \infty, \quad \text{when} \quad ||u||_{\mathbb{H}_2^2(0,T)} \nearrow \infty \ . \tag{3.13}$$

Thus, for some large enough $\alpha \in \mathbb{R}$, the set $\{u \in \mathcal{U} : J^{0,x}(u) \leq \alpha\}$ is non-empty, convex and weakly-compact. Thus, there exists an minimizer, which is unique due to the strict convexity of the cost function. □

Remarks on $\beta X^{\pi,\delta}$ in (3.4)

The presence of $\beta X^{\pi,\delta}$ term in (3.4) is not necessarily appropriate for every type of investors. For example, suppose that the investor is risk-neutral and β is positive. In this case, the investor may accumulate an extremely large long position which would make the security price significantly negative. The investor can receive positive cash flow by further increasing her long position which makes the system ill-defined. However, as we have seen in the above discussion, it does not cause any regularity problem under mild conditions regarding the penalty size on the outstanding position of the market maker. Although one may feel uneasy by the fact that the well-posedness of the model depends on the risk-averseness of the agent, we think that this term makes the model more realistic for the market maker. In fact, this term is expected to arise exactly because the other investors know that the relevant market maker has to operate with a rather stringent position limit. It is then natural for them to adjust their positioning so that they can manage a short/long squeeze from the market maker. From the view point of the market maker, it is being squeezed by the other investors as long as there exists an information leak about its position size.

4 Solving the Problem

4.1 A candidate solution

Let us prepare the optimality principle for the current problem.

Proposition 4.1. *(Optimality Principle) Let Assumptions A and B are satisfied. Then,* *(a) For all $x \in \mathbb{R}$, $(\pi, \delta) \in \mathcal{U}$ and $t \in [0, T]$, the process*

$$\left(V(s, X_s^{\pi,\delta}(t,x)) + \int_t^s \Big(\gamma_u |X_u^{\pi,\delta}(t,x)|^2 + X_u^{\pi,\delta}(t,x)\big(\beta_u b_u \Psi_u - l_u\big)\Big) du \right.$$
$$\left. + \int_t^s \Big(M_u \pi_u^2 + \lambda_u \eta_u \delta_u^2 + \big[(S_u + M_u \Theta_u)\pi_u + S_u \lambda_u \delta_u\big] + \big(S_u \Theta_u + \frac{\beta_u}{2}\Phi_{2,u}\big)\Big) du \right)_{s \in [t,T]}$$

is an $\mathbb{F}$-submartingale.
(b) (π^*, δ^*) is optimal if and only if

$$\left(V(s, X_s^{\pi^*,\delta^*}(t,x)) + \int_t^s \Big(\gamma_u |X_u^{\pi^*,\delta^*}(t,x)|^2 + X_u^{\pi^*,\delta^*}(t,x)\big(\beta_u b_u \Psi_u - l_u\big)\Big) du \right.$$
$$\left. + \int_t^s \Big(M_u {\pi^*}_u^2 + \lambda_u \eta_u {\delta^*}_u^2 + \big[(S_u + M_u \Theta_u){\pi^*}_u + S_u \lambda_u {\delta^*}_u\big] + \big(S_u \Theta_u + \frac{\beta_u}{2}\Phi_{2,u}\big)\Big) du \right)_{s \in [t,T]}$$

is an $\mathbb{F}$-martingale.

Proof. One can easily confirm it from the definition of the value function V, the fact that $V(T, X_T^{\pi,\delta}) = \xi |X_T^{\pi,\delta}|^2$ and the form of the cost function $J^{t,x}(\pi, \delta)$. See, for example, Proposition (A.1) of Mania & Tevzadze (2003) [28]. □

Our strategy is to derive the so-called stochastic HJB equation from the necessary condition so that the above optimality principle is satisfied by following the method proposed by Mania & Tevzadze [28]. After that, we are going to show that there exists a solution for the stochastic HJB equation and confirm that it actually satisfies the optimality principle. Then, we get one optimal solution. But the solution is also unique due to Proposition 3.1.

Let us assume that the $\mathbb{F}^W$ semimartingale $\Big(V(t,x)\Big)_{t \in [0,T]}$ has the following decomposition for every $x \in \mathbb{R}$:

$$V(s,x) = V(t,x) + \int_t^s a(u,x)du + \int_t^s Z(u,x)dW_u \tag{4.1}$$

where $a : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$, $Z : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}^d$ and $a(\cdot, x)$ as well as $Z(\cdot, x)$ are $\mathbb{F}^W$-adapted processes for all $x \in \mathbb{R}$. Let us suppose $V(t,x)$ are twice differentiable with respect to x. By applying Itô-Ventzell formula, we obtain

$$\begin{aligned} V(s, X_s^{\pi,\delta}(t,x)) &= V(t,x) + \int_t^s a(u, X_u^{\pi,\delta}(t,x))du + \int_t^s Z(u, X_u^{\pi,\delta}(t,x))dW_u \\ &+ \int_t^s V_x(u, X_u^{\pi,\delta}(t,x))\pi_u du + \int_t^s \int_K \Big(V(u, X_{u-}^{\pi,\delta}(t,x) + z) - V(u, X_{u-}^{\pi,\delta}(t,x))\Big)\mathcal{N}(du, dz) \\ &+ \int_t^s \Big(V(u, X_{u-}^{\pi,\delta}(t,x) + \delta_u) - V(u, X_{u-}^{\pi,\delta}(t,x))\Big) dH_u \ . \end{aligned} \tag{4.2}$$

Separating the local martingale parts, a necessary condition for the optimality principle is given by

$$
\begin{aligned}
& a(u,x) + \int_K \Big(V(u,x+z) - V(u,x) \Big) \Lambda(u,z)dz + \gamma_u x^2 + x(\beta_u b_u \Psi_u - l_u) \\
& + \Big(S_u \Theta_u + \frac{\beta_u}{2} \Phi_{2,u} \Big) + \inf_{\pi,\delta} \Bigg\{ \; V_x(u,x)\pi + \big(V(u,x+\delta) - V(u,x) \big) \lambda_u \\
& + M_u \pi^2 + \lambda_u \eta_u \delta^2 + (S_u + M_u \Theta_u)\pi + S_u \lambda_u \delta \; \Bigg\} = 0
\end{aligned} \tag{4.3}
$$

$d\mathbb{P} \otimes dt$-a.e. in $\Omega \times [0,T]$ for every $x \in \mathbb{R}$.

Exploiting the quadratic nature, let us suppose for every $t \in [0,T]$ and $x \in \mathbb{R}$,

$$
V(t,x) = V_2(t)x^2 + 2V_1(t)x + V_0(t) \tag{4.4}
$$

$$
Z(t,x) = Z_2(t)x^2 + 2Z_1(t)x + Z_0(t) \tag{4.5}
$$

where $V_2, V_1, V_0 : \Omega \times [0,T] \to \mathbb{R}$ and $Z_2, Z_1, Z_0 : \Omega \times [0,T] \to \mathbb{R}^d$ are $\mathbb{F}^W$-adapted processes. Then, (4.3) can be rewritten as

$$
\begin{aligned}
& a(u,x) + \big(2V_2(u)\Phi_u x + V_2(u)\Phi_{2,u} + 2V_1(u)\Phi_u \big) + \gamma_u x^2 + x\big(\beta_u b_u \Psi_u - l_u\big) + \Big(S_u \Theta_u + \frac{\beta_u}{2}\Phi_{2,u} \Big) \\
& + \inf_{\pi,\delta} \Bigg\{ \; M_u \Big(\pi + \frac{\big[V_2(u)x + V_1(u) + \frac{1}{2}(S_u + M_u\Theta_u)\big]}{M_u} \Big)^2 \\
& \quad + \lambda_u \big[V_2(u) + \eta_u\big] \Big(\delta + \frac{\big[V_2(u)x + V_1(u) + \frac{1}{2}S_u\big]}{V_2(u) + \eta_u} \Big)^2 \\
& \quad - \frac{1}{M_u} \Big(V_2(u)x + V_1(u) + \frac{1}{2}(S_u + M_u\Theta_u) \Big)^2 - \lambda_u \frac{\big[V_2(u)x + V_1(u) + \frac{1}{2}S_u\big]^2}{V_2(u) + \eta_u} \; \Bigg\} \\
& = 0 \quad d\mathbb{P} \otimes dt - a.e..
\end{aligned} \tag{4.6}
$$

For the well-posedness, we must have $V_2 + \eta > 0$ $d\mathbb{P} \otimes dt$-a.e..

Gathering each of (x^2, x^1, x^0)-proportional terms, one obtains the following result.

Lemma 4.1. *A "candidate" of the optimal solution and the corresponding value function for the market maker's problem (3.12) are given by*

$$
\pi_u^* = -\frac{1}{M_u} \Big(V_2(u) X_{u-}^{\pi^*,\delta^*}(t,x) + V_1(u) + \frac{1}{2}(S_u + M_u \Theta_u) \Big) \tag{4.7}
$$

$$
\delta_u^* = -\frac{\big[V_2(u) X_{u-}^{\pi^*,\delta^*}(t,x) + V_1(u) + \frac{1}{2}S_u\big]}{V_2(u) + \eta_u} \tag{4.8}
$$

for $u \in [t,T]$ and $V(t,x) = V_2(t)x^2 + 2V_1(t)x + V_0(t)$, respectively. Here, $X^{\pi^*,\delta^*}(t,x)$ is the *solution of*

$$X_s^{\pi^*,\delta^*}(t,x) = x + \int_t^s \int_K z\mathcal{N}(du,dz) + \int_t^s \pi_u^* du + \int_t^s \delta_u^* dH_u, \;\; s \in [t,T] \; . \tag{4.9}$$

$(V_2, Z_2), (V_1, Z_1)$ *and* (V_0, Z_0) *must be the well-defined solutions of the following three BSDEs*

$$V_2(t) = \xi + \int_t^T \Big\{ -\Big(\frac{1}{M_u} + \frac{\lambda_u}{V_2(u)+\eta_u}\Big) V_2(u)^2 + \gamma_u \Big\} du - \int_t^T Z_2(u) dW_u \tag{4.10}$$

$$\begin{aligned} V_1(t) = - \int_t^T \Bigg\{ & \; V_2(u)\Big(\frac{1}{M_u} + \frac{\lambda_u}{V_2(u)+\eta_u}\Big) V_1(u) - \frac{1}{2}\big(\beta_u b_u \Psi_u - l_u\big) \\ & + V_2(u) \left(\Big[\frac{1}{M_u} + \frac{\lambda_u}{V_2(u)+\eta_u}\Big] \frac{S_u}{2} - \frac{1}{2}\Theta_u - b_u \Psi_u \right) \Bigg\} du - \int_t^T Z_1(u) dW_u \end{aligned} \tag{4.11}$$

$$\begin{aligned} V_0(t) = - \int_t^T \Bigg\{ & \; \Big(\frac{1}{M_u} + \frac{\lambda_u}{V_2(u)+\eta_u}\Big)\Big(V_1(u) + \frac{S_u}{2}\Big)^2 - V_1(u)\big(\Phi_u + b_u\Psi_u\big) \\ & - V_2(u)\Phi_{2,u} - \frac{1}{2}(S_u\Theta_u + \beta_u \Phi_{2,u}) + \frac{1}{4} M_u \Theta_u^2 \Bigg\} du - \int_t^T Z_0(u) dW_u \; , \end{aligned} \tag{4.12}$$

satisfying

$$V_2 + \eta > 0 \tag{4.13}$$

$d\mathbb{P} \otimes dt$*-a.e. in* $\Omega \times [0,T]$.

4.2 Verification

We are now going to study each BSDE and show the existence of the candidate solution, and also confirm that it actually satisfies the optimality principle.

Proposition 4.2. *Under Assumptions A and B, the BSDE (4.10) has a unique solution in* $(V_2, Z_2) \in \mathbb{S}^p(0,T) \times \mathbb{H}_d^p(0,T)$ *for* $\forall \; p > 1$*, and in particular* $V_2(t)$ *satisfies for every* $t \in [0,T]$ *and* $\epsilon > 0$ *that:*

$$\begin{aligned} \frac{1}{\mathbb{E}\left[\frac{1}{\xi} + \int_t^T \Big(\frac{1}{M_s} + \frac{\lambda_s}{\eta_s}\Big) ds \Big| \mathcal{F}_t^W \right]} & \le V_2(t) \\ & \le \frac{1}{(T-t+\epsilon)^2} \mathbb{E}\left[\epsilon^2 \xi + \int_t^T \Big(M_s + (T-s+\epsilon)^2 \gamma_s\Big) ds \Big| \mathcal{F}_t^W \right] \; . \end{aligned} \tag{4.14}$$

Proof. Let us define the function as

$$f(t,y) = -\Big(\frac{1}{M_t} + \frac{\lambda_t}{y+\eta_t}\Big) y^2 + \gamma_t \; . \tag{4.15}$$

Firstly, let us consider the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s \vee 0) ds - \int_t^T Z_s dW_s \tag{4.16}$$

Due to Assumptions A, B and the definitions (3.10), ξ and $f(t, y \vee 0)$ with any fixed y are bounded. Furthermore, it is clear that $f(t, y \vee 0)$ is a decreasing function in y. Thus, (4.16) satisfies the standard monotone conditions for the BSDE. By Theorem 5.27 in [30] [4], there exists a unique solution $(Y, Z) \in \mathbb{S}^p(0,T) \times \mathbb{H}^p_d(0,T)$ for all $p > 1$. On the other hand, it is clear that we have a trivial solution $(Y, Z) = (0, 0)$ if $\xi = 0$ and $\gamma = 0$. Since the terminal value and the driver f is increasing in (ξ, γ), we actually have $Y \geq 0$ by the comparison theorem (See, for example, Proposition 5.33 in [30].). As a result, the BSDE (4.10) has a unique solution $(V_2, Z_2) \in \mathbb{S}^p(0,T) \times \mathbb{H}^p_d(0,T)$ for all $p > 1$, and in addition, V_2 is non-negative.

The derivation of the upper and lower bounds is an adaptation of Proposition 2.1 in Ankirchner, Jeanblanc & Kruse (2014) [5] for our problem. Let us start from the derivation of the upper bound. For all $y, k \in \mathbb{R}$,

$$y^2 - 2ky + k^2 \geq 0 \tag{4.17}$$

is satisfied. For an arbitrary constant $\epsilon > 0$, choosing $k = \dfrac{M_t}{T - t + \epsilon}$ yields

$$-\Big(\frac{1}{M_t} + \frac{\lambda_t}{y + \eta_t}\Big) y^2 \leq -\frac{1}{M_t} y^2 \leq -\frac{2}{T - t + \epsilon} y + \frac{M_t}{(T - t + \epsilon)^2} \tag{4.18}$$

for all $y \geq 0$.

With some abuse of notation, consider the next linear BSDE

$$Y^\epsilon_t = \xi + \int_t^T \left\{ -\frac{2}{T - s + \epsilon} Y^\epsilon_s + \frac{M_s}{(T - s + \epsilon)^2} + \gamma_s \right\} ds - \int_t^T Z^\epsilon_s dW_s \ . \tag{4.19}$$

This is a linear BSDE with a bounded Lipschitz constant. Due to the boundedness of ξ, M, γ, there exists a unique solution $(Y^\epsilon, Z^\epsilon) \in \mathbb{S}^p(0,T) \times \mathbb{H}^p_d(0,T)$ for all $p > 1$. By the inequality (4.18) and the comparison theorem, we have

$$V_2(t) \leq Y^\epsilon_t \tag{4.20}$$

for all $t \in [0, T]$ and $\epsilon > 0$. In addition, Y^ϵ can be solved as

$$\begin{aligned} Y^\epsilon_t &= \mathbb{E}\left[\xi e^{-\int_t^T \frac{2}{T-s+\epsilon} ds} + \int_t^T e^{-\int_t^s \frac{2}{T-u+\epsilon} du} \Big(\frac{M_s}{(T - s + \epsilon)^2} + \gamma_s \Big) ds \Big| \mathcal{F}^W_t \right] \\ &= \frac{1}{(T - t + \epsilon)^2} \mathbb{E}\left[\epsilon^2 \xi + \int_t^T \Big(M_s + (T - s + \epsilon)^2 \gamma_s \Big) ds \Big| \mathcal{F}^W_t \right] \end{aligned} \tag{4.21}$$

and hence we obtained the desired upper bound.

[4] One can simply put μ=l=0 in the theorem.

Now, let us study the lower bound. Put

$$\widetilde{V}_t := \mathbb{E}\left[\frac{1}{\xi} + \int_t^T \Big(\frac{1}{M_s} + \frac{\lambda_s}{\eta_s}\Big) ds \Big| \mathcal{F}_t^W\right]. \tag{4.22}$$

Due to the existence of a constant $c > 0$ such that $\xi, M, \eta \geq c$, it satisfies

$$\frac{1}{\bar{\xi}} \leq \widetilde{V} \leq \frac{1}{c}\Big(1 + T(1+\bar{\lambda})\Big) \ (:= \kappa) \tag{4.23}$$

where $\bar{\xi}, \bar{\lambda}$ are the upper bounds of ξ, λ, respectively. Therefore, there exists $\widetilde{Z} \in \mathbb{H}_d^p(0,T)$, $\forall p > 0$ such that

$$d\widetilde{V}_t = -\Big(\frac{1}{M_t} + \frac{\lambda_t}{\eta_t}\Big) dt + \widetilde{Z}_t dW_t \ . \tag{4.24}$$

Then the process $\widetilde{U}_t := 1/\widetilde{V}_t$, which has the bounds $1/\kappa \leq \widetilde{U} \leq \bar{\xi}$, satisfies

$$\widetilde{U}_t = \xi + \int_t^T \Big\{ -\Big(\frac{1}{M_s} + \frac{\lambda_s}{\eta_s}\Big)\widetilde{U}_s^2 - \frac{|\Gamma_s|^2}{\widetilde{U}_s}\Big\} ds - \int_t^T \Gamma_s dW_s \ , \tag{4.25}$$

where $\Gamma := -\dfrac{\widetilde{Z}}{\widetilde{V}^2}$. Here, the terminal value, the first term of the driver and the coefficient of $|\Gamma|^2$ are all bounded. Thus the comparison theorem for the quadratic BSDE (See, Theorem 2.6 in [25]), one sees $\widetilde{U}_t \leq V_2(t)$ and hence the desired result is obtained. □

Since the BSDEs for (V_1, Z_1) and (V_0, Z_0) are linear, one easily obtains the following results.

Proposition 4.3. *Under Assumptions A and B, there exist unique solutions* $(V_1, Z_1) \in \mathbb{S}^4(0,T) \times \mathbb{H}_d^4(0,T)$ *for (4.11), and* $(V_0, Z_0) \in \mathbb{S}^2(0,T) \times \mathbb{H}_d^2(0,T)$ *for (4.12), respectively.*

Proof. We denote by C some positive constant, which may change line by line. From Proposition 4.2, V_2 is uniformly bounded and hence so is the linear coefficient of V_1. In addition,

$$\begin{aligned}
&\mathbb{E}\left[\ \Big(\int_0^T \Big| V_2(s)\Big(\Big[\frac{1}{M_s} + \frac{\lambda_s}{V_2(s)+\eta_s}\Big]\frac{S_s}{2} - \frac{\Theta_s}{2} - b_s\Psi_s\Big) - \frac{1}{2}(\beta_s b_s \Psi_s - l_s)\Big| ds\Big)^4\right] \\
&\leq C\mathbb{E}\Big[1 + \int_0^T \Big(|S_s|^4 + |b_s|^4 + |l_s|^4\Big) ds\Big] < \infty
\end{aligned} \tag{4.26}$$

Thus, by Theorem 5.21 (see also Section 5.3.5) in [30], there exists a unique solution $(V_1, Z_1) \in \mathbb{S}^4(0,T) \times \mathbb{H}_d^4(0,T)$.

As for (4.12), it is easy to see that

$$\mathbb{E}\Bigg[\Big(\int_0^T\Big|\Big(\frac{1}{M_s}+\frac{\lambda_s}{V_2(s)+\eta_s}\Big)\Big(V_1(s)+\frac{S_s}{2}\Big)^2-V_1(s)(\Phi_s+b_s\Psi_s)-V_2(s)\Phi_{2,s}$$

$$-\frac{1}{2}(S_s\Theta_s+\beta_s\Phi_{2,s})+\frac{1}{4}M_s\Theta_s^2\Big|ds\Big)^2\Bigg]$$

$$\leq C\mathbb{E}\left[1+\int_0^T\Big(|V_1(s)|^4+|S_s|^4+|b_s|^4\Big)ds\right]<\infty, \tag{4.27}$$

where we have used $V_1\in\mathbb{S}^4(0,T)$ proved in the previous arguments. By the same reasoning, there exists a unique solution $(V_0,Z_0)\in\mathbb{S}^2(0,T)\times\mathbb{H}^2_d(0,T)$. □

In order to check the optimality condition, we also need the following property of X^{π^*,δ^*}.

Proposition 4.4. *Under Assumptions A and B, the process of the position size* $\Big(X_s^{\pi^*,\delta^*}(t,x)\Big)_{s\in[t,T]}$ *given by (4.9) belongs to* $\mathbb{S}^4(t,T)$.

Proof. Let us take the starting time 0 and write $X_s^{\pi^*,\delta^*}(0,x)$ as X_s^* for notational simplicity. We have

$$X_s^{\pi^*,\delta^*}(0,x)=x+\int_0^s\int_K z\widetilde{\mathcal{N}}(du,dz)-\int_0^s\frac{\big(V_2(u)X_{u-}^{\pi^*,\delta^*}(0,x)+V_1(u)+\frac{S_u}{2}\big)}{V_2(u)+\eta_u}d\widetilde{H}_u$$

$$-\int_0^s\Bigg\{\ V_2(u)\Big(\frac{1}{M_u}+\frac{\lambda_u}{V_2(u)+\eta_u}\Big)X_u^{\pi^*,\delta^*}(0,x)+\Big(\frac{1}{M_u}+\frac{\lambda_u}{V_2(u)+\eta_u}\Big)\Big(V_1(u)+\frac{S_u}{2}\Big)$$

$$+\frac{\Theta_u}{2}-\Phi_u\ \Bigg\}du \tag{4.28}$$

Under Assumptions A and B, there exists some positive constant C such that

$$|X_t^*|^4\leq C\Bigg[1+\int_0^t\Big(|X_u^*|^4+|V_1(u)|^4+|S_u|^4\Big)du$$

$$+\Big(\int_0^t\int_K z\widetilde{\mathcal{N}}(du,dz)\Big)^4+\Bigg(\int_0^t\frac{[V_2(u)X_{u-}^*+V_1(u)+\frac{S_u}{2}]}{V_2(u)+\eta_u}d\widetilde{H}_u\Bigg)^4\Bigg]\ , \tag{4.29}$$

for every $t\in[0,T]$. Let us define a sequence of $\mathbb{F}$-stopping times $(\tau_n)_{n\geq 0}$ by

$$\tau_n:=\inf\{t\geq 0:|X_t^*|>n\}\wedge T\ , \tag{4.30}$$

and denote the τ_n-stopped process of the position size as $\big(X_s^{*\tau_n}\big)_{s\geq 0}$. Since we already know that $X^*\in\mathbb{S}^2(0,T)$, it is clear that $\tau_n\to T$ a.s. as $n\to\infty$.

The BDG inequality (see, for example, Theorem 10.36 in [23] for general local martingales) and positivity of integrand yield

$$\mathbb{E}\Big[|X_t^{*\tau_n}|^4\Big] \leq C\mathbb{E}\Bigg[1 + \int_0^t \Big(|X_u^{*\tau_n}|^4 + |V_1(u)|^4 + |S_u|^4\Big)du + \Big(\int_0^t \int_K z^2 \mathcal{N}(du, dz)\Big)^2 + \Bigg(\int_0^t \Big|\frac{V_2(u)X_{u-}^{*\tau_n} + V_1(u) + \frac{S_u}{2}}{V_2(u) + \eta_u}\Big|^2 dH_u\Bigg)^2\Bigg] . \tag{4.31}$$

Using Itô-formula, the definition of the stopping time and the boundedness of λ, we obtain

$$\mathbb{E}\left[\Big(\int_0^t |X_{u-}^{*\tau_n}|^2 dH_u\Big)^2\right] = \mathbb{E}\left[\int_0^t |X_{u-}^{*\tau_n}|^4 dH_u + 2\int_0^t |X_{u-}^{*\tau_n}|^2\Big(\int_0^{u-} |X_{v-}^{*\tau_n}|^2 dH_v\Big)dH_u\right] \leq C\mathbb{E}\left[\int_0^t |X_u^{*\tau_n}|^4 du\right] , \tag{4.32}$$

which yields

$$\mathbb{E}\left[\Bigg(\int_0^t \Big|\frac{V_2(u)X_{u-}^{*\tau_n} + V_1(u) + \frac{S_u}{2}}{V_2(u) + \eta_u}\Big|^2 dH_u\Bigg)^2\right] \leq C\mathbb{E}\left[\int_0^t \Big(|V_1(u)|^4 + |S_u|^4\Big)du + \int_0^t |X_u^{*\tau_n}|^4 du\right] \tag{4.33}$$

since $V_1, S \in \mathbb{S}^4(0,T)$. Thus, from (4.31), we have

$$\mathbb{E}\Big[|X_t^{*\tau_n}|^4\Big] \leq C\mathbb{E}\left[1 + ||V_1||_T^4 + ||S||_T^4 + \int_0^t |X_u^{*\tau_n}|^4 du\right] \tag{4.34}$$

and hence, by the Gronwall lemma, for $\forall t \in [0,T]$,

$$\mathbb{E}\Big[|X_t^{*\tau_n}|^4\Big] \leq C\mathbb{E}\left[1 + ||V_1||_T^4 + ||S||_T^4\right] e^{CT} < \infty . \tag{4.35}$$

Passing to the limit $n \to \infty$, we see $\mathbb{E}\Big[|X_t^*|^4\Big] < C$ for every $t \in [0,T]$ with some positive constant C. Using the BDG inequality and the above estimate, we obtain from (4.29) that

$$\mathbb{E}\Big[||X^*||_T^4\Big] \leq C\mathbb{E}\left[1 + ||V_1||_T^4 + ||S||_T^4 + \int_0^T |X_u^*|^4 du\right] < \infty . \tag{4.36}$$

□

Corollary 4.1. *Under Assumptions A and B, the candidate solution* (π^*, δ^*) *in Lemma 4.1 is well-defined, unique and satisfies* $(\pi^*, \delta^*) \in \mathbb{S}^4(t,T) \times \mathbb{S}^4(t,T) \subset \mathcal{U}$.

Finally, we arrived the first main result of the paper.

Theorem 4.1. *Under Assumptions A and B, the candidate solution* (π^*, δ^*) *in Lemma 4.1 is, in fact, the unique optimal solution of the market maker's problem given by (3.12).*

Proof. It suffices to confirm that the optimality principle of Proposition 4.1 is indeed satisfied.

Firstly, we have to see

- $\left(\int_t^s Z(u, X_u^{\pi^*,\delta^*}(t,x))dW_u\right)_{s\in[t,T]}$
- $\left(\int_t^s \Big(V(u, X_{u-}^{\pi^*,\delta^*}(t,x)+\delta_u^*) - V(u, X_{u-}^{\pi^*,\delta^*}(t,x))\Big)d\widetilde{H}_u\right)_{s\in[t,T]}$
- $\left(\int_t^s \int_K \Big(V(u, X_{u-}^{\pi^*,\delta^*}(t,x)+z) - V(u, X_{u-}^{\pi^*,\delta^*}(t,x))\Big)\widetilde{\mathcal{N}}(du,dz)\right)_{s\in[t,T]}$

are all true $\mathbb{F}$-martingales. For notational simplicity, let us put $t=0$ and $X_s^* = X_s^{\pi^*,\delta^*}(0,x)$.

By the BDG inequality, Proposition 4.2, 4.3 and 4.4, there exists a positive constant C such that

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,T]}\left|\int_0^s Z(u,X_u^*)dW_u\right|\right] \le C\mathbb{E}\left[\left(\int_0^T |Z(s,X_s^*)|^2 ds\right)^{\frac{1}{2}}\right] \\
&\le C\mathbb{E}\left[\left(\int_0^T |Z_2(s)|^2|X_s^*|^4 ds\right)^{\frac{1}{2}} + \left(\int_0^T |Z_1(s)|^2|X_s^*|^2 ds\right)^{\frac{1}{2}} + \left(\int_0^T |Z_0(s)|^2 ds\right)^{\frac{1}{2}}\right] \\
&\le C\mathbb{E}\left[1 + ||X^*||_T^4 + \int_0^T \Big(|Z_2(s)|^2 + |Z_1(s)|^2 + |Z_0(s)|^2\Big)ds\right] < \infty
\end{aligned}
\tag{4.37}
$$

where in the second inequality, we have used the fact that

$$
(a+b+c)^{1/2} \le \sqrt{a} + \sqrt{b} + \sqrt{c}\ . \tag{4.38}
$$

for every $a, b, c \ge 0$. Thus, $\left(\int_0^s Z(u,X_u^*)dW_u\right)_{s\in[0,T]}$ is a martingale.

For the integrations by the counting and marked point processes, it is suffice to check that the integration by the corresponding compensator is in $\mathbb{L}^1(\Omega)$ (See, Corollary C4, Chapter VIII in [12].). Therefore, for the second term, we need to check

$$
\mathbb{E}\left[\int_0^T \big|V(u, X_u^* + \delta_u^*) - V(u, X_u^*)\big|\lambda_u du\right] < \infty \tag{4.39}
$$

In fact,

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T \big|V(u, X_u^* + \delta_u^*) - V(u, X_u^*)\big|\lambda_u du\right] \\
&= \mathbb{E}\left[\int_0^T \lambda_u\big|V_2(u)(2X_u^*\delta_u^* + (\delta_u^*)^2) + 2V_1(u)\delta_u^*\big|du\right] \\
&\le C\mathbb{E}\left[\int_0^T \Big(|X_u^*|^2 + |\delta_u^*|^2 + |V_1(u)|^2\Big)du\right] < \infty.
\end{aligned}
\tag{4.40}
$$

Similarly, for the third term,

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T\int_K \big|V(u,X_u^*+z)-V(u,X_u^*)\big|\Lambda(u,z)dudz\Big] \\
&=\mathbb{E}\Big[\int_0^T\int_K \big|V_2(u)(2X_u^*z+z^2)+2V_1(u)z\big|\Lambda(u,z)dudz\Big] \\
&\leq C\mathbb{E}\Big[\int_0^T\Big(|X_u^*|^2+|V_1(u)|^2+|\Phi_{2,u}|\Big)du\Big]<\infty
\end{aligned}
\tag{4.41}
$$

where we have used the boundedness of the compensator and the support K. The above facts combined with the construction of $a(t,x)$, strict positivity of M as well as $\lambda[V_2+\eta]$, guarantee that the optimality principle in Proposition 4.1 is indeed satisfied. □

5 The Property of the Terminal Position Size

Let us take a positive constant $1<L<\infty$ and set $\xi=L$, i.e., $\widetilde{\xi}=\beta_T/2+L$. We denote the corresponding solutions of the BSDEs (4.10), (4.11) and (4.12) by $(V_i^L,Z_i^L)_{\{i=1,2,3\}}$, respectively. In this section, we study the behavior of the terminal size $X_T^{\pi^*,\delta^*}$ according to the change of the penalty size L.

Assumption C
In this section, let us take the lower bound c in the assumption (b_3) in such a way that $c/(1+\bar{\lambda})<1$ and also $\widetilde{c}:=c/[\bar{M}(1+\bar{\lambda})]<1/2$. Obviously, we can always choose $c>0$ (or equivalently $\bar{M},\bar{\lambda}$) to satisfy these inequalities.

Lemma 5.1. *Under Assumptions A, B, C and $\xi=L$, the following inequalities hold for every $0\leq t\leq s\leq T$ with an L-independent positive constant C;*

$$
\begin{aligned}
&V_2^L(t)\leq C\frac{1}{T-t+\epsilon_L} \\
&\exp\Big(-\int_t^s r(u,V_2^L(u))du\Big)\leq\Big(\frac{T-s+\epsilon_L}{T-t+\epsilon_L}\Big)^{\widetilde{c}}
\end{aligned}
\tag{5.1}
$$

where $\epsilon_L:=\dfrac{1}{L}$, $\widetilde{c}:=\dfrac{c}{\bar{M}(1+\bar{\lambda})}$ *and* $r(t,y):=\Big(\dfrac{1}{M_t}+\dfrac{\lambda_t}{y+\eta_t}\Big)y$.

Proof. Since the inequality in Proposition 4.2 holds arbitrary $\epsilon>0$, one can choose $\epsilon=\epsilon_L=1/L$. Then one obtains

$$
\begin{aligned}
V_2^L(t)&\leq\frac{\epsilon_L}{(T-t+\epsilon_L)^2}+\frac{T-t}{(T-t+\epsilon_L)^2}\bar{M}+\frac{\bar{\gamma}}{3}\Big((T-t+\epsilon_L)-\frac{\epsilon_L^3}{(T-t+\epsilon_L)^2}\Big) \\
&\leq\frac{1}{T-t+\epsilon_L}\Big(1+\bar{M}+\frac{\bar{\gamma}}{3}(T+1)^2\Big) \\
&\leq\frac{C}{T-t+\epsilon_L}\ .
\end{aligned}
\tag{5.2}
$$

Similarly,

$$V_2^L(t) \geq \frac{1}{\mathbb{E}\Big[\frac{1}{L}+\int_t^T\Big(\frac{1}{M_s}+\frac{\lambda_s}{\eta_s}\Big)ds\Big|\mathcal{F}_t^W\Big]} \geq \frac{1}{\epsilon_L+\frac{(1+\bar{\lambda})}{c}(T-t)} \tag{5.3}$$

where $c>0$ is the lower bound given in (b_3). Thus,

$$\begin{aligned}\int_t^s r(u,V_2^L(u))du &\geq \int_t^s \frac{1}{\epsilon_L+\frac{1+\bar{\lambda}}{c}(T-u)}\frac{1}{\bar{M}}du \\ &= -\frac{c}{\bar{M}(1+\bar{\lambda})}\ln\Big(\frac{\epsilon_L+\frac{1+\bar{\lambda}}{c}(T-s)}{\epsilon_L+\frac{1+\bar{\lambda}}{c}(T-t)}\Big) \ . \end{aligned} \tag{5.4}$$

It yields

$$\exp\Big(-\int_t^s r(u,V_2^L(u))du\Big) \leq \left(\frac{\frac{c}{1+\bar{\lambda}}\epsilon_L+T-s}{\frac{c}{1+\bar{\lambda}}\epsilon_L+T-t}\right)^{\widetilde{c}} \ . \tag{5.5}$$

Note that for every $0\leq t\leq s\leq T$, $\Big(\frac{x\epsilon_L+T-s}{x\epsilon_L+T-t}\Big)^{\widetilde{c}}$ is a increasing function for $x\geq 0$. Thus, due to the arrangement of c, one obtains

$$\exp\Big(-\int_t^s r(u,V_2^L(u))du\Big) \leq \Big(\frac{T-s+\epsilon_L}{T-t+\epsilon_L}\Big)^{\widetilde{c}} \ . \tag{5.6}$$

□

We also have the following Lemma.

Lemma 5.2. *Under Assumptions A, B, C and $\xi=L$, there exists an L-independent positive constant C such that*

$$\mathbb{E}\Big[||V_1^L||_T^2+||X^{\pi^*,\delta^*}(0,x)||_T^2\Big] \leq C \ . \tag{5.7}$$

Proof. Let us put $A:\Omega\times[0,T]\to\mathbb{R}$ and $\alpha:\Omega\times[0,T]\to\mathbb{R}$ as

$$A_u := \Big(\frac{1}{M_u}+\frac{\lambda_u}{V_2^L(u)+\eta_u}\Big)\frac{S_u}{2}-\frac{1}{2}\Theta_u-b_u\Psi_u \tag{5.8}$$

$$\alpha_u := \frac{1}{2}\Big(\beta_u b_u\Psi_u-l_u\Big) \ . \tag{5.9}$$

Obviously, $A,\alpha\in\mathbb{S}^4(0,T)\subset\mathbb{S}^2(0,T)$, and whose $\mathbb{S}^2(0,T)$-norms can be dominated by L-

independent constants. It is straightforward to check that V_1^L can be written as

$$V_1^L(t) = -\mathbb{E}\left[\int_t^T e^{-\int_t^s r(u,V_2^L(u))du}\Big(V_2^L(s)A_s - \alpha_s\Big)ds\Big|\mathcal{F}_t^W\right] \ . \tag{5.10}$$

Thus, by Lemma 5.1, it satisfies the following inequality for $\forall t \in [0,T]$:

$$\begin{aligned}
|V_1^L(t)| &\le \mathbb{E}\left[\Big|\int_t^T e^{-\int_t^s r(u,V_2^L(u))du}(V_2^L(s)A_s - \alpha_s)ds\Big|\Big|\mathcal{F}_t^W\right] \\
&\le (T-t)\mathbb{E}\Big[||\alpha||_T\big|\mathcal{F}_t^W\Big] + \mathbb{E}\Big[||A||_T\big|\mathcal{F}_t^W\Big]\left(\int_t^T\Big(\frac{T-s+\epsilon_L}{T-t+\epsilon_L}\Big)^{\widetilde{c}}\frac{C}{T-s+\epsilon_L}ds\right) \\
&\le (T-t)\mathbb{E}\Big[||\alpha||_T\big|\mathcal{F}_t^W\Big] + C\mathbb{E}\Big[||A||_T\big|\mathcal{F}_t^W\Big]\frac{1}{\widetilde{c}}\Big(1-\Big[\frac{\epsilon_L}{T-t+\epsilon_L}\Big]^{\widetilde{c}}\Big) \\
&\le C\mathbb{E}\left[||\alpha||_T + ||A||_T\big|\mathcal{F}_t^W\right] \ .
\end{aligned} \tag{5.11}$$

Notice that $\Big(m_t := \mathbb{E}\Big[||\alpha||_T + ||A||_T\big|\mathcal{F}_t^W\Big]\Big)_{t\in[0,T]}$ is a square integrable martingale. Thus, from Doob's maximum inequality, one has

$$\begin{aligned}
\mathbb{E}\Big[||V_1^L||_T^2\Big] &\le C\mathbb{E}\left[\sup_{t\in[0,T]}\big|m_t\big|^2\right] \le 4C\mathbb{E}\Big[\big|m_T\big|^2\Big] \\
&\le C\mathbb{E}\Big[||\alpha||_T^2 + ||A||_T^2\Big]
\end{aligned} \tag{5.12}$$

where the right-hand side can be dominated by an L-independent constant.

Now, let us define another process $G : \Omega \times [0,T] \to \mathbb{R}$ as

$$G_u = \Big(\frac{1}{M_u} + \frac{\lambda_u}{V_2^L(u)+\eta_u}\Big)\Big(V_1^L(u) + \frac{S_u}{2}\Big) + \frac{\Theta_u}{2} - \Phi_u \tag{5.13}$$

which satisfies $G \in \mathbb{S}^4(0,T) \subset \mathbb{S}^2(0,T)$ and its $\mathbb{S}^2(0,T)$ norm can be dominated by an L-independent constant by the first part of the proof. From (4.7), (4.8) and (4.9), it is easy to see that

$$\begin{aligned}
X_t^* &= e^{-\int_0^t r(u,V_2^L(u))du}x - \int_0^t e^{-\int_s^t r(u,V_2^L(u))du}G_s ds \\
&+ \int_0^t\int_K e^{-\int_s^t r(u,V_2^L(u))du}z\widetilde{\mathcal{N}}(ds,dz) + \int_0^t e^{-\int_s^t r(u,V_2^L(u))du}\delta_s^* d\widetilde{H}_s
\end{aligned} \tag{5.14}$$

holds for every $t \in [0,T]$ with the notation $X_t^* := X_t^{\pi^*,\delta^*}(0,x)$. Using the fact that $r(\cdot, V_2^L(\cdot))$ is a positive process and the BDG inequality, we have, with some L-independent constant C,

$$\begin{aligned}
\mathbb{E}\Big[||X^*||_t^2\Big] &\le C\mathbb{E}\left[x^2 + ||G||_t^2 + \int_0^t\int_K z^2\mathcal{N}(ds,dz) + \int_0^t|\delta_s^*|^2 dH_s\right] \\
&\le C\mathbb{E}\Big[x^2 + ||G||_T^2 + ||\Phi_2||_T + ||V_1^L||_T^2 + ||S||_T^2\Big] + C\mathbb{E}\Big[\int_0^t ||X^*||_s^2 ds\Big] \ .
\end{aligned} \tag{5.15}$$

Let denote an L-independent constant dominating the first term by C'. Since we already know $X^* \in \mathbb{S}^4(0,T)$,

$$\mathbb{E}\Big[||X^*||_t^2\Big] \leq C' + C\int_0^t \mathbb{E}\Big[||X^*||_s^2\Big]ds \tag{5.16}$$

and hence by the Gronwall lemma,

$$\mathbb{E}\Big[||X^*||_T^2\Big] \leq C'e^{CT} \ . \tag{5.17}$$

Combining the first part, the claims were proved. □

Then, we can establish the following result.

Theorem 5.1. *Under Assumptions A, B, C and $\xi = L$, there exists an L-independent positive constant C satisfying*

$$\mathbb{E}\Big[\big|X_T^{\pi^*,\delta^*}(0,x)\big|^2\Big] \leq C\Big(\frac{\epsilon_L}{T+\epsilon_L}\Big)^{2\widetilde{c}} \tag{5.18}$$

and hence one can make the terminal position size arbitrarily close to zero by taking a large $L < \infty$ as the penalty.

Proof. From (5.14) and Lemma 5.1, we have

$$\begin{aligned}
\mathbb{E}\Big[\big|X_T^*\big|^2\Big] &\leq C\mathbb{E}\bigg[x^2\Big(e^{-\int_0^T r(u,V_2^L(u))du}\Big)^2 + ||G||_T^2\Big(\int_0^T e^{-\int_s^T r(u,V_2^L(u))du}ds\Big)^2 \\
&\quad + \int_0^T e^{-2\int_s^T r(u,V_2^L(u))du}\Big(\Phi_{2,s} + |\delta_s^*|^2\Big)ds\bigg] \\
&\leq Cx^2\Big(\frac{\epsilon_L}{T+\epsilon_L}\Big)^{2\widetilde{c}} \\
&+ C\mathbb{E}\Big[||X^*||_T^2 + ||V_1^L||_T^2 + ||G||_T^2 + ||S||_T^2 + ||\Phi_2||_T\Big]\int_0^T\Big(\frac{\epsilon_L}{T-s+\epsilon_L}\Big)^{2\widetilde{c}}ds \ .
\end{aligned} \tag{5.19}$$

Notice that the expectation in the second term is dominated by an L-independent constant by Lemma 5.2. Using the assumption $2\widetilde{c} < 1$, we have

$$\begin{aligned}
\mathbb{E}\Big[\big|X_T^*\big|^2\Big] &\leq C\left\{\Big(\frac{\epsilon_L}{T+\epsilon_L}\Big)^{2\widetilde{c}} + \frac{1}{1-2\widetilde{c}}(T+\epsilon_L)\Big(\frac{\epsilon_L}{T+\epsilon_L}\Big)^{2\widetilde{c}} - \frac{\epsilon_L}{1-2\widetilde{c}}\right\} \\
&\leq C\Big(\frac{\epsilon_L}{T+\epsilon_L}\Big)^{2\widetilde{c}}
\end{aligned} \tag{5.20}$$

with some L-independent positive constant, and hence obtained the desired result. □

Remark

Although we can discuss the limit of the singular terminal condition $L \to \infty$ as presented in [5], we can only apply their results to V_2^L. For V_1^L, there appears a singular drift term which is expected to create a discontinuity at the terminal point. This makes the detailed analysis difficult to carry out. However, as the previous result shows, we can make the terminal position size arbitrarily small by selecting large enough $L < \infty$ as the penalty. Therefore, the proposed strategy can still be used as the effective (the terminal position size is not exactly zero but can be arbitrarily small) liquidation strategy in the presence of incoming customer orders for a market maker.

6 An Extension to Portfolio Position Management

In the following sections, we are going to extend the previous framework so that we can deal with the optimal position management for a market maker in the presence of $n \in \mathbb{N}$ securities. As before, let us make the following assumptions.

Assumption A$'$
All the stochastic processes which do not jump by $\mathcal{N}^i$, H^i for $i \in \{1, \cdots, n\}$ are assumed to be $\mathbb{F}^W$-adapted and hence continuous. This $\mathbb{F}^W$ adaptedness includes all the stochastic processes defined below.

(a_1') $\boldsymbol{S} : \Omega \times [0,T] \to \mathbb{R}^n$ *is non-negative and* $\boldsymbol{S} \in \mathbb{S}^4_n(0,T)$.
(a_2') $\boldsymbol{b}, \boldsymbol{l} : \Omega \times [0,T] \to \mathbb{R}^n$ *and* $\boldsymbol{b}, \boldsymbol{l} \in \mathbb{S}^4_n(0,T)$.
(a_3') $\Lambda^i(\cdot,\cdot) : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ *for* $i \in \{1, \cdots, n\}$ *are such that* $\Lambda^i(t,\cdot)(\omega)$ *is a non-negative measurable function with bounded support* $K \subset \mathbb{R}$ *for every* $t \in [0,T]$ *and* $\omega \in \Omega$, *and that* $\Lambda^i(\cdot, z)$ *is a uniformly bounded* $\mathbb{F}^W$*-adapted process for every* $z \in K$.
(a_4') $\widetilde{\gamma} : \Omega \times [0,T] \to \mathbb{R}^{n\times n}$ *is uniformly bounded and takes values in the space of* $n \times n$ *symmetric positive-semidefinite matrices.*
(a_5') $M : \Omega \times [0,T] \to \mathbb{R}^{n\times n}$ *is uniformly bounded and takes values in the space of* $n \times n$ *symmetric positive-definite matrices.*
(a_6') $\widetilde{\xi} : \Omega \to \mathbb{R}^{n\times n}$ *is bounded,* $\mathcal{F}^W_T$*-measurable and takes values in the space of* $n \times n$ *symmetric positive-semidefinite matrices.*
(a_7') $\lambda^i, \widetilde{\eta}^i : \Omega \times [0,T] \to \mathbb{R}$ *for* $i \in \{1, \cdots, n\}$ *are uniformly bounded and strictly positive.*
(a_8') $\beta : \Omega \times [0,T] \to \mathbb{R}^{n\times n}$ *is uniformly bounded and takes values in the space of* $n \times n$ *symmetric matrices.*
(a_9') *There is no simultaneous jump among* $(\mathcal{N}^i, H^i)_{i\in\{1,\cdots,n\}}$.

6.1 The market description

We treat basically the same market described in Section 3.1 but now with n securities. The market maker's position for the securities starting $\boldsymbol{x} \in \mathbb{R}^n$ at time t is given by the following n-dimensional vector process:

$$\boldsymbol{X}^{\pi,\delta}_s(t,\boldsymbol{x}) = \boldsymbol{x} + \sum_{i=1}^n \int_t^s \int_K \boldsymbol{e}_i z \mathcal{N}^i(du,dz) + \int_t^s \boldsymbol{\pi}_u du + \sum_{i=1}^n \int_t^s \boldsymbol{e}_i \delta^i_u dH^i_u \tag{6.1}$$

where $\boldsymbol{e}_i$, $i \in \{1, \cdots, n\}$ is the unit $\mathbb{R}^n$ vector whose elements are all zero except the i-th element given by 1. $\boldsymbol{\pi} = (\pi^i)_{i\in\{1,\cdots,n\}}$ and $\boldsymbol{\delta} = (\delta^i)_{i\in\{1,\cdots,n\}}$ are $\mathbb{F}$-predictable trading strategies of the market maker in the exchange and in the dark pool, respectively. The superscript i is added to distinguish the corresponding security. $H^i, i \in \{1, \cdots, n\}$ is the counting process denoting the occurrence of the execution of the i-th security in the dark pool. $\mathcal{N}^i, i \in \{1, \cdots, n\}$ is the counting measure which describes the occurrence of an incoming customer order of the i-th security and its size.

We suppose that there exists a common bounded support $K \subset \mathbb{R}$ for the size of the incoming orders. We assume as before the existence of the compensators such that

$$\begin{aligned}
&\int_0^t \int_K \widetilde{\mathcal{N}}^i(ds,dz) = \int_0^t \int_K \Big(\mathcal{N}^i(ds,dz) - \Lambda^i(s,z)dzds\Big) \\
&\int_0^t d\widetilde{H}^i_s = \int_0^t \Big(dH^i_s - \lambda^i_s ds\Big)
\end{aligned} \tag{6.2}$$

for $t \in [0,T]$ are $\mathbb{F}$-martingales for every $i \in \{1, \cdots, n\}$. Let us also set the stochastic processes $\boldsymbol{\Phi} = (\Phi^i)_{i \in \{1,\cdots,n\}}$, $\boldsymbol{\Psi} = (\Psi^i)_{i \in \{1,\cdots,n\}}$ and $\boldsymbol{\Phi}_2 = (\Phi^i_2)_{i \in \{1,\cdots,n\}}$ representing the moments of the order size by

$$\Phi^i_t := \int_K z\Lambda^i(t,z)dz, \quad \Psi^i_t := \int_K |z|\Lambda^i(t,z)dz, \quad \Phi^i_{2,t} := \int_K z^2\Lambda^i(t,z)dz \ , \tag{6.3}$$

for $t \in [0,T]$ which are all uniformly bounded by Assumption A'.

We assume the price vector $\widetilde{\boldsymbol{S}}^{\pi,\delta}(t,\boldsymbol{x}) = \big(S^{\pi,\delta}_i(t,\boldsymbol{x})\big)_{i \in \{1,\cdots,n\}}$, which denotes the market price observed in the exchange under the impact of the market maker's strategy $(\boldsymbol{\pi},\boldsymbol{\delta})$ starting from the position size $\boldsymbol{x}$ at time t, is given by

$$\widetilde{\boldsymbol{S}}^{\pi,\delta}_s(t,\boldsymbol{x}) = \boldsymbol{S}_s + M_s\boldsymbol{\pi}_s - \beta_s\boldsymbol{X}^{\pi,\delta}_s(t,\boldsymbol{x}) \tag{6.4}$$

for $s \in [t,T]$. Here, M and β are not necessarily diagonal and hence they can induce direct as well as contagious stochastic linear price impacts from the continuous trading and also from the aggregate reactions of the other investors regarding the position size held by the market maker. It is natural to imagine that a high speed of trading or a big outstanding position of a certain security by a market maker can induce similar price actions among its closely related securities.

6.2 The market maker's problem

We model the cash flow in the interval $]t,T]$ to the market maker with strategy $(\boldsymbol{\pi},\boldsymbol{\delta})$ as

$$\begin{aligned}
&-\int_t^T \widetilde{\boldsymbol{S}}^{\pi,\delta}_s(t,\boldsymbol{x})^\top \boldsymbol{\pi}_s ds - \sum_{i=1}^n \int_t^T \int_K \widetilde{S}^{\pi,\delta}_{i,s-}(t,\boldsymbol{x})(1-\mathrm{sgn}(z)b^i_s)z\mathcal{N}^i(ds,dz) \\
&-\sum_{i=1}^n \int_t^T \Big(\big(\boldsymbol{S}_s - \beta_s\boldsymbol{X}^{\pi,\delta}_{s-}(t,\boldsymbol{x})\big)^i\delta^i_s + \widetilde{\eta}^i_s|\delta^i_s|^2\Big)dH^i_s + \int_t^T \boldsymbol{l}_s^\top \boldsymbol{X}^{\pi,\delta}_s(t,x)ds,
\end{aligned} \tag{6.5}$$

where the symbol $\top$ denotes the transposition. We consider the following market maker's problem:

$$\widetilde{V}(t,\boldsymbol{x}) = \operatorname*{ess\,inf}_{\boldsymbol{\pi},\boldsymbol{\delta}\in\mathcal{U}} \mathbb{E}\Bigg[(\boldsymbol{X}_T^{\pi,\delta})^\top \widetilde{\xi} \boldsymbol{X}_T^{\pi,\delta} + \int_t^T (\boldsymbol{X}_s^{\pi,\delta})^\top \widetilde{\gamma}_s \boldsymbol{X}_s^{\pi,\delta} ds$$
$$+ \int_t^T ((\widetilde{\boldsymbol{S}}_s^{\pi,\delta})^\top \boldsymbol{\pi}_s - \boldsymbol{l}_s^\top \boldsymbol{X}_s^{\pi,\delta}) ds + \sum_{i=1}^n \int_t^T \int_K \widetilde{S}_{i,s-}^{\pi,\delta} \big(1 - \mathrm{sgn}(z) b_s^i\big) z \mathcal{N}^i(ds,dz)$$
$$+ \sum_{i=1}^n \int_t^T \Big(\big(\boldsymbol{S}_s - \beta_s \boldsymbol{X}_{s-}^{\pi,\delta}\big)^i \delta_s^i + \widetilde{\eta}_s^i |\delta_s^i|^2 \Big) dH_s^i \,\Big|\mathcal{F}_t \Bigg], \tag{6.6}$$

where we have omitted the argument $(t, \boldsymbol{x})$ to save the space. By making $\widetilde{\xi}$ and $\widetilde{\gamma}$ proportional to the (stochastic) covariance matrix among the securities, the market maker can include the portfolio diversification effects for its position management. In the above modeling, the customer orders and the executions in the dark pool are assumed to occur independently for each security. However, it does not seem difficult to introduce simultaneous customer orders or the dark pool executions for an arbitrary subset of the securities by following the idea of dynamic Markov copula model studied by Bielecki, Cousin, Crépey & Herbertsson (2014a, 2014b) [9, 10]. If there exist strong clusterings among the customer orders or the dark pool executions, an extension to this direction may become relevant.

The set of admissible strategies $\mathcal{U}$ is defined below.

Definition 6.1. *We define the admissible strategies $\mathcal{U}$ by the set of $\mathbb{F}$-predictable processes $(\boldsymbol{\pi}, \boldsymbol{\delta})$ that belong to $\mathbb{H}_n^2(0,T) \times \mathbb{H}_n^2(0,T)$ and also Markovian with respect to the position size, i.e., they are expressed with some measurable functions (f^π, f^δ) by*

$$\boldsymbol{\pi}_s = f^\pi(s, \boldsymbol{X}_{s-}^{\pi,\delta}(t,\boldsymbol{x})), \quad \boldsymbol{\delta}_s = f^\delta(s, \boldsymbol{X}_{s-}^{\pi,\delta}(t,\boldsymbol{x})) \tag{6.7}$$

where, for $a \in \{\pi, \delta\}$, $f^a : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $f^a(\cdot, \boldsymbol{x})$ is an $\mathbb{F}^W$-adapted process for all $\boldsymbol{x} \in \mathbb{R}^n$.

Let us give the dynamics of the $\mathbb{F}^W$-adapted process β as

$$d\beta_t = \mu_t^\beta dt + \sum_{j=1}^d \big(\sigma_t^\beta\big)_j dW_t^j \tag{6.8}$$

and define

$$\xi := \widetilde{\xi} - \frac{\beta_T}{2}, \quad \gamma := \widetilde{\gamma} + \frac{\mu^\beta}{2}$$
$$\eta^i := \widetilde{\eta}^i + \frac{(\beta)_{i,i}}{2}, \quad \text{for } \ i \in \{1, \cdots, n\} \ . \tag{6.9}$$

Assumption B′
(b_1') *$\mu^\beta : \Omega \times [0,T] \to \mathbb{R}^{n\times n}$ and $(\sigma^\beta)_i, i \in \{1, \cdots, d\} : \Omega \times [0,T] \to \mathbb{R}^{n\times n}$ are uniformly bounded and symmetric.*

(b_2') ξ *is positive-semidefinite.*
(b_3') γ *is positive-semidefinite* $d\mathbb{P}\otimes dt$*-a.e..*
(b_4') *There exists a constant* $c>0$ *such that* $\lambda^i\eta^i \geq c$ $d\mathbb{P}\otimes dt$*-a.e. for every* $i\in\{1,\cdots,n\}$ *and also* $\boldsymbol{y}^\top M\boldsymbol{y}\geq c|\boldsymbol{y}|^2$ $d\mathbb{P}\otimes dt$*-a.e. for every* $\boldsymbol{y}\in\mathbb{R}^n$*.*

Definition 6.2. *The cost function for the market maker with a given position size* $\boldsymbol{x}\in\mathbb{R}^n$ *at* t *is*

$$J^{t,\boldsymbol{x}}(\boldsymbol{\pi},\boldsymbol{\delta})=\mathbb{E}\Bigg[\ (\boldsymbol{X}_T^{\pi,\delta})^\top\xi\boldsymbol{X}_T^{\pi,\delta}+\int_t^T\Big((\boldsymbol{X}_s^{\pi,\delta})^\top\gamma_s\boldsymbol{X}_s^{\pi,\delta}+(\boldsymbol{X}_s^{\pi,\delta})^\top\big(\beta_s(\boldsymbol{b\Psi})_s-\boldsymbol{l}_s\big)\Big)ds$$
$$+\int_t^T\Bigg\{\boldsymbol{\pi}_s^\top M_s\boldsymbol{\pi}_s+\big(\boldsymbol{S}_s+M_s\boldsymbol{\Theta}_s\big)^\top\boldsymbol{\pi}_s+\boldsymbol{S}_s^\top\boldsymbol{\Theta}_s+\sum_{i=1}^n\Big(\lambda_s^i\big[\eta_s^i(\delta_s^i)^2+S_s^i\delta_s^i\big]+\frac{(\beta_s)_{i,i}}{2}\Phi_{2,s}^i\Big)\Bigg\}ds\Big|\mathcal{F}_t^W\Bigg] \tag{6.10}$$

where $\boldsymbol{\Theta},\ \boldsymbol{b\Psi}:\Omega\times[0,T]\to\mathbb{R}^n$ *are defined by* $(\boldsymbol{\Theta}_s)^i:=\Phi_s^i-b_s^i\Psi_s^i$ *and* $(\boldsymbol{b\Psi})_s^i=b_s^i\Psi_s^i$ *for* $i\in\{1,\cdots,n\}$*. Here, the argument* $(t,\boldsymbol{x})$ *of the position size is omitted to save the space.*

Proposition 6.1. *Under Assumptions* A' *and* B'*, the market maker's problem (6.6) is equivalent to*

$$V(t,\boldsymbol{x})=\operatorname{ess}\inf_{(\boldsymbol{\pi},\boldsymbol{\delta})\in\mathcal{U}}J^{t,\boldsymbol{x}}(\boldsymbol{\pi},\boldsymbol{\delta}) \tag{6.11}$$

and it has a unique optimal solution $(\boldsymbol{\pi}^*,\boldsymbol{\delta}^*)\in\mathcal{U}$*.*

Proof. By using

$$\begin{aligned}-\int_t^T(\boldsymbol{X}_{s-}^{\pi,\delta})^\top\beta_s d\boldsymbol{X}_s^{\pi,\delta}&=-\frac{1}{2}(\boldsymbol{X}_T^{\pi,\delta})^\top\beta_T\boldsymbol{X}_T^{\pi,\delta}+\frac{1}{2}\boldsymbol{x}^\top\beta_t\boldsymbol{x}\\&+\int_t^T\Big(\frac{1}{2}(\boldsymbol{X}_s^{\pi,\delta})^\top\mu_s^\beta\boldsymbol{X}_s^{\pi,\delta}ds+\frac{1}{2}(\boldsymbol{X}_s^{\pi,\delta})^\top\sigma_s^\beta\boldsymbol{X}_s^{\pi,\delta}\cdot dW_s\Big)\\&+\sum_{i=1}^n\Big(\int_t^T\frac{1}{2}(\beta_s)_{i,i}(\delta_s^i)^2dH_s^i+\int_t^T\int_K\frac{1}{2}(\beta_s)_{i,i}z^2\mathcal{N}^i(ds,dz)\Big)\end{aligned}$$

and redefining the value function

$$V(t,\boldsymbol{x}):=\widetilde{V}(t,\boldsymbol{x})-\frac{1}{2}\boldsymbol{x}^\top\beta_t\boldsymbol{x} \tag{6.12}$$

one can prove it in exactly the same way as Proposition 3.1. □

7 Solving the Problem with Multiple Securities

7.1 A candidate solution

We derive a candidate solution for the market maker's problem. Firstly, let us rewrite the optimality principle for the problem with multiple securities.

Proposition 7.1. *(Optimality Principle) Let Assumptions A' and B' are satisfied. Then,*
(a) For all $\boldsymbol{x} \in \mathbb{R}^n$, $(\boldsymbol{\pi}, \boldsymbol{\delta}) \in \mathcal{U}$ and $t \in [0,T]$, the process

$$\Bigg(V(s, \boldsymbol{X}_s^{\pi,\delta}) + \int_t^s \Big((\boldsymbol{X}_u^{\pi,\delta})^\top \gamma_u \boldsymbol{X}_u^{\pi,\delta} + (\boldsymbol{X}_u^{\pi,\delta})^\top \big(\beta_u (\boldsymbol{b\Psi})_u - \boldsymbol{l}_u\big)\Big) du + \int_t^s \Bigg\{ \boldsymbol{\pi}_u^\top M_u \boldsymbol{\pi}_u$$
$$+ \big(\boldsymbol{S}_u + M_u \boldsymbol{\Theta}_u\big)^\top \boldsymbol{\pi}_u + \boldsymbol{S}_u^\top \boldsymbol{\Theta}_u + \sum_{i=1}^n \Big(\lambda_u^i \big[\eta_u^i (\delta_u^i)^2 + S_u^i \delta_u^i\big] + \frac{(\beta_u)_{i,i}}{2} \Phi_{2,u}^i \Big) \Bigg\} du \Bigg)_{s \in [t,T]} \tag{7.1}$$

is an $\mathbb{F}$-submartingale.
(b) $(\boldsymbol{\pi}^*, \boldsymbol{\delta}^*)$ is optimal if and only if

$$\Bigg(V(s, \boldsymbol{X}_s^{\pi^*,\delta^*}) + \int_t^s \Big((\boldsymbol{X}_u^{\pi^*,\delta^*})^\top \gamma_u \boldsymbol{X}_u^{\pi^*,\delta^*} + (\boldsymbol{X}_u^{\pi^*,\delta^*})^\top \big(\beta_u (\boldsymbol{b\Psi})_u - \boldsymbol{l}_u\big)\Big) du + \int_t^s \Bigg\{ (\boldsymbol{\pi}_u^*)^\top M_u (\boldsymbol{\pi}_u^*)$$
$$+ \big(\boldsymbol{S}_u + M_u \boldsymbol{\Theta}_u\big)^\top \boldsymbol{\pi}_u^* + \boldsymbol{S}_u^\top \boldsymbol{\Theta}_u + \sum_{i=1}^n \Big(\lambda_u^i \big[\eta_u^i ({\delta^*}_u^i)^2 + S_u^i {\delta^*}_u^i\big] + \frac{(\beta_u)_{i,i}}{2} \Phi_{2,u}^i \Big) \Bigg\} du \Bigg)_{s \in [t,T]} \tag{7.2}$$

is an $\mathbb{F}$-martingale.

Derivation of a candidate solution and the associated stochastic HJB equation is similar to the single security case. We assume that the $\mathbb{F}^W$ semimartingale $\Big(V(t, \boldsymbol{x})\Big)_{t \in [0,T]}$ has the following decomposition:

$$V(s, \boldsymbol{x}) = V(t, \boldsymbol{x}) + \int_t^s a(u, \boldsymbol{x}) du + \int_t^s Z(u, \boldsymbol{x}) dW_u \tag{7.3}$$

where $a : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $Z : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^d$ and $a(\cdot, \boldsymbol{x})$ as well as $Z(\cdot, \boldsymbol{x})$ are $\mathbb{F}^W$-adapted processes for all $\boldsymbol{x} \in \mathbb{R}^n$. We suppose that the value function can be decomposed, for every $t \in [0,T]$ and $\boldsymbol{x} \in \mathbb{R}^n$ as

$$V(t, \boldsymbol{x}) = \boldsymbol{x}^\top V_2(t) \boldsymbol{x} + 2x^\top V_1(t) + V_0(t) \tag{7.4}$$
$$Z(t, \boldsymbol{x}) = \boldsymbol{x}^\top Z_2(t) \boldsymbol{x} + 2x^\top Z_1(t) + Z_0(t) \tag{7.5}$$

where $V_2 : \Omega \times [0,T] \to \mathbb{R}^{n \times n}$, $V_1 : \Omega \times [0,T] \to \mathbb{R}^n$, $V_0 : \Omega \times [0,T] \to \mathbb{R}$, $Z_2 : \Omega \times [0,T] \to \mathbb{R}^{n \times n \times d}$, $Z_1 : \Omega \times [0,T] \to \mathbb{R}^{n \times d}$ and $Z_0 : \Omega \times [0,T] \to \mathbb{R}^d$ are all $\mathbb{F}^W$-adapted processes. In addition, V_2 and Z_2 (with respect to the first two indexes) are symmetric.

A lengthy but straightforward calculation shows that a necessary condition for the optimality principle is

$$a(u, \boldsymbol{x}) + \boldsymbol{x}^\top \gamma_u \boldsymbol{x} + \boldsymbol{x}^\top (\beta_u (\boldsymbol{b\Psi})_u - \boldsymbol{l}_u) + \boldsymbol{S}_u^\top \boldsymbol{\Theta}_u + \sum_{i=1}^n \frac{(\beta_u)_{i,i}}{2} \Phi_{2,u}^i$$
$$+ 2\boldsymbol{x}^\top V_2(u) \boldsymbol{\Phi}_u + 2V_1(u)^\top \boldsymbol{\Phi}_u + \sum_{i=1}^n [V_2(u)]_{i,i} \Phi_{2,u}^i$$

$$
\begin{aligned}
&+\inf_{\boldsymbol{\pi},\boldsymbol{\delta}}\Bigg\{\ \Big(\boldsymbol{\pi}+M_u^{-1}\big[V_2(u)\boldsymbol{x}+V_1(u)+\frac{1}{2}(\boldsymbol{S}_u+M_u\boldsymbol{\Theta}_u)\big]\Big)^\top M_u\\
&\qquad\qquad\times\Big(\boldsymbol{\pi}+M_u^{-1}\big[V_2(u)\boldsymbol{x}+V_1(u)+\frac{1}{2}(\boldsymbol{S}_u+M_u\boldsymbol{\Theta}_u)\big]\Big)\\
&\qquad+\sum_{i=1}^n\lambda_u^i\big([V_2(u)]_{i,i}+\eta_u^i\big)\Big(\delta^i+\frac{\big[V_2(u)\boldsymbol{x}+V_1(u)+\frac{1}{2}\boldsymbol{S}_u\big]^\top\boldsymbol{e}_i}{[V_2(u)]_{i,i}+\eta_u^i}\Big)^2\\
&-\Big[V_2(u)\boldsymbol{x}+V_1(u)+\frac{1}{2}(\boldsymbol{S}_u+M_u\boldsymbol{\Theta}_u)\Big]^\top M_u^{-1}\Big[V_2(u)\boldsymbol{x}+V_1(u)+\frac{1}{2}(\boldsymbol{S}_u+M_u\boldsymbol{\Theta}_u)\Big]\\
&-\sum_{i=1}^n\lambda_u^i\frac{\Big(\big[V_2(u)\boldsymbol{x}+V_1(u)+\frac{\boldsymbol{S}_u}{2}\big]^\top\boldsymbol{e}_i\Big)^2}{[V_2(u)]_{i,i}+\eta_u^i}\Bigg\}=0\quad d\mathbb{P}\otimes dt-a.e.,
\end{aligned}\tag{7.6}
$$

where we need $[V_2]_{i,i}+\eta^i>0$ $d\mathbb{P}\otimes dt$-a.e. for every $i\in\{1,\cdots,n\}$. As a result, we obtain the following.

Lemma 7.1. *A "candidate" of the optimal solution and the corresponding value function for the market maker's problem (6.11) are given by*

$$
\pi_u^*=-M_u^{-1}\Big(V_2(u)\boldsymbol{X}_{u-}^{\pi^*,\delta^*}(t,\boldsymbol{x})+V_1(u)+\frac{1}{2}\big(\boldsymbol{S}_u+M_u\boldsymbol{\Theta}_u\big)\Big)\tag{7.7}
$$

$$
(\delta_u^*)^i=-\frac{\big[V_2(u)\boldsymbol{X}_{u-}^{\pi^*,\delta^*}(t,\boldsymbol{x})+V_1(u)+\frac{1}{2}\boldsymbol{S}_u\big]^i}{[V_2(u)]_{i,i}+\eta_u^i},\quad\text{for}\quad i\in\{1,\cdots,n\}\tag{7.8}
$$

for $u\in[t,T]$ and $V(t,\boldsymbol{x})=\boldsymbol{x}^\top V_2(t)\boldsymbol{x}+2\boldsymbol{x}^\top V_1(t)+V_0(t)$, respectively. Here, $\boldsymbol{X}^{\pi^*,\delta^*}(t,\boldsymbol{x})$ is the solution of

$$
X_s^{\pi^*,\delta^*}(t,\boldsymbol{x})=\boldsymbol{x}+\sum_{i=1}^n\int_t^s\int_K\boldsymbol{e}_iz\mathcal{N}^i(du,dz)+\int_t^s\boldsymbol{\pi}_u^*du+\sum_{i=1}^n\int_t^s\boldsymbol{e}_i(\delta_u^*)^idH_u^i,\ \ s\in[t,T]\tag{7.9}
$$

(V_2,Z_2), (V_1,Z_1) and (V_0,Z_0) must be the well-defined solutions of the following three BSDEs

$$
V_2(t)=\xi+\int_t^T\Big\{-V_2(u)\Big[M_u^{-1}+\mathrm{diag}\Big(\frac{\lambda_u}{V_2(u)+\eta_u}\Big)\Big]V_2(u)+\gamma_u\Big\}du-\int_t^T Z_2(u)dW_u\tag{7.10}
$$

$$
\begin{aligned}
V_1(t)=-\int_t^T\Bigg\{&\ V_2(u)\Big[M_u^{-1}+\mathrm{diag}\Big(\frac{\lambda_u}{V_2(u)+\eta_u}\Big)\Big]V_1(u)-\frac{1}{2}\big(\beta_u(\boldsymbol{b\Psi})_u-\boldsymbol{l}_u\big)\\
&+V_2(u)\Big(\Big[M_u^{-1}+\mathrm{diag}\Big(\frac{\lambda_u}{V_2(u)+\eta_u}\Big)\Big]\frac{\boldsymbol{S}_u}{2}-\frac{1}{2}\boldsymbol{\Theta}_u-(\boldsymbol{b\Psi})_u\Big)\Bigg\}du-\int_t^T Z_1(u)dW_u
\end{aligned}\tag{7.11}
$$

$$
\begin{aligned}
V_0(t) = -\int_t^T \Bigg\{ & \Big(V_1(u) + \frac{\boldsymbol{S}_u}{2}\Big)^\top \Big[M_u^{-1} + \mathrm{diag}\Big(\frac{\lambda_u}{V_2(u) + \eta_u}\Big)\Big] \Big(V_1(u) + \frac{\boldsymbol{S}_u}{2}\Big) \\
& - \big(\boldsymbol{\Phi}_u + (\boldsymbol{b\Psi})_u\big)^\top V_1(u) - \sum_{i=1}^n [V_2(u)]_{i,i}\Phi^i_{2,u} \\
& - \frac{1}{2}\big(\boldsymbol{S}_u^\top \boldsymbol{\Theta}_u + \sum_{i=1}^n [\beta_u]_{i,i}\Phi^i_{2,u}\big) + \frac{1}{4}\boldsymbol{\Theta}_u^\top M_u \boldsymbol{\Theta}_u \Bigg\} du - \int_t^T Z_0(u) dW_u
\end{aligned} \tag{7.12}
$$

satisfying, for every $i \in \{1, \cdots, n\}$,

$$
[V_2]_{i,i} + \eta^i > 0 \tag{7.13}
$$

$d\mathbb{P} \otimes dt$*-a.e. in* $\Omega \times [0, T]$. *In the above,* $\mathrm{diag}\Big(\frac{\lambda_u}{V_2(u)+\eta_u}\Big)$ *is defined as a diagonal matrix whose* (i, i)*-th element* $i \in \{1, \cdots, n\}$ *is given by* $\dfrac{\lambda^i_u}{[V_2(u)]_{i,i} + \eta^i_u}$.

7.2 Verification

In the multiple-security setup, V_2 follows a non-linear matrix valued BSDE. Since there is no comparison theorem known for a multi-dimensional BSDE in general, we cannot apply the technique used in the single-security case. Interestingly however, we shall see V_2 is the backward stochastic Riccati differential equation (BSRDE) associated with a special type of stochastic linear quadratic control (SLQC) problem in a diffusion setup studied by Bismut (1976) [11].

Theorem 7.1. *Under Assumptions* A' *and* B'*, there exists a unique solution of* (V_2, Z_2) *for the BSDE (7.10). In particular,* V_2 *takes values in the space of* $n \times n$ *symmetric positive-semidefinite matrices and is a.s. uniformly bounded i.e., there exists a positive constant* C' *such that*

$$
\operatorname{ess\,sup}\Big(\sup_{t\in[0,T]} \big|V_2(t)\big|(\omega)\Big) \le C' \ , \tag{7.14}
$$

and $Z_2 \in \mathbb{H}^p_{n\times n\times d}(0, T)$ *for any* $p > 0$.

Proof. Let us introduce an n-dimensional Brownian motion w which is orthogonal to W and consider $\mathcal{F}'_t := \mathcal{F}^W_t \vee \mathcal{F}^w_t$ where $\mathcal{F}^w_t$ is the augmented filtration generated by w. We study an n-dimensional $\big(\mathbb{F}' := (\mathcal{F}'_t)_{t\ge0}\big)$-adapted vector process staring from $\boldsymbol{x} \in \mathbb{R}^n$ at time t which is controlled by the $2n$-dimensional vector process $\boldsymbol{\theta}$:

$$
\boldsymbol{X}^\theta_s(t, \boldsymbol{x}) = \boldsymbol{x} + \int_t^s C_u\boldsymbol{\theta}_u du + \sum_{j=1}^n \int_t^s D^j_u \boldsymbol{\theta}_u dw^j_u \ , \ s \in [t, T] \ . \tag{7.15}
$$

Here, $C : \Omega \times [0, T] \to \mathbb{R}^{n\times 2n}$ is defined by

$$
C_u := \big(\mathbb{I}_{n\times n} \quad \mathrm{diag}(\lambda^i_u)\big) \tag{7.16}
$$

for $u \in [0,T]$, where $\mathbb{I}_{n\times n}$ is the n-dimensional identity matrix, and $\mathrm{diag}(\lambda^i)$ is the n-dimensional diagonal matrix whose (i,i)-th element $i \in \{1,\cdots,n\}$ is given by λ^i. We use the same notation for the diagonal matrices below. $D^i : \Omega \times [0,T] \to \mathbb{R}^{n\times 2n}$ for $i \in \{1,\cdots,n\}$ has zero entry for all the elements except the $(i,n+i)$-th which is given by

$$[D^i_u]_{i,n+i} = \sqrt{\lambda^i_u} \tag{7.17}$$

for $u \in [0,T]$. We define the admissible strategies $\mathcal{U}'$ as the set of $2n$-dimensional $\mathbb{F}'$-adapted processes $\boldsymbol{\theta}$ that belong to $\mathbb{H}^2_{2n}(0,T)$.

Now, let us consider the following SLQC problem:

$$V'(t,\boldsymbol{x}) = \operatorname*{ess\,inf}_{\boldsymbol{\theta}\in\mathcal{U}'} \mathbb{E}\left[(\boldsymbol{X}^{\theta}_t)^\top \xi \boldsymbol{X}^{\theta}_T + \int_t^T \Big((\boldsymbol{X}^{\theta}_s)^\top \gamma_s \boldsymbol{X}^{\theta}_s + \boldsymbol{\theta}_s^\top N_s \boldsymbol{\theta}_s \Big) ds \Big| \mathcal{F}'_t \right] \tag{7.18}$$

where the argument $(t,\boldsymbol{x})$ is omitted from $\boldsymbol{X}$ to save the space, and $N : \Omega \times [0,T] \to \mathbb{R}^{2n\times 2n}$ is defined for $u \in [0,T]$ by

$$N_u = \begin{pmatrix} M_u & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathrm{diag}(\lambda^i_u \eta^i_u) \end{pmatrix}. \tag{7.19}$$

Then, by Proposition 5.1 in [11], the associated BSRDE on P (and its associated integrand Z_P of dW) such that $V'(t,\boldsymbol{x}) = \boldsymbol{x}^\top P(t)\boldsymbol{x}$ is given by

$$P(t) = \xi + \int_t^T \left\{ -P(u)C_u \Big(N_u + \sum_{i=1}^n (D^i_u)^\top P(u) D^i_u \Big)^{-1} C_u^\top P(u) + \gamma_u \right\} du - \int_t^T Z_P(u) dW_u \tag{7.20}$$

where the stochastic integration by dw vanishes because $W \perp w$ and that the terminal value ξ and all the processes included in the driver are $\mathbb{F}^W$-adapted. By noticing that

$$N_u + \sum_{i=1}^n (D^i_u)^\top P(u) D^i_u = \begin{pmatrix} M_u & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathrm{diag}\Big(\lambda^i_u([P(u)]_{i,i} + \eta^i_u)\Big) \end{pmatrix} \tag{7.21}$$

one can confirm that the BSDE of P is equal to that of V_2 given by (7.10) [5].

Under Assumptions A' and B', ξ is positive-semidefinite and bounded, γ is positive-semidefinite and uniformly bounded, C, D and N are uniformly bounded. In particular, there exists a constant $c > 0$ such that

$$\boldsymbol{y}^\top N_u \boldsymbol{y} \geq c|y|^2, \quad d\mathbb{P} \otimes dt - a.e. \tag{7.22}$$

for all $y \in \mathbb{R}^{2n}$. Thus, by Theorem 6.1 in [11], P (and hence V_2) has a unique solution, which is symmetric, positive-semidefinite and a.s. uniformly bounded. In particular, this implies $[P(u)]_{i,i} \geq 0$, $d\mathbb{P} \otimes dt$-a.e..

[5] It is not difficult to confirm the same BSRDE arises as the stochastic HJB equation by the same method we used in Lemma 7.1.

Since P is positive, one sees from (7.21),

$$0 < \boldsymbol{y}^\top \Big(N_u + \sum_{i=1}^n (D_u^i)^\top P(u) D_u^i\Big)^{-1} \boldsymbol{y} \leq \frac{|\boldsymbol{y}|^2}{c} \quad d\mathbb{P} \otimes dt - a.e. \tag{7.23}$$

for all $\boldsymbol{y} \in \mathbb{R}^{2n}$ and hence $\Big(N_u + \sum_{i=1}^n (D_u^i)^\top P(u) D_u^i\Big)^{-1}_{u \in [0,T]}$ is a uniformly bounded linear operator. Using the boundedness of P and the other variables, one sees

$$m_t = P(t) - P(0) - \int_0^t \left\{ P(u) C_u \Big(N_u + \sum_{i=1}^n (D_u^i)^\top P(u) D_u^i\Big)^{-1} C_u^\top P(u) - \gamma_u \right\} du \tag{7.24}$$

for $t \in [0,T]$ is a uniformly bounded martingale. Thus, from the BDG inequality, for any $p > 0$, there exists a positive constant C such that

$$\mathbb{E}\left[\Big(\int_0^T |Z_P(u)|^2 du\Big)^{p/2}\right] \leq C\mathbb{E}\Big[||m||_T^p\Big] < \infty \tag{7.25}$$

and hence Z_P (and so does Z_2) belongs to $\mathbb{H}^p_{n\times n\times d}(0,T)$ for $\forall p > 0$. □

For more general results on the SLQC problem and the associated BSRDE, we refer to Peng (1992) [31] and Tang (2003, 2014) [36, 37], where the assumption of the orthogonality "$w \perp W$" is removed.

The following results are obtained in exactly the same way in Proposition 4.3, 4.4 and Corollary 4.1.

Proposition 7.2. *Under Assumptions A' and B', there exist unique solutions $(V_1, Z_1) \in \mathbb{S}^4_n(0,T) \times \mathbb{H}^4_{n\times d}(0,T)$ for (7.11), and $(V_0, Z_0) \in \mathbb{S}^2(0,T) \times \mathbb{H}^2_d(0,T)$ for (7.12), respectively. Furthermore, the process for the position size $\Big(\boldsymbol{X}_s^{\pi^*,\delta^*}(t,\boldsymbol{x})\Big)_{s\in[t,T]}$ given by (7.9) belongs to $\mathbb{S}^4_n(t,T)$. The candidate solution (π^*, δ^*) in Lemma 7.1 is well-defined and satisfies $(\pi^*, \delta^*) \in \mathbb{S}^4_n(t,T) \times \mathbb{S}^4_n(t,T) \subset \mathcal{U}$.*

The above results establish the main theorem.

Theorem 7.2. *Under Assumptions A' and B', the candidate solution (π^*, δ^*) in Lemma 7.1 is, in fact, the unique optimal solution of the market maker's problem given by (6.11).*

Proof. The proof is the same as that of Theorem 4.1. □

7.3 Remark on the effective liquidation

It is natural to imagine that one can make the terminal position size arbitrarily small by increasing the size of the eigenvalues of ξ. Although it is intuitively clear, it is difficult to prove since we do not have an explicit expression for the upper/lower bound of V_2 any more.

Let us suppose, in the interval $[T-\epsilon, T]$ with some constant $\epsilon > 0$, that M, γ, ξ, β can be diagonalized by the common *constant* orthogonal matrix O. In addition, suppose the market maker stops accepting the customer orders and using the dark pool. Then, by considering the securities in the base $O^\top S$ and the corresponding positions $O^\top \boldsymbol{X}$, the market maker's problem can be decomposed into n single security liquidation problems. In this case, $\hat{V_2} := O^\top V_2 O$ becomes diagonal process in $[T-\epsilon, T]$ and $\hat{V_1} := O^\top V_1$ interacts with the only one corresponding element of $\hat{V_2}$. In this special situation, it is clear that the position can be made arbitrary small by the corresponding optimal strategy thanks to the arguments made in the single security case.

8 A Possible Approximation Scheme

In this section, we discuss a possible evaluation technique of V_2, which is the biggest obstacle to make the strategy implementable in actual applications.

8.1 A special case

Firstly, let us consider a special case where $\xi, \gamma, M, \eta, \lambda$ (and hence naturally so is β) are non-random. In this case, V_2 is a solution of the following matrix-valued ordinary differential equation (ODE):

$$\frac{dV_2(s)}{ds} = V_2(s)\left(M_s^{-1} + \mathrm{diag}\Big(\frac{\lambda_s}{V_2(s)+\eta_s}\Big)\right)V_2(s) - \gamma_s, \quad s \in [t,T] \tag{8.1}$$

$$V_2(T) = \xi \tag{8.2}$$

which is the same ODE studied by Kratz & Schöneborn [26]. It is not difficult to numerically solve this equation by the standard technique for ODEs. In contrast to the model in [26], we still need to evaluate V_1 to implement the optimal strategy.

For notational simplicity, let us put

$$F(s) := V_2(s)\left(M_s^{-1} + \mathrm{diag}\Big(\frac{\lambda_s}{V_2(s)+\eta_s}\Big)\right), \quad s \in [t,T], \tag{8.3}$$

which is a deterministic matrix process. Let us also consider another deterministic matrix process defined by the ODE

$$\frac{dY_{t,s}}{ds} = F(s)Y_{t,s}, \quad s \in [t,T] \tag{8.4}$$

where $Y_{t,t} = \mathbb{I}_{n\times n}$. Then, we have

$$\frac{dY_{t,s}^{-1}}{ds} = -Y_{t,s}^{-1}F(s), \quad s \in [t,T] \tag{8.5}$$

with $Y_{t,t}^{-1} = \mathbb{I}_{n\times n}$ and it is straightforward to obtain

$$V_1(s) = -Y_{t,s}\int_s^T Y_{t,u}^{-1}\mathbb{E}\left[\frac{1}{2}F(u)\boldsymbol{S}_u - V_2(u)\Big(\frac{1}{2}\boldsymbol{\Theta}_u + (\boldsymbol{b\Psi})_u\Big) - \frac{1}{2}\big(\beta_u(\boldsymbol{b\Psi})_u - \boldsymbol{l}_u\big)\Big|\mathcal{F}_s^W\right]du \tag{8.6}$$

for $s \in [t,T]$, which only requires the expressions of

$$\mathbb{E}\big[\boldsymbol{S}_u|\mathcal{F}_s^W\big],\ \mathbb{E}\big[\boldsymbol{\Phi}_u|\mathcal{F}_s^W\big],\ \mathbb{E}\big[(\boldsymbol{b\Psi})_u|\mathcal{F}_s^W\big],\ \mathbb{E}\big[\boldsymbol{l}_u|\mathcal{F}_s^W\big]\ . \tag{8.7}$$

These quantities can be evaluated analytically for simple models. Otherwise, one can apply the standard *small-diffusion* asymptotic expansion technique, which is developed by Yoshida (1992a) [39], Takahashi (1999) [35], Kunitomo & Takahashi (2003) [27] for the pricing of European contingent claims, and also Yoshida (1992b) [40] for statistical applications.

8.2 A general case with a Markovian factor process

Although it is impossible to solve V_2 analytically in a general setup, getting an explicit expression of its approximation is very important for successful implementation of the proposed scheme. A similar BSDE is also relevant for solving a different type of optimal liquidation problem treated in [5]. Furthermore, considering the wide spread applications of SLQC problems in various engineering issues, developing a successful approximation scheme for a general BSRDE should be a very important research topic in its own light.

We consider that a variant of technique proposed by Fujii & Takahashi (2012a) [17] looks very promising. The method was developed to analytically approximate a non-linear FBSDE by linearizing the driver in each approximation order, and its justification for a Lipschitz setup was recently given by Takahashi & Yamada (2013). We refer to Fujii & Takahashi (2012b) [18] for an example of explicit calculation, and Fujii & Takahashi (2014) [19] as an efficient Monte Carlo implementation where the analytical calculation is too cumbersome. See also Shiraya & Takahashi (2014) [34] and Crépey & Song (2014) [14] as an interesting application of the above perturbation method to the so-called credit valuation adjustment (CVA).

Although the justification in the current setup and the detailed numerical tests for a general BSRDE will be left for the future research, let us explain the main idea for the interested readers below. For simplicity, let us concentrate on the single security case. We assume that $\xi, \gamma, M, \lambda : \mathbb{R} \to \mathbb{R}$ are all smooth functions. We introduce the underlying factor process $X : \Omega \times [t,T] \to \mathbb{R}$ (do not confuse it with the position size process) which follows the SDE:

$$X_s = x + \int_t^s \mu(X_s)ds + \int_t^s \sigma(X_s)dW_s\ . \tag{8.8}$$

where $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}^d$ are smooth functions satisfying the standard Lipschitz conditions to guarantee the existence of a strong solution for X. An extension to a multi-dimensional setup is straightforward. Set a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$f(x,v) := -\left(\frac{1}{M(x)} + \frac{\lambda(x)}{v + \eta(x)}\right) v^2 + \gamma(x) \tag{8.9}$$

and consider to solve the BSDE

$$V_t = \xi(X_T) + \int_t^T f(X_s, V_s)ds - \int_t^T Z_s dW_s \tag{8.10}$$

which corresponds to (4.10).

We assume that the volatility of X is small and express it by introducing a small perturbation constant $\epsilon > 0$ as follows:

$$dX_s^\epsilon = \mu(X_s^\epsilon)ds + \epsilon\sigma(X_s^\epsilon)dW_s \quad s \in [t,T], \quad X_t^\epsilon = x \tag{8.11}$$

which is now a process depending on the parameter ϵ. The associated BSDE is given by

$$V_t^\epsilon = \xi(X_T^\epsilon) + \int_t^T f(X_s^\epsilon, V_s^\epsilon)ds - \int_t^T Z_s^\epsilon dW_s \ . \tag{8.12}$$

We assume that they can be expanded in terms of ϵ as follows:

$$\begin{aligned} X_s^\epsilon &= X_s^{[0]} + \epsilon X_s^{[1]} + \epsilon^2 X_s^{[2]} + \cdots \\ V_s^\epsilon &= V_s^{[0]} + \epsilon V_s^{[1]} + \epsilon^2 V_s^{[2]} + \cdots \end{aligned} \tag{8.13}$$

for $s \in [t,T]$.

For the zero-th order, $X^{[0]}$ and $V^{[0]}$ are given by the solutions of the ODEs:

$$\frac{dX_s^{[0]}}{ds} = \mu(X_s^{[0]}) \quad s \in [t,T], \quad X_t^{[0]} = x \tag{8.14}$$

$$\frac{dV_s^{[0]}}{ds} = -f(X_s^{[0]}, V_s^{[0]}) \quad s \in [t,T], \quad V_T^{[0]} = \xi(X_T^{[0]}) \tag{8.15}$$

which corresponds to the special case discussed in the previous subsection. As long as $\xi(X_T^{[0]}), M(X^{[0]}), \gamma(X^{[0]}), \eta(X^{[0]}), \lambda(X^{[0]})$ satisfy the boundedness and positivity assumptions we made before, the above Riccati ODE should have a positive bounded solution and obviously $Z^{[0]} = 0$.

In the first order, we have the linear FBSDE system:

$$dX_s^{[1]} = \partial_x\mu^0(s)X_s^{[1]}ds + \sigma^0(s)dW_s \quad s \in [t,T], \quad X_t^{[1]} = 0 \tag{8.16}$$

$$V_t^{[1]} = \partial_x\xi^0(T)X_T^{[1]} + \int_t^T \left\{ \partial_v f^0(s)V_s^{[1]} + \partial_x f^0(s)X_s^{[1]} \right\} ds - \int_t^T Z_s^{[1]} dW_s \tag{8.17}$$

where we have used the short hand notation:

$$\mu^0(s) := \mu(X_s^{[0]}), \ \ \sigma^0(s) := \sigma(X_s^{[0]}), \ \ \xi^0(T) := \xi(X_T^{[0]}), \ \ f^0(s) := f(X_s^{[0]}, V_s^{[0]}) \ . \tag{8.18}$$

which are all deterministic functions. We have also used $\partial_x := \partial/\partial x$, $\partial_v := \partial/\partial v$. It is straightforward to confirm that

$$V_s^{[1]} = y(s)X_s^{[1]}, \quad s \in [t,T] \tag{8.19}$$

where $y : [t,T] \to \mathbb{R}$ is the solution of the linear ODE:

$$\begin{aligned} \frac{dy(s)}{ds} &= -\Big(\partial_x\mu^0(s) + \partial_v f^0(s)\Big)y(s) - \partial_x f^0(s) \quad s \in [t,T] \\ y(T) &= \partial_x\xi^0(T). \end{aligned} \tag{8.20}$$

In particular, we see that $V_t^{[1]} = 0$.

In the second order expansion, one can find

$$\begin{aligned} dX_s^{[2]} &= \Big(\partial_x\mu^0(s)X_s^{[2]} + \frac{1}{2}\partial_x^2\mu^0(s)(X_s^{[1]})^2\Big)ds + X_s^{[1]}\partial_x\sigma^0(s)dW_s, \quad s \in [t,T] \\ X_t^{[2]} &= 0 \end{aligned} \tag{8.21}$$

and

$$\begin{aligned} V_t^{[2]} &= \partial_x\xi^0(T)X_T^{[2]} + \frac{1}{2}\partial_x^2\xi^0(T)(X_T^{[1]})^2 + \int_t^T \Big\{\partial_v f^0(s)V_s^{[2]} + \partial_x f^0(s)X_s^{[2]} \\ &+ \frac{1}{2}\partial_v^2 f^0(s)(V_s^{[1]})^2 + \partial_{x,v}f^0(s)X_s^{[1]}V_s^{[1]} + \frac{1}{2}\partial_x^2 f^0(s)(X_s^{[1]})^2\Big\}ds - \int_t^T Z_s^{[2]}dW_s, \end{aligned} \tag{8.22}$$

which is also a simple linear BSDE. In this case, one has the solution with the following from:

$$V_s^{[2]} = y_2(s)X_s^{[2]} + y_1(s)(X_s^{[1]})^2 + y_0(s), \quad s \in [t,T]. \tag{8.23}$$

Here, $y_i, i \in \{2,1,0\} : [t,T] \to \mathbb{R}$ are defined as the solution of the next linear ODE system for $s \in [t,T]$:

$$\begin{aligned} \frac{dy_2(s)}{ds} &= -\Big(\partial_x\mu^0(s) + \partial_v f^0(s)\Big)y_2(s) - \partial_x f^0(s) \\ \frac{dy_1(s)}{ds} &= -\Big(2\partial_x\mu^0(s) + \partial_v f^0(s)\Big)y_1(s) \\ &\quad -\Big\{\frac{1}{2}\partial_x^2\mu^0(s)y_2(s) + \frac{1}{2}\partial_v^2 f^0(s)(y(s))^2 + \partial_{x,v}f^0(s)y(s) + \frac{1}{2}\partial_x^2 f^0(s)\Big\} \\ \frac{dy_0(s)}{ds} &= -\partial_v f^0(s)y_0(s) - (\sigma^0(s)\sigma^0(s)^\top)y_1(s) \end{aligned} \tag{8.24}$$

with the terminal conditions:

$$y_2(T) = \partial_x\xi^0(T), \quad y_1(T) = \frac{1}{2}\partial_x^2\xi^0(T), \quad y_0(T) = 0\ . \tag{8.25}$$

The above results can be obtained by applying Itô formula to (8.23) and comparing its drift to that of (8.22). See Fujii (2015) [16] as a related idea of expansion of BSDEs. These procedures, at least formally, can be repeated to an arbitrary higher order and provide the analytic expressions with the coefficients to be fixed by linear ODEs. Justification and numerical experiments for the current problem as well as for more general BSRDEs will be detailed elsewhere in the future research.

8.3 A determination of the bid/offer spreads

Before closing the section, let us comment on a possible determination of the bid/offer spreads $\boldsymbol{b}$. Although we have assumed that the market maker do not dynamically control the bid/offer spreads to give a bias to the customer orders, it is important of course to use a sustainable spread size for its market making business. Suppose, for example, the spread size b^i is proportional to the volatility $|\sigma^i|$ of the i-th security as

$$b_s^i = a|\sigma_s^i| \tag{8.26}$$

for $i \in \{1, \cdots, n\}$. The market maker can obtain the cost function or the distribution of its revenue by running the simulation based on the optimal strategy $(\boldsymbol{\pi}^*, \boldsymbol{\delta}^*)$ for each choice of a. Even if the intensity (and/or distribution) of the customer orders is a non-linear function of $(b^i)_{i\in\{1,\cdots,n\}}$, it is just an optimization on the single parameter a. This information may be used to set an appropriate size of a.

9 Concluding Remarks

In this paper, we discussed the optimal position management strategy for a maker maker who faces uncertain in- and out-flow of customer orders. The optimal strategy is represented by the solution of the stochastic Hamilton-Jacobi-Bellman equation which is decomposed into three (one non-linear and two linear) BSDEs. We provided the verification of the solution using the standard BSDE techniques for the single-security case and an interesting connection to a special type of SLQC problem for the multiple-security case. We also gave a possible approximation technique of the relevant BSRDE by perturbation, which still needs a justification and numerical experiments in the future.

Removing the assumption of $\mathbb{F}^W$-adaptedness of the relevant parameters looks an interesting extension of the proposed framework. This situation will arise when one introduces simultaneous jumps in the parameters, such as M, and the executions in the dark pool. In this case, the driver of the resultant BSRDE depends on the martingale coefficient of the counting process. As long as we know, the existence and uniqueness of the solution for the corresponding BSRDE have not yet been proved.

It looks also interesting to combine a stochastic filtering for the intensity of customer orders. An introduction of a hidden Markov process which affects the intensity is likely to help for modeling possible herding behavior among the customer orders. See a related work Fujii & Takahashi (2015) [20] on the mean-variance hedging problem for fund and insurance managers.

Acknowledgement

The author gratefully acknowledges the helpful comments and discussions from Akihiko Takahashi. The author is also grateful for Kenichiro Shiraya and Taiga Saito for useful discussions. This research is partially supported by Center for Advanced Research in Finance (CARF).

References

[1] Alfonsi, A., Fruth, A. and Schied, A., (2008), "Constrained portfolio liquidation in a limit order book model," *Banach Cent. Publ.* **83***, 9-25.*

[2] Alfonsi, A., Fruth, A. and Schied, A., (2010), "Optimal execution strategies in limit order books with general shape functions," *Quantitative Finance,* **10***(2), 143-157.*

[3] Almgren, R. and Chriss, N., (1999), "Value under liquidation," *Risk* **12***, 61-63.*

[4] Almgren, R. and Chriss, N., (2000), "Optimal execution of portfolio transactions, " *Journal of Risk,* **3***, 5-39.*

[5] Ankirchner, S., Jeanblanc, M. and Kruse, T. , (2014), "BSDEs with Singular Terminal Condition and Control Problems with Constraints," *SIAM J. Control Optim., 52(2), 893-913.*

[6] Ankirchner, S. and Kruse, T., (2013), "Optimal Trade Execution under Price-Sensitive Risk Preferences," *Quantitative Finance, Vol. 13, No. 9, 1395-1409.*

[7] Bank, P. and Baum, D., (2004), "Hedging and portfolio optimization in financial markets with a large trader," *Math. Finance,* **14***, 1-18.*

[8] Bertsimas, D. and Lo, A., (1998),"Optimal control of execution costs," *Journal of Financial Markets,* **1***, 1-50.*

[9] Bielecki, T.R., Cousin, A., Crépey, S. and Herbertsson, A., (2014a), "A bottom-up dynamic model of portfolio credit risk with stochastic intensities and random recoveries," *Communications in Statistics-Theory and Methods,* **43** *(7), 1362-1389.*

[10] Bielecki, T.R., Cousin, A., Crépey, S. and Herbertsson, A., (2014b), "Dynamic hedging of portfolio credit risk in a Markov copula model," *Journal of Optimization Theory and Applications, Vol. 161, Issue 1, 90-102.*

[11] Bismut, J., (1976), "Linear Quadratic Optimal Stochastic Control with Random Coefficients," *SIAM J. Control and Optimization, Vol. 14, No. 3, 419-444.*

[12] Bremaud, P., (1981),"Point Processes and Queues," *Springer Series in Statistics, Berlin.*

[13] Cetin, U., Jarrow, R. and Protter, P., (2004), "Liquidity Risk and Arbitrage Pricing Theory," *Finance Stoch.* **8***, 311-341.*

[14] Crépey, S. and Song, S., (2014), "Counterparty Risk and Funding: Immersion and Beyond," *Working paper, Évry University.*

[15] Fruth, A. and Schöneborn, T., (2014), "Optimal trade execution and price manipulation in order books with time-varying liquidity," *Math. Finance, Vol. 24, No. 4, 651-695.*

[16] Fujii, M., (2015), "A polynomial scheme of asymptotic expansion for backward SDEs and option pricing," *Forthcoming in Quantitative Finance.*

[17] Fujii, M. and Takahashi, A., (2012a), "Analytical approximation for non-linear FBSDEs with Perturbation Scheme, " *International Journal of Theoretical and Applied Finance, Vol. 15, Issue 5, 1250034 (24).*

[18] Fujii, M. and Takahashi, A., (2012b), "Perturbative expansion of FBSDE in an incomplete market with stochastic volatility," *Quarterly Journal of Finance,* **2**, *1250015(24).*

[19] Fujii, M. and Takahashi, A., (2014), "Perturbative Expansion Technique for Non-linear FBSDEs with Interacting Particle Method," *Forthcoming in Asia-Pacific Financial Markets, Online version: DOI 10.1007/s10690-015-9201- 7.*

[20] Fujii, M. and Takahashi, A., (2015), "Optimal hedging for fund and insurance managers with partially observable investment flows," *Quantitative Finance, Vol. 15, Issue 3, 535-551.*

[21] Gatheral, J. and Schied, A., (2013), "Dynamical models of market impact and algorithms for order execution," *J. Fouque and J. Langsam eds, 'Handbook on Systemic Risk,' Cambridge University Press, Chapter 22.*

[22] Gökay, S., Roch, A. and Soner, H.M., (2011), "Liquidity models in continuous and discrete time," *G. di Nunno & B. Oksendal, eds, 'Advanced Mathematical Methods for Finance,' Springer-Verlag, pp. 333-366.*

[23] He, S., Wang, J. and Yan, J., (1992), "Semimartingale Theory and Stochastic Calculus," *CRC press inc., Beijing, China.*

[24] Jarrow, R., (1992), "Market manipulation, bubbles, corners and short squeeze," *J. Financ. Quant. Anal.,* **27** *(3), 311-336.*

[25] Kobylanski, M., (2000), "Backward stochastic differential equations and partial differential equations with quadratic growth," *Annals of Probability, 28: 558-602.*

[26] Kratz, P. and Schöneborn, (2013), "Portfolio Liquidation in Dark Pools in Continuous Time," *Forthcoming in Mathematical Finance.*

[27] Kunitomo, N. and Takahashi, A., 2003, "On validity of the Asymptotic Expansion Approach in Contingent Claim Analysis," *The Annals of Applied Probability, 13, no. 3, 914-952.*

[28] Mania, M. and Tevzadze, R., (2003), "Backward Stochastic PDE and Imperfect Hedging," *International Journal of Theoretical and Applied Finance, Vol. 7, No. 7, 663-692.*

[29] Obizhaeva, A. and Wang, J., (2013), "Optimal trading strategy and supply/demand dynamics," *Journal of Financial Markets,* **16**, *1-32.*

[30] Pardoux, E. and Rascanu, A., (2014), "Stochastic Differential Equations, Backward SDEs, Partial Differential Equations," *Springer International Publishing Switzerland.*

[31] Peng, S., (1992), "Stochastic Hamilton-Jacobi-Bellman equations," *SIAM J. Control and Optimization, Vol. 30, No. 2, pp. 284-304.*

[32] Roch, A., (2011), "Liquidity risk, price impacts and the replication problem," *Finance Stoch.* **15***, 3., 399-419.*

[33] Schied, A. & Schönborn, T., (2009), "Risk Aversion and the Dynamics of Optimal Liquidation Strategies in Illiquid Markets, " *Finance Stoch.* **13***, 181-204.*

[34] Shiraya, K. and Takahashi, A., (2014), "Price Impacts of Imperfect Collateralization," *CARF working paper, CARF-F-355.*

[35] Takahashi, A., (1999), "An Asymptotic Expansion Approach to Pricing Contingent Claims," *Asia-Pacific Financial Markets, 6, 115-151.*

[36] Tang, S., (2003), "General linear quadratic optimal stochastic control problems with random coefficients: Linear stochastic Hamilton systems and backward stochastic Riccati equations," *SIAM J. Control and Optimization, Vol 42., No. 1, pp. 53-75.*

[37] Tang, S., (2014), "Dynamic programming for general linear quadratic optimal stochastic control with random coefficients," *Working paper, arXiv:1407.5031.*

[38] Takahashi, A. and Yamada, T., (2013), "On an Asymptotic Expansion of Forward-Backward SDEs with a Perturbation Driver," CARF working paper series. CARF-F-326.

[39] Yoshida, N., (1992a), "Asymptotic Expansion for Statistics Related to Small Diffusions," *J. Japan Statist. Soc., Vol. 22, No. 2, 139-159.*

[40] Yoshida, N., (1992b), "Asymptotic Expansions of Maximum Likelihood Estimators for Small Diffusions via the Theory of Malliavin-Watanabe," *Probability Theory and Related Fields, 92, 275-311.*