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Asymptotic Properties of the Estimator of the Long-run Coefficient in a Dynamic Model with Integrated Regressors and Serially Correlated Errors

Zonglu He
Risshikan University

Koichi Maekawa
Hiroshima University

Michael McAleer
University of Western Australia

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Zonglu He[1)], Koichi Maekawa[2)†] and Michael McAleer[3)]

[1)]Risshikan University

[2)]Hiroshima University

[3)]University of Western Australia

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Abstract

In this paper we examine the asymptotic properties of the estimator of the long-run coefficient (LRC) in a dynamic regression model with integrated regressors and serially correlated errors. We show that the OLS estimators of the regression coefficients are inconsistent but the OLS-based estimator of the LRC is superconsistent. Furthermore, we propose an alternative consistent estimator of the LRC, compare

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†Correspondence author: Koichi Maekawa, Faculty of Economics, Hiroshima University, 1-2-1 Kagamiyama, Higashi Hiroshima, 739-8525 Japan. Tel. +81 824 247264, fax. +81 824 247212, e-mail maekawa@hiroshima-u.ac.jp

the two estimators through a Monte Carlo experiment, and find that the proposed estimator is MSE-superior to the OLS-based estimator.

JEL Classification Numbers: C13, C15, C22

1 Introduction

An autoregressive distributed lag model with serially correlated disturbances represents an important class of dynamic regression model in econometrics. Such a model containing lagged dependent and lagged independent variables with lag-orders p and q, respectively, is called an autoregressive distributed lag model (henceforth, denoted by $ADL(p,q)$). An ADL model can be written as

$$y_t = c + A(L)y_t + B(L)z_t + u_t, \tag{1}$$

where $A(L)$ and $B(L)$ are the polynomials of the lag operator L, defined by

$$\begin{aligned} A(L) &= \alpha_1 L + \cdots + \alpha_p L^p, \\ B(L) &= \beta_0 + \beta_1 L + \cdots + \beta_q L^q. \end{aligned}$$

The long-run effect of z on y is given by the long-run coefficient, defined as

$$\delta = \frac{B(1)}{1 - A(1)} = \frac{\sum_{j=0}^{q} \beta_j}{1 - \sum_{i=1}^{p} \alpha_i}.$$

If (1) is regarded as a consumption function, with consumption y and income z, δ is the long-run marginal propensity to consume (abbreviated as $LRMPC$).

Since many economic time series are nonstationary processes, such as integrated or cointegrated processes, we need to develop the asymptotic theory for nonstationary $ADL(p,q)$ models. Furthermore, such models with serially correlated disturbances are important in both theory and practice.

Maekawa, Yamamoto, Takeuchi and Hatanaka (1996, abbreviated as MYTH) dealt with the $ADL(1,0)$ model with an integrated regressor and serially correlated disturbances: namely, $y_t = \alpha y_{t-1} + \beta z_t + u_t$, where z_t is integrated of order 1. When u_t is assumed to be a stationary $AR(1)$ process, MYTH showed that $\hat{\alpha}$ and $\hat{\beta}$ are $\sqrt{T}-$*inconsistent* but asymptotically normally distributed.

This paper proceeds as follows. Section 2 presents the model and the assumptions. Section 3 derives the asymptotic distributions of the OLS estimators of the regression coefficients and the long-run coefficient. Section 4 proposes an alternative estimator of the long-run coefficient and investigates its asymptotic properties. Section 5 compares the small sample distributional properties of two estimators of the long-run coefficient by performing Monte Carlo experiments for the most simple case of the model. Section 6 summarizes the main results of the paper and provides some concluding comments. Detailed derivations and proofs are tedious and hence are largely omitted from the paper. Instead, an outline of the proofs and derivations are provided. Their details are given in a Supplement to this paper, and are available from the authors upon request.

2 ADL (p,q) Model and OLS-Estimator

First we specify the model (1) as follows:

$$\begin{aligned} y_t &= c + A(L)y_t + B(L)z_t + u_t, \\ u_t &= C(L)v_t, \\ z_t &= z_{t-1} + \varepsilon_t, \\ t &= 1, 2, \cdots, T, \end{aligned} \tag{2}$$

where $v_t \sim i.i.d.N(0, \sigma_1^2)$, $\varepsilon_t \sim i.i.d.N(0, \sigma_2^2)$, v_t and ε_t are independent, all the roots of the characteristic equation $1 - A(L) = 0$ lie outside the unit circle, and $C(L) = \sum_{i=0}^{\infty} c_i L^i$, with $\sum_{i=0}^{\infty} |c_i| < \infty$.

As is well known, we have

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} v_t \Rightarrow B_1(r), \frac{1}{\sqrt{T}}\sum_{t=1}^{[Tr]} \varepsilon_t \Rightarrow B_2(r),\ r \in [0, 1],$$

where $B_1(r)$ and $B_2(r)$ are Brownian motions, $\Rightarrow$ signifies weak convergence, and $[x]$ denotes the integer part of x.

To derive the asymptotic properties of the OLS estimators of the coefficients and the OLS-based estimator for LRC in model (2), we introduce the following vectors:

$$\mathbf{y}' = (y_1, \cdots, y_T),$$

$$\begin{aligned}
\mathbf{y}'_{-i} &= (y_{-(i-1)}, \cdots, y_0, y_1, \cdots, y_{T-i}),\ i = 1, \cdots, p, \\
\mathbf{z}' &= (z_1, \cdots, z_T), \\
\mathbf{z}'_{-j} &= (z_{-(j-1)}, \cdots, z_0, z_1, \cdots, z_{T-j}),\ j = 1, \cdots, q, \\
\mathbf{1}' &= \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}_{1\times T}, \\
Z &= \begin{pmatrix} \mathbf{y}_{-1} & \mathbf{y}_{-2} & \cdots & \mathbf{y}_{-p} & \mathbf{z} & \mathbf{z}_{-1} & \cdots & \mathbf{z}_{-q} & \mathbf{1} \end{pmatrix}, \\
\Gamma' &= \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} & c \end{pmatrix}, \\
\boldsymbol{\alpha} &= \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_p \end{pmatrix}, \\
\boldsymbol{\beta} &= \begin{pmatrix} \beta_0 & \beta_1 & \cdots & \beta_q \end{pmatrix}, \\
\mathbf{u}' &= \begin{pmatrix} u_1 & \cdots & u_T \end{pmatrix}.
\end{aligned}$$

Using this notation, model (2) can be rewritten as follows:

$$\mathbf{y} = Z\Gamma + \mathbf{u}, \tag{3}$$

and the OLS estimator of (3) is given as

$$\hat{\Gamma} - \Gamma = (Z'Z)^{-1}Z'\mathbf{u}. \tag{4}$$

Obviously, if the exogenous variable z_t is a stationary process, the asymptotic distribution of the estimator of (3) is straightforward to obtain. However, it is difficult to apply the same method to integrated exogenous variables. We can show that the regressors in (2) are cointegrated (see in (8) below), and hence we have $(Z'Z)^{-1} \xrightarrow{p} (0/0)_{(p+q+2)\times(p+q+2)}$ as $T \to \infty$. In other words, cointegrated regressors have an effect similar to multicollinearity which reduces rank$(Z'Z)$ as $T \to \infty$. We make an appropriate transformation given by (9) below for the right side of the model (4) to avoid the indeterminacy in $(Z'Z)^{-1}$. To do this, we first show that y_{t-1} and z_t are cointegrated.

Note that (2) can be rewritten as

$$y_{t-1} = \tilde{c} + \varphi(L)z_{t-1} + \phi(L)v_{t-1}, \tag{5}$$

where

$$\begin{aligned}
\tilde{c} &= \frac{c}{1 - A(1)} = \frac{c}{1 - \sum\limits_{i=1}^{p} \alpha_i}, \\
\varphi(L) &= \frac{B(L)}{1 - A(L)} = \sum_{i=0}^{\infty} \varphi_i L^i,
\end{aligned}$$

$$\phi(L) = \frac{C(L)}{1-A(L)} = \sum_{i=0}^{\infty} \phi_i L^i.$$

It is easy to see that

$$\delta = \frac{\sum\limits_{j=0}^{q} \beta_j}{1-\sum\limits_{i=1}^{p} \alpha_i} = \varphi(1). \tag{6}$$

We can prove that

$$\sum_{i=0}^{\infty} i \mid \varphi_i \mid < \infty, \quad \sum_{i=0}^{\infty} \mid \phi_i \mid < \infty. \tag{7}$$

(see Appendix A in the Supplement to this paper.) Without loss of generality, we assume that the initial values for $z_{-i}, i = 0, 1, \cdots, p-1$, are zero. Using the Beveridge-Nelson decomposition yields

$$\begin{aligned}
\varphi(L) z_{t-1} &= \varphi(1) z_{t-1} - \sum_{i=0}^{\infty} \left(\varphi_{i+1} + \varphi_{i+2} + \cdots\right) \varepsilon_{t-i-1} \\
&= \delta z_t - \left[\delta \varepsilon_t + \sum_{i=0}^{\infty} \left(\varphi_{i+1} + \varphi_{i+2} + \cdots\right) \varepsilon_{t-i-1}\right].
\end{aligned}$$

Define $\gamma_i^* = \varphi_i + \varphi_{i+1} + \cdots$, $i = 0, 1, 2, \cdots$. Then $\delta = \gamma_0^*$ and $\varphi(L) z_{t-1} = \delta z_t - \sum\limits_{i=0}^{\infty} \gamma_i^* \varepsilon_{t-i}$. We thus have the following expansion:

$$y_{t-1} = \tilde{c} + \delta z_t + a_t, \tag{8}$$

where $a_t = \sum\limits_{i=0}^{\infty} \phi_i v_{t-i-1} - \sum\limits_{i=0}^{\infty} \gamma_i^* \varepsilon_{t-i} \sim I(0)$ and $z_t \sim I(1)$, implying that y_{t-1} and z_t are cointegrated, so that our model has cointegrated regressors. This could be called "*stochastic multicollinearity*". From (8) we see that y_{t-1} can be written as the sum of an integrated process and a stationary process. This formula will be used to obtain the asymptotic distributions of $\hat{\Gamma}' = \left(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, \hat{c}\right)$ and $\hat{\delta}$.

To write the system compactly, we introduce the following notation:

$$\mathbf{a}'_{-i} = \begin{pmatrix} a_{-(i-1)} & \cdots & a_0 & a_1 & \cdots & a_{T-i} \end{pmatrix}, \; i = 0, 1, \cdots, p-1,$$

$$
\begin{array}{rcl}
\epsilon'_{-j} &=& \left(\begin{array}{cccccc} \varepsilon_{-(j-1)} & \cdots & \varepsilon_0 & \varepsilon_1 & \cdots & \varepsilon_{T-j} \end{array} \right), \ j = 0, 1, \cdots, q-1, \\
W &=& \left(\begin{array}{cccc} \mathbf{a}_0 & \mathbf{a}_{-1} & \cdots & \mathbf{a}_{-p+1} \end{array} \right), \\
V &=& \left(\begin{array}{cccc} \epsilon_0 & \epsilon_{-1} & \cdots & \epsilon_{-q+1} \end{array} \right), \\
X &=& \left(\begin{array}{cccc} W & V & \mathbf{1} & \mathbf{z} \end{array} \right).
\end{array}
$$

Without loss of generality, we assume $p \leqslant q-1$. To avoid the indeterminacy in $(Z'Z)^{-1}$ we introduce an inverse transformation

$$
X = ZG, \tag{9}
$$

where

$$
\begin{array}{rcl}
G &=& \left(\begin{array}{cc} H & H^{(1)} \\ H^{(2)} & \mathbf{e}'_2 \end{array} \right), \\
H &=& \left(\begin{array}{cc} I_p & \mathbf{0}_{p\times q} \\ H_1 & H_2 \end{array} \right), \\
H^{(1)} &=& \left(\begin{array}{cc} \mathbf{0}_{p\times 1} & \mathbf{0}_{p\times 1} \\ \mathbf{0}_{(q+1)\times 1} & \mathbf{e}_{q+1} \end{array} \right), H^{(2)} = \left(\begin{array}{cc} -\tilde{c}\mathbf{1}'_p & \mathbf{0}_{1\times q} \end{array} \right), \\
H_1 &=& \left(\begin{array}{c} -\delta I_p \\ \mathbf{0}_{(q-p+1)\times p} \end{array} \right), H_2 = \left(\begin{array}{ccccc} 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \\ 0 & \cdots & 0 & 0 & -1 \end{array} \right)_{(q+1)\times q},
\end{array}
$$

where I_p is an identity matrix of order p, and $\mathbf{e}_i$ is the first element vector of the identity matrix I_i for $i = 2, q+1$.

Using this transformation, we have

$$
\hat{\Gamma} - \Gamma = (Z'Z)^{-1}Z'\mathbf{u} = (G'^{-1}X'XG^{-1})^{-1}G^{-1\prime}X'\mathbf{u} = G(X'X)^{-1}X'\mathbf{u},
$$

where

$$
X'X = \left(\begin{array}{cccc} W'W & W'V & W'\mathbf{1} & W'\mathbf{z} \\ V'W & V'V & V'\mathbf{1} & V'\mathbf{z} \\ \mathbf{1}'W & \mathbf{1}'V & T & \mathbf{1}'\mathbf{z} \\ \mathbf{z}'W & \mathbf{z}'V & \mathbf{z}'\mathbf{1} & \mathbf{z}'\mathbf{z} \end{array} \right), X'\mathbf{u} = \left(\begin{array}{c} W'\mathbf{u} \\ V'\mathbf{u} \\ \mathbf{1}'\mathbf{u} \\ \mathbf{z}'\mathbf{u} \end{array} \right).
$$

Let

$$
\begin{aligned}
A_T &= \begin{pmatrix} \frac{1}{T}W'W & \frac{1}{T}W'V \\ \frac{1}{T}V'W & \frac{1}{T}V'V \end{pmatrix}, B_T = \begin{pmatrix} \frac{1}{T}W'\mathbf{1} & \frac{1}{T}W'\mathbf{z} \\ \frac{1}{T}V'\mathbf{1} & \frac{1}{T}V'\mathbf{z} \end{pmatrix}, \\
C_T &= \begin{pmatrix} C_{1,T} & C_{2,T} \end{pmatrix}, C_{1,T} = \begin{pmatrix} \frac{1}{\sqrt{T}}\mathbf{1}'W \\ \frac{1}{T^{3/2}}\mathbf{z}'W \end{pmatrix}, C_{2,T} = \begin{pmatrix} \frac{1}{\sqrt{T}}\mathbf{1}'V \\ \frac{1}{T^{3/2}}\mathbf{z}'V \end{pmatrix}, \\
D_T &= \begin{pmatrix} 1 & \frac{1}{T}\mathbf{1}'\mathbf{z} \\ \frac{1}{T^2}\mathbf{z}'\mathbf{1} & \frac{1}{T^2}\mathbf{z}'\mathbf{z} \end{pmatrix}, B_{1,T} = \begin{pmatrix} \frac{1}{\sqrt{T}}W'\mathbf{u} \\ \frac{1}{\sqrt{T}}V'\mathbf{u} \end{pmatrix}, B_{2,T} = \begin{pmatrix} \frac{1}{\sqrt{T}}\mathbf{1}'\mathbf{u} \\ \frac{1}{T\sqrt{T}}\mathbf{z}'\mathbf{u} \end{pmatrix}.
\end{aligned}
\tag{10}
$$

The normalized OLS-estimator can be written as

$$
\sqrt{T}\left(\hat{\Gamma} - \Gamma\right) = G \begin{pmatrix} A_T & B_T \\ \frac{1}{\sqrt{T}}C_T & D_T \end{pmatrix}^{-1} \begin{pmatrix} B_{1,T} \\ B_{2,T} \end{pmatrix}. \tag{11}
$$

3 Asymptotic Distributions of $\hat{\Gamma}$ and $\hat{\delta}$

We derive the asymptotic distributions of $\hat{\boldsymbol{\alpha}}$, $\hat{\boldsymbol{\beta}}$, $\hat{c}$ by using the functional central limit theorem and the continuous mapping theorem (see Phillips, 1987). The theoretical results are given as follows:

Theorem 1 *(**$\sqrt{T}$-inconsistency** **of** $\hat{\boldsymbol{\alpha}}$, $\hat{\boldsymbol{\beta}}$, $\hat{\mathbf{c}}$) In model (2), we have*

(*a*)

$$
\begin{pmatrix} \sqrt{T}\left(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha} - \boldsymbol{\alpha}^*\right) \\ \sqrt{T}\left(\boldsymbol{\beta} - \boldsymbol{\beta} - \boldsymbol{\beta}^*\right) \end{pmatrix} \Rightarrow N\left(\mathbf{0}, \boldsymbol{\Sigma}\right);
$$

(*b*)

$$
\sqrt{T}\left(\hat{c} - c - c^{**}\right) \Rightarrow N\left(0, \Sigma^*\right) + f\left[B_1(r), B_2(r)\right],
$$

where $\boldsymbol{\alpha}^*$, $\boldsymbol{\beta}^*$, c^{**} are, respectively, the biases of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, c; $\boldsymbol{\Sigma}$ and *Σ^* are matrices of non-random elements; $f\left[B_1(r), B_2(r)\right]$ is a functional of $B_1(r)$ and $B_2(r)$. The expressions for $\boldsymbol{\alpha}^*$ and $\boldsymbol{\beta}^*$ are given below. The precise expressions for c^{**}, $\boldsymbol{\Sigma}$, Σ^* and $f(\cdot)$ are lengthy and unnecessary to follow the discussion here, but are given in Appendix C of the Supplement to the paper.*

Proof. We only sketch the outline of the proof which consists of the following steps. [For the precise proof, see Appendix C in the Supplement to the paper.]

Step 1. After lengthy manipulation, we can show that

$$\begin{aligned} &\sqrt{T}\begin{pmatrix} \hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha} \\ \hat{\boldsymbol{\beta}}-\boldsymbol{\beta} \end{pmatrix} \\ &= HA_T^{-1}B_{1,T}+H^{(1)}\left(D_T^{-1}B_{2,T}-\frac{1}{\sqrt{T}}D_T^{-1}C_TA_T^{-1}B_{1,T}\right)+o_p(1), \quad (12) \end{aligned}$$

where

$$H^{(1)}\left(D_T^{-1}B_{2,T}-\frac{1}{\sqrt{T}}D_T^{-1}C_TA_T^{-1}B_{1,T}\right)=o_p(1),$$

and

$$\sqrt{T}\begin{pmatrix} \hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha} \\ \hat{\boldsymbol{\beta}}-\boldsymbol{\beta} \end{pmatrix}=HA_T^{-1}B_{1,T}+o_p(1). \quad (13)$$

Step 2. We can show that

$$\begin{aligned} A_T &\Rightarrow E\begin{pmatrix} W'W & W'V \\ V'W & V'V \end{pmatrix}\equiv\Sigma_1 \text{ as } T\to\infty, \\ B_{1,T} &= \sqrt{T}\begin{pmatrix} P_{wu} \\ \mathbf{0} \end{pmatrix}+u+o_p(1),\ u\sim N(\mathbf{0},\boldsymbol{\Sigma}_2). \end{aligned}$$

Step 3. Substituting the above two formulae into (13) yields

$$\sqrt{T}\begin{pmatrix} \hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha} \\ \hat{\boldsymbol{\beta}}-\boldsymbol{\beta} \end{pmatrix}=\sqrt{T}\begin{pmatrix} \boldsymbol{\alpha}^* \\ \boldsymbol{\beta}^* \end{pmatrix}+v+o_p(1),\ v\sim N(\mathbf{0},\Sigma)$$

with

$$\begin{pmatrix} \boldsymbol{\alpha}^* \\ \boldsymbol{\beta}^* \end{pmatrix}=H\Sigma_1^{-1}\begin{pmatrix} P_{wu} \\ \mathbf{0} \end{pmatrix},\ P_{wu}=\begin{pmatrix} \sum_{i=0}^{\infty}\phi_ic_{i+1} \\ \sum_{i=0}^{\infty}\phi_ic_{i+2} \\ \vdots \\ \sum_{i=0}^{\infty}\phi_ic_{i+p} \end{pmatrix},\Sigma{=}H\Sigma_1^{-1}\Sigma_2\Sigma_1^{-1}H',$$

which produces Theorem 1(a).

Step 4. Similarly, we can show that

$$\sqrt{T}\left(\hat{c}-c\right)=H^{(2)}A_T^{-1}B_{1,T}+\mathbf{e}_2'\left(D_T^{-1}B_{2,T}-\frac{1}{\sqrt{T}}D_T^{-1}C_TA_T^{-1}B_{1,T}\right)+o_p\left(1\right), \tag{14}$$

where

$$\begin{aligned} H^{(2)}A_T^{-1}B_{1,T} &= \sqrt{T}c^{**}+w+o_p\left(1\right),\ w\sim N\left(0,\Sigma^*\right), \\ \mathbf{e}_2'\left(D_T^{-1}B_{2,T}-\frac{1}{\sqrt{T}}D_T^{-1}C_TA_T^{-1}B_{1,T}\right) &\Rightarrow f\left[B_1\left(r\right),B_2\left(r\right)\right]\ \text{ as } T\to\infty, \end{aligned}$$

implying that Theorem 1(b) holds. ■

Now consider the OLS-based estimator for δ defined in (6):

$$\hat{\delta}=\frac{\hat{B}\left(1\right)}{1-\hat{A}\left(1\right)}=\frac{\sum_{j=\mathbf{0}}^{q}\hat{\beta}_j}{1-\sum_{i=1}^{p}\hat{\alpha}_i}, \tag{15}$$

which can be rewritten as

$$\sqrt{T}\left(\hat{\delta}-\delta\right)=\frac{\left(\delta\mathbf{1}_p'\ \mathbf{1}_{q+1}'\right)\begin{pmatrix}\sqrt{T}\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}\right)\\ \sqrt{T}\left(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}\right)\end{pmatrix}}{1-\mathbf{1}'\hat{\boldsymbol{\alpha}}}. \tag{16}$$

By virtue of (13), we can write

$$\left(\delta\mathbf{1}_p'\ \mathbf{1}_{q+1}'\right)\begin{pmatrix}\sqrt{T}\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}\right)\\ \sqrt{T}\left(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}\right)\end{pmatrix}=\left(\delta\mathbf{1}_p'\ \mathbf{1}_{q+1}'\right)HA_T^{-1}B_{1,T}+o_p\left(1\right).$$

Note that

$$\left(\delta\mathbf{1}_p'\ \mathbf{1}_{q+1}'\right)H=\left(\delta\mathbf{1}_p'\ \mathbf{1}_{q+1}'\right)\begin{pmatrix}I_p & \mathbf{0}_{p\times q}\\ H_1 & H_2\end{pmatrix}=\mathbf{0}_{1\times(p+q)}. \tag{17}$$

This means that the asymptotic distribution of $\sqrt{T}\left(\hat{\delta}-\delta\right)$ degenerates as $T\to\infty$. Therefore, we cannot use $\sqrt{T}$ as the normalizer as the denominator degenerates in obtaining the asymptotic distribution of $\left(\hat{\delta}-\delta\right)$ as $T\to\infty$.

Instead, normalize $\left(\hat{\delta}-\delta\right)$ by T to calculate the asymptotic distribution of $\hat{\delta}$ directly, and we have the following result:

Theorem 2 *The asymptotic distribution of* $T\left(\hat{\delta}-\delta\right)$, *defined in (15), is given by*

$$T\left(\hat{\delta}-\delta\right) \Rightarrow \frac{f_a}{f_b}$$

where

$$\begin{aligned} f_a &= C(1)\left[\int_0^1 B_2(r)dB_1(r) - B_1(1)\int_0^1 B_2(r)dr\right] + \\ &\left[\int_0^1 B_2(r)dr\left(\phi(1)B_1(1) - \left(\delta+\sum_{i=0}^{\infty} i\varphi_i\right)B_2(1)\right)\mathbf{1}_p' - f_{WZ}'\right]\Sigma^{11}P_{wu} \\ &+\left[\int_0^1 B_2(r)drB_2(1)\mathbf{1}_q' - f_{\epsilon Z}'\right]\Sigma^{21}P_{wu} \end{aligned}$$

and

$$f_b = (1-\mathbf{1}'\boldsymbol{\alpha}-\mathbf{1}'\boldsymbol{\alpha}^*)\left\{\int_0^1 B_2^2(r)dr - \left[\int_0^1 B_2(r)dr\right]^2\right\}.$$

In the above expressions, f_{WZ}' and $f_{\epsilon Z}'$ *are functionals of the Brownian motion* B_1 *and* B_2; *and* Σ^{11} *and* Σ^{21} *are matrices of non-random elements. Their precise expressions are unnecessary here, but are given in Appendix B of the Supplement to the paper.* ■

Proof. Here we only give a sketch of the proof. [For the precise proof, see Appendix D in the Supplement to the paper.]

Step 1. We normalize $\left(\hat{\delta}-\delta\right)$ by T :

$$T\left(\hat{\delta}-\delta\right) = \frac{\left(\delta\mathbf{1}_p'\ \mathbf{1}_{q+1}'\right)\begin{pmatrix} T\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}\right) \\ T\left(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}\right)\end{pmatrix}}{1-\mathbf{1}'\hat{\boldsymbol{\alpha}}}. \tag{18}$$

Step 2. From (12) and (17) it can be written as

$$
\begin{aligned}
& \left(\delta \mathbf{1}_p' \; \mathbf{1}_{q+1}'\right) \begin{pmatrix} T\left(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}\right) \\ T\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \end{pmatrix} \\
= \; & (0,1)\left(\sqrt{T} D_T^{-1} B_{2,T} - D_T^{-1} C_T A_T^{-1} B_{1,T}\right) + o_p(1). \qquad (19)
\end{aligned}
$$

Step 3. Applying the functional central limit theorem and the continuous mapping theorem to (19) , we can show that as $T \to \infty$

$$
(0,1)\left(\sqrt{T} D_T^{-1} B_{2,T} - D_T^{-1} C_T A_T^{-1} B_{1,T}\right) \Rightarrow \frac{f_a}{\int_0^1 B_2^2(r)dr - \left[\int_0^1 B_2(r)dr\right]^2}
$$

and

$$
1 - \mathbf{1}'\hat{\boldsymbol{\alpha}} \xrightarrow{p} 1 - \mathbf{1}'\boldsymbol{\alpha} - \mathbf{1}'\boldsymbol{\alpha}^*.
$$

It follows from (18) that Theorem 2 holds. ■

Remark 1 *From Theorems 1 and 2, the asymptotic distributions of* $\sqrt{T}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha} - \boldsymbol{\alpha}^*)$ *and* $\sqrt{T}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta} - \boldsymbol{\beta}^*)$ *are normal, but the asymptotic distributions of* $\sqrt{T}\left(\hat{c} - c - c^{**}\right)$ *and* $T(\hat{\delta} - \delta)$ *are non-standard. Furthermore, although the OLS estimators* $\hat{c}$, $\hat{\boldsymbol{\alpha}}$ *and* $\hat{\boldsymbol{\beta}}$ *in our model are* $\sqrt{T}-$*inconsistent, the OLS-based estimator* $\hat{\delta}$ *is* $T-$*consistent, or superconsistent.*

Next we examine how the serial correlation in u_t affects the asymptotic properties of $\hat{c}$, $\hat{\boldsymbol{\alpha}}$, $\hat{\boldsymbol{\beta}}$, and $\hat{\delta}$ in the simplest case of $ADL(1,0)$, i.e.,

$$
\begin{array}{c}
y_t = c + \alpha y_{t-1} + \beta z_t + u_t, \;\; \mid \alpha \mid < 1 \\
u_t = \rho u_{t-1} + v_t, \mid \rho \mid < 1 \\
z_t = z_{t-1} + \varepsilon_t, \\
t = 1, 2, \cdots, T,
\end{array} \qquad (20)
$$

where $v_t \sim i.i.d.N(0,\sigma_1^2)$ and $\varepsilon_t \sim i.i.d.N(0,\sigma_2^2)$ are assumed to be independent. When $c = 0$, (20) reduces to the model in MYTH. In this case, the asymptotic distributions of $\hat{c}$, $\hat{\alpha}$ and $\hat{\beta}$ are given in the following corollaries. The proofs are given in an earlier version of this paper, which is obtainable from the authors upon request.

Corollary 1 *In model (20), we have*
(a) $\text{plim}(\hat{\alpha} - \alpha) = \alpha^*$;
(b) $\text{plim}(\hat{\beta} - \beta) = \beta^*$;
(c) $\text{plim}(\hat{c} - c) = c^*$;
(d) $\sqrt{T}(\hat{\alpha} - \alpha - \alpha^*) \Rightarrow N(0, \tilde{\sigma}_1^2)$;
(e) $\sqrt{T}(\hat{\beta} - \beta - \beta^*) \Rightarrow \frac{\beta}{1-\alpha} N(0, \tilde{\sigma}_1^2)$;
(f) $\sqrt{T}(\hat{c} - c - c^*) \Rightarrow N(0, \tilde{\sigma}_2^2) + \frac{1}{1-\rho} \frac{B_1(1)\int_0^1 B_2^2(r)dr - \int_0^1 B_2(r)dr[B_2(r)dB_1(r) - (1-\rho)\gamma^* Q_2]}{\int_0^1 B_2^2(r)dr - [\int_0^1 B_2(r)dr]^2}$,

in which
$\alpha^* = \frac{P}{Q_3}$, $\beta^* = -\frac{\beta}{1-\alpha}\alpha^*$, $c^* = -\frac{c}{1-\alpha}\alpha^*$,
$\tilde{\sigma}_1^2 = \lambda_1^2 \sigma_1^2 \sigma_2^2 + \lambda_2^2 \sigma_1^4$,
$\lambda_1 = -\frac{\beta}{(1-\alpha)^2(1-\rho)Q_3}$, $\lambda_2 = [\frac{1}{(1-\alpha)(1-\rho)^2} - \frac{\rho}{(1-\alpha\rho)(1-\rho^2)}]\frac{1}{Q_3}$,
$\tilde{\sigma}_2^2 = \mu_1^2 \sigma_1^2 \sigma_2^2 + \mu_2^2 \sigma_1^4 + \mu_3^2 \sigma_1^2$,
$\mu_1 = -\frac{c}{1-\alpha}\lambda_1$, $\mu_2 = -\frac{c}{1-\alpha}\lambda_2$, $\mu_3 = (\frac{c}{1-\alpha})^2 \frac{1}{(1-\rho)Q_3}$,
$P = \frac{\rho\sigma_1^2}{(1-\alpha\rho)(1-\rho^2)}$,
$Q_2 = (\frac{1}{1-\alpha})(\frac{1}{1-\rho})\int_0^1 B_2(r)dB_1(r) - \frac{1}{2}(\frac{\beta}{1-\alpha})(\frac{1}{1-\alpha})[B_2^2(1) + \sigma_2^2]$,
$Q_3 = (\frac{1}{1-\alpha^2})(\frac{\beta}{1-\alpha})^2\sigma_2^2 + \frac{(1+\alpha\rho)\sigma_1^2}{(1-\alpha^2)(1-\alpha\rho)(1-\rho^2)}$. ■

Note that $\hat{c}$, $\hat{\alpha}$ and $\hat{\beta}$ are $\sqrt{T}-$consistent only if there is no serial correlation in u_t, that is, $\rho = 0$ in model (20).

Corollary 2 *In model* (20), *the asymptotic distribution of* $\hat{\delta}$ *collapses to the following:*
$T(\hat{\delta} - \delta) \Rightarrow$

$$\frac{\frac{1}{1-\rho}\int_0^1 B_2(r)dB_1(r) - \{\frac{1}{1-\rho}B_1(1) + \frac{\beta}{1-\alpha}\mu B_2(1)\}\int_0^1 B_2(r)dr + \frac{1}{2}\frac{\beta}{1-\alpha}\mu\{B_2^2(1) + \sigma_2^2\}}{(1-\alpha)\{\int_0^1 B_2^2(r)dr - (\int_0^1 B_2(r)dr)^2\}}, \tag{21}$$

where $\mu = \frac{(1+\alpha)\rho\sigma_1^2}{(1-\alpha\rho)(1-\rho^2)(\frac{\beta}{1-\alpha})^2\sigma_2^2 + (1-\rho)\sigma_1^2}$. ■

The following corollary shows that $\int_0^1 B_2(r)dr$ occurs only if $c \neq 0$.

Corollary 3 *When the constant term $c = 0$ in model (20), we have*

$$T(\hat{\delta} - \delta) \Rightarrow \frac{\frac{1}{1-\rho}\int_0^1 B_2(r)dB_1(r) + \frac{1}{2}\frac{\beta}{1-\alpha}\mu\{B_2^2(1) + \sigma_2^2\}}{(1-\alpha)\int_0^1 B_2^2(r)dr}. \tag{22}$$

■

Remark 2 *Comparing (21) and (22), the asymptotic distributions depend on the existence of the constant term, c, but not on its value.*

Since the long-run relationship in model (20) is given by

$$y_{t-1} = \frac{\beta}{1-\alpha}z_t + \frac{c}{1-\alpha} + a_t + o_p(1), \tag{23}$$

it is possible to estimate $\delta = \beta/(1-\alpha)$ by simply regressing y_{t-1} on z_t. We have the following corollary:

Corollary 4 *The asymptotic distribution of the OLS−estimator $\hat{\delta}'$ based on* (23) *is given by:*

$$T(\hat{\delta}' - \delta) \Rightarrow$$

$$\frac{\frac{1}{1-\rho}\int_0^1 B_2(r)dB_1(r) - [\frac{1}{1-\rho}B_1(1) - \frac{\beta}{1-\alpha}B_2(1)]\int_0^1 B_2(r)dr - \frac{1}{2}\frac{\beta}{1-\alpha}[B_2^2(1) + \sigma_2^2]}{(1-\alpha)[\int_0^1 B_2^2(r)dr - (\int_0^1 B_2(r)dr)^2]}. \tag{24}$$

Remark 3 *The asymptotic distributions given in (21) and (24) differ slightly from each other (note that μ in (21) is not included in (24)).*

4 An Alternative Estimator of the Long-run Coefficient

In this section, we introduce an alternative estimator of δ in (2). In the previous section, it was shown that the serial correlation in u_t causes the inconsistency of the OLS estimators $\hat{c}$, $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$. To obtain a consistent estimator, we transform the model to eliminate the serial correlation before applying OLS. Assume

$$C^*(L) = 1/C(L) = 1 - c_1^*L - c_2^*L^2 - \cdots - c_l^*L^l.$$

Then, the model (2) can be rewritten as:

$$\begin{aligned} y_t &= \mu^* + A^*(L)y_t + B^*(L)z_t + v_t, \\ u_t &= C(L)v_t, \\ z_t &= z_{t-1} + \varepsilon_t, \\ t &= 1, 2, \cdots, T, \end{aligned} \tag{25}$$

where

$$\begin{aligned} \mu^* &= cC^*(1), \\ A^*(L) &= 1 - C^*(L) + C^*(L)A(L) = a_1^*L + a_2^*L^2 + \cdots + a_{p+l}^*L^{p+l}, \\ B^*(L) &= C^*(L)B(L) = b_0^* + b_1^*L + \cdots + b_{q+l}^*L^{q+l}. \end{aligned}$$

In this model, the long-run effect of z on y is defined as above, namely $B^*(1)/[1 - A^*(1)]$. It is straightforward to show that

$$\frac{B^*(1)}{1 - A^*(1)} = \frac{C^*(1)B(1)}{1 - [1 - C^*(1) + C^*(1)A(1)]} = \frac{B(1)}{1 - A(1)} = \delta.$$

We propose an alternative estimator, defined by

$$\check{\delta} = \frac{\check{B}^*(1)}{1 - \check{A}^*(1)} = \frac{\sum_{j=0}^{q+l} \check{b}_j^*}{1 - \sum_{i=1}^{p+l} \check{a}_i^*}, \tag{26}$$

where $\check{a}_i^*, i = 1, \cdots, p+l$, and $\check{b}_j^*, j = 0, 1, \cdots, q+l$, are the OLS estimators for model (25). As this transformation assumes that $v_t \sim i.i.d.N(0, \sigma_1^2)$, it is independent of y_{t-1} and z_t, so that the OLS estimators $\check{a}_i^*$ and $\check{b}_j^*$, $i = 1, 2, \cdots, p+l$, $j = 0, 1, \cdots, q+l$, are consistent. Therefore, $\check{\delta}$ is also a consistent estimator. We would expect $\check{\delta}$ to have better distributional properties than $\hat{\delta}$ in small samples because the OLS estimators, $\hat{\alpha}_i$ and $\hat{\beta}_i$, are inconsistent.

Using a similar method to that employed in Section 3, we present the following Theorem.

Theorem 3 *For model (25), the asymptotic distribution of the estimator, $\check{\delta}$, defined in (26) is given as:*

$$T\left(\check{\delta} - \delta\right) \Rightarrow \frac{C(1)}{1 - A(1)} \cdot \frac{\int_0^1 B_2(r)dB_1(r) - B_1(1)\int_0^1 B_2(r)dr}{\int_0^1 B_2^2(r)dr - \left[\int_0^1 B_2(r)dr\right]^2}.$$

Proof. See Appendix E in the Supplement to the paper. ■

5 Simulation Experiments

Although we have shown that $\hat{\delta}$ and $\check{\delta}$ are T-consistent, and both $T(\check{\delta}-\delta)$ and $T(\hat{\delta}-\delta)$ converge to non-standard distributions, we do not know the small sample properties of $\hat{\delta}$ and $\check{\delta}$.

To investigate their finite sample performance, we conduct some Monte Carlo experiments by using the model given by

$$\begin{aligned} y_t &= a_0 + a_1 y_{t-1} + a_2 y_{t-2} + a_3 z_t + a_4 z_{t-1} + v_t, \\ z_t &= z_{t-1} + \varepsilon_t, t = 1, 2, \cdots, T. \end{aligned}$$

In the experiments, we fix the parameters as follows:

$$a_0 = c(1-\rho), a_1 = \alpha + \rho, a_2 = -\alpha\rho, a_3 = \beta, a_4 = -\beta\rho,$$

and

$$\sigma_2^2 = 1.0, \sigma_1^2 = 0.25, c = 1, \alpha = 0.38, \beta = 0.4, \delta = 0.645.$$

We specify the other parameters as $\rho = 0.0, 0.5, 0.8$; $T = 50, 100, 500$; and calculate $\check{d} = (\check{\delta}-\delta)/s_{\check{\delta}}$ and $\hat{d} = (\hat{\delta}-\delta)/s_{\hat{\delta}}$ 5000 times for each parameter combination, where $s_{\check{\delta}}$ and $s_{\hat{\delta}}$ represent the estimated standard errors.

Figures 1 through 9 are the empirical distributions obtained from the experiments. From these figures we observe that $\check{\delta}$ is almost unbiased, but $\hat{\delta}$ is slightly biased in small samples. The bias does not vary greatly as the sample size increases, but increases with ρ.

To compare the performance of $\check{\delta}$ and $\hat{\delta}$, we calculate the sample mean squared errors (MSE):

$$\check{e} = \frac{1}{5000}\sum_{i=1}^{5000}(\check{\delta}_i - \delta)^2, \ \hat{e} = \frac{1}{5000}\sum_{i=1}^{5000}(\hat{\delta}_i - \delta)^2.$$

The calculated values of $\check{e}$ and $\hat{e}$ are given in Table 1.

It can be seen that the difference between $\check{e}$ and $\hat{e}$ becomes large as ρ increases. Moreover, the values of $\hat{e}$ are generally larger than those of $\check{e}$ if $\rho \neq 0$, the difference becoming small as the sample size increases. Judging by the MSE criterion, $\check{\delta}$ is superior to $\hat{\delta}$.

6 Concluding Remarks

In this paper, we developed an asymptotic theory for the estimators in a general autoregressive distributed lag model with serially correlated disturbances and integrated regressors. It was found that the OLS estimators $\hat{c}$, $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ for the regression coefficients are $\sqrt{T}$*-inconsistent* but have asymptotic normal distributions, and that the OLS-based estimator $\hat{\delta}$ for the long-run coefficient is T-consistent, i.e., superconsistent, but with a nonstandard asymptotic distribution. Therefore, standard statistical inference which relies on asymptotic normality for the regression coefficients and the long-run coefficient can be misleading. Furthermore, we proposed an alternative estimator $\check{\delta}$ for the long-run coefficient, obtained by transforming the original model to eliminate the serial correlation in the disturbances, and examined the asymptotic properties of the proposed estimator. Monte Carlo experiments showed that the proposed estimator $\check{\delta}$ is MSE*-superior* to the OLS-based estimator.

Table 1. Comparison of $\check{\delta}$ and $\hat{\delta}$ by MSE

ρ	MSE	T=50	T=100	T=500
0.0	$\hat{e}$	0.003146	0.000772	0.000003
	$\check{e}$	0.003593	0.000813	0.000003
0.5	$\hat{e}$	0.013747	0.003642	0.000150
	$\check{e}$	0.013722	0.003041	0.000115
0.8	$\hat{e}$	0.101997	0.027711	0.001292
	$\check{e}$	0.116559	0.018894	0.000689

Note. $\hat{e} = \frac{1}{5000}\sum_{i=1}^{5000}(\hat{\delta}_i - \delta)^2$ and $\check{e} = \frac{1}{5000}\sum_{i=1}^{5000}(\check{\delta}_i - \delta)^2$.

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Figure 1. ($T = 50$, $\rho = 0.0$)

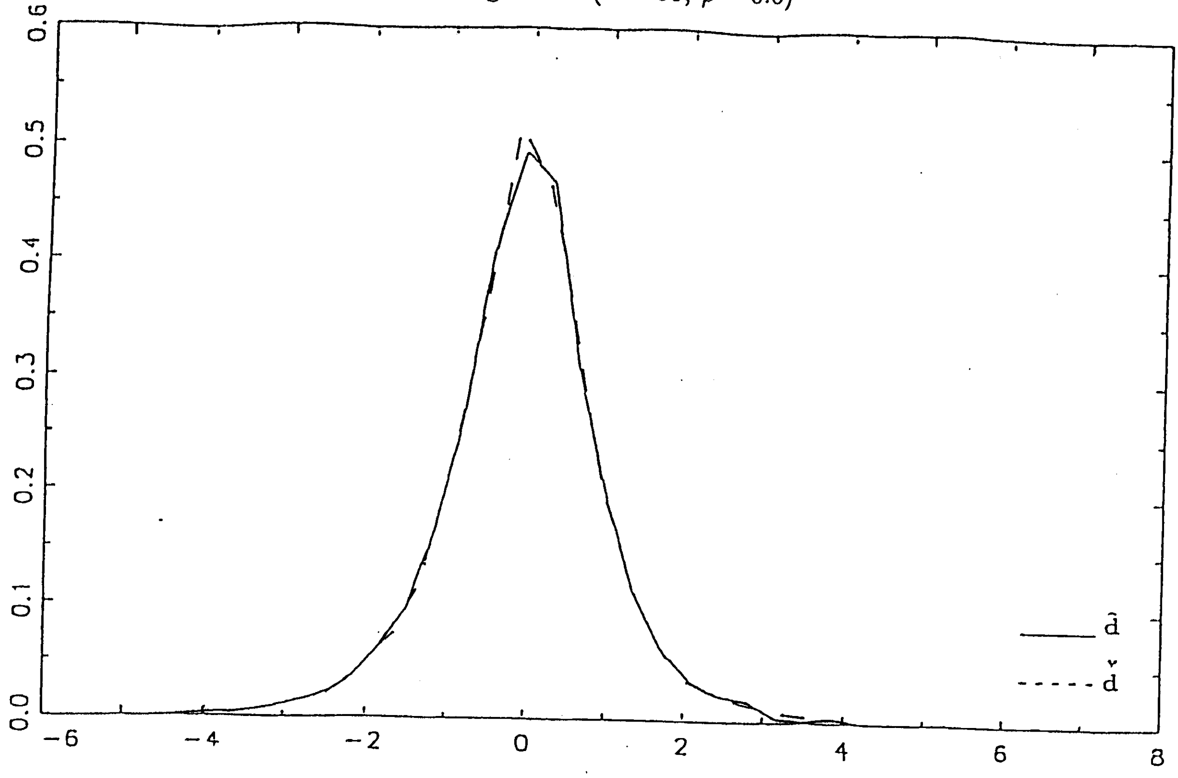


Figure 2. ($T = 50$, $\rho = 0.5$)

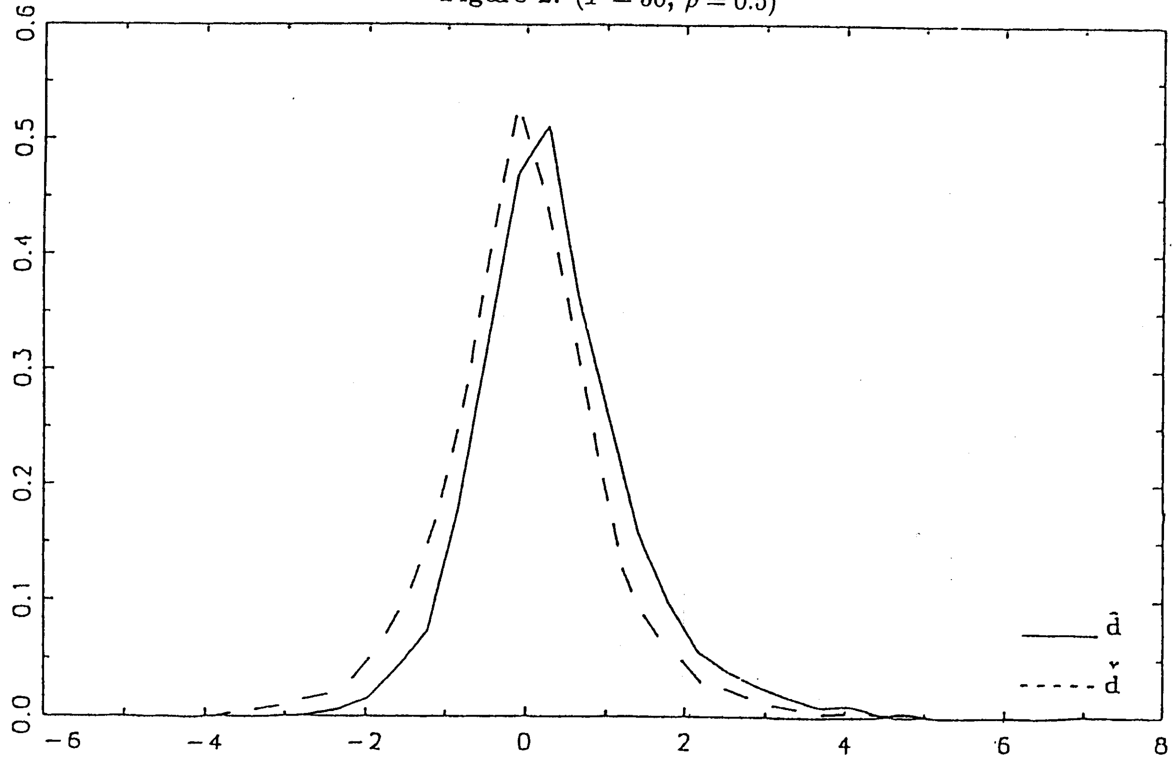


Figure 3. ($T = 50$, $\rho = 0.8$)

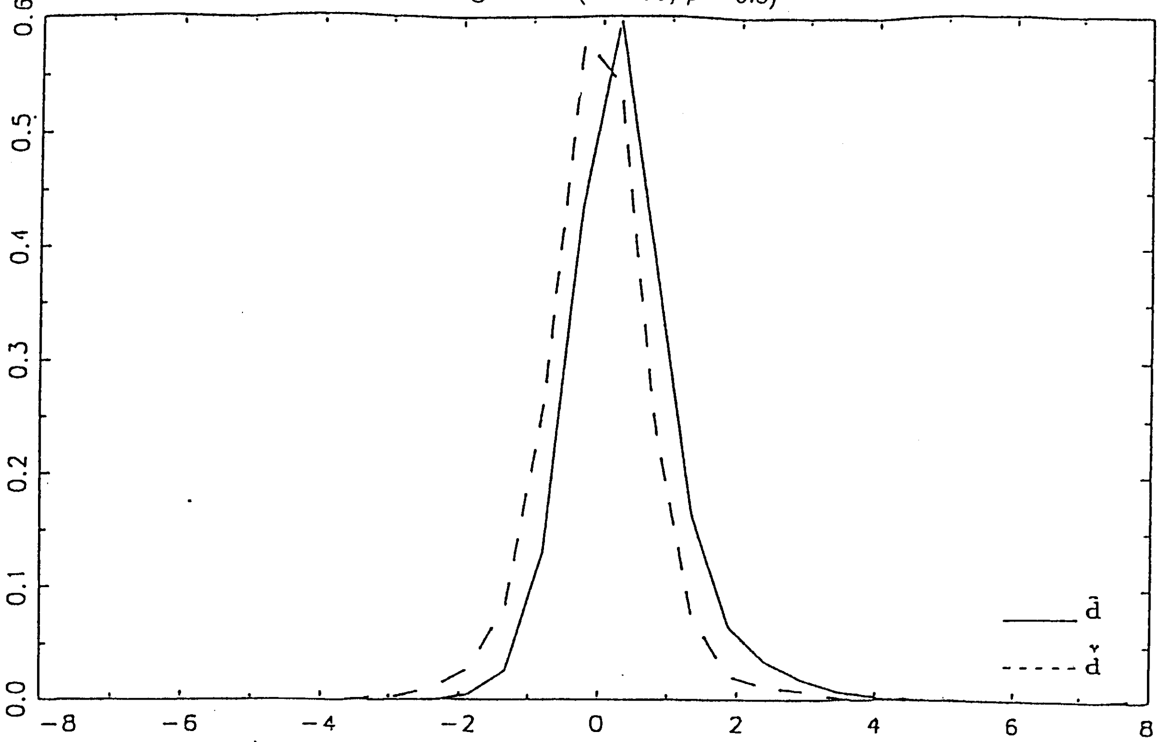


Figure 4. ($T = 100$, $\rho = 0.0$)

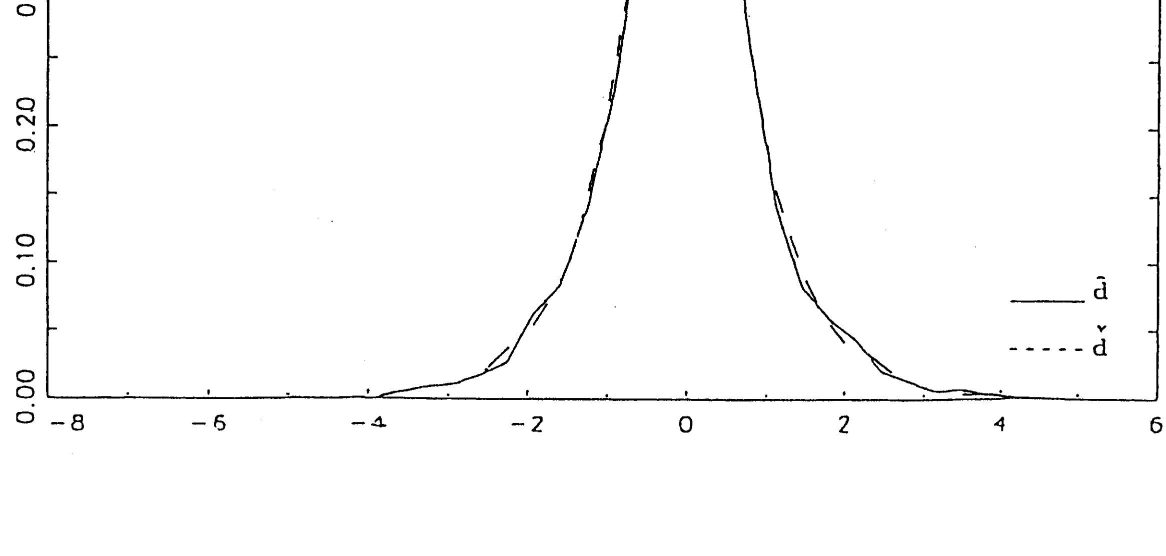


Figure 5. ($T = 100$, $\rho = 0.5$)

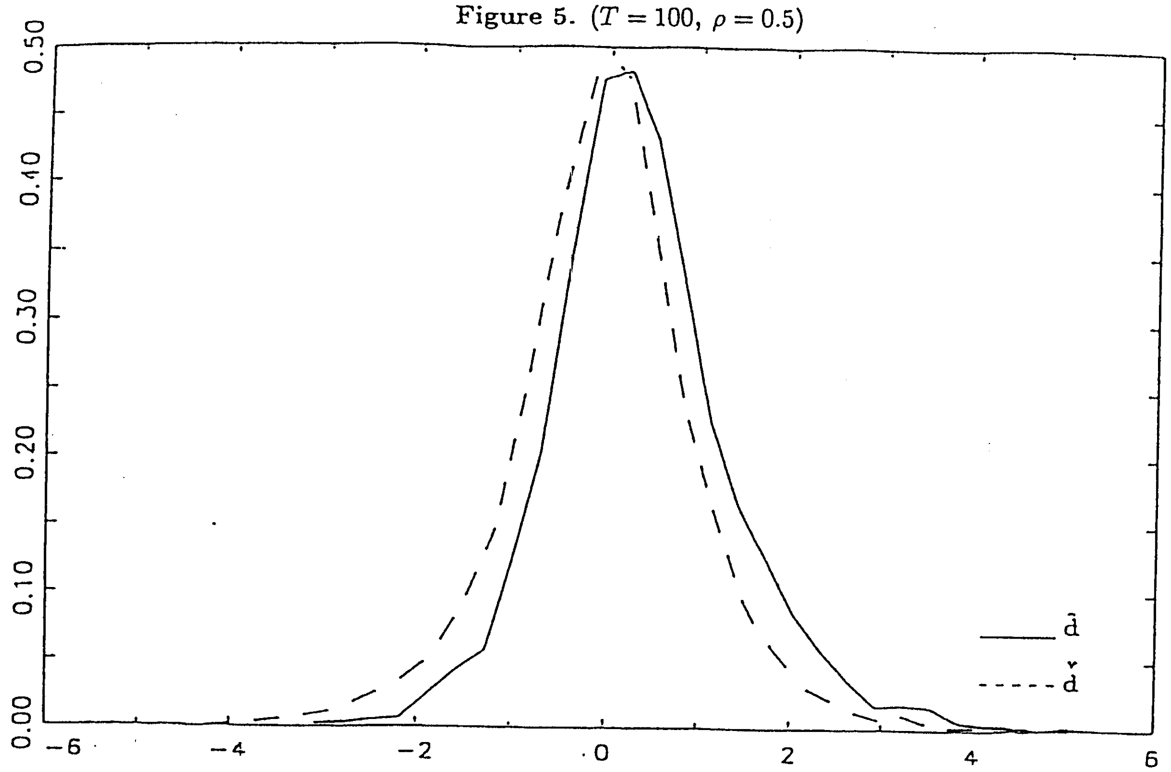


Figure 6. ($T = 100$, $\rho = 0.8$)

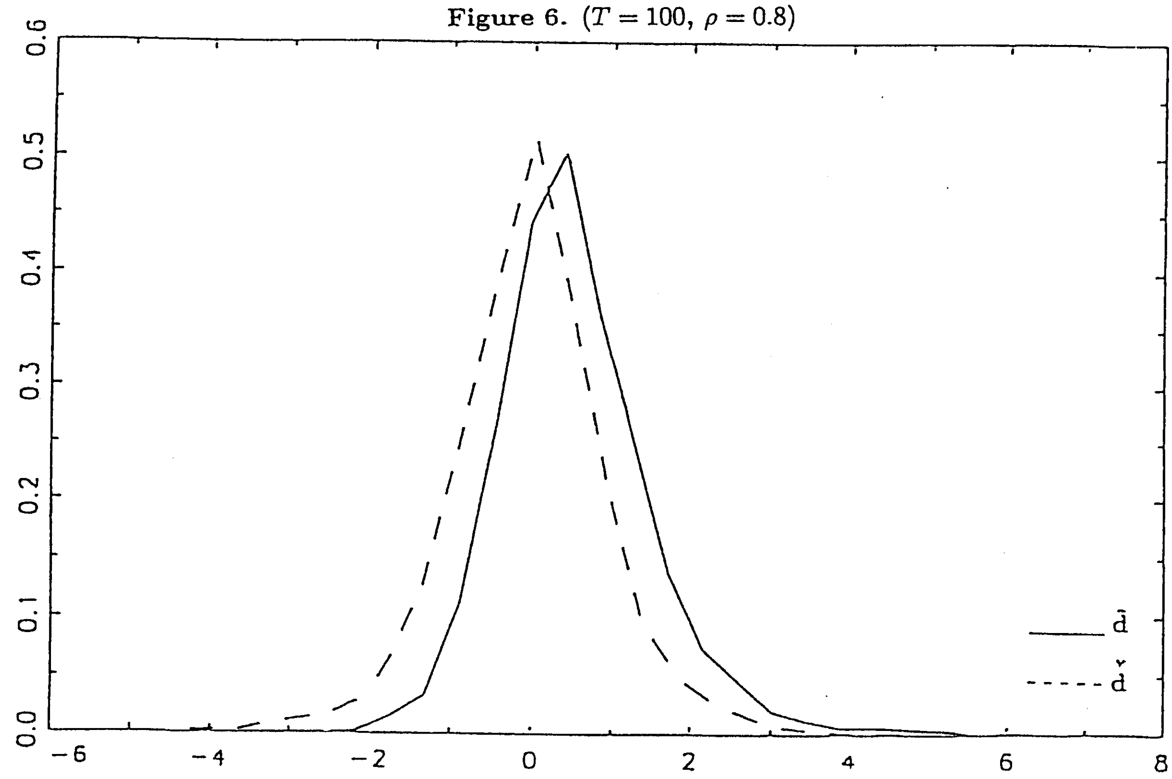


Figure 7. $(T = 500,\ \rho = 0.0)$

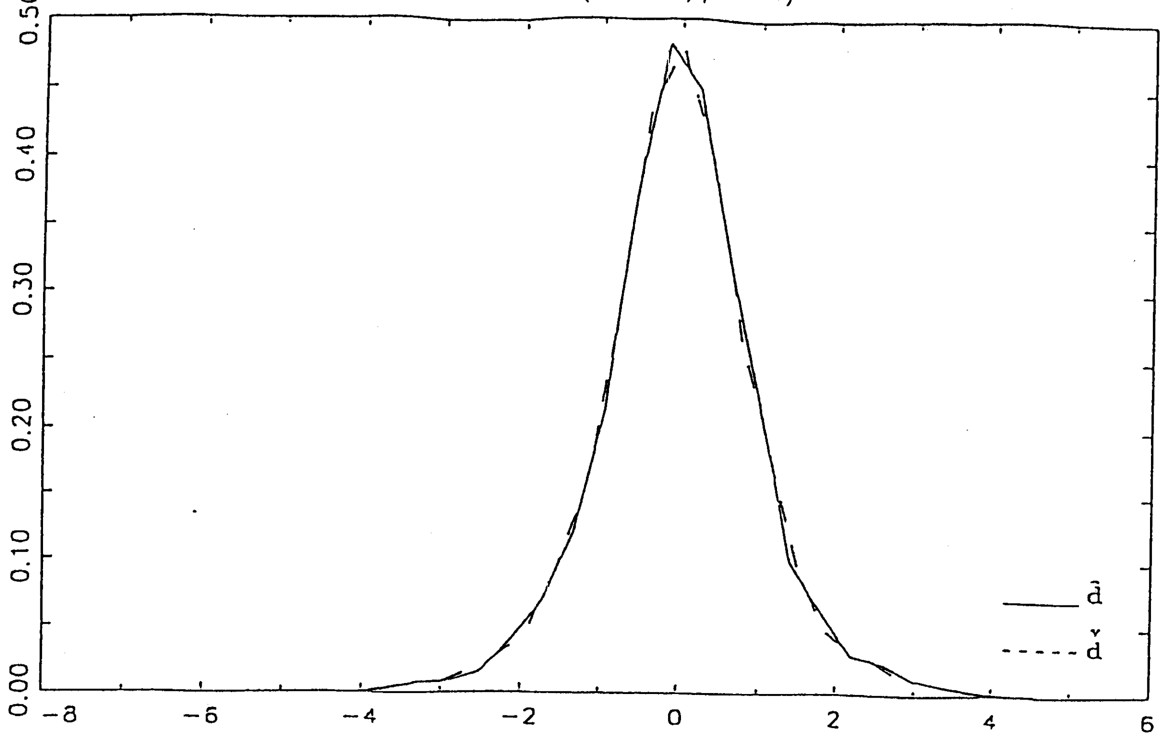


Figure 8. $(T = 500,\ \rho = 0.5)$

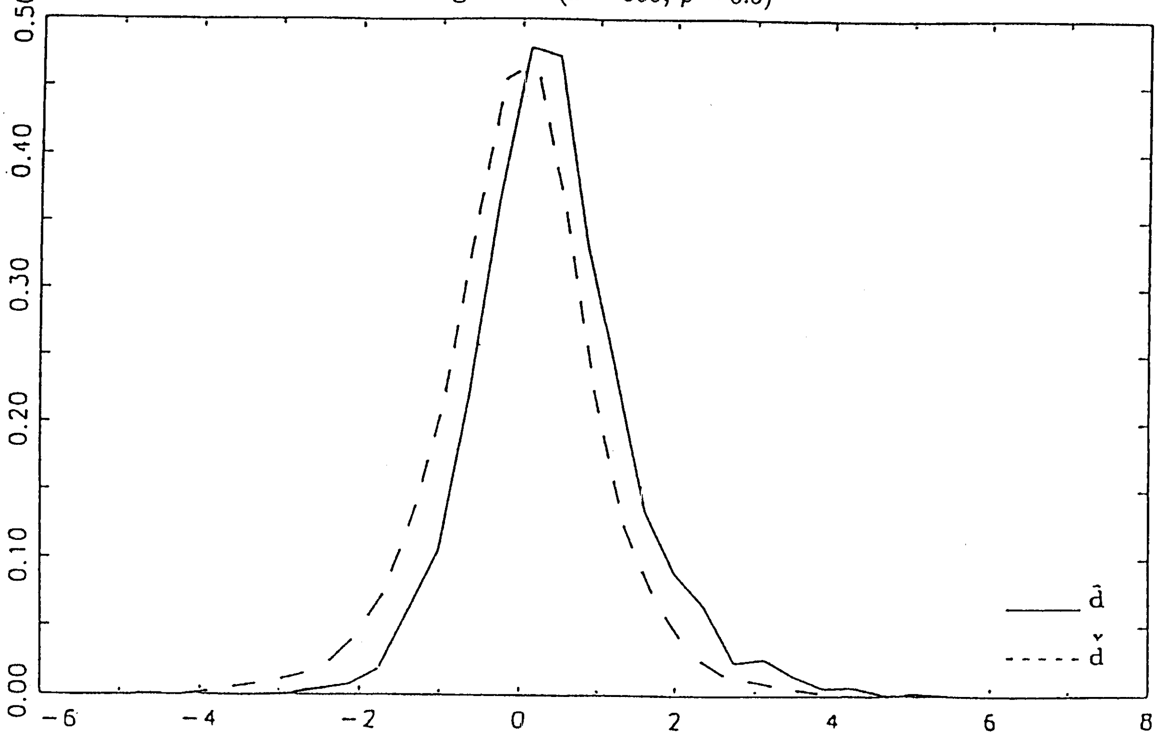
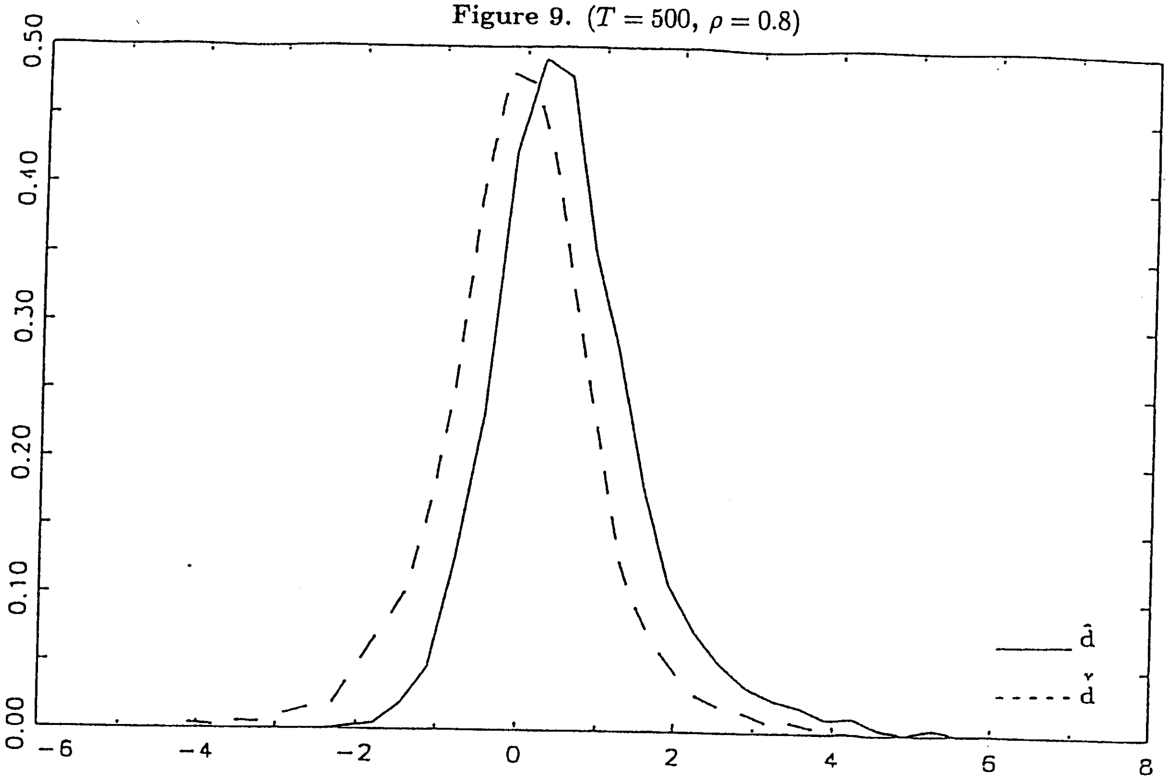


Figure 9. ($T = 500$, $\rho = 0.8$)



Supplement to "Asymptotic Properties of the Estimator of the Long-run Coefficient in a Dynamic Model with Integrated Regressors and Serially Correlated Errors"

Zonglu He[1)], Koichi Maekawa[2)], and Michael McAleer[3)]
[1)]Risshikan University
[2)]Hiroshima University
[3)]University of Western Australia

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This supplement is written for readers who are interested in the proofs and derivations for "Asymptotic Properties of the Estimator of the Long-run Coefficient in a Dynamic Model with Integrated Regressors and Serially Correlated Errors", by K. Maekawa, M. McAleer and Z. He (2002) (henceforth MMH).

Appendix A

Proof of (7) **in MMH.** First, we give the proof of $\sum_{i=0}^{\infty} i\left|\varphi_i\right| < \infty$.

Suppose that $1/\lambda_k$, $k = 1, 2, \cdots, p$, are the roots of $1 - A(x) = 0$. Then we can write

$$\frac{1}{1 - A(L)} = \frac{1}{(1-\lambda_1 L)(1-\lambda_2 L)\cdots(1-\lambda_p L)} = \sum_{k=1}^{p} \frac{\kappa_k}{1-\lambda_k L}$$

and

$$\varphi(L) \equiv \sum_{i=0}^{\infty} \varphi_i L^i = \frac{B(L)}{1-A(L)} = \sum_{k=1}^{p} \frac{\kappa_k B(L)}{1-\lambda_k L} = \sum_{i=0}^{\infty}\sum_{j=0}^{q} \beta_j \left(\sum_{k=1}^{p} \kappa_k \lambda_k^i\right) L^{i+j}.$$

Denote $\kappa_i^* = \sum_{k=1}^{p} \kappa_k \lambda_k^i$ for $i = 1, 2, \cdots$, $\kappa_i^* = 0$ for $i < 0$. It is easily seen that

$$\varphi_i = \beta_0 \kappa_i^* + \beta_1 \kappa_{i-1}^* + \cdots + \beta_q \kappa_{i-q}^*.$$

We thus have

$$\sum_{i=0}^{\infty} i\left|\varphi_i\right| \le \left|\beta_0\right| \sum_{i=0}^{\infty} i\left|\kappa_i^*\right| + \left|\beta_1\right| \sum_{i=0}^{\infty} i\left|\kappa_{i-1}^*\right| + \cdots + \left|\beta_q\right| \sum_{i=0}^{\infty} i\left|\kappa_{i-q}^*\right|,$$

where

$$\sum_{i=0}^{\infty} i\left|\kappa_i^*\right| \leq \left|\kappa_1\right| \sum_{i=0}^{\infty} i\left|\lambda_1\right|^i + \cdots + \left|\kappa_p\right| \sum_{i=0}^{\infty} i\left|\lambda_p\right|^i < \infty$$

since $|\lambda_k| < 1$ for $k = 1, 2, \cdots, p$. We can similarly prove that $\sum_{i=0}^{\infty} i\left|\kappa_{i-j}^*\right| < \infty, j = 1, 2, \cdots, q$. It follows that

$$\sum_{i=0}^{\infty} i\left|\varphi_i\right| < \infty$$

holds. We can prove $\sum_{i=0}^{\infty} \mid \phi_i \mid < \infty$ in the same manner. ■

Appendix B

To obtain the asymptotic distributions of the OLS-estimator (11) in MMH, the following lemma is useful. In Lemma 1, $(a_k)_{k=0,1,2,\cdots,p-1}$ denotes $(a_0, a_1, a_2, \cdots, a_{p-1})'$ and $(a_k)_{k=1,2,\cdots,q}$ denotes $(a_1, a_2, \cdots, a_q)'$, and so on.

Lemma 1. For the series $\{x_t\}$ and $\{y_t\}$ in model (2) in MMH, we have

(a) $\frac{1}{\sqrt{T}}\mathbf{1}'\mathbf{u} \Rightarrow C(1)B_1(1)$;

(b) $\frac{1}{T}\mathbf{z}'\mathbf{u} \Rightarrow C(1)\int_0^1 B_2(r)dB_1(r)$;

(c) $\frac{1}{T^{3/2}}\mathbf{1}'\mathbf{z} \Rightarrow \int_0^1 B_2(r)dr$;

(d) $\frac{1}{T^2}\mathbf{z}'\mathbf{z} \Rightarrow \int_0^1 B_2^2(r)dr$;

(e) $\frac{1}{\sqrt{T}}W'\mathbf{u} = \sqrt{T}P_{wu} + N_{wu} + o_p(1)$,

where

$$P_{wu} = \sigma_1^2 \left(\sum_{i=0}^{\infty} \phi_i c_{i+k}\right)_{k=1,2,\cdots,p},$$

$$N_{wu} = \left(\sum_{\substack{j \neq i+k+1 \\ i,j=0,1,\cdots,\infty}} \phi_i c_j \frac{1}{\sqrt{T}} \sum_{t=1}^{T} v_{t-i-k-1} v_{t-j}\right)_{k=0,1,\cdots,p-1} + \left(-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \gamma_i^* c_j \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \varepsilon_{t-i-k} v_{t-j}\right)_{k=0,1,\cdots,p-1};$$

(f) $\frac{1}{\sqrt{T}}V'\mathbf{u}=\left(\sum\limits_{j=0}^{\infty}c_j\frac{1}{\sqrt{T}}\sum\limits_{t=0}^{T}\varepsilon_{t-k}v_{t-j}\right)_{k=0,1,\cdots,q-1}\equiv N_{\varepsilon u};$

(g) $\frac{1}{\sqrt{T}}W'\mathbf{1}\Rightarrow\left[\phi\left(1\right)B_1\left(1\right)-\left(\delta+\sum\limits_{i=0}^{\infty}i\varphi_i\right)B_2\left(1\right)\right]\mathbf{1}_p;$

(h) $\frac{1}{\sqrt{T}}V'\mathbf{1}\Rightarrow B_2\left(1\right)\mathbf{1}_q;$

(i) $\frac{1}{T}W'\mathbf{z}\Rightarrow\phi\left(1\right)\int_0^1 B_2\left(r\right)dB_1\left(r\right)-\frac{1}{2}\left(\delta+\sum\limits_{i=0}^{\infty}i\varphi_i\right)\left[B_2^2\left(1\right)+\sigma_2^2\right]\equiv f_{WZ};$

(j) $\frac{1}{T}V'\mathbf{z}\Rightarrow\frac{1}{2}\left[B_2^2\left(1\right)+\sigma_2^2\right]\mathbf{1}_q\equiv f_{\epsilon Z};$

(k) $\frac{1}{T}\begin{pmatrix}W'W & W'V\\ V'W & V'V\end{pmatrix}\xrightarrow{p}\begin{pmatrix}E\left(W'W\right) & E\left(W'V\right)\\ E\left(V'W\right) & E\left(V'V\right)\end{pmatrix}\equiv\Sigma_1,$

where

$$E\left(W'W\right)=\begin{pmatrix}\gamma_0^{(a)} & \gamma_1^{(a)} & \cdots & \gamma_{p-1}^{(a)}\\ \gamma_1^{(a)} & \gamma_0^{(a)} & \ddots & \vdots\\ \vdots & \ddots & \ddots & \gamma_1^{(a)}\\ \gamma_{p-1}^{(a)} & \cdots & \gamma_1^{(a)} & \gamma_0^{(a)}\end{pmatrix},$$

$$\gamma_k^{(a)}=E\left(\mathbf{a}'\mathbf{a}_{-k}\right)=\sigma_1^2\sum_{i=0}^{\infty}\phi_i\phi_{i+k}+\sigma_2^2\sum_{i=0}^{\infty}\gamma_i^*\gamma_{i+k}^*,k=0,1,\cdots,p-1,$$

$$E\left(W'V\right)=-\sigma_2^2\begin{pmatrix}\gamma_0^* & \gamma_1^* & \cdots & \gamma_{q-1}^*\\ 0 & \gamma_0^* & \ddots & \vdots\\ \vdots & \ddots & \ddots & \gamma_1^*\\ 0 & \cdots & 0 & \gamma_0^*\end{pmatrix},$$

$$E\left(V'V\right)=\sigma_2^2I_q;$$

(l) $D_T^{-1}B_{2,T}=\begin{pmatrix}f_{DB_2}+o_p\left(1\right)\\ O_p\left(\frac{1}{\sqrt{T}}\right)\end{pmatrix},$

where

$$f_{DB_2}=C\left(1\right)\frac{B_1(1)\int_0^1 B_2^2(r)dr-\int_0^1 B_2(r)dr\int_0^1 B_2(r)dB_1(r)}{\int_0^1 B_2^2(r)dr-\left[\int_0^1 B_2(r)dr\right]^2};$$

(m) $D_T^{-1}C_T = \begin{pmatrix} f_{DW} + o_p(1) & f_{DV} + o_p(1) \\ O_p\left(\frac{1}{\sqrt{T}}\right) & O_p\left(\frac{1}{\sqrt{T}}\right) \end{pmatrix},$

where

$$f_{DW} = \frac{\int_0^1 B_2^2(r)dr\left[\phi(1)B_1(1) - (\delta + \gamma^*(1))B_2(1)\right]\mathbf{1}_p' - \int_0^1 B_2(r)dr f_{WZ}'}{\int_0^1 B_2^2(r)dr - \left[\int_0^1 B_2(r)dr\right]^2},$$

$$f_{DV} = \frac{\int_0^1 B_2^2(r)dr B_2(1)\mathbf{1}_q' - \int_0^1 B_2(r)dr f_{\epsilon Z}'}{\int_0^1 B_2^2(r)dr - \left[\int_0^1 B_2(r)dr\right]^2};$$

(n) $\frac{1}{\sqrt{T}}D_T^{-1}C_T A_T^{-1}B_{1,T} = \begin{pmatrix} \left(f_{DW}\Sigma^{11} + f_{DV}\Sigma^{21}\right)P_{wu} + o_p(1) \\ O_p\left(\frac{1}{\sqrt{T}}\right) \end{pmatrix},$

$\begin{pmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22} \end{pmatrix} = \Sigma_1^{-1};$

(o) $A_T^{-1}B_T D_T^{-1}B_{2,T} = o_p(1);$

(p) $\frac{1}{\sqrt{T}}A_T^{-1}B_T D_T^{-1}C_T A_T^{-1}B_{1,T} = o_p(1).$

Proof.

(a) Omitted.

(b) Let $\xi_t = \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \cdots$. We have

$$\begin{aligned} \frac{1}{T}\mathbf{z}'\mathbf{u} &= \frac{1}{T}\sum_{t=1}^{T}\xi_t\left(\sum_{j=0}^{\infty}c_j v_{t-j}\right) \\ &= \sum_{j=0}^{\infty}c_j\frac{1}{T}\sum_{t=1}^{T}\xi_t v_{t-j} \\ &\Rightarrow C(1)\int_0^1 B_2(r)\,dB_1(r), \end{aligned}$$

by making use of Proposition 17.3 in Hamilton (1994). Thus, we have shown result (b).

(c) Omitted.

(d) Omitted.

(e) It can be seen that for $k=0,1,\cdots,p-1,$

$$\begin{aligned}
\frac{1}{\sqrt{T}}\mathbf{a}_{-k}'\mathbf{u} &= \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\sum_{i=0}^{\infty}\phi_i v_{t-i-k-1}-\sum_{i=0}^{\infty}\gamma_i^*\varepsilon_{t-i-k}\right)\sum_{j=0}^{\infty}c_j v_{t-j} \\
&= \sqrt{T}\sum_{i=0}^{\infty}\phi_i c_{i+k+1}\frac{1}{T}\sum_{t=1}^{T}v_{t-i-k-1}^2 \\
&\quad+\sum_{\substack{j\neq i+k+1\\ i,j=0,1,\cdots,\infty}}\phi_i c_j\frac{1}{\sqrt{T}}\sum_{t=1}^{T}v_{t-i-k-1}v_{t-j} \\
&\quad-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\gamma_i^* c_j\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{t-i-k}v_{t-j}.
\end{aligned}$$

Then, we can write

$$\begin{aligned}
\tfrac{1}{\sqrt{T}}\ & W'\mathbf{u} \\
= \ & \sqrt{T}\sigma_1^2\left(\sum_{i=0}^{\infty}\phi_i c_{i+k}\right)_{k=1,2,\cdots,p} \\
&+\left(\sum_{\substack{j\neq i+k\\ i,j=0,1,\cdots,\infty}}^{\infty}\phi_i c_j\frac{1}{\sqrt{T}}\sum_{t=1}^{T}v_{t-i-k}v_{t-j}\right)_{k=1,2,\cdots,p} \\
&+\left(-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\gamma_i^* c_j\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{t-i-k}v_{t-j}\right)_{k=0,1,\cdots,p-1}+o_p(1) \\
\equiv \ & \sqrt{T}P_{wu}+N_{wu}+o_p(1)\,,\ \text{say,}
\end{aligned}$$

as claimed in result (e).

(f) to (h) can be shown by the same manner as in (e).

(i) From the definitions of a_t and ξ_t, we can write

$$\frac{1}{T}\mathbf{a}'\mathbf{z}=\sum_{i=0}^{\infty}\phi_i\frac{1}{T}\sum_{t=1}^{T}\xi_t v_{t-i-1}-\sum_{i=0}^{\infty}\gamma_i^*\frac{1}{T}\sum_{t=1}^{T}\xi_t\varepsilon_{t-i}. \tag{27}$$

Note that

$$\sum_{i=0}^{\infty}\phi_i\frac{1}{T}\sum_{t=1}^{T}\xi_t v_{t-i-1}\Rightarrow\sum_{i=0}^{\infty}\phi_i\int_0^1 B_2(r)\,dB_1(r) \tag{28}$$

and

$$\sum_{i=0}^{\infty}\gamma_i^*\frac{1}{T}\sum_{t=1}^{T}\xi_t\varepsilon_{t-i} \Rightarrow \frac{\sum\limits_{i=0}^{\infty}\gamma_i^*}{2}\left[B_2^2(1)+\sigma_2^2\right]$$

$$= \frac{\delta+\sum\limits_{i=0}^{\infty}i\varphi_i}{2}\left[B_2^2(1)+\sigma_2^2\right]. \tag{29}$$

Substituting (28) and (29) into (27) yields

$$\frac{1}{T}\mathbf{a}'\mathbf{z}\Rightarrow\phi(1)\int_0^1 B_2(r)\,dB_1(r)-\frac{1}{2}\left(\delta+\sum_{i=0}^{\infty}i\varphi_i\right)\left[B_2^2(1)+\sigma_2^2\right],$$

which implies that result (i) holds.

(j) Obviously, it is seen that

$$\frac{1}{T}V'\mathbf{z}\Rightarrow\frac{1}{2}\left[B_2^2(1)+\sigma_2^2\right]\mathbf{1}_q.$$

(k) It is seen that

$$\frac{1}{T}W'W\xrightarrow{p}E(W'W)=\begin{pmatrix}\gamma_0^{(a)} & \gamma_1^{(a)} & \cdots & \gamma_{p-1}^{(a)}\\ \gamma_1^{(a)} & \gamma_0^{(a)} & \ddots & \vdots\\ \vdots & \ddots & \ddots & \gamma_1^{(a)}\\ \gamma_{p-1}^{(a)} & \cdots & \gamma_1^{(a)} & \gamma_0^{(a)}\end{pmatrix},$$

where $\gamma_k^{(a)}=E(\mathbf{a}'\mathbf{a}_{-k})=\sigma_1^2\sum\limits_{i=0}^{\infty}\phi_i\phi_{i+k}+\sigma_2^2\sum\limits_{i=0}^{\infty}\gamma_i^*\gamma_{i+k}^*$ for $k=0,1,\cdots,p-1$.

Also

$$\frac{1}{T}W'V\xrightarrow{p}E(W'V)=-\sigma_2^2\begin{pmatrix}\gamma_0^* & \gamma_1^* & \cdots & \gamma_{q-1}^*\\ 0 & \gamma_0^* & \ddots & \vdots\\ \vdots & \ddots & \ddots & \gamma_1^*\\ 0 & \cdots & 0 & \gamma_0^*\end{pmatrix}$$

and

$$\frac{1}{T}V'V\xrightarrow{p}E(V'V)=\sigma_2^2I_q.$$

(l) It is easy to see that

$$\begin{aligned} D_T^{-1}B_{2,T} &= \frac{1}{|D_T|}\begin{pmatrix} \frac{1}{T^2}\mathbf{z}'\mathbf{z} & -\frac{1}{T}\mathbf{z}'\mathbf{1} \\ -\frac{1}{T^2}\mathbf{z}'\mathbf{1} & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{T}}\mathbf{1}'\mathbf{u} \\ \frac{1}{T^{3/2}}\mathbf{z}'\mathbf{u} \end{pmatrix} \\ &= \frac{1}{|D_T|}\begin{pmatrix} \frac{1}{T^2}\mathbf{z}'\mathbf{z}\frac{1}{\sqrt{T}}\mathbf{1}'\mathbf{u}-\frac{1}{T^{3/2}}\mathbf{z}'\mathbf{u}\frac{1}{T}\mathbf{z}'\mathbf{1} \\ \frac{1}{\sqrt{T}}\left(-\frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{\sqrt{T}}\mathbf{1}'\mathbf{u}+\frac{1}{T}\mathbf{z}'\mathbf{u}\right) \end{pmatrix} \\ &= \begin{pmatrix} C(1)\frac{B_1(1)\int_0^1 B_2^2(r)dr-\int_0^1 B_2(r)dr\int_0^1 B_2(r)dB_1(r)}{\int_0^1 B_2^2(r)dr-\left[\int_0^1 B_2(r)dr\right]^2} \\ O_p\left(\frac{1}{\sqrt{T}}\right) \end{pmatrix} \\ &\equiv \begin{pmatrix} f_{DB_2}+o_p(1) \\ O_p\left(\frac{1}{\sqrt{T}}\right) \end{pmatrix}. \end{aligned}$$

(m) It can be written as

$$\begin{aligned} & D_T^{-1}C_T \\ = {} & \frac{1}{|D_T|}\begin{pmatrix} \frac{1}{T^2}\mathbf{z}'\mathbf{z} & -\frac{1}{T}\mathbf{z}'\mathbf{1} \\ -\frac{1}{T^2}\mathbf{z}'\mathbf{1} & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{T}}\mathbf{1}'W & \frac{1}{\sqrt{T}}\mathbf{1}'V \\ \frac{1}{T^{3/2}}\mathbf{z}'W & \frac{1}{T^{3/2}}\mathbf{z}'V \end{pmatrix} \\ = {} & \frac{1}{|D_T|}\begin{pmatrix} \frac{1}{T^2}\mathbf{z}'\mathbf{z}\frac{1}{\sqrt{T}}\mathbf{1}'W - \frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{T}\mathbf{z}'W & \frac{1}{T^2}\mathbf{z}'\mathbf{z}\frac{1}{\sqrt{T}}\mathbf{1}'V - \frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{T}\mathbf{z}'V \\ O_p\left(\frac{1}{\sqrt{T}}\right) & O_p\left(\frac{1}{\sqrt{T}}\right) \end{pmatrix}. \end{aligned}$$

Using Lemma 1(a) to (k) yields

$$D_T^{-1}C_T = \begin{pmatrix} f_{DW}+o_p(1) & f_{DV}+o_p(1) \\ O_p\left(\frac{1}{\sqrt{T}}\right) & O_p\left(\frac{1}{\sqrt{T}}\right) \end{pmatrix},$$

as claimed in the result (m).

(n) Using the result (m) produces

$$\begin{aligned} & \frac{1}{\sqrt{T}}D_T^{-1}C_TA_T^{-1}B_{1,T} \\ = {} & \begin{pmatrix} f_{DW}+o_p(1) & f_{DV}+o_p(1) \\ O_p\left(\frac{1}{\sqrt{T}}\right) & O_p\left(\frac{1}{\sqrt{T}}\right) \end{pmatrix}\Sigma_1^{-1}\begin{pmatrix} P_{wu} \\ \mathbf{0}_{(q+1)\times 1} \end{pmatrix}+o_p(1) \\ = {} & \begin{pmatrix} f_{DW}+o_p(1) & f_{DV}+o_p(1) \\ O_p\left(\frac{1}{\sqrt{T}}\right) & O_p\left(\frac{1}{\sqrt{T}}\right) \end{pmatrix}\begin{pmatrix} \Sigma^{11} \\ \Sigma^{21} \end{pmatrix}P_{wu}+o_p(1) \\ = {} & \begin{pmatrix} \left(f_{DW}\Sigma^{11}+f_{DV}\Sigma^{21}\right)P_{wu}+o_p(1) \\ O_p\left(\frac{1}{\sqrt{T}}\right) \end{pmatrix}. \end{aligned}$$

Thus, the result (n) has been proved.

(o) and (p) Similarly, we can prove the results (o) and (p). ■

Appendix C

Proof of Theorem 1 in MMH. On the right side of (11), we have

$$
\begin{aligned}
&\begin{pmatrix} A_T & B_T \\ \frac{1}{\sqrt{T}}C_T & D_T \end{pmatrix}^{-1} \begin{pmatrix} B_{1,T} \\ B_{2,T} \end{pmatrix} \\
= &\begin{pmatrix} A_T^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} B_{1,T} \\ B_{2,T} \end{pmatrix} + \begin{pmatrix} -A_T^{-1}B_T \\ E \end{pmatrix} \\
&\times \begin{pmatrix} D_T & -\frac{1}{\sqrt{T}}C_T A_T^{-1} B_T \end{pmatrix}^{-1} \begin{pmatrix} -\frac{1}{\sqrt{T}}C_T A_T^{-1} & E \end{pmatrix} \begin{pmatrix} B_{1,T} \\ B_{2,T} \end{pmatrix}.
\end{aligned}
$$

Note that

$$
\frac{1}{\sqrt{T}}C_T A_T^{-1} B_T = O_p \begin{pmatrix} \frac{1}{\sqrt{T}} & 1 \\ \frac{1}{T} & \frac{1}{\sqrt{T}} \end{pmatrix} \text{ and } D_T = O_p \begin{pmatrix} 1 & \sqrt{T} \\ \frac{1}{\sqrt{T}} & 1 \end{pmatrix},
$$

we have $\begin{pmatrix} D_T & -\frac{1}{\sqrt{T}}C_T A_T^{-1} B_T \end{pmatrix}^{-1} \doteqdot D_T^{-1}$, so that

$$
\begin{aligned}
&\begin{pmatrix} A_T & B_T \\ \frac{1}{\sqrt{T}}C_T & D_T \end{pmatrix}^{-1} \begin{pmatrix} B_{1,T} \\ B_{2,T} \end{pmatrix} \\
\doteqdot &\left[\begin{pmatrix} A_T^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \begin{pmatrix} -A_T^{-1}B_T \\ I \end{pmatrix} D_T^{-1} \begin{pmatrix} -\frac{1}{\sqrt{T}}C_T A_T^{-1} & I \end{pmatrix} \right] \begin{pmatrix} B_{1,T} \\ B_{2,T} \end{pmatrix} \\
\doteqdot &\begin{pmatrix} A_T^{-1}B_{1,T} - A_T^{-1}B_T D_T^{-1} B_{2,T} + \frac{1}{\sqrt{T}} A_T^{-1} B_T D_T^{-1} C_T A_T^{-1} B_{1,T} \\ D_T^{-1}B_{2,T} - \frac{1}{\sqrt{T}} D_T^{-1} C_T A_T^{-1} B_{1,T} \end{pmatrix} \\
\equiv &\begin{pmatrix} C^{(1)} \\ C^{(2)} \end{pmatrix}, \text{ say.}
\end{aligned}
$$

Therefore, we see that

$$
\begin{pmatrix} \sqrt{T}\,(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}) \\ \sqrt{T}\,(\boldsymbol{\beta} - \boldsymbol{\beta}) \end{pmatrix} = HC^{(1)} + H^{(1)}C^{(2)} + o_p(1), \tag{30}
$$

$$
\sqrt{T}\,(\hat{c} - c) = H^{(2)}C^{(1)} + \mathbf{e}_2'C^{(2)} + o_p(1), \tag{31}
$$

where

$$C^{(1)} = A_T^{-1}B_{1,T} - A_T^{-1}B_T D_T^{-1}B_{2,T} + \frac{1}{\sqrt{T}}A_T^{-1}B_T D_T^{-1}C_T A_T^{-1}B_{1,T},$$
$$C^{(2)} = D_T^{-1}B_{2,T} - \frac{1}{\sqrt{T}}D_T^{-1}C_T A_T^{-1}B_{1,T}.$$

Applying Lemma 1 in Appendix B to (30) yields

$$\begin{aligned}&\sqrt{T}\begin{pmatrix} \hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha} \\ \hat{\boldsymbol{\beta}} - \boldsymbol{\beta} \end{pmatrix} \\ = \;& HA_T^{-1}B_{1,T} + H^{(1)}\left(D_T^{-1}B_{2,T} - \frac{1}{\sqrt{T}}D_T^{-1}C_T A_T^{-1}B_{1,T}\right) + o_p(1), \quad (32)\end{aligned}$$

where

$$H^{(1)}\left(D_T^{-1}B_{2,T} - \frac{1}{\sqrt{T}}D_T^{-1}C_T A_T^{-1}B_{1,T}\right) = o_p(1)$$

and hence

$$\sqrt{T}\begin{pmatrix} \hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha} \\ \hat{\boldsymbol{\beta}} - \boldsymbol{\beta} \end{pmatrix} = HA_T^{-1}B_{1,T} + o_p(1).$$

Applying Lemma 1 in Appendix B to the above formula, we have

$$\begin{pmatrix} \sqrt{T}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}) \\ \sqrt{T}(\boldsymbol{\beta} - \boldsymbol{\beta}) \end{pmatrix} = H\Sigma_1^{-1}\left[\sqrt{T}\begin{pmatrix} P_{wu} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} N_{wu} \\ N_{\varepsilon u} \end{pmatrix}\right]. \quad (33)$$

First, we give the asymptotic distribution of $\begin{pmatrix} N_{wu} & N_{\varepsilon u} \end{pmatrix}'$.

Note that both $\{\varepsilon_{t-i}v_{t-j}\}$ and $\{v_{t-i-k-1}v_{t-j}\}$ for $j \neq i+k+1$ are martingale difference sequences for fixed i, j, and finite integer number k. Noting that $\sum_{i=0}^{\infty}|\gamma_i^*| < \infty$, $\sum_{j=0}^{\infty}|c_j| < \infty$, and $\sum_{i=0}^{\infty}|\phi_i| < \infty$, we can prove that the asymptotic distributions of the elements in the vector $\begin{pmatrix} N_{wu} & N_{\varepsilon u} \end{pmatrix}'$ are given by

$$\begin{aligned}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\gamma_i^* c_j \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{t-i-k}v_{t-j} &\Rightarrow N\left(0, \sigma_1^2\sigma_2^2\left(\sum_{i=0}^{\infty}\gamma_i^*\sum_{j=0}^{\infty}c_j\right)^2\right) \quad (34)\\ \text{for } k &= 0, 1, \cdots, p-1,\end{aligned}$$

$$
\begin{aligned}
\sum_{\substack{j\neq i+k+1\\ i,j=0,1,\cdots,\infty}} \phi_i c_j \frac{1}{\sqrt{T}} \sum_{t=1}^{T} v_{t-i-k-1} v_{t-j} &\Rightarrow N\left(0, \sigma_1^4 \left(\sum_{\substack{j\neq i+k+1,\\ i,j=0,1,\cdots,\infty}} \phi_i c_j \right)^2 \right) \quad (35)\\
\text{for } k &= 0,1,\cdots,p-1,\\
\sum_{j=0}^{\infty} c_j \frac{1}{\sqrt{T}} \sum_{t=0}^{T} \varepsilon_{t-k} v_{t-j} &\Rightarrow N\left(0, \sigma_1^2\sigma_2^2 \left(\sum_{j=0}^{\infty} c_j \right)^2 \right) \quad (36)\\
\text{for } k &= 0,1,\cdots,q-1.
\end{aligned}
$$

Next, we prove the above results by following the same line as the proof of Theorem 6.3.3 of White (1984). First, we consider the case $k=0$. Let

$$
\begin{aligned}
Y_{tl} &= \sum_{i=0}^{l}\sum_{j=0}^{l} \gamma_i^* c_j \varepsilon_{t-i} v_{t-j},\\
W_{tl} &= \sum_{i=0}^{l}\sum_{j=l+1}^{\infty} \gamma_i^* c_j \varepsilon_{t-i} v_{t-j} + \sum_{i=l+1}^{\infty}\sum_{j=0}^{l} \gamma_i^* c_j \varepsilon_{t-i} v_{t-j} \quad (37)\\
&\quad + \sum_{i=l+1}^{\infty}\sum_{j=l+1}^{\infty} \gamma_i^* c_j \varepsilon_{t-i} v_{t-j}\\
&\equiv W_{tl}^{(1)} + W_{tl}^{(2)} + W_{tl}^{(3)}, \text{ say,}
\end{aligned}
$$

and define the normalized sums

$$
S_{lT} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} Y_{tl},\ \ D_{lT} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} W_{tl}. \quad (38)
$$

For fixed l, we have

$$
\begin{aligned}
\gamma_{W^{(1)}}(s) &= E\left(\sum_{i=0}^{l}\sum_{j=l+1}^{\infty} \gamma_i^* c_j \varepsilon_{t-i} v_{t-j} \sum_{i'=0}^{l}\sum_{j'=l+1}^{\infty} \gamma_{i'}^* c_{j'} \varepsilon_{t-i'} v_{t-j'} \right)\\
&= (\sigma_1\sigma_2)^2 \sum_{i=0}^{l} \gamma_i^* \gamma_{i+s}^* \sum_{j=l+1}^{\infty} c_j c_{j+s}.
\end{aligned}
$$

Similarly, it is seen that

$$
\gamma_{W^{(2)}}(s) = (\sigma_1\sigma_2)^2 \sum_{i=l+1}^{\infty} \gamma_i^* \gamma_{i+s}^* \sum_{j=0}^{l} c_j c_{j+s},
$$

$$\gamma_{W^{(3)}}(s) = (\sigma_1\sigma_2)^2 \sum_{i=l+1}^{\infty} \gamma_i^* \gamma_{i+s}^* \sum_{j=l+1}^{\infty} c_j c_{j+s}.$$

Then,

$$Var(D_{lT}) = \frac{1}{T} \sum_{s=-(T-1)}^{T-1} (T - |s|) \sum_{i=1}^{3} \gamma_{W^{(i)}}(s),$$

where, for any given $\varepsilon > 0$ there exists l_0 such that $l > l_0$, we have

$$\begin{aligned}
& \frac{1}{T} \sum_{s=-(T-1)}^{T-1} (T - |s|) \left|\gamma_{W^{(1)}}(s)\right| \\
\leqslant \; & (\sigma_1\sigma_2)^2 \left(\sum_{i=0}^{l} \gamma_i^{*2} \sum_{j=l+1}^{\infty} c_j^2 + 2 \sum_{s=1}^{T-1} \sum_{i=1}^{l} \left|\gamma_i^* \gamma_{i+s}^*\right| \sum_{j=l+1}^{\infty} \left|c_j c_{j+s}\right| \right) \\
\leqslant \; & c\,(\sigma_1\sigma_2)^2 \left(\sum_{j=l+1}^{\infty} c_j^2 + 2 \sum_{s=1}^{T-1} \sum_{j=l+1}^{\infty} \left|c_j c_{j+s}\right| \right) \\
\leqslant \; & K \left(\sum_{j=l+1}^{\infty} |c_j| \right)^2 < \varepsilon,
\end{aligned}$$

providing of $\sum_{i=0}^{\infty} |\gamma_i^*| < \infty$ and $\sum_{j=0}^{\infty} |c_j| < \infty$. It is similarly shown that

$$\frac{1}{T} \sum_{s=-(T-1)}^{T-1} (T - |s|) \sum_{i=2}^{3} \gamma_{W^{(i)}}(s) < \varepsilon,$$

implying that

$$Var(D_{lT}) \to 0 \text{ as } l \to \infty.$$

It follows from Chebyshev's inequality that

$$D_{lT} \xrightarrow{p} 0 \text{ as } l \to \infty \tag{39}$$

uniformly in T. Note that $\{\varepsilon_{t-i} v_{t-j}\}$ is a martingale difference sequence for fixed i and j. Therefore, for fixed l, we have

$$\begin{aligned}
S_{lT} &= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} Y_{tl} = \sum_{i=0}^{l} \sum_{j=0}^{l} \gamma_i^* c_j \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \varepsilon_{t-i} v_{t-j} \\
&\Rightarrow \sigma_1\sigma_2 \sum_{i=0}^{l} \gamma_i^* \sum_{j=0}^{l} c_j N(0,1) \text{ as } T \to \infty
\end{aligned} \tag{40}$$

by the continuous mapping theorem and the central limit theorem. But

$$\sigma_1\sigma_2 \sum_{i=0}^{l} \gamma_i^* \sum_{j=0}^{l} c_j N(0,1) \to N\left(0, \sigma_1^2\sigma_2^2 \left(\sum_{i=0}^{\infty} \gamma_i^* \sum_{j=0}^{\infty} c_j\right)^2\right) \text{ as } l \to \infty. \tag{41}$$

By virtue of Lemma 6.3.1 of White (1984) and (39), (40), and (41), we have

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \sum_{i,j=0}^{\infty} \gamma_i^* c_j \varepsilon_{t-i} v_{t-j} \Rightarrow N\left(0, \sigma_1^2\sigma_2^2 \left(\sum_{i=0}^{\infty} \gamma_i^* \sum_{j=0}^{\infty} c_j\right)^2\right) \text{ as } T \to \infty.$$

It can be similarly proved that for $k = 1, 2, \cdots, p-1$,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \sum_{i,j=0}^{\infty} \gamma_i^* c_j \varepsilon_{t-i-k} v_{t-j} \Rightarrow N\left(0, \sigma_1^2\sigma_2^2 \left(\sum_{i=0}^{\infty} \gamma_i^* \sum_{j=0}^{\infty} c_j\right)^2\right) \text{ as } T \to \infty,$$

as claimed in (34). We can similarly prove (35) and (36).

Define

$$\mathbf{x}_t = \begin{pmatrix} \sum\limits_{\substack{j \neq i+1 \\ i,j=0,1,\cdots,\infty}} \phi_i c_j v_{t-i-1} v_{t-j} - \sum\limits_{j=0}^{\infty}\sum\limits_{j=0}^{\infty} \gamma_i^* c_j \varepsilon_{t-i} v_{t-j} \\ \sum\limits_{\substack{j \neq i+2 \\ i,j=0,1,\cdots,\infty}} \phi_i c_j v_{t-i-2} v_{t-j} - \sum\limits_{j=0}^{\infty}\sum\limits_{j=0}^{\infty} \gamma_i^* c_j \varepsilon_{t-i-1} v_{t-j} \\ \vdots \\ \sum\limits_{\substack{j \neq i+p \\ i,j=0,1,\cdots,\infty}} \phi_i c_j v_{t-i-p} v_{t-j} - \sum\limits_{j=0}^{\infty}\sum\limits_{j=0}^{\infty} \gamma_i^* c_j \varepsilon_{t-i-p+1} v_{t-j} \\ \sum\limits_{j=0}^{\infty} c_j \varepsilon_t v_{t-j} \\ \sum\limits_{j=0}^{\infty} c_j \varepsilon_{t-1} v_{t-j} \\ \vdots \\ \sum\limits_{j=0}^{\infty} c_j \varepsilon_{t-q+1} v_{t-j} \end{pmatrix}.$$

The covariance matrix of the vector $\left(\begin{array}{cc} N_{wu} & N_{\varepsilon u} \end{array} \right)'$ is indicated by $\Sigma_2 \equiv E\left(\mathbf{x}_t \mathbf{x}_t'\right)$. Then, we have

$$\left(\begin{array}{c} N_{wu} \\ N_{\varepsilon u} \end{array} \right) \Rightarrow N\left(0, \Sigma_2\right) \text{ as } T \to \infty.$$

Let

$$\left(\begin{array}{c} \boldsymbol{\alpha}^* \\ \boldsymbol{\beta}^* \end{array} \right) = H\Sigma_1^{-1} \left(\begin{array}{c} P_{wu} \\ \mathbf{0} \end{array} \right), \ \Sigma = H\Sigma_1^{-1}\Sigma_2\Sigma_1^{-1}H'.$$

Substituting the above two formulae into (33) yields

$$\left(\begin{array}{c} \sqrt{T}\left(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha} - \boldsymbol{\alpha}^*\right) \\ \sqrt{T}\left(\boldsymbol{\beta} - \boldsymbol{\beta} - \boldsymbol{\beta}^*\right) \end{array} \right) \Rightarrow N\left(\mathbf{0}, \boldsymbol{\Sigma}\right).$$

Similarly, applying Lemma 1 in Appendix B to (31) leads to

$$\begin{aligned} &\sqrt{T}\left(\hat{c} - c\right) = \\ &H^{(2)} A_T^{-1} B_{1,T} + \mathbf{e}_2' \left(D_T^{-1} B_{2,T} - \frac{1}{\sqrt{T}} D_T^{-1} C_T A_T^{-1} B_{1,T} \right) + o_p(1), \end{aligned}$$

where

$$\begin{aligned} H^{(2)} A_T^{-1} B_{1,T} &= \sqrt{T} H^{(2)} \Sigma_1^{-1} \left(\begin{array}{c} P_{wu} \\ \mathbf{0}_{(q+1)\times 1} \end{array} \right) + H^{(2)} \Sigma_1^{-1} N\left(0, \Sigma_2\right) + o_p(1) \\ &\equiv \sqrt{T} c^{**} + N\left(0, \Sigma^*\right) + o_p(1), \text{ say,} \end{aligned}$$

and

$$\mathbf{e}_2' \left(D_T^{-1} B_{2,T} - \frac{1}{\sqrt{T}} D_T^{-1} C_T A_T^{-1} B_{1,T} \right) \Rightarrow f_{DB_2} - \left(f_{DW} \Sigma^{11} + f_{DV} \Sigma^{21} \right) Pwu.$$

Thus, it is seen that

$$\sqrt{T}\left(\hat{c} - c - c^{**}\right) \Rightarrow N\left(0, \Sigma^*\right) + f\left[B_1(r), B_2(r)\right],$$

with $f\left[B_1(r), B_2(r)\right] = f_{DB_2} - \left(f_{DW}\Sigma^{11} + f_{DV}\Sigma^{21}\right) Pwu$, where f_{DB_2}, f_{DW}, f_{DV}, Σ^{11}, Σ^{21}, and Pwu are given in Lemma 1 in Appendix B. We have thus completed the proof of Theorem 1 in MMH. ■

Appendix D

Proof of Theorem 2 in MMH. We have already (19) in MMH:

$$\begin{aligned}
&\left(\delta \mathbf{1}_p' \ \mathbf{1}_{q+1}'\right) \begin{pmatrix} T\left(\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}\right) \\ T\left(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}\right) \end{pmatrix} \\
= \ & (0,1)\left(\sqrt{T} D_T^{-1} B_{2,T} - D_T^{-1} C_T A_T^{-1} B_{1,T}\right) + o_p(1).
\end{aligned}$$

We evaluate each term in (19) as follows:

First, applying Lemma 1 in Appendix B, we have

$$\begin{aligned}
& \begin{pmatrix} 0 & 1 \end{pmatrix} \sqrt{T} D_T^{-1} B_{2,T} \\
= \ & \frac{1}{|D_T|}\left(-\frac{1}{T^{3/2}} \mathbf{z}'\mathbf{1}\frac{1}{\sqrt{T}}\mathbf{1}'\mathbf{u} + \frac{1}{T}\mathbf{z}'\mathbf{u}\right) \\
\Rightarrow \ & C(1)\,\frac{\int_0^1 B_2(r)dB_1(r) - B_1(1)\int_0^1 B_2(r)dr}{\int_0^1 B_2^2(r)dr - \left[\int_0^1 B_2(r)dr\right]^2}.
\end{aligned} \tag{42}$$

Next, applying Lemma 1 we have

$$\begin{aligned}
& D_T^{-1} C_T A_T^{-1} B_{1,T} \\
= \ & \frac{1}{|D_T|} \times \\
& \begin{pmatrix} \frac{1}{T^2}\mathbf{z}'\mathbf{z}\frac{1}{\sqrt{T}}\mathbf{1}'W - \frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{T}\mathbf{z}'W & \frac{1}{T^2}\mathbf{z}'\mathbf{z}\frac{1}{\sqrt{T}}\mathbf{1}'V - \frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{T}\mathbf{z}'V \\ \frac{1}{\sqrt{T}}\left(-\frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{\sqrt{T}}\mathbf{1}'W + \frac{1}{T}\mathbf{z}'W\right) & \frac{1}{\sqrt{T}}\left(-\frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{\sqrt{T}}\mathbf{1}'V + \frac{1}{T}\mathbf{z}'V\right) \end{pmatrix} \\
& \times \begin{pmatrix} \sqrt{T}\Sigma^{11}P_{wu} + \Sigma^{11}N_{wu} + \Sigma^{12}N_{\varepsilon u} \\ \sqrt{T}\Sigma^{21}P_{wu} + \Sigma^{21}N_{wu} + \Sigma^{22}N_{\varepsilon u} \end{pmatrix} + o_p(1) \\
\equiv \ & \frac{1}{|D_T|}\begin{pmatrix} [c] \\ [d] \end{pmatrix} + o_p(1), \text{ say,}
\end{aligned}$$

where

$$\begin{aligned}
[d] \ = \ & \left(-\frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{\sqrt{T}}\mathbf{1}'W + \frac{1}{T}\mathbf{z}'W\right)\Sigma^{11}P_{wu} + \\
& \left(-\frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{\sqrt{T}}\mathbf{1}'V + \frac{1}{T}\mathbf{z}'V\right)\Sigma^{21}P_{wu} + o_p(1).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\begin{pmatrix} 0 & 1 \end{pmatrix} D_T^{-1} C_T A_T^{-1} B_{1,T} &= \frac{1}{|D_T|}[d] + o_p(1) \\
&= \frac{1}{|D_T|}\left(-\frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{\sqrt{T}}\mathbf{1}'W + \frac{1}{T}\mathbf{z}'W\right)\Sigma^{11}P_{wu} + \\
&\quad \frac{1}{|D_T|}\left(-\frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{\sqrt{T}}\mathbf{1}'V + \frac{1}{T}\mathbf{z}'V\right)\Sigma^{21}P_{wu} + o_p(1) \\
&\equiv \frac{-e_T}{\int_0^1 B_2^2(r)dr - \left[\int_0^1 B_2(r)dr\right]^2} + o_p(1). \qquad (43)
\end{aligned}
$$

By the application of Lemma 1, we see that

$$
\begin{aligned}
e_T &\Rightarrow \left\{\int_0^1 B_2(r)dr \left[\begin{array}{c} \phi(1)B_1(1) - \\ \left(\delta + \sum_{i=0}^{\infty} i\varphi_i\right)B_2(1) \end{array}\right]\mathbf{1}_p' - f_{WZ}'\right\}\Sigma^{11}P_{wu} \\
&\quad + \left[\int_0^1 B_2(r)dr B_2(1)\mathbf{1}_q' - f_{\epsilon Z}'\right]\Sigma^{21}P_{wu} \\
&\equiv e, \text{ say.}
\end{aligned}
$$

It follows from (19) in MMH, (42), and (43) that

$$
\begin{aligned}
&\left(\delta\mathbf{1}_p' \; \mathbf{1}_{q+1}'\right)\begin{pmatrix} T(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}) \\ T\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \end{pmatrix} \\
&\Rightarrow \frac{C(1)\left[\int_0^1 B_2(r)dB_1(r) - B_1(1)\int_0^1 B_2(r)dr\right] + e}{\int_0^1 B_2^2(r)dr - \left[\int_0^1 B_2(r)dr\right]^2},
\end{aligned}
$$

as claimed in Theorem 2 in MMH. ■

Appendix E

Proof of Theorem 3 in MMH. Note that the estimators $\check{\Gamma} - \Gamma$ in (26) and $\hat{\Gamma} - \Gamma$ in (15) of MMH have the same construction if p and q are replaced with $p + l$ and $q + l$, respectively. Thus, we can prove Theorem 3 by the same principle as the proof of Theorem 2. First, we introduce similar notation to Section 2, e.g., $C_T^*,\ A_T^*,\ B_{1,T}^*, B_{2,T}^* \leftrightarrow C_T, A_T, B_{1,T}, B_{2,T}$ in (10) of MMH such that

$$
\begin{aligned}
& \delta T\left(\sum_{i=1}^{p+l}\breve{a}_i^* - \sum_{i=1}^{p+l}a_i^*\right) + T\left(\sum_{i=0}^{q+l}\breve{b}_i^* - \sum_{i=0}^{q+l}b_i^*\right) \\
= & \sqrt{T}\begin{pmatrix} 0 & 1 \end{pmatrix} C^{*(2)} \\
= & \begin{pmatrix} 0 & 1 \end{pmatrix}\left(\sqrt{T}D_T^{-1}B_{2,T}^* - D_T^{-1}C_T^*A_T^{*-1}B_{1,T}^*\right).
\end{aligned}
$$

It is seen that

$$
\begin{aligned}
& \begin{pmatrix} 0 & 1 \end{pmatrix}\sqrt{T}D_T^{-1}B_{2,T}^* \\
= & \frac{1}{|D_T|}\left(-\frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{\sqrt{T}}\mathbf{1}'\mathbf{v} + \frac{1}{T}\mathbf{z}'\mathbf{v}\right) \\
\Rightarrow & \frac{\int_0^1 B_2(r)dB_1(r) - B_1(1)\int_0^1 B_2(r)dr}{\int_0^1 B_2^2(r)dr - \left[\int_0^1 B_2(r)dr\right]^2}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
& D_T^{-1}C_T^*A_T^{*-1}B_{1,T}^* \\
= & \frac{1}{|D_T|}\times \\
& \begin{pmatrix} \frac{1}{T^2}\mathbf{z}'\mathbf{z}\frac{1}{\sqrt{T}}\mathbf{1}'W^* - \frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{T}\mathbf{z}'W^* & \frac{1}{T^2}\mathbf{z}'\mathbf{z}\frac{1}{\sqrt{T}}\mathbf{1}'V^* - \frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{T}\mathbf{z}'V^* \\ \frac{1}{\sqrt{T}}\left(-\frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{\sqrt{T}}\mathbf{1}'W^* + \frac{1}{T}\mathbf{z}'W^*\right) & \frac{1}{\sqrt{T}}\left(-\frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{\sqrt{T}}\mathbf{1}'V^* + \frac{1}{T}\mathbf{z}'V^*\right) \end{pmatrix} \\
& \times\begin{pmatrix} A^{11}N_{wv}^* + A^{12}N_{\varepsilon v}^* \\ A^{21}N_{wv}^* + A^{22}N_{\varepsilon v}^* \end{pmatrix} + o_p(1) \\
\equiv & \frac{1}{|D_T|}\begin{pmatrix} [c_1] \\ [d_1] \end{pmatrix} + o_p(1)\,, \text{ say,}
\end{aligned}
$$

where

$$
\begin{aligned}
[d_1] = & \frac{1}{\sqrt{T}}\left(-\frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{\sqrt{T}}\mathbf{1}'W^* + \frac{1}{T}\mathbf{z}'W^*\right)\left(A^{11}N_{wv}^* + A^{12}N_{\varepsilon v}^*\right) + \\
& \frac{1}{\sqrt{T}}\left(-\frac{1}{T^{3/2}}\mathbf{z}'\mathbf{1}\frac{1}{\sqrt{T}}\mathbf{1}'V^* + \frac{1}{T}\mathbf{z}'V^*\right)\left(A^{21}N_{wv}^* + A^{22}N_{\varepsilon v}^*\right) + o_p(1) \\
= & o_p(1)\,.
\end{aligned}
$$

Therefore,

$$\left(\begin{array}{cc} 0 & 1 \end{array} \right) D_T^{-1} C_T^* A_T^{*-1} B_{1,T} = \frac{1}{|D_T|} [d_1] \xrightarrow{p} 0.$$

Then,

$$\begin{array}{l} \delta T \left(\sum_{i=1}^{p+l} \breve{a}_i^* - \sum_{i=1}^{p+l} a_i^* \right) + T \left(\sum_{i=0}^{q+l} \breve{b}_i^* - \sum_{i=0}^{q+l} b_i^* \right) \\ \Rightarrow \quad \frac{\int_0^1 B_2(r) dB_1(r) - B_1(1) \int_0^1 B_2(r) dr}{\int_0^1 B_2^2(r) dr - \left[\int_0^1 B_2(r) dr \right]^2} \end{array}$$

holds, and hence

$$\begin{aligned} T\left(\breve{\delta} - \delta\right) &= \frac{\delta T \left(\sum_{i=1}^{p+l} \breve{a}_i^* - \sum_{i=1}^{p+l} a_i^* \right) + T \left(\sum_{i=0}^{q+l} \breve{b}_i^* - \sum_{i=0}^{q+l} b_i^* \right)}{1 - \sum_{i=1}^{p+l} \breve{a}_i^*} \\ &\Rightarrow \frac{\int_0^1 B_2(r) dB_1(r) - B_1(1) \int_0^1 B_2(r) dr}{\left(1 - \sum_{i=1}^{p+l} a_i^* \right) \left\{ \int_0^1 B_2^2(r) dr - \left[\int_0^1 B_2(r) dr \right]^2 \right\}} \\ &= \frac{C(1)}{1 - A(1)} \cdot \frac{\int_0^1 B_2(r) dB_1(r) - B_1(1) \int_0^1 B_2(r) dr}{\int_0^1 B_2^2(r) dr - \left[\int_0^1 B_2(r) dr \right]^2}, \end{aligned}$$

where the last line is obtained by making use of the relationship $C^*(1) = 1/C(1)$ and $1 - A^*(1) = C^*(1)[1 - A(1)]$. We have thus obtained Theorem 3 in MMH. ■

References

[1] Hamilton, J.D., (1994), *Time Series Analysis,* Princeton, NJ, Princeton University Press.

[2] White, H., (1984), *Asymptotic Theory for Econometricians*, Academic Press, INC.