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An Axiomatic Approach to ε-contamination

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Abstract

Suppose that an economic agent is $(1-\varepsilon)\times 100\%$ certain that uncertainty she faces is characterized by a particular probability measure, but that she has a fear that, with $\varepsilon \times 100\%$ chance, her conviction is completely wrong and she is left perfectly ignorant about the true measure in the present as well as in the future. This situation is often called "ε-contamination of confidence." The purpose of this paper is to provide a simple set of behavioral axioms under which the decision-maker's preference is represented by the Choquet expected utility with the ε-contamination of confidence.

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1. Introduction

Suppose that an economic agent is $(1-\varepsilon)\times 100\%$ certain that uncertainty she faces is characterized by a particular probability measure, but that she has a fear that, with $\varepsilon \times 100\%$ chance, her conviction is completely wrong and she is left perfectly ignorant about the true measure in the present as well as in the future. This situation is often called "ε-contamination of confidence."

The ε-contamination is a special case of Knightian uncertainty or ambiguity in which the decision-maker faces not a single probability measure but a set of probability measures. Since it is analytically tractable, a number of authors have examined the ε-contamination or its variants in search behavior (Nishimura and Ozaki, 2001), portfolio choice (Chen and Epstein, 2002), learning (Nishimura and Ozaki, 2002) and voting (Chu and Liu, 2002).

The purpose of this paper is to provide a simple set of behavioral axioms under which the decision-maker's preference is represented by the Choquet expected utility with the ε-contamination of confidence. It turns out that a natural extension of the Anscombe and Aumann theory (Anscombe and Aumann, 1963) leads to the ε-contamination representation of preferences.

2. Preliminaries

Let (S, Σ) be a measurable space, where S is the set of states of the world and Σ is an algebra on it. Let Y be a mixture space. We call an element of Y a lottery. As a concrete example, X may be taken as a set of prizes and Y may be taken as the set of simple probability measures on $(X, 2^X)$. Then, Y will be clearly a mixture space with the operation in a vector space. Given $y, y' \in Y$ and $\lambda \in [0,1]$, we denote by $\lambda y + (1-\lambda)y'$ the "compound" lottery.[1] A *simple lottery act* is a Y-valued Σ-measurable function on S whose range is a finite subset of Y. We henceforth call it a lottery act, or more simply, an act. The set of simple lottery acts is denoted by L_0. A lottery act whose range is a singleton is referred to as a *constant act* and the set of constant acts is denoted by L_c.

[1]Here, $\lambda y + (1-\lambda)y'$ should be understood as the element of Y to which (y, y', λ) is mapped by the operation which makes Y a mixture space, and hence, it does not necessarily mean the outcome of the operation in a vector space. Accidentally, it does when Y is the set of simple probability measures on $(X, 2^X)$.

The decision-maker's preference is given by a binary relation $\succ$ on L_0. The two binary relations, $\succeq$ and $\sim$, are defined from $\succ$ by: $\succeq \Leftrightarrow \not\prec$ and $\sim \Leftrightarrow [\not\succ$ and $\not\prec]$. A binary relation $\succ$ is a *preference order* by definition if it is asymmetric and negatively transitive.[2] We define a binary relation over Y by restricting $\succ$ on L_c and denote it by the same symbol $\succ$, that is,

$$(\forall y, y' \in Y) \quad y \succ y' \;\Leftrightarrow\; (\exists f, g \in L_c) \;\; (\forall s \in S) \; f(s) = y,\, g(s) = y' \text{ and } f \succ g\,.$$

We say that two acts, f and g, are *comonotonic* if $(\forall s, t \in S)\; [\, f(s) \succ f(t) \Rightarrow g(t) \not\succ g(s)\,]$.

In the following discussion, the "worst-limit" constant act as well as the "best-limit" one plays a crucial role. Given $f \in L_0$, let $Y_{\min f}$ be the subset of Y representing f's worst-limit costant act, defined by

$$Y_{\min f} = \{\, y \in Y \,|\, (\forall s)\; y \preceq f(s) \text{ and } (\exists s)\; y = f(s)\,\}\,.$$

Since f is a simple act, $Y_{\min f}$ is nonempty when $\succ$ is a preference order. We henceforth denote by $y_{\min f}$ an arbitrary element of $Y_{\min f}$. Similarly, $Y_{\max f}$, representing f's best-limit constant act, is defined by

$$Y_{\max f} = \{\, y \in Y \,|\, (\forall s)\; y \succeq f(s) \text{ and } (\exists s)\; y = f(s)\,\}\,,$$

and its arbitrary element is denoted by $y_{\max f}$.

Given $f, g \in L_0$ and $\lambda \in [0,1]$, a "compound" lottery act $\lambda f + (1-\lambda) g \in L_0$ is defined by $(\forall s)\; (\lambda f + (1-\lambda)g)(s) = \lambda f(s) + (1-\lambda) g(s)$. By this operation, L_0 turns out to be a mixture space. For a notational ease, we sometimes use the following notation:

$$f_\lambda g \equiv \lambda f + (1-\lambda) g\,.$$

Finally, a special case of the above "compound" lottery act will turn to be important. Define

$$(y_{\min f})_\lambda\,(y_{\max f}) \equiv \lambda y_{\min f} + (1-\lambda) y_{\max f}\,,$$

that is, a "compound" act of the worst-limit act with "probability" λ and the best-limit act with "probability" $1-\lambda$. We hereafter call it the λ-*worst-limit* $1-\lambda$-*best-limit compound act*.

[2] A binary relation $\succ$ is *asymmetric* if $(\forall f, g \in L_0)\; f \succ g \Rightarrow g \not\succ f$, and it is *negatively transitive* if $(\forall f, g, h \in L_0)\; [f \not\succ g$ and $g \not\succ h] \Rightarrow f \not\succ h$.

3. Axioms and Main Results

We consider the following set of axioms which may be imposed on a binary relation defined on L_0. In the axioms, f, g and h denote arbitrary elements in L_0 and λ denotes an arbitrary real number such that $\lambda \in (0,1]$. The first five axioms (A1 through A5) are the same as those of Schmeidler (1989). The sixth and the seventh are new in the literature.

A1 (Ordering) $\succ$ is a preference order on L_0 .

A2 (Comonotonic-independence) If f, g, h are pairwise comonotonic, then

$$f \succ g \;\Rightarrow\; \lambda f + (1-\lambda)h \succ \lambda g + (1-\lambda)h\,.$$

A3 (Continuity) If $f \succ g$ and $g \succ h$, then

$$(\exists \alpha, \beta \in (0,1))\;\; \alpha f + (1-\alpha)h \succ g \;\text{ and }\; g \succ \beta f + (1-\beta)h\,.$$

A4 (Monotonicity) $[\,(\forall s \in S)\; f(s) \succeq g(s)\,] \;\Rightarrow\; f \succeq g$.

A5 (Non-degeneracy) $(\exists f, g \in L_0)\;\; f \succ g$.

As shown in Schmeidler (1989), these five axioms as a whole characterize the preference which is represented by the Choquet expected utility with respect to some capacity.[3]

The next axiom requires that any simple lottery f is dominated by some compound lottery of its worst-limit and best-limit constant acts. In the axiom, ε is a real number such that $\varepsilon \in [0,1)$. The axiom requires that the given relation should hold with respect to this ε. Therefore, whether the axiom is satisfied or not depends on ε, and hence, it is labeled A6(ε), rather than A6.

A6(ε) (Dominance of the ε-worst-limit $1-\varepsilon$-best-limit compound act)

$$(y_{\min f})_{\varepsilon}\,(y_{\max f})\,[= (1-\varepsilon)y_{\max f} + \varepsilon y_{\min f}] \succeq f.$$

[3] For related axiomatizations, see Gilboa (1987) and Gilboa and Schmeidler (1989).

Under A1 through A6(ε), it can be shown (see Lemma 5 of Section 4) that all $f \in L_0$ has the following ε-contamination equivalence:

$$(\forall f \in L_0)(\exists y_f \in L_c) \;\; f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}\,, \tag{1}$$

where ε is the one with which A6(ε) holds. This property shows that all simple lottery acts have their own equivalent compound act consisting of its worst-limit constant act with "probability" ε and some constant act y_f with "probability" $1-\varepsilon$.

Clearly, y_f defined in (1) is one way of representing f. We hereafter call it *f's equivalent constant act in ε-contamination equivalence.*

The next axiom concerns ordering among these equivalent constant acts in ε-contamination equivalence. In the axiom, ε is a real number such that $\varepsilon \in [0,1)$. By the same reason given for A6(ε), we label it A7(ε), rather than A7.

A7(ε) (Irrelevance of the worst limit in ordering among equivalent constant acts in ε-contamination equivalence) Both of the following hold:

A7(ε)-1 (Affine irrelevance) If there exist y_f, y_g, $y_{f_\lambda g} \in L_c$ such that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$, $g \sim (1-\varepsilon)y_g + \varepsilon y_{\min g}$ and $f_\lambda g \sim (1-\varepsilon)y_{f_\lambda g} + \varepsilon y_{\min f_\lambda g}$, then $y_{f_\lambda g} \sim \lambda y_f + (1-\lambda)y_g$; and

A7(ε)-2 (Monotone irrelevance) If $(\forall s)\;\; f(s) \succeq g(s)$ and there exist y_f, $y_g \in L_c$ such that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$ and $g \sim (1-\varepsilon)y_g + \varepsilon y_{\min g}$, then $y_f \succeq y_g$.

Axiom A7(ε)-1 means that if y_f, y_g and $y_{f_\lambda g}\left[= y_{\{\lambda f+(1-\lambda)g\}}\right]$ are the equivalent acts of f, g and $f_\lambda g\,[= \lambda f + (1-\lambda)\,g]$ in ε-contamination equivalence, respectively, then $y_{f_\lambda g}\left[= y_{\{\lambda f+(1-\lambda)g\}}\right] = \lambda y_f + (1-\lambda)y_g$, regardless of characteristics of the worst-limits $y_{\min f}$, $y_{\min g}$ and $y_{\min f_\lambda g}$. Similarly, Axiom A7(ε)-2 implies that if $f(s) \succeq g(s)$ for all s, then the equivalent act of f and that of g in ε-contamination equivalence, that is, y_f and y_g, should satisfy $y_f \succeq y_g$, regardless of characteristics of the worst-limits $y_{\min f}$ and $y_{\min g}$. These two axioms imply that the worst limits are irrelevant in ordering among equivalent constant acts in ε-contamination equivalence.

Axioms A6(ε) and A7(ε) are closely related to the axioms of Anscombe and Aumann (1963), especially their independence axiom. In fact they can be considered as a natural extension of the Anscombe-Aumann theory to the case in which the decision-maker has a fear of the worst outcome with the possibility of ε all the time. We will turn to this issue in the next section.

The main result of this paper is the following theorem and corollary. The proof is relegated to Section 5.

Theorem 1. *A binary relation $\succ$ defined on L_0 satisfies A1-A5, A6(ε) and A7(ε) if and only if there exist a unique finitely additive probability measure μ on (S, Σ), an affine function $u : Y \to \mathbb{R}$, which is unique up to a positive affine transformation, and $\varepsilon \in [0, 1)$ such that*

$$f \succ g \;\Leftrightarrow\; (1-\varepsilon)\int_S u(f(s))\,d\mu(s) + \varepsilon \min_{s\in S} u(f(s)) > (1-\varepsilon)\int_S u(g(s))\,d\mu(s) + \varepsilon \min_{s\in S} u(g(s))\,.$$

Let $\mathcal{M} = \mathcal{M}(S, \Sigma)$ be the set of finitely additive probability measures (probability charges) on (S, Σ), let $\varepsilon \in [0, 1)$, and let $\mu \in \mathcal{M}$. Let us now define *ε-contamination of* μ, $\{\mu\}^{\varepsilon}$, which is a subset of $\mathcal{M}$, by

$$\{\mu\}^{\varepsilon} = \{\, (1-\varepsilon)\mu + \varepsilon q \mid q \in \mathcal{M} \,\}\ .$$

Then, it follows that

$$\begin{aligned}(\forall f \in L_0) \qquad \int_S u(f(s))\,d\{\mu\}^{\varepsilon}(s) &\equiv \min\left\{ \int_S u(f(s))\,dp(s) \,\middle|\, p \in \{\mu\}^{\varepsilon} \right\} \\ &= (1-\varepsilon)\int_S u(f(s))\,d\mu(s) + \varepsilon \min_{s\in S} u(f(s))\,.\end{aligned}$$

Therefore, the following corollary is immediate.

Corollary 1. *A binary relation $\succ$ defined on L_0 satisfies A1-A5, A6(ε) and A7(ε) if and only if there exist a unique finitely additive probability measure μ on (S, Σ), an affine function $u : Y \to \mathbb{R}$, which is unique up to a positive affine transformation, and $\varepsilon \in [0, 1)$ such that*

$$f \succ g \;\Leftrightarrow\; \int_S u(f(s))\,d\{\mu\}^{\varepsilon}(s) > \int_S u(g(s))\,d\{\mu\}^{\varepsilon}(s)\,.$$

4. Relation to the Anscombe-Aumann Theory

Consider the following axiom which strengthens Axiom A2.

AA2 (Independence) $f \succ g \;\Rightarrow\; \lambda f + (1-\lambda)h \succ \lambda g + (1-\lambda)h$.

Note that in AA2, f, g and h are *not* assumed to be pairwise comonotonic. The next theorem is well-known.

Theorem 2 (Anscombe and Aumann, 1963). *A binary relation $\succ$ defined on L_0 satisfies A1, AA2, A3, A4 and A5 if and only if there exist a unique finitely additive probability measure μ on (S, Σ) and an affine function $u : Y \to \mathbb{R}$, which is unique up to a positive affine transformation, such that*

$$f \succ g \;\Leftrightarrow\; \int_S u(f(s))\, d\mu(s) > \int_S u(g(s))\, d\mu(s)\,. \tag{2}$$

We now show that Axioms A1-A5 and the following A6(0) and A7(0), which are special cases of A6(ε) and A7(ε) by setting $\varepsilon = 0$, are necessary and sufficient for the Anscombe-Aumann axioms (A1, AA2, A3-A5).

A6(0) $y_{\max f} \succeq f$;

A7(0)-1 If there exist y_f, y_g, $y_{f_\lambda g} \in L_c$ such that $f \sim y_f$, $g \sim y_g$ and $f_\lambda g \sim y_{f_\lambda g}$, then $y_{f_\lambda g} \sim \lambda y_f + (1-\lambda) y_g$; and

A7(0)-2 If $(\forall s)$ $f(s) \succeq g(s)$ and there exist y_f, $y_g \in L_c$ such that $f \sim y_f$ and $g \sim y_g$, then $y_f \succeq y_g$.

Proposition 1. *(A1, AA2, A3, A4, A5) $\Rightarrow$ (A6(0), A7(0)).*

Proof. Assume that A1, AA2, A3, A4 and A5 are satisfied. It is immediate that A4 implies A6(0) and that A1 and A4 imply A7(0)-2. Then, we only need to prove that A7(0)-1 holds. The proof will be complete if we show the following claim:

$$(\forall f, g, h \in L_0) \quad f \sim g \;\Rightarrow\; \lambda f + (1-\lambda)h \sim \lambda g + (1-\lambda)h\,, \tag{3}$$

for it follows from (3) that

$$\lambda y_f + (1-\lambda)y_g \sim \lambda y_f + (1-\lambda)g \sim \lambda f + (1-\lambda)g = f_\lambda g \sim y_{f_\lambda g}\,.$$

However, A1, AA2 and A3 imply (3) by Kreps (1988, p.46, Lemma 5.6(c)). ■

We also have its "converse."

Proposition 2. *(A1, A2, A3, A4, A5, A6(0), A7(0)) ⇒ AA2.*

Proof. Assume that A1, A2, A3, A4, A5, A6(0) and A7(0) are satisfied. Then, Lemma 5 (Section 5) proves that $(\forall f \in L_0)(\exists y_f \in L_c)\ f \sim y_f$ (simply let $\varepsilon = 0$ there). Let $y_f, y_g, y_h, y_{f_\lambda h}, y_{g_\lambda h} \in L_c$ be such that $f \sim y_f$, $g \sim y_g$, $h \sim y_h$, $f_\lambda h \sim y_{f_\lambda h}$ and $g_\lambda h \sim y_{g_\lambda h}$, and let $f \succ g$. Then, A1 implies that $y_f \succ y_g$. Since any pair of constant acts is comonotonic, A2 implies that $\lambda y_f + (1-\lambda)y_h \succ \lambda y_g + (1-\lambda)y_h$. Finally, A1 and A7(0)-1 imply that $f_\lambda h \sim y_{f_\lambda h} \succ y_{g_\lambda h} \sim g_\lambda h$, which completes the proof. ■

By combining these two propositions, we have

Proposition 3. *(A1, A2, A3, A4, A5, A6(0), A7(0)) ⇔ (A1, AA2, A3, A4, A5).*

By this proposition, we immediately have that the set of axioms, (A1, A2, A3, A4, A5, A6(0), A7(0)), characterizes the preference which is represented by (2). This shows that (A1, A2, A3, A4, A5, A6(ε), A7(ε)) can be considered as an extension of the Anscombe-Aumann theory to the case where the decision-maker considers the possibility of the worst outcome with the possibility of ε all the time (ε-contamination).

The similarity of our axioms with those of Anscombe and Aumann is utilized in the proof of the main theorem, which we now turn to.

5. Proof

The necessity of the axioms in Theorem 1 can be easily verified. We prove the sufficiency of them in this section.

Let $\varepsilon \in [0, 1)$ be such that Axioms A6(ε) and A7(ε) hold with it. We henceforth suppress "(ε)" and simply write as A6 and A7. Throughout the section, we always assume that Axioms A1-A5 are satisfied.

5.1. Definition of $\succ^*$ and Preliminary Lemmas

We define a binary relation $\succ^*$ on L_0 induced by $\succ$ as follows:

$$f \succ^* g \Leftrightarrow \left[\begin{array}{l} f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f} \ \text{ and } \ g \sim (1-\varepsilon)y_g + \varepsilon y_{\min g} \\ \Rightarrow \ y_f \succ y_g \end{array} \right],$$

where y_f and y_g are arbitrary elements of L_c. By definition, $f \succ^* g$ holds true whenever there does not exist such a y_f and/or y_g.

Clearly, y_f $[y_g]$ is, if it exists (existence will be proved later in Lemma 5), f $[g]$'s equivalent constant act in ε-contamination equivalence. Thus, the binary relation $\succ^*$ is induced by the original preferences over these equivalent constant acts. In this subsection, we show this induced binary relation is a preference order by showing it is asymmetric (Lemma 6) and negatively transitive (Lemma 3). We also prove non-degeneracy of the binary relation $\succ^*$ (Lemma 4).

A binary relation on Y is naturally induced from $\succ^*$ as its restriction on L_c and it is denoted by the same symbol, $\succ^*$. Also, we define $\succeq^*$ and $\sim^*$ from $\succ^*$ by the same manner as we did for $\succ$. Then, the following lemma holds.

Lemma 1. *Let $f \in L_0$ and $y_f \in L_c$. If $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$, then $f \sim^* y_f$.*

Proof. Suppose that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$. It always holds that $y_f \sim (1-\varepsilon)y_f + \varepsilon y_{\min y_f}$ since $y_{\min y_f} = y_f$. Furthermore, $y_f \not\succ y_f$ since $\succ$ is asymmetric. Therefore, by the definition of $\succ^*$, it follows that $f \not\succ^* y_f$. Similarly, $y_f \not\succ^* f$. Therefore, $f \sim^* y_f$. ■

Lemma 2. *The two binary relations, $\succ$ and $\succ^*$, coincide on L_c.*

Proof. Let $y, y' \in Y$. First, assume that $y \succ^* y'$. Note that $y \sim (1-\varepsilon)y + \varepsilon y_{\min y}$ and $y' \sim (1-\varepsilon)y' + \varepsilon y'_{\min y'}$ hold since $(\forall y \in Y)\ y_{\min y} = y$. Hence, it follows from the definition of $\succ^*$ that $y \succ y'$.

Second, assume that $y \succ y'$. Let $\bar{y}$ and $\bar{y}'$ be arbitrary constant acts such that (a) $y \sim (1-\varepsilon)\bar{y} + \varepsilon y_{\min y}$ and (b) $y' \sim (1-\varepsilon)\bar{y}' + \varepsilon y'_{\min y'}$. Such $\bar{y}$ and $\bar{y}'$ certainly exist (for example, set $\bar{y} = y$ and $\bar{y}' = y'$). From (a), it holds that $(1-\varepsilon)y + \varepsilon y_{\min y} \sim (1-\varepsilon)\bar{y} + \varepsilon y_{\min y}$. Therefore, A2 implies that $y \sim \bar{y}$ (recall that any pair of constant acts is comonotonic). Similarly, it holds from (b) that $y' \sim \bar{y}'$. Finally, A1 and the assumption that $y \succ y'$ show that $\bar{y} \succ \bar{y}'$, which in turn shows that $y \succ^* y'$ by the definition of $\succ^*$. ■

Lemma 3. *The binary relation $\succ^*$ is negatively transitive.*

Proof. Assume that $f \not\succ^* g$ and $g \not\succ^* h$. Then, there exist constant acts y_f and y_g such that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$, $g \sim (1-\varepsilon)y_g + \varepsilon y_{\min g}$ and $y_f \not\succ y_g$, and there exist constant acts y'_g and y_h such that $g \sim (1-\varepsilon)y'_g + \varepsilon y'_{\min g}$, $h \sim (1-\varepsilon)y_h + \varepsilon y_{\min h}$ and $y'_g \not\succ y_h$. It then holds that

$$(1-\varepsilon)y_g + \varepsilon y_{\min g} \sim g \sim (1-\varepsilon)y'_g + \varepsilon y'_{\min g} \sim (1-\varepsilon)y'_g + \varepsilon y_{\min g}$$

where the last indifference relation holds since $y_{\min g} \sim y'_{\min g}$. (See Kreps, 1988, p.46, Lemma 5.6(c). Note that $\succ$ satisfies all the axioms of the mixture-space theorem (Herstein and Milnor, 1954) on L_c and hence (3) holds on L_c.) Therefore, A2 implies that $y_g \sim y'_g$ (recall that any pair of constant acts are comonotonic). Hence, A1 implies that $y_f \not\succ y_h$, which shows that $f \not\succ^* h$. ■

Lemma 4. $(\exists f, g \in L_0)\ \ f \succ^* g$.

Proof. From A4 and A5, it follows that $(\exists y, y' \in Y)\ y \succ y'$. Since $\succ^*$ and $\succ$ coincide on L_c (Lemma 2), $y \succ^* y'$. ■

So far, we have not assumed any additional axioms beyond A1-A5. The following lemmas need Axiom A6.

Lemma 5. *Assume that Axiom A6 holds. Then,*

$$(\forall f \in L_0)(\exists y_f \in L_c) \;\; f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}\,.$$

Proof. Let $f \in L_0$. Then,

$$y^* \equiv (1-\varepsilon)y_{\max f} + \varepsilon y_{\min f} \succeq f \succeq y_{\min f} \equiv y_*\,,$$

where the first and second orderings hold true by A6 and by A4, respectively. In the rest of proof, we assume that

$$y^* \succ f \succ y_* \tag{4}$$

since the lemma would follow immediately otherwise.

This paragraph shows that

$$0 \le a < b \le 1 \;\Rightarrow\; by^* + (1-b)y_* \succ ay^* + (1-a)y_*\,. \tag{5}$$

Let $y \equiv by^* + (1-b)y_*$. Then, it follows from A2 that $y \succ y_*$ (recall that any pair of constant acts is comonotonic), and hence, that

$$\begin{aligned} y &= (1-(a/b))y + (a/b)y \succ (1-(a/b))y_* + (a/b)y \\ &= (1-(a/b))y_* + (a/b)(by^* + (1-b)y_*) = ay^* + (1-a)y_*\,, \end{aligned}$$

which shows the claim.

Define $a^* \in [0,1]$ by

$$a^* = \sup\{\, a \in [0,1] \mid f \succeq ay^* + (1-a)y_* \,\}\,.$$

The set defining a^* is nonempty by (4) and hence a^* is well-defined. We complete the proof in three steps.

(a) Assume that $a^*y^* + (1-a^*)y_* \succ f$. Then, since $a^*y^* + (1-a^*)y_* \succ f \succ y_*$ by (4), A3 implies that $(\exists b \in (0,1))\; b(a^*y^* + (1-a^*)y_*) + (1-b)y_* = ba^*y^* + (1-ba^*)y_* \succ f$. Since $a^* \neq 0$ by the assumption of (a), it holds that $ba^* < a^*$. It then follows from the definition of a^* that $(\exists a' \in (ba^*, a^*))\; f \succeq a'y^* + (1-a')y_*$. Then, (5) implies that $f \succ ba^*y^* + (1-ba^*)y_*$, which is a contradiction.

(b) Assume that $f \succ a^*y^* + (1-a^*)y_*$. Then, since $y^* \succ f \succ a^*y^* + (1-a^*)y_*$ by (4), A3 implies that $(\exists b \in (0,1))\ f \succ (1-b)y^* + b(a^*y^* + (1-a^*)y_*) = (1-b(1-a^*))y^* + b(1-a^*)y_*$. Since $(1-b(1-a^*)) > a^*$, the definition of a^* implies that $(1-b(1-a^*))y^* + b(1-a^*)y_* \succ f$, which is a contradiction.

(c) By (a) and (b), only remaining possibility is: $f \sim a^*y^* + (1-a^*)y_*$. On the other hand,

$$
\begin{aligned}
a^*y^* + (1-a^*)y_* &= a^*((1-\varepsilon)y_{\max f} + \varepsilon y_{\min f}) + (1-a^*)((1-\varepsilon)y_{\min f} + \varepsilon y_{\min f}) \\
&= (1-\varepsilon)(a^*y_{\max f} + (1-a^*)y_{\min f}) + \varepsilon y_{\min f}\,.
\end{aligned}
$$

Therefore, to define $y_f = a^*y_{\max f} + (1-a^*)y_{\min f}$ completes the proof. ■

Lemma 6. *Assume that Axiom A6 holds. Then, the binary relation $\succ^*$ is asymmetric.*

Proof. Assume that $f \succ^* g$. Also suppose that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$ and that $g \sim (1-\varepsilon)y_g + \varepsilon y_{\min g}$. The existence of constant acts, y_f and y_g, is guaranteed by Lemma 5. Then, it follows from the definition of $\succ^*$ that $y_f \succ y_g$ and the asymmetry of $\succ$ implies that $y_g \not\succ y_f$. Hence, the definition of $\succ^*$ implies that $g \not\succ^* f$. ■

5.2. $\succ^*$ and Anscombe-Aumman Axioms

In this subsection, we show that the binary relation $\succ^*$ satisfies axioms postulated in Anscombe and Aumann (1963). For concreteness, we first list the Anscombe-Aumann axioms below. In these axioms, f, g and h denote arbitrary elements in L_0 and λ denotes an arbitrary number such that $\lambda \in (0,1]$.

AA1* (Ordering) $\succ^*$ is a preference order on L_0 .

AA2* (Independence) $f \succ^* g \Rightarrow \lambda f + (1-\lambda)h \succ^* \lambda g + (1-\lambda)h$.

AA3* (Continuity) If $f \succ^* g$ and $g \succ^* h$, then

$$(\exists \alpha, \beta \in (0,1))\ \ \alpha f + (1-\alpha)h \succ^* g \ \text{ and } \ g \succ^* \beta f + (1-\beta)h\,.$$

AA4* (Monotonicity) $[(\forall s \in S)\ f(s) \succeq^* g(s)] \ \Rightarrow\ f \succeq^* g$.

AA5* (Non-degeneracy) $(\exists f, g \in L_0)\ f \succ^* g$.

It should be noted that we have already proved that $\succ^*$ satisfies Axioms AA1* and AA5* when $\succ$ satisfies Axiom A6 (Lemmas 3, 4 and 6). The following lemmas show that the other axioms are also satisfied with Axioms A7-1 and A7-2 as well as Axiom A6.

Lemma 7. *Assume that Axioms A6 and A7-1 hold. Then, $\succ^*$ satisfies Axioms AA2* and AA3*.*

Proof. (AA2*) Assume that $f \succ^* g$ and let $y_{f_\lambda h}$ and $y_{g_\lambda h}$ be any constant acts such that $\lambda f + (1-\lambda)h \sim (1-\varepsilon)y_{f_\lambda h} + \varepsilon y_{\min f_\lambda h}$ and $\lambda g + (1-\lambda)h \sim (1-\varepsilon)y_{g_\lambda h} + \varepsilon y_{\min g_\lambda h}$. Such $y_{f_\lambda h}$ and $y_{g_\lambda h}$ exist by Lemma 5. We show that $y_{f_\lambda h} \succ y_{g_\lambda h}$, which completes the proof by the definition of $\succ^*$.

By the assumption that $f \succ^* g$ and Lemma 5, there exist constant acts y_f, y_g and y_h such that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$, $g \sim (1-\varepsilon)y_g + \varepsilon y_{\min g}$, $h \sim (1-\varepsilon)y_h + \varepsilon y_{\min h}$ and $y_f \succ y_g$. Since any pair of constant acts is comonotonic, A2 implies that $\lambda y_f + (1-\lambda)y_h \succ \lambda y_g + (1-\lambda)y_h$. On the other hand, A7-1 implies that $\lambda y_f + (1-\lambda)y_h \sim y_{f_\lambda h}$ and $\lambda y_g + (1-\lambda)y_h \sim y_{g_\lambda h}$. Therefore, A1 shows that $y_{f_\lambda h} \succ y_{g_\lambda h}$.

(AA3*) Assume that $f \succ^* g$ and $g \succ^* h$ and let y_f, y_g and y_h be any constant acts such that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$, $g \sim (1-\varepsilon)y_g + \varepsilon y_{\min g}$ and $h \sim (1-\varepsilon)y_h + \varepsilon y_{\min h}$. Such y_f, y_g and y_h exist by Lemma 5. By the assumption that $f \succ^* g$ and $g \succ^* h$ and the definition of $\succ^*$, it follows that $y_f \succ y_g$ and $y_g \succ y_h$. Then, A3 implies that there exists $\alpha \in (0,1)$ such that $\alpha y_f + (1-\alpha)y_h \succ y_g$. Let $y_{f_\alpha h}$ be any constant act such that $\alpha f + (1-\alpha)h \sim (1-\varepsilon)y_{f_\alpha h} + \varepsilon y_{\min f_\alpha h}$. Such a $y_{f_\alpha h}$ exists by Lemma 5. Then, A7-1 implies that $y_{f_\alpha h} \sim \alpha y_f + (1-\alpha)y_h$. Therefore, A1 shows that $y_{f_\alpha h} \succ y_g$, which in turn shows that $\alpha f + (1-\alpha)h \succ^* g$ by the definition of $\succ^*$. A similar proof applies for the existence of β. ■

Lemma 8. *Assume that Axioms A6 and A7-2 hold. Then, $\succ^*$ satisfies Axioms AA4*.*

Proof. Suppose that $(\forall s \in S)\ f(s) \succeq^* g(s)$. Since $\succ^*$ and $\succ$ coincide on L_c (Lemma 2), it follows that $(\forall s \in S)\ f(s) \succeq g(s)$. Let y_f and y_g be constant acts such that $f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}$ and $g \sim (1-\varepsilon)y_g + \varepsilon y_{\min g}$. Such y_f and y_g exist by Lemma 5. Then, A7-2 implies that $y_f \succeq y_g$, or equivalently, $y_g \not\succ y_f$. Therefore, it follows from the definition of $\succ^*$ that $g \not\succ^* f$, implying that $f \succeq^* g$. ■

5.3. Completion of Proof

Assume that all the axioms in the theorem hold. By Lemmas 3, 4, 6, 7 and 8, the binary relation $\succ^*$ satisfies AA1*-AA5*. Therefore, Anscombe and Aumann's theorem (1963) shows that there exist a unique finitely additive probability measure μ on (S, Σ) and an affine function $u : Y \to \mathbb{R}$, which is unique up to a positive affine transformation, such that

$$f \succ^* g \Leftrightarrow \int_S u(f(s))\, d\mu(s) > \int_S u(g(s))\, d\mu(s)\,. \tag{6}$$

Define $J^* : L_0 \to \mathbb{R}$ by

$$(\forall f \in L_0) \quad J^*(f) = \int_S u(f(s))\, d\mu(s)\,. \tag{7}$$

Note that when f is a constant act such that $(\exists y \in Y)(\forall s \in S)\ f(s) = y$, then $J^*(f) = u(y)$.

Define $J : L_0 \to \mathbb{R}$ by

$$(\forall f \in L_0) \quad J(f) = u((1-\varepsilon)y_f + \varepsilon y_{\min f})$$

where $y_f \in L_c$ is a constant act such that

$$f \sim (1-\varepsilon)y_f + \varepsilon y_{\min f}\,. \tag{8}$$

The existence of such a y_f is guaranteed by Lemma 5. Note that u represents $\succ^*$ on L_c by (6) and that $\succ^*$ and $\succ$ coincide on L_c by Lemma 2. It then follows that u represents $\succ$ on L_c. This shows that J is well-defined and represents $\succ$ on L_0.

Finally, we have

$$J(f) = u((1-\varepsilon)y_f + \varepsilon y_{\min f})$$

$$
\begin{aligned}
&= (1-\varepsilon)u(y_f) + \varepsilon u(y_{\min f}) \\
&= (1-\varepsilon)J^*(y_f) + \varepsilon \min_s u(f(s)) \\
&= (1-\varepsilon)J^*(f) + \varepsilon \min_s u(f(s)) \\
&= (1-\varepsilon)\int_S u(f(s))\, d\mu(s) + \varepsilon \min_s u(f(s))\,,
\end{aligned}
$$

where the first equality holds by the definition of J; the second equality holds by u's affinity; the third equality holds by the definition of J^* and the fact that u represents $\succ$ on Y; the fourth equality holds by (8), Lemma 1 and the fact that J^* represents $\succ^*$; and the last equality holds by (7). Since J represents $\succ$ on L_0, the proof is complete. ■

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