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Random Matching Models and Money:
The Global Structure and Approximation of the Set of Stationary Equilibria

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Abstract

Random matching models with different states are an important class of dynamic games; for example, money search models, job search models, and some games in biology are special cases. In this paper, we investigate the basic structure of the models: the existence of equilibria, the global structure of the set of equilibria, and the approximation and computation of equilibria. Under conditions which are typically satisfied in monetary models, the equilibrium condition can be considered as a non-linear complementarity problem with some new feature.

Keywords: Random Matching Model, Money, Stationary Equilibria, Non-linear Comlementarity Problem.

Journal of Economic Literature Classification Number: C61, C62, C63, C72, C73, D51, E40.

1 Introduction

In this paper, we study the basic structure of random matching models with a finite number of states and a continuum of agents. Random matching models are an important class of dynamic games; for example, money search models, job search models, and some games in biology are special cases. In the models, each matched pair of agents play a game of which action spaces and payoffs depend on their states. The states follow a Markov process, i.e.,

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the probability distribution on states in the next period is determined by the current states and actions.

In spite of its importance, the structure of random matching models has not been fully explored in the literature. For example, no existence theorem in a general framework has been known. In this paper, we investigate the basic structure of the model: the existence of equilibria, the global structure of the set of equilibria, and the approximation and computation of equilibria.

The existence of equilibria is one of the most important questions in any economic model. In some specific random matching models, the existence of equilibria has been proved by directly calculating equilibria. (See, for example, Green and Zhou (1998) and Kiyotaki and Wright (1989).) However, to the best of our knowledge, no general existence theorem has been proved in the literature. In this paper, first we present an existence theorem in a general framework. The existence theorems in stochastic games, e.g., Duffie et al. (1994), cannot be applied to random matching models, since the number of players is typically finite in stochastic games.

Second, assuming some condition that is typically satisfied in random matching models with money, we show some remarkable features about the set of (monetary) equilibria. More precisely, under that condition, the set of equilibria is at least one-dimensional, and adding some transversality condition, it is a one-dimensional manifold with endpoints in which all agents are at the same state. Moreover, under these conditions, the equilibrium condition can be considered as a non-linear complementarity problem having one degree of freedom and with a new feature. The equilibrium condition can be written as follows:

$$\begin{aligned} f_n(x, y) &= 0, \quad n = 1, 2, \dots , N, \\ y_m g_m(x, y) &= 0, \quad m = 1, 2, \dots , M, \\ x_n &\geq 0, \quad n = 1, 2, \dots , N, \\ g_m(x, y) \geq 0, y_m &\geq 0, \quad m = 1, 2, \dots , M. \end{aligned}$$

The first system, the condition for stationary probability distribution, has one redundant equation and seems to be a standard system of equations, while the second and the fourth system, the condition for dynamic optimization, look as a standard complementarity problem. However, because of the random matching structure, the whole system can be seen as a complementarity problem. Suppose some x_n becomes zero when we follow a one-dimensional set of equilibria. Then, by the random matching structure, $f_n(x, y) = 0$ becomes an identity and precisely one positive y_m simultaneously becomes zero, and thus we can follow the set of equilibria further by setting $x_n = 0$ and $y_m = 0$ and relaxing $g_m(x, y) \geq 0$. In this way a connected component of equilibria can be obtained by finding an endpoint and following the one-dimensional manifold by some simplicial or predictor-corrector algorithm. (See, for example Allgower and Georg (1990).)

We also consider the case that the transversality condition does not necessarily hold.

In this case, a higher dimensional set of equilibria may exist. We present a method to follow the set of equilibria approximately by perturbing the system. As is well known, some perturbation of a system of equations can make the system regular. For example, adding constant terms to the system, the regularity directly follows from Sard theorem. However, in our case, such perturbation does not work and some sophisticated perturbation is needed. By the perturbation, the set of approximated equilibria becomes a one-dimensional manifold and one of its connected components can be followed by finding an endpoint and using some path-following algorithm.

The paper is organized as follows. In Section 2, we describe the basic model and present an existence theorem. In Section 3, we present an example of the random matching model with money, a special case of the basic model. In Section 4, assuming some transversality condition and some special condition, being typically satisfied in monetary model, we completely characterize the global structure of the set of equilibria. Finally, in Section 5, we consider the case that the transversality condition is not satisfied and approximate the set of equilibria by perturbing the system.

2 The Basic Model and the Existence of Stationary Markov Perfect Equilibria

Time is discrete starting from 0. There are a finite number of states denoted by $n = 0, 1, \ldots, N$. Let $\mathcal{N} = \{0, 1, \ldots, N\}$ denote the set of states. We assume that there are infinitely lived agents with a nonatomic mass of measure one. A probability measure on $\mathcal{N}$ is denoted by a vector vector $h = (h(0), h(1), \ldots, h(N))$ with all $h(n) \geq 0$ and $\sum_{n=0}^{N} h(n) = 1$, where $h(n)$ is the proportion of agents at state n.

An agent at state n chooses an action in the set $A^n = \{a_1^n, \ldots, a_{k_n}^n\}$. The number $K = \sum_{n=0}^{N} k_n$ denotes the total number of actions. Let $\beta_{nj} \geq 0$ satisfying $\sum_{j=1}^{k_n} \beta_{nj} = 1$ be the proportion of the agents choosing action a_j^n among the agents at state n, and let $\beta = (\beta_{01}, \ldots, \beta_{nj}, \ldots, \beta_{Nk_N})$. The number $h(n, j) = \beta_{nj} h(n)$ will denote the proportion of agents choosing action a_j^n.

At each time period, ordered pairs of agents are matched. Suppose an (n, j) agent, who is at state n and chooses action a_j^n, is matched to an (n', j') agent, who is at state n' and chooses action $a_{j'}^{n'}$. Then in the next period the first agent's and the second agent's states will be $f_1((n, j), (n', j'))$ and $f_2((n, j), (n', j'))$, respectively. That is both f_1 and f_2 map an ordered pair $((n, j), (n', j'))$ to elements in $\mathcal{N}$.

Example 1 When one agent chooses an action and the other agent chooses an action after observing the first agent's action, we may order the pair $((n, j), (n', j'))$ in such a way that the (n, j) agent chooses first and the (n', j') chooses second.

Example 2 A special case occurs when it holds that

$$f_1((n,j),(n',j')) = f_2((n',j'),(n,j)) \quad \text{for all} \quad (n,j) \quad \text{and} \quad (n',j').$$

In these models there is no ordering in the pair of agents being matched.

We assume a random matching structure; the proportion of matching between agents with (n,j) and (n',j') is equal to $\mu h(n,j)h(n',j')$, where $\mu \in (0,1]$ is exogenously given, and with probability $\frac{1}{2}$ one of the two agents becomes the first agent and the other one becomes the second agent in the ordering. Therefore, in each period a proportion $\frac{1}{2}\mu h(n,j)h(n',j')$ of agents move from state n to state $f_1((n,j),(n',j'))$ and from state n' to state $f_2((n,j),(n',j'))$.

Let $u_1((n,j),(n',j'))$ and $u_2((n,j),(n',j'))$ be the one-period utilities of an (n,j) agent and an (n',j') agent, respectively, when the first agent is matched to the second agent. Let $\alpha \in \mathbb{R}_+$ be the discount factor and let $\gamma \in \mathbb{R}^L$ be the vector of parameters of the model including α and μ.

We adopt a Bellman equation approach. Let $V(n)$ be the value of state n, $n = 0,1,\ldots,N$, and let $V = (V(0),V(1),\ldots,V(N))$. The variables in the model are denoted by $x = (V,h,\beta)$. Let

$$B_n^1 = \{((i,j),(i',j')) \mid f_1((i,j),(i',j')) = n\},$$
$$B_n^2 = \{((i,j),(i',j')) \mid f_2((i,j),(i',j')) = n\}.$$

Then the value of action $a_j^n \in A^n$ at state $n = 0,1,\ldots,N$ is defined as:

$$W_{nj}(x;\gamma) = U((n,j),h,\beta;\gamma) + \alpha \sum_{n'=0}^{N} q_{n'}((n,j),h,\beta;\mu)V(n'),$$

where

$$U((n,j),h,\beta;\gamma) = \frac{1}{2}\sum_{(n',j')} h(n',j')\left(u_1((n,j),(n',j')) + u_2((n',j'),(n,j))\right),$$

the one-period expected utility of an (n,j) agent, and

$$q_{n'}((n,j),h,\beta;\mu) = \sum_{\{(n'',j'') \mid ((n,j),(n'',j''))\in B^1_{n'}\}} \frac{1}{2}\mu h(n'',j'') + \sum_{\{(n'',j'') \mid ((n'',j''),(n,j))\in B^2_{n'}\}} \frac{1}{2}\mu h(n'',j''),$$

the probability of state n' in the next period conditional on $((n,j),h,\beta)$. We will define stationary Markov perfect equilibria in our framework. First, by dynamic optimization, the following conditions should be satisfied:

1. $\sum_{j=1}^{k_n} \beta_{nj} = 1, \quad n = 0,1,\ldots,N,$

2. $V(n) = \max_j W_{nj}(x;\gamma),\ \ n = 0, 1, \ldots, N,$

3. $\beta_{nj} \geq 0$ for all (n, j), and $\beta_{nj'} > 0$ implies $j' \in \arg\max_j W_{nj}(x;\gamma),\ \ n = 0, 1, \ldots, N.$

The second condition is the Bellman equation and the third condition means that only the best responses are used in an equilibrium. The above conditions are equivalent to:

$$\begin{aligned}
\beta_{nj} \geq 0, j = 1, \ldots, k_n, \quad \sum_{j=1}^{k_n} \beta_{nj} - 1 = 0, \quad n = 0, 1, \ldots, N, \\
V(n) - W_{nj}(x;\gamma) = 0, \ \text{if} \ \ \beta_{nj} > 0, n = 0, 1, \ldots, N, \\
V(n) - W_{nj}(x;\gamma) \geq 0, \ \text{if} \ \ \beta_{nj} = 0, n = 0, 1, \ldots, N.
\end{aligned}$$

Indeed, since $\sum_{j=1}^{k_n} \beta_{nj} - 1 = 0$, there exists a j' such that $\beta_{nj'} > 0$ and thus for all such j' it holds that $V(n) = W_{nj'}(x;\gamma)$ and $W_{nj'}(x;\gamma) \geq W_{nj}(x;\gamma)$ for all j. This implies the above three conditions. The converse clearly holds.

The (gross) outflow $O_n(h, \beta;\gamma)$ and the (gross) inflow $I_n(h, \beta;\gamma)$ at state n are given by

$$O_n(h, \beta;\gamma) = \textstyle\sum_{j,i',j'} \frac{1}{2}\mu h(n, j)h(i', j') + \sum_{i,j,j'} \frac{1}{2}\mu h(i, j)h(n, j'),$$

and

$$\begin{aligned}
I_n(h, \beta;\gamma) = \ \ & \textstyle\sum_{((i,j),(i',j')) \in B_n^1} \frac{1}{2}\mu h(i, j)h(i', j') \\
& + \textstyle\sum_{((i,j),(i',j')) \in B_n^2} \frac{1}{2}\mu h(i, j)h(i', j').
\end{aligned}$$

Note that "gross" means that the cases of $f_1 = n$ and $f_2 = n$ are included in the definition. Needless to say, such terms appear both in O_n and in I_n. Thus the condition for stationarity is

$$\begin{aligned}
O_0(h, \beta;\gamma) &= I_0(h, \beta;\gamma) \\
O_1(h, \beta;\gamma) &= I_1(h, \beta;\gamma) \\
&\vdots \\
O_N(h, \beta;\gamma) &= I_N(h, \beta;\gamma) \\
\sum_{n=0}^{N} h(n) &= 1.
\end{aligned}$$

Since any term in $O_n(h, \beta;\gamma)$ should be also in $I_{n'}(h, \beta;\gamma)$ for some n',

$$\sum_{n=0}^{N} O_n(h, \beta;\gamma) = \sum_{n=0}^{N} I_n(h, \beta;\gamma)$$

is an identity and thus $O_0(h, \beta;\gamma) = I_0(h, \beta;\gamma)$ is redundant.

Definition 1 The tuple $x^* = (V^*, h^*, \beta^*)$, with $h^*(n) \geq 0$ for $n = 0, 1, \ldots, N$, and $\beta^*_{nj} \geq 0$ for all (n, j), is a stationary Markov perfect equilibrium if

$$\begin{aligned} O_n(h^*, \beta^*; \gamma) - I_n(h^*, \beta^*; \gamma) = 0, \;\; n = 1, 2, \ldots, N, \\ \sum_{n=0}^{N} h^*(n) - 1 = 0, \\ \sum_{j=1}^{k_n} \beta^*_{nj} - 1 = 0, \;\; n = 0, 1, \ldots, N, \\ V^*(n) - W_{nj}(x^*; \gamma) = 0, \;\; \text{if} \;\; \beta^*_{nj} > 0, \\ V^*(n) - W_{nj}(x^*; \gamma) \geq 0, \;\; \text{if} \;\; \beta^*_{nj} = 0. \end{aligned}$$

Below, we present an existence theorem under the following assumption.

Assumption 1 For the discount factor α it holds that $\alpha \in (0, 1)$.

Theorem 1 Under Assumption 1, there exists a stationary Markov perfect equilibrium.

Proof:
For given γ, $V(n)$, $n = 0, 1, \ldots, N$, can be expressed as functions of (h, β) by solving the following system of linear equations:

$$V(n) = \sum_{j=1}^{k_n} \beta_{nj} \left(U((n, j), h, \beta; \gamma) + \alpha \sum_{n'=0}^{N} q_{n'}((n, j), h, \beta; \mu) V(n') \right), \quad n = 0, 1, \ldots, N.$$

Clearly, by Assumption 1 the system is regular. Thus these functions are continuous functions of (h, β) and will be denoted by $\tilde{V}_n(h, \beta; \gamma)$, $n = 0, 1, \ldots, N$.[1] Also $W_{nj}(x; \gamma)$ can be expressed as a continuous function of (h, β), to be denoted by $\tilde{W}_{nj}(h, \beta; \gamma), n = 0, 1, \ldots, N, j = 1, \ldots, k_n$.

For given γ, let $g : S^{N+1} \times \Pi_{n=0}^{N} S^{k_n} \to \mathbb{R}^{N+1} \times \Pi_{n=0}^{N} \mathbb{R}^{k_n}$ be defined by

$$g(h, \beta) = \begin{pmatrix} g^1(h, \beta) \\ (g^2_n(h, \beta))_{n=0}^{N} \end{pmatrix} = \begin{pmatrix} (-O_n(h, \beta; \gamma) + I_n(h, \beta; \gamma))_{n=0}^{N} \\ (\tilde{V}_n(h, \beta; \gamma) e^{k_n} - \tilde{W}_n(h, \beta; \gamma))_{n=0}^{N} \end{pmatrix},$$

where $S^m = \{x \in \mathbb{R}^m_+ | \sum_i x_i = 1\}$ and e^m is the m-vector of ones.

Since the domain of g is a non-empty, convex, compact set and g itself is a continuous function of (h, β), there exists a stationary point (h^*, β^*) of g, i.e.,

$$\begin{aligned} h^\top g^1(h^*, \beta^*) \leq h^{*\top} g^1(h^*, \beta^*), &\quad \text{for all} \;\; h \in S^{N+1}, \\ \beta_n^\top g^2_n(h^*, \beta^*) \leq \beta_n^{*\top} g^2_n(h^*, \beta^*), &\quad \text{for all} \;\; \beta_n \in S^{k_n}, n = 0, 1, \ldots, N. \end{aligned}$$

[1]Note that, since $\alpha \in (0, 1)$, $\tilde{V}_n(h, \beta; \gamma), n = 0, 1, \ldots, N$, can be obtained by iteratively substituting $V(n), n = 0, 1, \ldots, N$, in the above system.

Take $d(t) = (1-t)h^* + te(n)$, where $e(n)$ denotes an n-th unit vector of appropriate length. If $h^*(n) > 0$, then $d(t) \in S^{N+1}$ for any t close to zero. Hence, if $h^*(n) > 0$,

$$g_n^1(h^*, \beta^*) = h^{*\top} g^1(h^*, \beta^*).$$

If $h^*(n) = 0$ then $d(t) \in S^{N+1}$ for small positive t. Hence, since $I_n(h^*, \beta^*; \gamma)) \geq 0 = O_n(h^*, \beta^*; \gamma)$ when $h^*(n) = 0$, we obtain that if $h^*(n) = 0$,

$$0 \leq g_n^1(h^*, \beta^*) \leq h^{*\top} g^1(h^*, \beta^*).$$

Since $\sum_{n=0}^{N} g_n^1(h^*, \beta^*) = 0$, we obtain $g_n^1(h^*, \beta^*) = 0$ for all $n = 0, 1, \ldots, N$, and so $O_n(h^*, \beta^*; \gamma) = I_n(h^*, \beta^*; \gamma)$ for all $n = 0, 1, \ldots, N$.

Next take $d(t) = (1-t)\beta_n^* + te(j)$. If $\beta_{nj}^* > 0$, then $d(t) \in S^{k_n}$ for any t close to zero. Hence, if $\beta_{nj}^* > 0$, then

$$g_{nj}^2(h^*, \beta^*) = \beta_n^{*\top} g_n^2(h^*, \beta^*),$$

and so

$$\tilde{W}_{nj}(h^*, \beta^*; \gamma) = \beta_n^{*\top} \tilde{W}_n(h^*, \beta^*; \gamma).$$

If $\beta_{nj}^* = 0$, then $d(t) \in S^{k_n}$ for small positive t. Hence, if $\beta_{nj}^* = 0$, then

$$\tilde{W}_{nj}(h^*, \beta^*; \gamma) \leq \beta_n^{*\top} \tilde{W}_n(h^*, \beta^*; \gamma).$$

Therefore

$$\tilde{W}_{nj}(h^*, \beta^*; \gamma) = \max_{j'} \tilde{W}_{nj'}(h^*, \beta^*; \gamma) \quad \text{if} \quad \beta_{nj}^* > 0.$$

Consequently, for all $n = 0, 1, \ldots, N$,

$$W_{nj}(x^*; \gamma) = V^*(n) \quad \text{if} \quad \beta_{nj}^* > 0$$

and

$$W_{nj}(x^*; \gamma) \leq V^*(n) \quad \text{if} \quad \beta_{nj}^* = 0.$$

∎

The theorem states that if the discount factor α lies between 0 and 1, there exists at least one stationary Markov perfect equilibrium.

Now we investigate the case that, for each matched pair, the sum of their states does not change. That is we assume the following.

Assumption 2 For all (n, j) and (n', j'),

$$f_1((n,j),(n',j')) + f_2((n,j),(n',j')) = n + n'.$$

This assumption is satisfied in random matching models with money. In such models, n stands for the amount of money an agent has, i.e. any agent at state n has n units of money. For a matched pair $((n,j),(n',j'))$, an (n,j) agent is a (potential) seller and an (n',j') agent is a (potential) buyer, and a possible transaction is made using money. Since the amount of money the seller pays is the same as the buyer gets, the total amount of money does not change. For details, see the next sections and Green and Zhou (1988), Kamiya and Shimizu (2002), and Zhou (1999).

In this case, the following lemma holds.[2]

Lemma 1 Under Assumption 2,

$$\sum_{n=0}^{N} nO_n(h,\beta;\gamma) = \sum_{n=0}^{N} nI_n(h,\beta;\gamma). \tag{1}$$

Proof:

Consider an ordered pair $((n,j),(n',j'))$. By the matchings between them, a proportion $\frac{1}{2}\mu h(n,j)h(n',j')$ of agents move from n to $f_1((n,j),(n',j'))$, and the same proportion moves from n' to $f_2((n,j),(n',j'))$. Corresponding to the moves, the following terms appear in the left hand side (LHS) and in the right hand side (RHS) of (1):

the LHS	the RHS
$\frac{1}{2}n\mu h(n,j)h(n',j')$	$\frac{1}{2}f_1((n,j),(n',j')))\mu h(n,j)h(n',j')$
$\frac{1}{2}n'\mu h(n,j)h(n',j')$	$\frac{1}{2}f_2((n,j),(n',j')))\mu h(n,j)h(n',j')$

Because of Assumption 2 the sum of the terms in the LHS is equal to the one in the RHS. Since this holds for every pair $((n,j),(n',j'))$, formula (1) holds. ∎

From the lemma it follows that

$$\sum_{n=0}^{N} n(O_n(h,\beta;\gamma) - I_n(h,\beta;\gamma)) = 0. \tag{2}$$

Since also $\sum_{n=0}^{N}(O_n(h,\beta;\gamma) - I_n(h,\beta;\gamma)) = 0$, without loss of generality, we can first delete $O_0(h,\beta;\gamma) - I_0(h,\beta;\gamma) = 0$ and then, by (2), we can delete $O_1(h,\beta;\gamma) - I_1(h,\beta;\gamma) = 0$. Thus the distribution h is stationary if and only if $O_n(h,\beta;\gamma) - I_n(h,\beta;\gamma) = 0$, $n = 2,\ldots,N$,

[2] Kamiya and Shimizu (2002) presented a similar theorem.

and $\sum_{n=0}^{N} h(n) - 1 = 0$ hold. This means that the tuple $x^* = (V^*, h^*, \beta^*)$ is a stationary Markov perfect equilibrium if

$$
\begin{aligned}
O_n(h^*, \beta^*; \gamma) - I_n(h^*, \beta^*; \gamma) = 0, \;\; n = 2, \dots, N, \\
\sum_{n=0}^{N} h^*(n) - 1 = 0, \\
\sum_{j=1}^{k_n} \beta^*_{nj} - 1 = 0, \;\; n = 0, 1, \dots, N, \\
V^*(n) - W_{nj}(x^*; \gamma) = 0, \;\; \text{if } \; \beta^*_{nj} > 0, \\
V^*(n) - W_{nj}(x^*; \gamma) \geq 0, \;\; \text{if } \; \beta^*_{nj} = 0.
\end{aligned}
$$

Suppose the inequalities are strict and for n, $n = 0, \dots, N$, let A^*_n be the set of actions at state n being chosen in the equilibrium with positive probability. Then the total number of variables, $V(n)$, $h(n)$, and $\beta_{nj}, j \in A^*_n$, $n = 0, 1, \dots, N$, is $(N+1) + (N+1) + \sum_{n=0}^{N} \#A^*_n$, and the number of equalities is $(N-1) + 1 + (N+1) + \sum_{n=0}^{N} \#A^*_n$. Thus the dimension of the set of equilibria is typically at least equal to one. Moreover, if some transversality conditions are satisfied, the dimension is equal to one. In Section 4, we show this rigorously.

3 An Example

In this section, we present a discrete time version of Zhou (1999)'s model with an exogenously given upper bound of money holdings and we completely characterize some of the sets of stationary Markov perfect equilibria.

3.1 A discrete time version of Zhou's model

There are $k \geq 3$ types of agents with equal fraction and the same number of types of goods. Only one unit of good i can be produced and held by a type $i-1$ (mod. k) agent. The production cost is $c > 0$. A type i agent obtains utility $u > 0$ only when she consumes one unit of good i. If a type $i-1$ agent is matched to a type i agent, then the first agent is a (potential) seller and the second agent is a (potential) buyer. The first agent refuses to trade or posts a take-it-or-leave-it price offer and the second agent can accept the offer or not. For example, when $k = 3$ a type 1 agent produces one unit of good 2 being desired by a type 2 agent, a type 2 agent produces one unit of good 3 being desired by a type 3 agent, and a type 3 agent produces one unit of good 1 being desired by a type 1 agent. Moreover, if a type 1 agent meets a type 2 agent, then the former becomes a (potential) seller and the latter one a (potential) buyer. Similar arguments apply to the cases of a matching of a type 2 agent and a type 3 agent, and of a matching of a type 3 agent and a type 1 agent.

Fiat money is assumed to be divisible. We confine our attention to the case that, for a given $p > 0$, the support of money holding distribution is $\{0, p, 2p, \ldots, Np\}$, where N is exogenously given. The number p is determined by $\sum_{n=0}^{N} pnh(n) = M$, where $M > 0$ is an exogenously given supply of money. Thus, without abuse of notation, a distribution h on $\mathcal{N}$ is identified with a distribution on $\{0, p, 2p, \ldots, Np\}$. We will focus on symmetric stationary Markov perfect equilibria for which the strategies that agents with an identical money holding take are time-invariant and type-invariant. Therefore, we will hereafter discuss a generic type i. A strategy of an agent (of any type) is defined as a set of two correspondences, an offer strategy $\omega : \mathcal{N} \to \mathcal{N} \cup \{\mathrm{NT}\}$ and a reservation price strategy $\rho : \mathcal{N} \to \mathcal{N}$, where NT stands for no trade. For $n \in \mathcal{N}$, $\omega(n)$ is a set of prices and/or no trade that an agent with money holding np offers when she is a (potential) seller. More precisely, if $o \in \omega(n)$ and $o \neq \mathrm{NT}$, then op is an offer price, and if $o \in \omega(n)$ and $o = \mathrm{NT}$, then she does not sell, no matter what the buyer's reservation price is. A seller with money holding n offers one of the elements in $\omega(n)$. Since for any agent his money holding cannot exceed the amount N, the offer price is at most equal to $(N - n)p$. For $n \in \mathcal{N}$, $r \in \rho(n)$ is a reservation price, below which offers are accepted and above which they are rejected. In fact, we will show that, by the perfectness condition, $\rho(n)$ gives the maximum price that a buyer is willing to defray for the consumption good, and so ρ becomes a function rather than a correspondence. Since the reservation price of a buyer cannot exceed her money holdings, ρ should satisfy the following feasibility condition:

$$\rho(n) \leq n, \;\; n = 0, 1, \ldots, N. \tag{3}$$

For a money holding np, an offer price op, and a reservation price rp, $H(n, o, r)$ denotes a stationary distribution defined on $\mathcal{N} \times (\mathcal{N} \cup \{\mathrm{NT}\}) \times \mathcal{N}$. From H, the stationary distribution of offer prices, Ω, and the stationary distribution of reservation prices, R, are given by

$$\Omega(x) = H(\{(n, o, r) | o \neq \mathrm{NT}, o \leq x\}) \tag{4}$$
$$R(x) = H(\{(n, o, r) | r < x\}). \tag{5}$$

Let $V : \mathcal{N} \to \mathbb{R}$ be the value function. That is, $V(n)$ is the maximum value of discounted utility achievable by the agent's current money holding np. At every moment, a type i agent with money holding np is a (potential) buyer when meeting a type $i-1$ agent, which occurs with probability $1/k$. Transaction does not occur and money holding does not change if the seller chooses NT or her offer op exceeds the type i's reservation price rp. If the partner's offer price op is not more than reservation price rp, then transaction occurs and the type i agent derives utility u from consumption and enters in the next trading opportunity with money holding $(n - o)p$. The probability that type i with money holding np is a seller and meets a type $i + 1$ agent is also $1/k$. Transaction does not occur if the type i chooses NT or her offer op is greater than the partner's reservation price rp. If type i's offer op does not exceed rp, then transaction occurs and she faces the next matching opportunity with money holding $(n+o)p$. Then, using α, Ω, and R, the Bellman equation is given by

$$V(n) = \frac{1}{k} \max_{r \in \{0,1,\dots,n\}} \left\{ \sum_{x=0}^{r} (u + \alpha V(n-x))(\Omega(x) - \Omega(x-1)) + (1 - \Omega(r))\alpha V(n) \right\}$$
$$+ \frac{1}{k} \max_{o \in \{0,1,\dots,N-n\} \cup \{\mathrm{NT}\}} \left\{ R(o)\alpha V(n) + (1 - R(o))(-c + \alpha V(n+o)) \right\}$$
$$+ (1 - \frac{2}{k})\alpha V(n), n = 0, 1, \dots, N, \tag{6}$$

where $\Omega(-1) = 0$. Thus

$$V(n) = \frac{1}{(1-\alpha)k + 2\alpha} \left(\max_{r \in \{0,1,\dots,n\}} \left\{ \sum_{x=0}^{r} (u + \alpha V(n-x))(\Omega(x) - \Omega(x-1)) + (1 - \Omega(r))\alpha V(n) \right\} \right.$$
$$\left. + \max_{o \in \{0,1,\dots,N-n\} \cup \{\mathrm{NT}\}} \left\{ R(o)\alpha V(n) + (1 - R(o))(-c + \alpha V(n+o)) \right\} \right),$$
$$n = 0, 1, \dots, N. \tag{7}$$

In terms of $V(n)$, it is optimal for a buyer to accept offer op if $u + \alpha V(n-o) \geq \alpha V(n)$. Then the perfectness condition with respect to the reservation price is as follows:

$$\rho(n) = \max\{r \in \{0, \dots, n\} \,|\, u + \alpha V(n-r) \geq \alpha V(n)\}. \tag{8}$$

That is, agent of type i's reservation price is her full value for good i, and thus it is a function of n.

From the above, the action space and the matching technology can be written as follows:

- $A^n = \{a^n_{(o,r)} | o = 0, 1, \dots, N-n, \mathrm{NT}, \text{ and } r = 0, 1, \dots, n\}$. An action $a^n_{(o,r)}$ means that an agent with n offers op or no trade NT when she is a seller, and she accepts the partner's offer only if the offer price is less than or equal to rp when she is a buyer.

- f_1 and f_2 are given by

$$f_1((n,(o,r)),(n',(o',r'))) = \begin{cases} n + o & \text{if } o \neq \mathrm{NT}, \text{ and } o \leq r' \\ n & \text{otherwise} \end{cases}$$

$$f_2((n,(o,r)),(n',(o',r'))) = \begin{cases} n' - o & \text{if } o \neq \mathrm{NT}, \text{ and } o \leq r' \\ n' & \text{otherwise.} \end{cases}$$

- The proportion of matching between agents with (n, j) and (n', j') is equal to $\frac{2}{k}h(n,j)h(n',j')$, and with probability $\frac{1}{2}$ one of the two agents becomes the first agent and the other one becomes the second agent.

3.2 The Case of $N = 1$

We completely characterize the set of equilibria in the case of $N = 1$, $k = 3$, $u = \frac{15}{4}, c = \frac{3}{4}$, and $\alpha = \frac{3}{4}$. Note that in what follows an agent with np units of money is called an agent with n. The set of equilibria consists of the following two pieces (see Figure 1):

1. The strategy of the first piece is

   - agents with 0 offer p and agents with 1 accept any price less than or equal to p.

   Thus the conditions for a stationary Markov perfect equilibrium are

$$\begin{aligned} h(0) + h(1) - 1 &= 0 \\ V(0) - (\frac{9}{4})^{-1} \left(h(1)(-\frac{3}{4} + \frac{3}{4}V(1)) + \frac{3}{4}h(0)V(0) + \frac{3}{4}V(0) \right) &= 0 \\ V(1) - (\frac{9}{4})^{-1} \left(\frac{3}{4}V(1) + h(0)(\frac{15}{4} + \frac{3}{4}V(0)) + \frac{3}{4}h(1)V(1) \right) &= 0 \\ h(1)(-\frac{3}{4} + \frac{3}{4}V(1)) + \frac{3}{4}h(0)V(0) - V(0) &\geq 0 \\ h(0)(\frac{15}{4} + \frac{3}{4}V(0)) + \frac{3}{4}h(1)V(1) - V(1) &\geq 0, \end{aligned}$$

   where the last two inequalities are the incentive constraints for agents with 0 and with 1, respectively. From the first three equations in the above system, $V(0)$ and $V(1)$ can be solved as a function of $h(0)$ as follows:

$$\begin{aligned} V(0) &= -\frac{1}{2} + \frac{5}{2}h(0) - 2h(0)^2 \\ V(1) &= \frac{9}{2}h(0) - 2h(0)^2. \end{aligned}$$

   At $h(0) = 1$, all incentives are satisfied. When $h(0)$ is decreased, the incentive of agents without money becomes binding first at $h(0) = \frac{1}{4}$, because $h(1)\left(-\frac{3}{4} + \frac{3}{4}V(1)\right) + \frac{3}{4}h(0)V(0) = V(0)$ holds for $h(0) = \frac{1}{4}$.

2. The strategy of the second piece is

   - agents with 0 are indifferent between offering p and NT, i.e., a proportion $a \in [0, 1]$ of agents with 0 offers p and a proportion $1 - a$ chooses NT, and agents with 1 accept any offer less than or equal to p.

Thus the conditions for a stationary Markov perfect equilibrium are

$$\begin{aligned}
h(0)+h(1)-1&=0\\
V(0)-(\frac{9}{4})^{-1}\left(h(1)(-\frac{3}{4}+\frac{3}{4}V(1))+\frac{3}{4}h(0)V(0)+\frac{3}{4}V(0)\right)&=0\\
\frac{3}{4}V(0)-h(1)(-\frac{3}{4}+\frac{3}{4}V(1))+\frac{3}{4}h(0)V(0)&=0\\
V(1)-(\frac{9}{4})^{-1}\left(\frac{3}{4}V(1)+ah(0)(\frac{15}{4}+\frac{3}{4}V(0))+(h(1)+ah(0))\frac{3}{4}V(0)\right)&=0\\
h(0)(\frac{15}{4}+\frac{3}{4}V(0))+\frac{3}{4}h(1)V(1)-V(1)&\geq 0,
\end{aligned}$$

where the last inequality is the incentive constraint for the agents with 1. From the first four equations in the above system, $V(0)$, $V(1)$, and a can be solved as a function of $h(0)$ as follows:

$$V(0)=0, V(1)=1, a=\frac{1}{4h(0)}. \tag{9}$$

At $h(0)=\frac{1}{4}$, the endpoint of the first piece, all incentive constraints for the agents with 0 are satisfied. When $h(0)$ is increased, all incentive constraints for the agents with 0 remain satisfied until $h(0)=1$. That is the solution path reaches an endpoint.

In this example the set of stationary Markov perfect equilibria consists of one one-dimensional manifold and both endpoints satisfy $h(0)=1$.

3.3 The Case of $N=2$

Next, we consider the case $N=2$, $k=3$, $u=\frac{15}{4}, c=\frac{3}{4}$, and $\alpha=\frac{3}{4}$. In this case, there are two connected components of solutions, but we only investigate one of them. The connected set consists of six pieces as follows. The other component looks quite similar. For simplicity, we do not present the Bellman equations.

The first piece:

- the strategies: agents with 0 offer p, agents with 1 offer p and their reservation price is p, and the reservation price of agents with 2 is $2p$.
- the starting point: $(h(0),h(1),h(2))=(1,0,0)$.
- the endpoint: $(h(0),h(1),h(2))=(0.62321,0.26452,0.11227)$.

- at the endpoint, agents with 1 become indifferent between offering p and NT.

The second piece:

- the strategies: agents with 0 offer p, agents with 1 are indifferent between offering p and NT and their reservation price is p, and the reservation price of agents with 2 is $2p$.
- the starting point: the endpoint of the first piece.
- the endpoint: $(h(0), h(1), h(2)) = (0.84307, 0.15693, 0)$.
- at the endpoint, $h(2)$ becomes 0 and the proportion of agents with 1 offering 1 becomes 0.

The third piece:

- the strategies: agents with 0 offer p, agents with 1 choose NT and their reservation price is p, and the reservation price of agents with 2 is $2p$.
- the starting point: the endpoint of the second piece.
- the endpoint: $(h(0), h(1), h(2)) = (0.25, 0.75, 0)$.
- at the endpoint, agents with 0 become indifferent between offering 1, 2, and NT.

At the endpoint of the third piece, we should analyze the case that three actions are used. However, we first analyze the case that agents with 0 offer p or $2p$.

The forth piece:

- the strategies: agents with 0 offer p or $2p$, agents with 1 choose NT and their reservation price is p, and the reservation price of agents with 2 is $2p$.
- the starting point: the endpoint of the third piece.
- the endpoint: $(h(0), h(1), h(2)) = (0.58333, 0.41667, 0)$.
- at the endpoint, agents with 1 become indifferent between offering p and NT.

Next, we investigate the general case at the endpoint of the third piece. There exists the following two-dimensional solution set.

Piece A:

- the strategies: agents with 0 choose p, $2p$, or NT, agents with 1 choose NT and their reservation price is p, and the reservation price of agents with 2 is $2p$.
- the piece has four extreme points: two endpoints of the fourth piece, one extreme point at $h = (1, 0, 0)$, where the proportions of offering p and $2p$ are $\frac{1}{4}$ and 0, respectively, and an other extreme point at $h = (1, 0, 0)$, where the proportions of offering p and $2p$ are $\frac{1}{4}$ and $\frac{1}{3}$, respectively. (See Figure 2.)

Remark 1 In piece A, although there are three best actions for agents with 0, one indifference condition is redundant; namely, the indifference between offering $2p$ and NT is written as

$$\frac{3}{4}V(0) = h(2)\left(\frac{3}{4}V(2) - \frac{3}{4}\right) + (h(0) + h(1))\frac{3}{4}V(0).$$

However, by $h(2) = 0$ and $h(0) + h(1) + h(2) = 1$, the above equation is in fact an identity. Thus there are two free variables in the system of equations and the dimension of the set of equilibria is two.

The fifth piece:

- the strategies: agents with 0 offer p or $2p$, agents with 1 offer p or NT and their reservation price is p, and the reservation price of agents with 2 is $2p$.
- the starting point: the endpoint of the forth piece.
- the endpoint: $(h(0), h(1), h(2)) = (0.61193, 0.23247, 0.15226)$.
- at the endpoint, the proportion of agents with 1 choosing NT becomes 0.

The sixth piece:

- the strategies: agents with 0 offer p or $2p$, agents with 1 offer p and their reservation price is p, and the reservation price of agents with 2 is $2p$.
- the starting point: the endpoint of the fifth piece.

- the endpoint: $(h(0), h(1), h(2)) = (1, 0, 0)$.
- at the endpoint, the path has returned to $h(0) = 1$.

In this example the connected sets of stationary Markov perfect equilibria are not one-dimensional. We showed the existence of a two-dimensional piece on one of the sets of equilibria. However, if certain transversality conditions are satisfied, the dimension of each piece is one, as we will show in the next section.

4 The Random Matching Model with Conservation Law

In this section we discuss conditions under which the dimension of stationary Markov perfect equilibria is equal to one. Let B denote the power set of $\{(n, j) \mid j = 1, \dots, k_n, n = 0, 1, \dots, N\}$ and let $\hat{B} = \{b \in B \mid \forall n, \exists (n, j) \in b\}$. An element $b \in \hat{B}$ can be considered to be a set of actions potentially used in an equilibrium. For given $b \in \hat{B}$, let

$$\Omega^b = \{(\beta_{nj})_{(n,j)\in b} \mid \beta_{nj} > 0 \quad \text{for all} \quad (n, j) \in b\}.$$

Let $x^b = (V, h, \beta^b)$, for some $\beta^b \in \Omega^b$. For given $b \in \hat{B}$ and $(n, j) \in b$, $W^b_{nj}(x^b; \gamma)$ is defined from $W_{nj}(x; \gamma)$ by setting $\beta_{n'j'} = 0$ for all $(n', j') \notin b$. In parallel with this, $I^b_n(h, \beta^b; \gamma)$ and $O^b_n(h, \beta^b; \gamma)$ are defined.

Let

$$\begin{aligned} \Gamma = \ & \{(b, J) \in \hat{B} \times 2^{\mathcal{N}} \mid h(n) = 0, \ n \notin J, \text{ and } h(n) > 0, \ n \in J, \\ & \text{imply} \ \ I^b_n(h, \beta^b; \gamma) = 0 \text{ for all } n \notin J, \beta^b \in \Omega^b\}. \end{aligned}$$

Note that if $I^b_n(h, \beta^b; \gamma) = 0$ for some (h, β^b) such that $h(n) > 0$ for $n \in J$ and $h(n) = 0$ for $n \notin J$, then by the random matching structure $I^b_n(h, \beta^b; \gamma) = 0$ holds for all (h, β^b) such that $h(n) > 0$ for $n \in J$ and $h(n) = 0$ for $n \notin J$. Notice that every stationary Markov perfect equilibrium corresponds to some $(b, J) \in \Gamma$ and therefore we only need to consider elements in Γ. For every $(b, J) \in \Gamma$ it holds that $O^b_n - I^b_n = 0$, $n \notin J$, is an identity. Therefore, these identities can be deleted from the system of equations. Note that if $J = \{n'\}$, i.e., a singleton, then $\sum_{n=0}^{N} h(n) - 1 = 0$ can also be deleted. Let $J = \{j_1, \dots, j_m\}$, where $j_1 < \cdots < j_m$.

For $(b, J) \in \Gamma$, let

$$g^{(b,J)}(x^b; \gamma) = \begin{cases} O^b_n(h, \beta^b; \gamma) - I^b_n(h, \beta^b; \gamma) & n \in J \setminus \{j_1, j_2\}, \\ \sum_{n=0}^{N} h(n) - 1 & \text{if } \#J \neq 1 \\ \sum_{\{j \mid (n,j)\in b\}} \beta^b_{nj} - 1 & n = 0, 1, \dots, N, \\ V(n) - W^b_{nj}(x^b; \gamma) & (n, j) \in b, \\ V(n) - W^b_{nj}(x^b; \gamma) & (n, j) \notin b, \end{cases}$$

where $J \setminus \{j_1, j_2\} = \emptyset$, if J is a singleton.

Let

$$E_\gamma^{(b,J)} = \{x^b \mid h(n) > 0, n \in J, h(n) = 0, n \notin J, g^{(b,J)}(x^b; \gamma) \in \underbrace{\{0\} \times \cdots \times \{0\}}_{N+m+\#b} \times \underbrace{\mathbb{R}_+ \times \cdots \times \mathbb{R}_+}_{K-\#b}\},$$

be the set of stationary Markov perfect equilibria for given $(b, J) \in \Gamma$. Moreover, for $(b, J) \in \Gamma$, let

$$C^{(b,J)} = \underbrace{\{0\} \times \cdots \times \{0\}}_{N+m+\#b} \times \underbrace{\mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++}}_{K-\#b},$$

and, for $(n, j) \notin b$,

$$C^{(b,J),(n,j)} = \underbrace{\{0\} \times \cdots \times \{0\}}_{N+m+\#b} \times \underbrace{\mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++} \times \{0\} \times \mathbb{R}_{++} \times \cdots \times \mathbb{R}_{++}}_{K-\#b},$$

where the last $\{0\}$ corresponds to the component $V(n) - W_{nj}^b(x^b; \gamma)$. Moreover, for $(n, j), (n', j') \notin b$ such that $(n, j) \neq (n', j')$, let

$$C^{(b,J),(n,j),(n',j')} = \underbrace{\{0\} \times \cdots \times \{0\}}_{N+m+\#b} \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \{0\} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{K-\#b},$$

where the last two $\{0\}$s correspond to the components $V(n) - W_{nj}^b(x^b; \gamma)$ and $V(n') - W_{n'j'}^b(x^b; \gamma)$.

Assumption 3 For every $(b, J) \in \Gamma$, $g^{(b,J)}$, with domain restricted to $h(n) > 0$ for $n \in J$ and $h(n) = 0$ for $n \notin J$ when $\#J > 1$ and to $h(\bar{n}) = 1$ and $h(n) = 0$ for $n \notin J$ when $J = \{\bar{n}\}$, transversely intersects $C^{(b,J)}$, $C^{(b,J),(n,j)}$, $C^{(b,J),(n,j),(n',j')}$ for all $(n, j), (n', j') \notin b$ such that $(n, j) \neq (n', j')$.

Theorem 2 Under Assumptions 1, 2, and 3, each connected component of the set of stationary Markov perfect equilibria is homeomorphic either to a circle or to the unit interval $[0, 1]$. Moreover, the two endpoints of a component, if they exist, are at $h(n) = 1$ and $h(n') = 1$ for some n and n'.

Intuitively, for given $(b, J) \in \Gamma$ such that $\#J \neq 1$, the system for stationary Markov perfect equilibria contains $N + m + \#b$ equations and $N + m + 1 + \#b$ variables. Thus, under the transversality condition, the set of equilibria is a one-dimensional manifold. At the endpoints, one of the following four cases may occur:

- $\beta_{nj} = 0$ for some $(n, j) \in b$;
- $V(n) - W^b_{nj}(x^b; \gamma) = 0$ for some $(n, j) \notin b$;
- $h(\bar{n}) = 0$ for some $\bar{n} \in J$ and $h(n) < 1$ for all $n \in J$;
- $h(n) = 1$ for some $n \in J$.

In the first three cases, the next lemma implies that the endpoint is also an endpoint of precisely one other connected component of some $E^{(b', J')}_\gamma$. On the other hand, if $|J| = 1$, the system for stationary Markov perfect equilibria contains $N + m + \#b$ equations and $N + m + \#b$ variables. Thus generically the equilibria are determinate. Because of this, the paths in $E^{(b,J)}_\gamma$ can be linked for different (b, J) in Γ to form loops and paths with two different endpoints in the spaces $h(n) = 1$ and $h(n') = 1$ for some n and n'.

First, we prove the following lemma.

Lemma 2 Under Assumptions 1, 2, and 3, if a connected component of $E^{(b,J)}_\gamma$ for a given $(b, J) \in \Gamma$ has an endpoint in the space of $h(\bar{n}) = 0$ for some $\bar{n} \in J$ and $h(n)$ does not become equal to 1 for some $n \in J$, then some $\beta_{n'j'}$, $(n', j') \in b$, becomes zero at the endpoint. Moreover, this $\beta_{n'j'}$ is unique.

Proof:
Let J' be the subset of J for which $h(i)$, $i \in J'$, becomes zero at the endpoint. Since $h(n)$ does not become equal to 1 for some $n \in J$, the endpoint is a solution of the system deleting the variables $h(i)$, $i \in J' \cup J^c$, and the equations $O^b_i - I^b_i = 0, i \in J' \cup J^c$.[3] Consider the set of solutions in the space of $h(i) > 0$, $i \in J, h(i) = 0, i \notin J$. At the endpoint of the set of solutions, all terms in $I^b_i, i \in J'$, become zero, since $O^b_i, i \in J'$, become zero. There are two cases:

1. For at least one term $\frac{1}{2}\beta_{n'j'}h(n')\beta_{n''j''}h(n'')$ in I^b_i such that $i \in J'$, either $\beta_{n'j'}$ or $\beta_{n''j''}$ becomes zero. As in Kamiya and Shimizu (2002), because of Assumption 3, only one incentive constraint becomes binding in the space of $h(i) > 0, i \in J \setminus J', h(i) = 0, i \in J' \cup J^c$, and thus only one of them included in such terms can be zero.
2. For all terms $\frac{1}{2}\beta_{n'j'}h(n')\beta_{n''j''}h(n'')$ in I^b_i such that $i \in J'$, at least one of n' and n'' is in J', and $\beta_{n'j'}$ and $\beta_{n''j''}$ remain positive.

In the first case, the lemma holds. In the second case, all of such terms $\frac{1}{2}\beta_{n'j'}h(n')\beta_{n''j''}h(n'')$ are included in at least one O^b_i, $i \in J'$. Thus $\sum_{i \in J'} O^b_i - \sum_{i \in J'} I^b_i = 0$ is an identity in the space of $h(i) > 0$, $i \in J$. This contradicts Assumption 3 so that the second case does not occur. ∎

[3] J^c denotes the complement of J.

Lemma 2 shows the new feature in the complementarity problem with respect to the variables h and β. If at an endpoint some $h(n)$ becomes zero, then $I_n - O_n = 0$ becomes an identity and simultaneously $\beta_{n'j'}$ becomes 0 for some unique (n', j').

Proof of Theorem 2: For given $(b, J) \in \Gamma$, $E_\gamma^{(b,J)}$ is a one-dimensional manifold because of the assumptions. (See Kamiya and Shimizu (2002).) Therefore it consists of loops and paths with two endpoints. From Lemma 2 and the assumptions it follows that at each endpoint, exactly one of the following possibilities occurs:

- $\beta_{nj} = 0$ for precisely one $(n, j) \in b$ and $h(n') > 0$ for all $n' \in J$;
- $V(n) - W_{nj}^b(x^b; \gamma) = 0$ for precisely one $(n, j) \notin b$;
- $h(\bar{n}) = 0$ for some $\bar{n} \in J$ and $\beta_{nj} = 0$ for precisely one $(n, j) \in b$;
- $h(n) = 1$ for some $n \in J$.

In the first case, the endpoint is also an endpoint of a unique path in $E_\gamma^{(b',J)}$, where $b' = b \setminus \{(n, j)\}$. In the second case, the endpoint is also an endpoint of a unique path in $E_\gamma^{(b',J)}$, where $b' = b \cup \{(n, j)\}$. In the third case, the endpoint is also an endpoint of a unique path in $E_\gamma^{(b',J')}$, where $b' = b \setminus \{(n, j)\}$ and $J' = J \setminus \{\bar{n} \in J | h(\bar{n}) = 0\}$. In the fourth case, we can delete all equations $O_n - I_n = 0$, $n \in J$, and $\sum_n h(n) - 1 = 0$, so that the system has the same number of equations as number of variables. Therefore, from the assumptions, it follows that the endpoint is not an endpoint of any other path in some $E_\gamma^{(b',J')}$. Because of this, the paths in $E_\gamma^{(b,J)}$ can be linked for different $(b, J) \in \Gamma$ to form loops and paths with two different endpoints in the spaces $h(n) = 1$ and $h(n') = 1$ for some n and n'. ∎

In random matching models with money, $h(n) = 1$ for some $n \neq 0$ typically does not hold in monetary equilibria. In such models, a stationary Markov perfect equilibrium is a monetary equilibrium if $f_1((n, j), (n', j')) \neq n$ holds for some $(n, j), (n', j') \in b$, where b is the set of actions in the equilibrium. For more details, see Green and Zhou (1998), Kamiya and Shimizu (2002), and Zhou (1999). Thus, under Assumption 3, the global structure of the set of equilibria is described as in the following corollary.

Corollary 1 Suppose Assumption 3 holds and $h(n) = 1$ for some $n \neq 0$ does not hold in any stationary Markov perfect equilibrium. Then each connected component of the set of stationary Markov perfect equilibria is homeomorphic to either a circle or to $[0, 1]$. Moreover, the two endpoints of a component, if they exist, are at $h(0) = 1$.

Since, in random matching models with money, at $h(0) = 1$, the Bellman equation is typically quite simple, it is often easy to compute an equilibrium in the space of $h(0) = 1$.

Then following the set of equilibria starting at this equilibrium by some path-following method, we eventually reach another equilibrium with $h(0) = 1$. In this way, we can generate a continuum of equilibria of random matching models with conservation law.

5 Approximation of the Set of Equilibria

In this section we consider the case that Assumption 3 does not necessarily hold. When Assumption 3 holds, every connected set of equilibria is one-dimensional and in case it is not a loop it can be followed from one endpoint at $h(n) = 1$ for some n to the other endpoint at $h(n') = 1$ for some n'. However, as the forth piece in the example in the previous section shows, a higher-dimensional piece may exist if Assumption 3 is not satisfied. Below, we present a method to follow the set of equilibria approximately by perturbing the system. As is well known, certain perturbations of a system of equations can make the system regular. For example, adding constant terms to the system, regularity may directly follow from Sard theorem. However, in our case, such perturbation does not work; in the set of approximated equilibria, the endpoints may not be at $h(0) = 1$ and it may not be the case that some $\beta_{n^0 j^0}$ becomes zero when $h(n)$ becomes zero.

5.1 The Perturbation Method

In this subsection, we assume the structure of monetary economies. Let

$$B^* = \{b \in \hat{B} | \exists n, \exists \beta^b \in \Omega^b, \exists h, I^b_n(h, \beta^b; \gamma) > 0\},$$

the set of actions that are candidates for monetary equilibria, i.e., transactions potentially occur for $b \in B^*$. First, we assume the following as in the examples of Section 3.

Assumption 4 For all $b \in B^*$, $h(n) = 1$ for some $n \neq 0$ does not hold in any stationary Markov perfect equilibrium x^b.

For simplicity, we also assume that if no transaction is made, then for every agent the value is equal to zero.

Assumption 5 In stationary Markov perfect equilibria, $V(n) = 0$, $n = 0, 1, \ldots, N$, if $b \notin B^*$.

The above assumptions are satisfied in the example in Subsection 3.1 if the cost parameter c is small enough.

Theorem 3 In the example in Subsection 3.1, Assumptions 4 and 5 hold if $c < u\alpha^N / \sum_{i=0}^{N-1} \alpha^i$.

Proof:
First, Assumption 5 clearly holds. Suppose x^* is a stationary Markov perfect equilibrium such that $\exists \bar{n} \neq 0, h(\bar{n}) = 1$. Let b^* be the set of indices (n, j) used in x^*. Recall that j can be denoted by (o^n, r^n), where o^n is an offer price and r^n is a reservation price. Then

$$f_1((\bar{n}, j), (\bar{n}, j')) = 0 \text{ holds for all } (\bar{n}, j), (\bar{n}, j') \in b^*. \tag{10}$$

Thus by the assumptions $V(\bar{n}) = 0$ holds and thus $V(n) = 0$ holds for all $n \leq \bar{n}$. This implies that the reservation price of agents with $\bar{n}p$ is $\bar{n}p$. Indeed, $u = u + V(\bar{n} - q) > V(\bar{n}) = 0$ for any positive integer $q \leq \bar{n}$.

Let $\tilde{n} = \bar{n} + \max\{o \mid (o, \bar{n}) \in b^*\}$. Note that by $b^* \in B^*$ there exists $(o, \bar{n}) \in b^*$ satisfying $o \neq NT$. We consider the following strategy:

1. $\omega(n) = \{1\}$ for $n = \bar{n}, \ldots, \tilde{n} - 1$ and $\omega(n)$ =NT otherwise;

2. $\rho(n) = n$ for all n.

The payoff of this strategy at $\bar{n}$ is at least equal to $u\alpha^{(\tilde{n}-1)-\bar{n}} - c\sum_{i=0}^{(\tilde{n}-1)-\bar{n}-1} \alpha^i > 0$. This contradicts $V(\bar{n}) = 0$. ∎

As in Matsui and Shimizu (2001), if there exists infinitesimally small cost on holding money, the following condition holds.

Assumption 6 For all $b \in B^*$, $h(0) = 0$ does not hold in any stationary Markov perfect equilibrium x^b.

For $b \in B^*$, let $\hat{\beta}_n^b$ and $\hat{h}$ be obtained by deleting the first element of β_n^b and h, respectively. Let $\hat{\beta}^b = (\hat{\beta}_0^b, \hat{\beta}_1^b, \ldots, \hat{\beta}_N^b)$ and $\hat{x}^b = (V, \hat{h}, \hat{\beta}^b)$. For given $\epsilon_n > 0, n = 1, 2, \ldots, N$, substituting $\beta_{n1}^b = 1 - \sum_{\{j|(n,j)\in b, j\neq 1\}} \hat{\beta}_{nj}^b$ and $h(0) = 1 - \sum_{n=1}^N \hat{h}(n)$, let

$$\begin{aligned}
\hat{D}_0^b(\hat{h}, \hat{\beta}^b; \gamma, \epsilon) &= O_0^b(h, \beta^b; \gamma) - I_0^b(h, \beta^b; \gamma) + \textstyle\sum_{j=1}^N \frac{\epsilon_j}{N}\hat{h}(j) - \epsilon_1\hat{h}(1),\\
\hat{D}_n^b(\hat{h}, \hat{\beta}^b; \gamma, \epsilon) &= O_n^b(h, \beta^b; \gamma) - I_n^b(h, \beta^b; \gamma) + \epsilon_n\hat{h}(n) - \epsilon_{n+1}\hat{h}(n+1),\ n = 1, 2, \ldots, N-1,\\
\hat{D}_N^b(\hat{h}, \hat{\beta}^b; \gamma, \epsilon) &= O_N^b(h, \beta^b; \gamma) - I_N^b(h, \beta^b; \gamma) + \epsilon_N\hat{h}(N) - \textstyle\sum_{j=1}^N \frac{\epsilon_j}{N}\hat{h}(j).
\end{aligned}$$

Remark 2 The above ϵ-perturbation corresponds to the reallocation of assets in such a way that agents with $n \neq 0$ give one unit to agents with $n' \neq N$. More precisely, the proportion $\epsilon_n h(n)$ of agents with n give $\frac{1}{N}$ units to each agent with $n' = 0, 1, \ldots, N-1$. Then the excess outflow at each state n becomes

$$O_0^b(h, \beta^b; \gamma) + \sum_{j=1}^N \frac{\epsilon_j}{N}\hat{h}(j) - I_0^b(h, \beta^b; \gamma) - \epsilon_1\hat{h}(1), n = 0,$$

$$O_n^b(h,\beta^b;\gamma)+\epsilon_n\hat{h}(n)+\sum_{j=1}^{N}\frac{\epsilon_j}{N}\hat{h}(j)-I_n^b(h,\beta^b;\gamma)-\epsilon_{n+1}\hat{h}(n+1)-\sum_{j=1}^{N}\frac{\epsilon_j}{N}\hat{h}(j),$$

$$n=1,2,\dots,N-1,$$

and

$$O_N^b(h,\beta^b;\gamma)-I_N^b(h,\beta^b,\gamma)+\epsilon_N\hat{h}(N)-\sum_{j=1}^{N}\frac{\epsilon_j}{N}\hat{h}(j), n=N.$$

Thus they coincide with $\hat{D}_n^b(\hat{h},\hat{\beta}^b;\gamma,\epsilon), n=0,1,\dots,N-1,N$. Note that $\sum_{n=0}^{N}\hat{D}_n^b(\hat{h},\hat{\beta}^b;\gamma,\epsilon)=0$ and $\sum_{n=0}^{N}n\hat{D}_n^b(\hat{h},\hat{\beta}^b;\gamma,\epsilon)=0$ hold.

Let

$$\kappa(\hat{h},\hat{\beta}^b,\epsilon)=\left(\hat{D}_n^b(\hat{h},\hat{\beta}^b;\gamma,\epsilon)\right)_{n=2}^{N}$$

.

Lemma 3 Under Assumptions 4 and 6, for small enough $\epsilon>0$, $\kappa(\hat{h},\hat{\beta}^b,\epsilon)=0$ implies $h(0)>0$ and $\hat{h}(n)>0$ for all $n\geq 1$, unless $h(0)=1$.

Proof:
If $\hat{h}(n)=0$ for some $n\geq 2$, then $O_n^b(h,\beta^b;\gamma)=0$ holds and together with $\kappa_n(\hat{h},\hat{\beta}^b,\epsilon)=0$ it follows that $\hat{h}(n+1)=0$. By induction, $\hat{h}(N)=0$ holds. On the other hand, when $\hat{h}(N)=0$ it follows from $\kappa_N(\hat{h},\hat{\beta}^b,\epsilon)=0$ that $\hat{h}(j)=0, j=1,2,\dots,N$. Thus $h(0)=1$ holds. For the case of $\hat{h}(1)=0$, since $\sum_{n=0}^{N}\hat{D}_n^b(\hat{h},\hat{\beta}^b;\gamma,\epsilon)=0$ and $\sum_{n=0}^{N}n\hat{D}_n^b(\hat{h},\hat{\beta}^b;\gamma,\epsilon)=0$, a similar argument applies. Note that $h(0)=0$ cannot be a solution for small enough ϵ. ∎

For $b\in B^*$, let

$$f^b(\hat{x}^b,\epsilon,\sigma)=\begin{pmatrix} f_1^b(\hat{x}^b,\epsilon,\sigma)\\ f_2^b(\hat{x}^b,\epsilon,\sigma)\\ f_3^b(\hat{x}^b,\epsilon,\sigma)\end{pmatrix}=$$

$$\begin{pmatrix} \kappa(\hat{h},\hat{\beta}^b,\epsilon)\\ (V(n)-W_{nj}^b(\hat{x}^b;\gamma)+\sigma_{nj})_{(n,j)\in b}\\ (V(n)-W_{nj}^b(\hat{x}^b;\gamma)+\sigma_{nj})_{(n,j)\notin b}\end{pmatrix}.$$

Definition 2 For given $b \in B^*$, $\epsilon > 0$ and σ, $\hat{x}^b$ is an (ϵ, σ)-approximated equilibrium if

$$\begin{array}{l} f_1^b(\hat{x}^b, \epsilon, \sigma) = 0 \\ f_2^b(\hat{x}^b, \epsilon, \sigma) = 0 \\ f_3^b(\hat{x}^b, \epsilon, \sigma) \geq 0. \end{array}$$

Let

$$\varphi^b(V, \hat{\beta}^b, \sigma) = f_2^b(V, (0, \ldots, 0), \hat{\beta}^b, \epsilon, \sigma).$$

The following lemma directly follows from Sard theorem.

Lemma 4 Let $b \in B^*$ be given. $D_{(V,\hat{\beta}^b)}\varphi^b$ is of full rank at almost every σ. Moreover, the number of solutions to $\varphi^b(V, \hat{\beta}^b, \sigma) = 0$ for such σ is finite.

Let the set of solutions for some given σ be $(V^i, \hat{\beta}^{bi}), i = 1, \ldots, \ell$. The Jacobian matrix of f^b with respect to $(V, \hat{h}, \hat{\beta}, \epsilon, \sigma)$ is equal to

$$Df^b = \begin{pmatrix} 0 & D_{\hat{h}}\kappa & \bigcirc & D_\epsilon \kappa & 0 \\ \bigcirc & \bigcirc & \bigcirc & 0 & I \end{pmatrix},$$

where $\bigcirc$ represents some nonzero matrix and I is the identity matrix.

Lemma 5 Suppose Assumptions 4 and 6 hold. Let $b \in B^*$ be given. Then, if $h(0) < 1$, $D_{\hat{x}^b} f^b(\cdot, \epsilon, \sigma)$ is of full rank for almost every (ϵ, σ), where $\epsilon > 0$ is small enough, i.e., $f^b(\cdot, \epsilon, \sigma)$ is transversal to $C^{(b,\mathcal{N})}$, $C^{(b,\mathcal{N})(n,j)}$, and $C^{(b,\mathcal{N})(n,j)(n^0,j^0)}$ for all $(n, j), (n', j') \notin b$ for almost every (ϵ, σ), where $\epsilon > 0$ is small enough.

Proof:
The first $N - 1$ columns of the Jacobian matrix $D_\epsilon \kappa$ are equal to

$$\begin{pmatrix} 0 & \hat{h}(2) & -\hat{h}(3) & 0 & \cdots & 0 \\ 0 & 0 & \hat{h}(3) & -\hat{h}(4) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\hat{h}(N-1) \\ 0 & 0 & 0 & 0 & \cdots & \hat{h}(N-1) \\ -\frac{1}{N}\hat{h}(1) & -\frac{1}{N}\hat{h}(2) & -\frac{1}{N}\hat{h}(3) & -\frac{1}{N}\hat{h}(4) & \cdots & -\frac{1}{N}\hat{h}(N-1) \end{pmatrix}.$$

The absolute value of the determinant of this matrix is $\Pi_{n=1}^{N-1} \frac{1}{N}\hat{h}(n)$. Thus, by Lemma 3, the above matrix is of full rank if $h(0) < 1$ and $\epsilon > 0$ is small enough. Since the lower part of Df^b has I, Df^b is of full rank. ∎

Lemma 6 Let $b \in B^*$ be given. Then, if $h(0) = 1$, $D_{\hat{x}^b} f^b(\cdot, \epsilon, \sigma)$ is of full rank and thus $f^b(\cdot, \epsilon, \sigma)$ is transversal to $C^{(b,\mathcal{N})}$, $C^{(b,\mathcal{N})(n,j)}$, and $C^{(b,\mathcal{N})(n,j)(n^{\prime},j^{\prime})}$ for all $(n,j),(n',j') \notin b$ for almost every (ϵ, σ).

Proof:
The first $N-1$ columns of the Jacobian matrix $D_{\hat{h}}\kappa$ can be written as

$$D_{\tilde{h}}((O_n^b - I_n^b)_{n=2}^N) + \begin{pmatrix} 0 & \varepsilon_2 & -\varepsilon_3 & 0 & \cdots & 0 \\ 0 & 0 & \varepsilon_3 & -\varepsilon_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\varepsilon_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & \varepsilon_{N-1} \\ -\frac{1}{N}\varepsilon_1 & -\frac{1}{N}\varepsilon_2 & -\frac{1}{N}\varepsilon_3 & -\frac{1}{N}\varepsilon_4 & \cdots & -\frac{1}{N}\varepsilon_{N-1} \end{pmatrix},$$

where $\tilde{h} = (\hat{h}(1), \hat{h}(2), \ldots, \hat{h}(N-1))$. Let the first matrix evaluated at $(V^i, \hat{\beta}^{bi})$ be denoted by A^i and the second matrix, a function of ϵ, by denoted by $E(\epsilon)$.

First note that the absolute value of the determinant of the second matrix is $\Pi_{n=1}^{N-1} \frac{1}{N}\varepsilon_n \neq 0$ for $\epsilon > 0$. Next, let

$$F^i(\epsilon) = \det(A^i + E(\epsilon)).$$

Choose an $\epsilon^* > 0$. Then

$$F^i(\frac{1}{\chi}\epsilon^*) = \frac{1}{\chi^{N-1}} \det(\chi A^i + E(\epsilon^*)).$$

Since $\det(\cdot)$ is a continuous function and $\det E(\epsilon^*) \neq 0$, then, for small enough χ, $\det(\chi A^i + E(\epsilon^*)) \neq 0$. Thus it follows that $F^i(\frac{1}{\chi}\epsilon^*) \neq 0$. Since $F^i(\epsilon)$ is an analytic function (polynomial), then either (i) $\{\epsilon \mid F^i(\epsilon) = 0\}$ is a set of measure zero or (ii) $F^i(\epsilon) = 0$ for all ϵ. By $F^i(\frac{1}{\chi}\epsilon^*) \neq 0$, (i) holds.

Thus the above matrix is of full rank at $h(0) = 1$ for almost every ϵ. Since the lower part of Df^b has I, then Df^b is of full rank at $h(0) = 1$ for almost every ϵ. ∎

By the above lemmata, $f^b(\cdot, \epsilon, \sigma)$ is transversal to these sets for almost every (ϵ, σ), where $\epsilon > 0$ is small enough.

To sum up, for given $b \in B^*$, for almost every (ϵ, σ), with $\epsilon > 0$ small enough, f^b is transversal to $C^{(b,\mathcal{N})}$, $C^{(b,\mathcal{N})(n,j)}$, and $C^{(b,\mathcal{N})(n,j)(n^{\prime},j^{\prime})}$ for all $(n,j),(n',j') \notin b$ and $D_{(V,\hat{\beta})}\varphi^b$ is of full rank. Applying the same argument for any $b \in B^*$, the above holds for all $b \in B^*$ and almost every (ϵ, σ), where ϵ is small enough. Thus, for almost every (ϵ, σ) with $\epsilon > 0$ small enough, the approximated solution path is a one-dimensional manifold such that

1. $h(n) > 0, n = 0, 1, \ldots, N$, unless $h(0) = 1$;

2. the solution path transversely intersects the boundary at $h(0) = 1$.

Thus the set of approximated equilibria can be generated by finding an endpoint at $h(0) = 1$ and following the one-dimensional manifold by some simplicial or predictor-corrector algorithm, see, for example, Allgower and Georg (1990). Moreover, by Assumption 5, a nonmonetary equilibrium cannot be an endpoint of the path if $\sigma_{nj} > 0$ for all (n, j).

Remark 3 It might happen that, at $h(0) = 1$, some indifference condition becomes an identity in the original system. For example, in the sixth piece of the example in Subsection 3.3, agents with 0 offer p or $2p$ and thus

$$(1 - h(0))(-c + \alpha V(1)) + h(0)V(0) = h(2)(-c + V(2)) + (1 - h(2))V(0)$$

holds. At $h(0) = 1$, the above equation becomes $V(0) = V(0)$, an identity. If we add constant terms on both sides of the equation, there is generically no solution. Thus the approximation might be bad around $h(0) = 1$. In order to overcome this difficulty, we may use the following general approximated system. Let

$$f^b(\hat{x}^b, \epsilon, \delta, \xi, \sigma) = \begin{pmatrix} f_1^b(\hat{x}^b, \epsilon, \delta, \xi, \sigma) \\ f_2^b(\hat{x}^b, \epsilon, \delta, \xi, \sigma) \\ f_3^b(\hat{x}^b, \epsilon, \delta, \xi, \sigma) \end{pmatrix} =$$

$$\begin{pmatrix} \kappa(\hat{h}, \hat{\beta}^b, \epsilon) \\ (V(n) - W_{nj}^b(\hat{x}^b; \gamma) + \sum_{n'=0}^{N} \delta_{nn'} V(n') + \sum_{n'=0}^{N} \sum_{k=2}^{\#A^n} \xi_{nn'k} \hat{\beta}_{n'k}^b + \sigma_{nj})_{(n,j)\in b} \\ (V(n) - W_{nj}^b(\hat{x}^b; \gamma) + \sum_{n'=0}^{N} \delta_{nn'} V(n') + \sum_{n'=0}^{N} \sum_{k=2}^{\#A^n} \xi_{nn'k} \hat{\beta}_{n'k}^b + \sigma_{nj})_{(n,j)\notin b} \end{pmatrix}.$$

The latter system typically does not contain an identity, not even at $h(0) = 1$.

5.2 Zhou's Model Revisited

In this subsection, we investigate the case of $N = 2$, $k = 3$, $u = \frac{15}{4}, c = \frac{3}{4}$, and $\alpha = \frac{3}{4}$. As illustrated in Subsection 3.3, the set of equilibria contains a two-dimensional piece. However, by using the perturbation above, the set of approximated equilibria becomes one-dimensional. We use the following perturbed system:

$$\epsilon_1 = \epsilon_2 = 1 \times 10^{-5}, \sigma_{0\mathrm{NT}} = 4.8 \times 10^{-5}, \sigma_{01} = 5.1 \times 10^{-5}, \sigma_{02} = 4.3 \times 10^{-5},$$

$$\sigma_{1\mathrm{NT}} = 4.9 \times 10^{-5}, \sigma_{11} = 4.3 \times 10^{-5}, \sigma_{2\mathrm{NT}} = 4.6 \times 10^{-5},$$

where σ_{nj} is the constant term for offering jp or NT at state n. Since in the original system the incentive constraints for the reservation prices never become binding, we do not need to perturb these. As in Remark 3, the approximation is not very good around the endpoint of the sixth piece. In fact, there should exist another short piece adjacent to the sixth piece. However, we could not numerically identify it because the perturbation is too small.

Below, we present the (first) six approximated pieces corresponding to the pieces in the original system; the strategies of each piece are the same as those of the corresponding piece. (See Figure 3.) Note that the approximation is close except around the end of the sixth piece.

The first piece:

- the starting point: $(h(0), h(1), h(2)) = (1, 0, 0)$.
- the endpoint: $(h(0), h(1), h(2)) = (0.62351, 0.26438, 0.11211)$.
- at the endpoint, agents with 1 become indifferent between offering p and NT.

The second piece:

- the starting point: the endpoint of the first piece.
- the endpoint: $(h(0), h(1), h(2)) = (0.84299, 0.15699, 5 \times 10^{-5})$.
- at the endpoint, the proportion of agents with 1 offering p becomes 0.

The third piece:

- the starting point: the endpoint of the second piece.
- the endpoint: $(h(0), h(1), h(2)) = (0.25001, 0.74999, 4.9997 \times 10^{-5})$.
- at the endpoint, agents with 0 become indifferent between offering p, $2p$, and NT.

The forth piece:

- the starting point: the endpoint of the third piece.
- the endpoint: $(h(0), h(1), h(2)) = (0.58347, 0.41645, 8.3252 \times 10^{-5})$.
- at the endpoint, agents with 1 become indifferent between offering p and NT.

The fifth piece:

- the starting point: the endpoint of the forth piece.
- the endpoint: $(h(0), h(1), h(2)) = (0.61067, 0.23378, 0.15555)$.
- at the endpoint, the proportion of agents with 1 choosing NT becomes 0.

The sixth piece:

- the strategies: agents with 0 offer p or $2p$, agents with 1 offer p and their reservation price is p, and the reservation price of agents with 2 is $2p$.
- the starting point: the endpoint of the fifth piece.
- the endpoint: it is numerically hard to find the endpoint around $h(0) = 1$.

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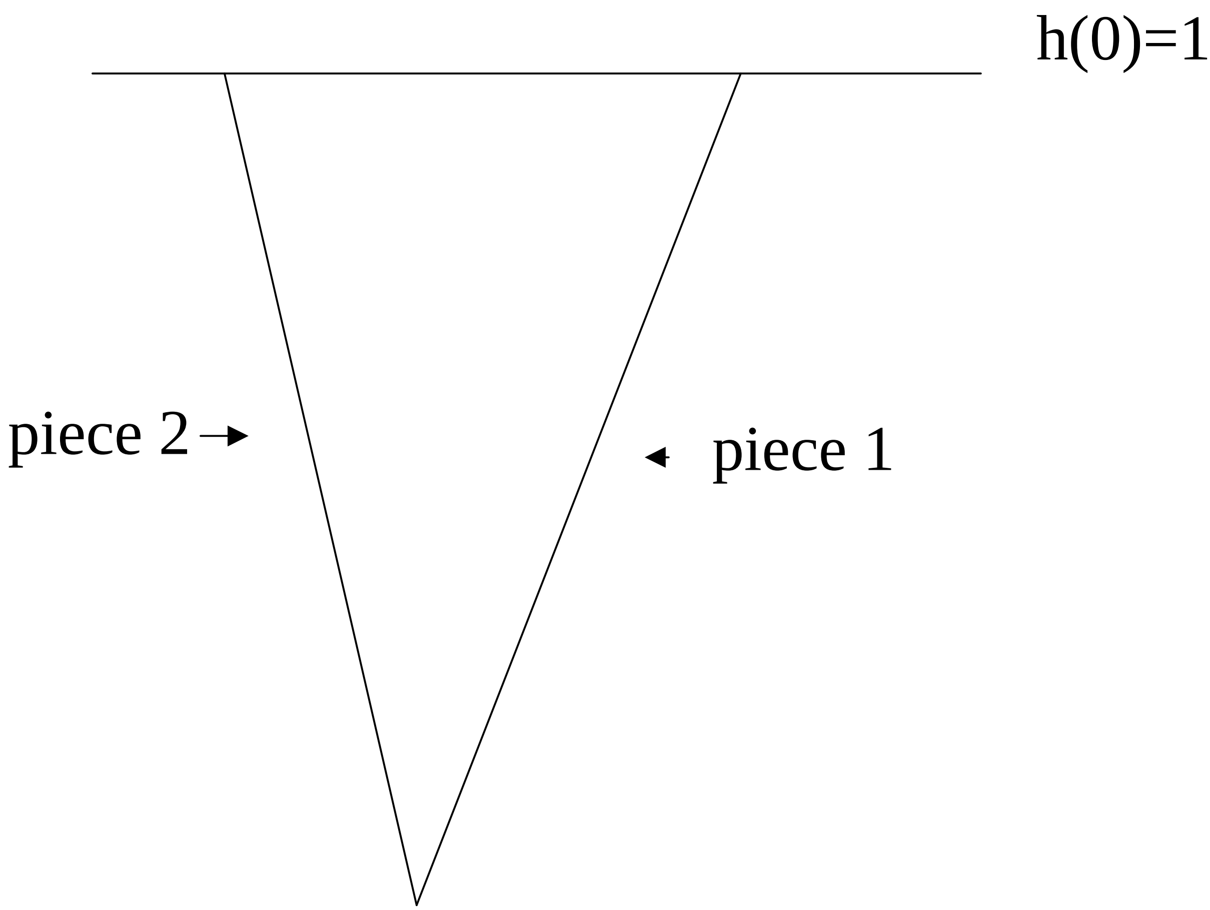

Figure 1

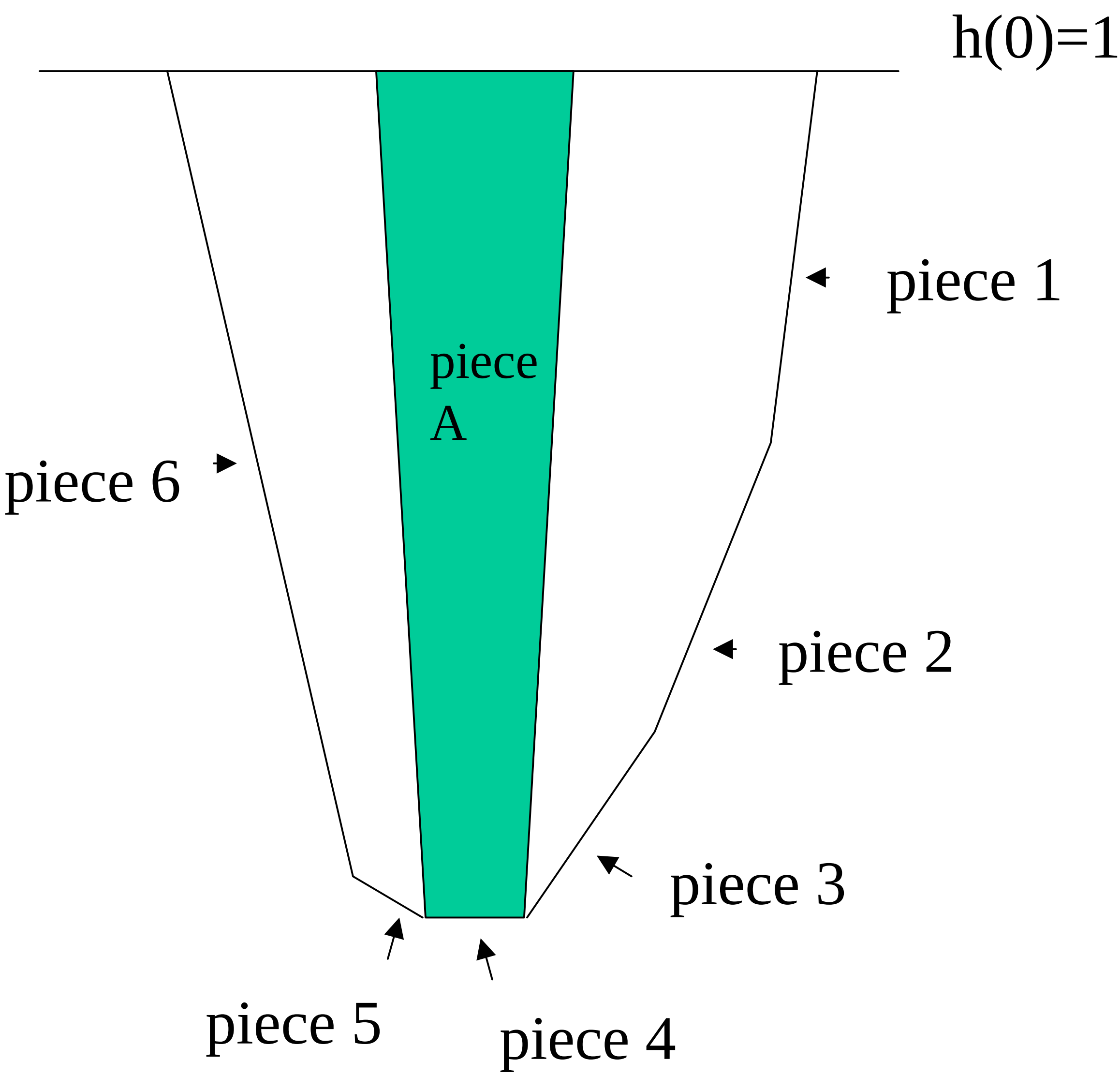

Figure 2

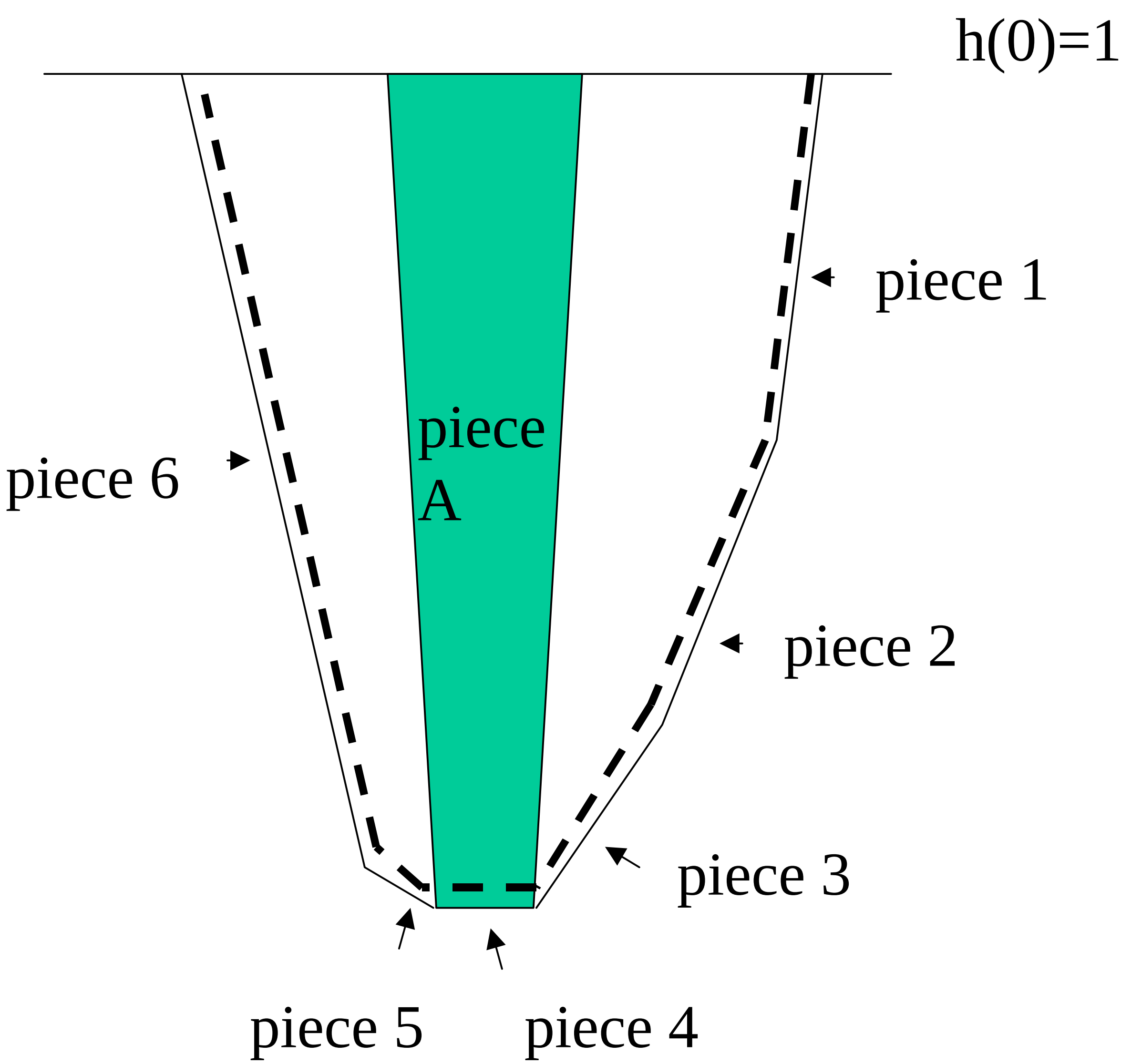

Figure 3