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Abstract

This paper investigates repeated games with perfect monitoring, where the number of repetition is finite, and the discount factor is far less than unity. Players can make a side payment contract, but their liability is severely limited. The history of play may not necessarily be verifiable. With positive interest rate of the contractible asset, we show that, in spite of limited liability and verifiability, efficiency is sustainable in that there exist a contract and an efficient perfect equilibrium in its associated game, and that efficiency is even uniquely sustainable if there exists the unique one-shot Nash equilibrium. In partnership games, efficiency is uniquely and approximately sustainable, even if the interest rate equals zero. In partnership games with two players and positive interest rate, efficient sustainability is robust to renegotiation-proofness on the terms of explicit contracting as well as implicit agreements.

Key Words: Finitely Repeated Games, Discounting, Side Payment Contracts, Limited Liability, Limited verifiability, Efficiency, Uniqueness, Renegotiation-Proofness.

+ The earlier version is Matsushima (2000). There exist many important points in the current version that have not been investigated by the earlier version. For example, the earlier version did not investigate the discounting case, the positive interest rate case, unique and approximate efficiency, and renegotiation-proofness on the terms of explicit contracting.

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1. Introduction

This paper investigates repeated games with perfect monitoring, where the number of repetition is finite but sufficiently large, and the discount factor may be far less than unity. At the beginning of the initial period, players can make a budget-balancing side payment contract. The contract requires each player to invest in a safe asset, and may require a deviant to give the whole return from this asset to the others. In the initial period, however, players' limited liability is so severe that each player can invest only a small monetary amount. Moreover, the realized history of play may not necessarily be verifiable to the court, and therefore, the contract may not necessarily be enforceable.

We show that, in spite of the limited liability and verifiability above, when the interest rate of the asset is positive, efficiency is sustainable in the sense that there exists a contract and an efficient perfect equilibrium in its associated game. Benoit and Krishna (1985) and Friedman (1985) showed that in standard models of finitely repeated game with no side payments, there might exist an approximately efficient perfect equilibrium when there exists multiple one-shot Nash equilibria. Radner (1980), Chu and Geanakoplos (1988), and Conlon (1996) showed that when players are irrational, there might exist an approximately efficient perfect equilibrium, even if there exists the unique one-shot Nash equilibrium.[1] In contrast to these works, our result is on 'exact' efficiency as opposed to 'approximate', and does not assume no discounting. In fact, our efficiency result holds under the same condition on the discount factor as that under which efficiency is sustainable in infinitely repeated games by using trigger strategies.

It is well known that if there exists the unique one-shot Nash equilibrium, then the repetition of the one-shot Nash equilibrium play is the unique perfect equilibrium in finitely repeated games. In sharp contrast to this, we show that if there exists the unique one-shot Nash equilibrium and the interest rate is positive, then efficiency is even uniquely sustainable in the sense that there exists a contract, in the game associated with which, there exists the unique perfect equilibrium, and it induces an efficient payoff vector. We also show that, in partnership games, even if the interest rate equals zero, efficiency is uniquely and approximately sustainable in the sense that there exists a contract, in the game associated with which, there exists a perfect equilibrium, and it induces an approximately efficient payoff vector.

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[1] There exists a huge volume of works on repeated games presenting theoretical foundations to the widely accepted view that long-term relationships facilitate collusion more than do in short-term relationships. For the survey on repeated games, see Pearce (1992).

The logic behind collusive behavior in this paper is different from that in standard models. In standard models, players are confronted with the same subgame in every period, and this subgame has multiple perfect equilibria. If a player deviates, her opponents will retaliate from the next period by playing an unfavorable equilibrium to her. Because of this move to the unfavorable equilibrium, each player hesitates to deviate. In contrast to this orthodoxy, the present paper adopts an alternative basis for collusion to occur. Subgames of a finitely repeated game with side payments differ across past histories because the history-contingent contract influences the payoff structures of these subgames. These subgames each have their own respective perfect equilibria. If a player deviates, all players will be confronted in the next period with the subgame whose perfect equilibrium is unfavorable to the deviant. Because of this move to the unfavorable subgame, each player hesitates to deviate. Since collusive behavior can be described as a unique perfect equilibrium, the predictive power in this paper is much stronger than that in standard models. The fact that not only the whole game but also every subgame satisfies the uniqueness implies that, in every period, players have no room to renegotiate the terms of implicit agreement and improve their welfare. This point contrasts with the fact that renegotiation-proofness on the terms of implicit agreement has long been a controversial issue in the repeated game literature.[2]

There exists a sizeable literature dealing with the agency problem with moral hazard, seeking to clarify whether a single-period relationship attains the first-best allocation through the writing of explicit contracts.[3] This literature commonly makes the assumption that it is difficult for the court to verify players' action choices, but that there exists a public signal that is randomly determined according to a probability distribution conditional on player's action choices, and this signal is verifiable. Hence, players can agree to write an explicit contract that depends not on their action choices but on the realization of this signal. A large proportion of this literature was devoted to investigating the single-agent problem, while several works such as Holmstrom (1982), Williams and Radner (1989), and Legros and Matsushima (1991) investigated multi-agent relationships. Legros and Matsushima showed a necessary and sufficient condition under which there exists a budget-balancing side payment contract that induces players to choose a

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[2] See, for example, Bernheim and Rey (1989), Farrell and Maskin (1989) and Pearce (1987) on infinitely repeated games, and Benoit and Krishna (1988) on finitely repeated games.

[3] For the references, see the surveys by Hart and Holmstrom (1987), Dutta and Radner (1994), and Salanie (1997).

collusive action profile as a Nash equilibrium.[4] This contract, however, requires that players pay a large fine as the penalty for deviation. In contrast, the present paper assumes that players establish a long-term relationship, but that only small fines, which may be close to zero, exist in totality. The paper shows that the establishment of a long-term relationship dramatically economizes on monetary fines without harming players' incentive to collude.

As such, this paper may offer an important economic implication within the field of law and economics. In real situations of labor contracting with moral hazard, it is practically difficult to establish measures of performance that are always verifiable to the court. It is also unrealistic to expect that a limitedly liable worker will be able to pay a large fine when the fact that she has neglected her duty could be disclosed to the public. From this, it is widely believed that in real situations the legal enforcement of explicit contracts plays only a limited role in resolving issues of moral hazard. Many economists, such as MacLeod and Malcomson (1989), have emphasized that the self-enforcement of implicit contracts instead plays a more crucial role than legal enforcement, which are thought of formally as a perfect equilibrium in an infinitely repeated game. In contrast, the present paper shows that even if workers' liability is severely limited and their performances are hardly verifiable, the role of legal enforcement is still very crucial and even indispensable for workers' incentives.

This paper is also in contrast with the literature of the reputational theory of finitely repeated games, which assumes incomplete information on players' types.[5] Several works such as Kreps and Wilson (1982) and Milgrom and Roberts (1982) provided their respective examples of a chain-store game in which there exists a unique perfect Bayesian equilibrium and it induces an approximately efficient allocation. However, these results depend crucially on their own specifications of the incomplete information structure, while it is hard to tell about how players can determine the well-behaved incomplete information structure in advance of their repeated play. The present paper, however, assumes complete information and is based on a more plausible scenario of a contracting process, in which players will collectively agree to write a side payment contract that guarantees both uniqueness and, exact or approximate, efficiency.[6]

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[4] These works did not consider the uniqueness of equilibrium. Ma (1988), and Ma, Moore and Turnbull (1988) investigated unique implementation in multi-agent relationships.

[5] For the survey of this literature, see Fudenberg (1992).

[6] For example, one of the partners, called the principal, decides a side payment contract, which maximizes her own payoff given the constraint that there exists a unique perfect equilibrium and

Finally, we investigate partnership games with two players, where we assume that the interest rate is positive. We show that there may exist a combination of a contract and an efficient perfect equilibrium in its associated game that is renegotiation-proof in the sense that in any period, the induced payoff vector in the subgame is always efficient irrespective of the past history, and therefore, players never agree to breach and renegotiate the contract. Hence, we may not need to assume that the contract has full commitment power in the sense that players cannot breach and renegotiate it once the repeated game starts, and therefore, our efficiency results may be robust to renegotiation on the terms of explicit contracting as well as the terms of implicit agreement. Rey and Salanie (1990), and Fudenberg, Holmstrom, and Milgrom (1990) showed that in a long-term principal-agent relationship, renegotiable short-term contracts would implement efficiency. In contrast to the present paper, however, these works depend crucially on the assumption that large side payments are possible.

The organization of the paper is as follows. Section 2 defines the model. Section 3 shows that efficiency is sustainable, or uniquely sustainable, when the interest rate is positive. Section 4 shows sufficient conditions under which efficiency is approximately sustainable, or uniquely and approximately sustainable, even if the interest rate equals zero. Section 5 defines partnership games with zero interest rates, and shows that efficiency is uniquely and approximately sustainable. Section 6 shows that in partnership games with two players and zero interest rates, there exists a renegotiation-proof combination of a contract and an efficient perfect equilibrium. Finally, Section 7 concludes.

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that this perfect equilibrium induces a payoff vector that is no worse than the payoff vector that partners can receive outside if they fail to establish their partnership. Another scenario is that all players agree to a side payment contract according to the Nash bargaining solution at the beginning of period 1 given the constraint that there exists a unique perfect equilibrium and that the payoff vector that partners can receive outside is regarded as the threat point.

2. The Model

Let $N = \{1,...,n\}$ denote the finite set of players. The component game is given by $G \equiv (A_i, u_i)_{i \in N}$, where A_i is the set of actions for player $i \in N$, $A \equiv \prod_{i \in N} A_i$, and $u_i : A \to R$ is the instantaneous payoff function for player i. We assume that there exists a Nash equilibrium action profile $a^e \in A$ in G. Let $a^* \in A$ denote a payoff vector that Pareto-dominates a^e and efficient in that $u(a^*) > u(a^e)$, and there exists no $a \in A$ such that $u(a) \neq u(a^*)$ and $u(a) \geq u(a^*)$.

A finitely repeated game with side payments is given by $\Gamma \equiv \Gamma(T, m, \tau, \delta)$. Players T times repeatedly play the component game G, where $T > 0$ is a positive integer. Let $\delta \in [0,1]$ denote the common discount factor. We assume that monitoring is perfect in that at the end of every period t, players can observe all players' action choices $a(t) \equiv (a_1(t),...,a_n(t)) \in A$. We assume that public randomization devices are possible in that at the end of every period t, every player observes a public signal $\lambda(t)$ that is drawn according to the uniform distribution function on the interval $[0,1]$.[7] A history up to period $t \in \{1,...,T\}$ is denoted by $h(t) \equiv (a(1), \lambda(1),..., a(t), \lambda(t))$. Let $h(0)$ denote the null history. A history up to the final period T is called a *complete* history. The set of histories up to period t is denoted by $H(t)$.

Fix a positive real number $M > 0$ and a nonnegative real number $r \geq 0$ arbitrarily, where we assume that r is less than or equals the discount rate associated with δ, i.e.,

$$(1) \qquad r \leq \frac{1-\delta}{\delta}.$$

At the beginning of the initial period 1, players agree to write a *side payment contract* denoted by (m, τ), where $m = (m_1,...,m_n)$, $m_i : H(T) \to R$, and $\tau : H(T) \to \{1,...,T\}$.[8] We assume that m is *budget balancing*, i.e.,

$$\sum_{i \in N} m_i(h(T)) = 0 \text{ for all } h(T) \in H(T).$$

We assume that each player's liability is limited in ways that for every $i \in N$, and every $h(T) \in H(T)$,

$$m_i(h(T)) \geq -(1+r)^{\tau(h(T))-1} M.$$

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[7] We use public randomization devices only in the study of unique sustainability or unique and approximate sustainability.

[8] We assume from Section 3 to Section 5 that a side payment contract has full commitment power in the sense that in every period players cannot breach and renegotiate it. In Section 6, we will drop this assumption and take the possibility of renegotiation on the terms of explicit contracting into account.

and for every $h'(T) = (a'(1),...,a'(T)) \in H(T)$, if $a(t) = a(t')$ for all $t \in \{1,...,\tau(h(T))\}$, then

$$m_i(h'(T)) = m_i(h(T)) \text{ and } \tau(h'(T)) = \tau(h(T)).$$

Given the complete history $h(T) \in H(T)$, players will make transfer payments at the end of period $\tau(h(T))$ and each player $i \in N$ receives the monetary amount $m_i(h(T))$ at this time. Hence, the long-run payoff for player i when $h(T)$ occurs is given by

$$v_i(h(T);m,\tau) = v_i(h(T)) \equiv \frac{1-\delta}{1-\delta^T}\{\sum_{t=1}^{T}\delta^{t-1}u_i(a(t)) + \delta^{\tau(h(T))-1}m_i(h(T))\}.$$ [9]

Let $v(h(T)) \equiv (v_1(h(T)),\ldots,v_n(h(T)))$. The upper bound of fines, up to which each player is able to pay when the transfer payments are made in period t, is given by

$$(1+r)^{t-1}M.$$

An interpretation is put as follows. At the beginning of every period $t \in \{1,...,\tau(h(T))\}$, each player is required to invest $(1+r)^{t-1}M$ dollars in a safe asset, which provides her with a return of $(1+r)^t M$ dollars at the end of this period with certain. After period $\tau(h(T))+1$, i.e., after the transfers are made, no player is required to invest in any such assets. Another interpretation is put as follows. Fix a positive real number $\varepsilon \in (0,1]$ arbitrarily. At the beginning of every period $t \in \{1,...,\tau(h(T))\}$, she can invest $\frac{(1+r)^{t-1}M}{\varepsilon}$ dollars in the asset. The realized history up to period $\tau(h(T))$, however, is not necessarily verifiable to the court, i.e., can be verified only with probability ε.[10] Hence, the expected value of the fine that each player will pay at the end of period $\tau(h(T))$ is at most $(1+r)^{\tau(h(T))-1}M$.

We must note that the problem is rather trivial when the inequality (1) does not hold. If $r > \frac{1-\delta}{\delta}$, the present value of the return at the end of the final period is $\delta^{T-1}(1+r)^{T-1}M$, which diverges to infinity as T increases, and therefore, implies that players' liability is practically unlimited even in the initial period when the number of repetition is sufficiently large.

A strategy for player i in Γ is defined by $s_i: \bigcup_{t=0}^{T-1} H(t) \to A_i$. Let S_i denote the set of strategies for player i. Let $S \equiv \prod_{i\in N} S_i$ and $s(h(t)) \equiv (s_1(h(t)),\ldots,s_n(h(t))) \in A$. The expected long-run payoff for player i induced by a strategy profile $s = (s_1,...,s_n) \in S$ is denoted by

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[9] Here, we assume that utilities are quasi-linear.

[10] Here, for simplicity, we assume that the probability of the history up to period t being verifiable does not depend on t. This assumption, however, is redundant.

$$v_i(s;m,\tau) = v_i(s) \equiv E[v_i(h(T)) \mid s],$$

where $E[\cdot \mid s]$ implies the expectation conditional on the play of the strategy profile s. Let $v(s) \equiv (v_1(s),...,v_n(s))$. The expected long-run payoff for player i induced by $s \in S$ when $h(t) \in H(t)$ occurs is denoted by

$$v_i(s,h(t);m,\tau) = v_i(s,h(t)) \equiv E[v_i(h(T)) \mid s,h(t)],$$

where $E[\cdot \mid s,h(t)]$ implies the expectation conditional on the history $h(t)$ and the play of s after $h(t)$. Let $v_i(s) \equiv v_i(s,h(0))$. A strategy profile $s \in S$ is said to be a *perfect equilibrium in* Γ if for every $t \in \{1,...,T\}$, every $h(t-1) \in H(t-1)$, and every $i \in N$,

$$v_i(s,h(t-1)) \geq v_i(s/s_i',h(t-1)) \text{ for all } s_i' \in S_i.$$

3. Efficiency

We specify a side payment contract, denoted by (m^+,τ^+), as follows. We need no public randomization devices. For every $h(T)\in H(T)$, let $\tau^+(h(T))$ be the period in which there exist the first deviants from the efficient action profile a^* when $h(T)$ occurs, i.e.,

$$a(\tau^+(h(T)))\neq a^*\text{, and } a(t)=a^* \text{ for all } t<\tau^+(h(T)).$$

When there exist no deviants in every period, let $\tau^+(h(T))=T$. Let $n^+(h(T))$ denote the number of the first deviants, i.e., the number of player $i\in N$ choosing $a_i(\tau^+(h(T)))\neq a_i^*$. For every $i\in N$, and every $h(T)\in H(T)$, let

$$m_i^+(h(T))=(\frac{n^+(h(T))-1}{n-1}-1)(1+r)^{\tau^+(h(T))-1}M \text{ if } a_i(\tau^+(h(T)))\neq a_i^*,$$

and

$$m_i^+(h(T))=\frac{n^+(h(T))}{n-1}(1+r)^{\tau^+(h(T))-1}M \text{ if } a_i(\tau^+(h(T)))=a_i^*,$$

Hence, each player will be fined if and only if she is the first deviant from a^*. Note that (m^+,τ^+) satisfies the assumptions in Section 2.

We specify a 'trigger' strategy profile $s^+\in S$ as follows. For every $t\in\{1,...,T\}$, and every $h(t-1)\in H(t-1)$,

$$s^+(h(t-1))=a^* \text{ if } a(\tau)=a^* \text{ for all } \tau<t,$$

and

$$s^+(h(t-1))=a^e \text{ if } a(\tau)\neq a^* \text{ for some } \tau<t.$$

Hence, in every period, players choose a^* if they have chosen a^* in all the previous periods, while they choose the Nash equilibrium action profile a^e otherwise. Note that

$$v(s^+)=u(a^*),$$

i.e., the strategy profile s^+ induces the efficient payoff vector $u(a^*)$.

The following theorem shows a sufficient condition under which s^+ is a perfect equilibrium in $\Gamma(T,m^+,\tau^+,\delta)$.

Theorem 1: *If for every $i\in N$,*

(2) $$(1+r)^{T-1}M\geq\max_{a_i\in A_i} u_i(a^*/a_i)-u_i(a^*),$$

and

(3) $$M+\frac{(1-\delta^{T-1})\delta}{1-\delta}\{u_i(a^*)-u_i(a^e)\}\geq\max_{a_i\in A_i} u_i(a^*/a_i)-u_i(a^*),$$

then s^+ is a perfect equilibrium in $\Gamma(T,m^+,\tau^+,\delta)$.

Proof: Fix $i \in N$, $t \in \{1,...,T\}$, and $h(t-1) \in H(t-1)$ arbitrarily. If $a(\tau) \neq a^*$ for some $\tau < t$, then, according to s_i^+, player i always continues choosing a_i^e after $h(t-1)$. Since a^e is a one-shot Nash equilibrium, it follows that

$$v_i(s^+, h(t-1)) \geq v_i(s^+ / s_i, h(t-1)) \text{ for all } s_i \in S_i .$$

Suppose that $a(\tau) = a^*$ for all $\tau < t$. Then, it follows

$$v_i(s^+, h(t-1)) = u_i(a^*) .$$

Fix $a_i \in A_i / \{a_i^*\}$ arbitrarily, and let s_i be the strategy satisfying that

$$s_i(h(t-1)) = a_i ,$$

and for every $t' \in \{1,...,T\}$, and every $h'(t'-1) \in H(t'-1)$,

$$s_i(h'(t'-1)) = s_i^+(h'(t'-1)) \text{ if } h'(t'-1) \neq h(t-1) .$$

Hence, s_i assigns the same action as s_i^+ to any history other than $h(t-1)$. Note

$$\begin{aligned} & v_i(s^+, h(t-1)) - v_i(s^+ / s_i, h(t-1)) \\ & = \frac{(1-\delta)\delta^{t-1}}{1-\delta^T}[(1+r)^{t-1}M + \frac{(1-\delta^{T-t})\delta}{1-\delta}\{u_i(a^*) - u_i(a^e)\} \\ & - u_i(a^* / a_i) + u_i(a^*)] . \end{aligned}$$

Let

$$\begin{aligned} & R_i(t) \equiv (1+r)^{t-1}M + \frac{(1-\delta^{T-t})\delta}{1-\delta}\{u_i(a^*) - u_i(a^e)\} \\ & - \max_{a_i \in A_i} u_i(a^* / a_i) + u_i(a^*) . \end{aligned}$$

The inequalities (2) and (3) imply $R_i(T) \geq 0$ and $R_i(1) \geq 0$, respectively. Note

$$R_i(t+1) - R_i(t) = \delta^{T-t}\{(1+r)^{t-1}\delta^{t-1}\frac{rM}{\delta^T} - u_i(a^*) + u_i(a^e)\} .$$

The inequality (1) implies that $(1+r)^{t-1}\delta^{t-1}$ is non-increasing with respect to t, and therefore, for every $t \in \{1,....,T-2\}$,

$$R_i(t+2) < R_i(t+1) \text{ if } R_i(t+1) < R_i(t) .$$

This implies that if $R_i(T) \geq 0$ and $R_i(1) \geq 0$, then $R_i(t) \geq 0$ for all $t \in \{1,....,T\}$. Hence,

$$R_i(t) \geq 0 \text{ for all } i \in N \text{ and all } t \in \{1,...,T\} ,$$

and therefore,

$$v_i(s^+, h(t-1)) - v_i(s^+ / s_i, h(t-1)) \geq 0 .$$

Hence, we have proved that s^+ is a perfect equilibrium in $\Gamma(T, m^+, \tau^+, \delta)$.

Q.E.D.

The inequalities (2) imply that in the one-shot game with side payments, each player's one-shot gain from deviation is less than or equals the monetary amount $(1+r)^{T-1}M$ that the contract requires her to pay. This corresponds to the incentive constraints required in the final period T of the repeated game $\Gamma(T, m^+, \tau^+, \delta)$. The inequalities (3) imply that given that all players play according to the trigger strategy profile s^{+} from period 2, each player's one-shot gain from deviation in period 1 is less than or equals the future loss from the collapse of their implicit collusion plus the monetary amount M that the contract requires her to pay at the end of period 1. This corresponds to the incentive constraints required in period 1. Theorem 1 implies that the incentive constraints in the initial and finial periods above are sufficient for the perfect equilibrium property in all periods.

The inequalities (1) and (2) imply

(4) $$M \geq \delta^{T-1}\{\max_{a_i \in A_i} u_i(a^* / a_i) - u_i(a^*)\}.$$

If δ is close to 1, and M is so small that

$$M < \max_{a_i \in A_i} u_i(a^* / a_i) - u_i(a^*),$$

then the inequality (4) does not hold, and therefore, s^{+} is not a perfect equilibrium in $\Gamma(T, m^{+}, \tau^{+}, \delta)$. This implies that when players are sufficiently patient, it might be difficult to achieve efficiency in $\Gamma(T, m^{+}, \tau^{+}, \delta)$. This is in contrast to standard models of infinitely repeated game where implicit collusion is easier as the discount factor is closer to unity.

Efficiency is said to be *sustainable with respect to* (δ, M, r) if for every sufficiently large T, there exist (m, τ) and a perfect equilibrium s in $\Gamma(T, m, \tau, \delta)$ such that $v(s) = u(a^*)$. The following corollary is straightforward from Theorem 1.

Corollary 2: *Efficiency is sustainable with respect to* (δ, M, r) *if* $r > 0$*, and for every* $i \in N$*,*

$$M + \frac{\delta}{1-\delta}\{u_i(a^*) - u_i(a^e)\} \geq \max_{a_i \in A_i} u_i(a^* / a_i) - u_i(a^*).$$

Hence, efficiency is sustainable with respect to (δ, M, r) *irrespective of* $M > 0$ if $r > 0$, and for every $i \in N$,

(5) $$\frac{\delta}{1-\delta}\{u_i(a^*) - u_i(a^e)\} \geq \max_{a_i \in A_i} u_i(a^* / a_i) - u_i(a^*).$$

Whenever the interest rate r is positive, then efficiency is sustainable even if the discount factor δ is far less than unity and each player's liability in the initial period is severely limited, i.e., M is close to zero.

Suppose that $u(a^e)$ is not only a Nash equilibrium payoff vector but also the *minimax* payoff vector satisfying that $u_i(a^e) = \min_{a_{-i} \in A_{-i}} \{\max_{a_i \in A_i} u_i(a)\}$ for all $i \in N$. Then, the inequalities (5) are necessary conditions for the existence of perfect equilibria in the infinitely repeated game with no side payments that induces the efficient payoff vector

$u(a^*)$. Hence, we can conclude that efficiency is sustainable in finitely repeated games with small side payments under the same condition as that in infinitely repeated games with no side payments.

The following corollary is also straightforward from Theorem 1.

Corollary 3: *If $r = 0$, and M is large enough to satisfy that for every $i \in N$,*

(6) $$M \geq \max_{a_i \in A_i} u_i(a^* / a_i) - u_i(a^*),$$

then efficiency is sustainable with respect to $(\delta, M, 0)$ irrespective of $\delta \in [0,1]$.

Corollary 3 implies that whenever M is more than or equals the one-shot gain from deviation from a^*, then efficiency is sustainable irrespective of the length of repetition T even if all players are myopic and the interest rate is zero, i.e., $(\delta, r) = (0,0)$. Hence, we can conclude that the establishment of a long-term relationship, together with the use of the side payment contract that fines only the first deviants, dramatically economizes on monetary fines without harming players' incentive to collude.

The following theorem shows a sufficient condition under which s^+ is the unique perfect equilibrium in $\Gamma(T, m^+, \tau^+, \delta)$ irrespective of T and δ.

Theorem 4: *If a^e is the unique Nash equilibrium in G, and for every $i \in N$,*

(7) $$M > \max_{a \in A} \{u_i(a) - u_i(a / a_i^*)\},$$

then s^+ is the unique perfect equilibrium in $\Gamma(T, m^+, \tau^+, \delta)$ irrespective of T and δ.[11]

Proof: Let s be a perfect equilibrium in $\Gamma(T, m^+, \tau^+, \delta)$. Since a^e is the unique Nash equilibrium in G, it follows that for every $t \in \{1,...,T\}$, and every $h(t-1) \in H(t-1)$,

$$s(h(t-1)) = a^e \text{ whenever } a(\tau) \neq a^* \text{ for some } \tau < t.$$

From the inequality (7), it follows that for every $h(T-1) \in H(T-1)$, if $a(\tau) = a^*$ for all $\tau < T$, then the choices of a^* are strictly dominant in the final period T, and therefore,

$$s(h(T-1)) = a^*.$$

Fix $t \in \{1,...,T\}$ arbitrarily, where we assume that for every $t' \in \{t+1,...,T\}$, and every $h(t'-1) \in H(t'-1)$,

$$s(h(t'-1)) = a^* \text{ if } a(\tau) = a^* \text{ for all } \tau < t'.$$

Fix $h(t-1) \in H(t-1)$ arbitrarily, where we assume that

$$a(\tau) = a^* \text{ for all } \tau < t.$$

[11] Note that the inequalities (7) are more restrictive than the inequalities (6).

Fix $i \in N$ and $a \in A / \{a^*\}$ arbitrarily, where we assume $a_i \neq a_i^*$. Suppose

$$s(h(t-1)) = a.$$

Then, from the inequalities (7),

$$v_i(s / s_i^+, h(t-1)) - v_i(s, h(t-1))$$
$$\geq \frac{(1-\delta)\delta^{t-1}}{1-\delta^T}\{u_i(a / a_i^*) - u_i(a) + (1+r)^{t-1}M\} > 0.$$

This is a contradiction, and therefore,

$$s(h(t-1)) = a^*.$$

Hence, we have proved that s^+ is the unique perfect equilibrium in $\Gamma(T, m^+, \tau^+, \delta)$.

Q.E.D.

Next, we specify a side payment contract (m^{++}, τ^{++}) as follows. Here, we do use public randomization devices. Let $n^{++}(a)$ denote the number of players i choosing $a_i \neq a_i^*$. Let $\tau^{++}(h(T))$ be the first period t in which there exist deviants from a^* and the realized public signal $\lambda(t)$ is less than the number of the deviants divided by n, i.e.,

$$\lambda(\tau) \geq \frac{n^{++}(a(\tau))}{n} \text{ for all } \tau < t,$$

and

$$\lambda(t) < \frac{n^{++}(a(t))}{n},$$

If there exists no such t, let $\tau^{++}(h(T)) = T$. For every $h(T) \in H(T)$, let

$$m_i^{++}(h(T)) = (\frac{n^{++}(a(\tau^{++}(h(T)))) - 1}{n-1})M \text{ if } a_i(\tau^{++}(h(T))) \neq a_i^*,$$

and

$$m_i^{++}(h(T)) = \frac{n^{++}(a(\tau^{++}(h(T))))}{n-1}M \text{ if } a_i(\tau^{++}(h(T))) = a_i^*.$$

If no agent has been fined in the past and $\tilde{n}$ agents deviate in the present, then they will be fined with probability $\frac{\tilde{n}}{n}$. Note that (m^{++}, τ^{++}) satisfies the assumptions in Section 2. We specify a 'modified' trigger strategy profile $s^{++} \in S$ as follows. For every $t \in \{1, ..., T\}$, and every

$$s_i^{++}(h(t-1)) = a_i^* \text{ if } \lambda(\tau) > \frac{n^{++}(a(\tau))}{n} \text{ for all } \tau < t,$$

and

$$s_i^{++}(h(t-1)) = a_i^e \;\; \lambda(\tau) \leq \frac{n^{++}(a(\tau))}{n} \text{ for some } \tau < t.$$

Note that $v(s^{++}) = u(a^*)$, i.e., the strategy profile s^{++} induces the efficient payoff vector

$u(a^*)$.

The following theorem shows a sufficient condition under which s^{++} is the unique perfect equilibrium in $\Gamma(T, m^{++}, \tau^{++}, \delta)$.

Theorem 5: *If a^e is the unique Nash equilibrium in G, and for every $i \in N$,*

(8) $$(1+r)^{T-1} M > n \max_{a \in A} \{u_i(a) - u_i(a / a_i^*)\},$$

and

(9) $$M + \frac{(1-\delta^{T-1})\delta}{1-\delta} \{u_i(a^*) - u_i(a^e)\} \geq n \max_{a \in A} \{u_i(a) - u_i(a / a_i^*)\},$$

then, the strategy profile s^{++} is the unique perfect equilibrium in $\Gamma(T, m^{++}, \tau^{++}, \delta)$.

Proof: Let s be a perfect equilibrium in $\Gamma(T, m^{++}, \tau^{++}, \delta)$. Since a^e is the unique Nash equilibrium in G, it follows that for every $i \in N$, every $t \in \{1, ..., T\}$, and every $h(t-1) \in H(t-1)$,

$$s(h(t-1)) = a^e \text{ whenever } \lambda(\tau) < \frac{n^{++}(a(\tau))}{n} \text{ for some } \tau < t.$$

From the inequality (8), it follows that for every $h(T-1) \in H(T-1)$, if $\lambda(\tau) \geq \frac{n^{++}(a(\tau))}{n}$ for all $\tau < t$, then the choices of a^* are strictly dominant in period T, and therefore,

$$s(h(T-1)) = a^*.$$

Fix $t \in \{1, ..., T\}$ arbitrarily. Suppose that for every $t' \in \{t+1, ..., T\}$, and every $h(t'-1) \in H(t'-1)$,

$$s(h(t'-1)) = a^* \text{ if } \lambda(\tau) \geq \frac{n^{++}(a(\tau))}{n} \text{ for all } \tau < t'.$$

Fix $h(t-1) \in H(t-1)$ arbitrarily, where we assume

$$a(\tau) = a^* \text{ for all } \tau < t.$$

Fix $i \in N$ and $a \in A / \{a^*\}$ arbitrarily, where we assume $a_i \neq a_i^*$. Suppose

$$s(h(t-1)) = a.$$

Note

$$\begin{aligned} &v_i(s / s_i^{++}, h(t-1)) - v_i(s, h(t-1)) \\ &\geq \frac{(1-\delta)\delta^{t-1}}{n(1-\delta^T)} [(1+r)^{t-1} M + \frac{(1-\delta^{T-t})\delta}{1-\delta} \{u_i(a^*) - u_i(a^e)\} \\ &- n\{u_i(a) - u_i(a / a_i^*)\}]. \end{aligned}$$

Let

$$B_i(t) \equiv (1+r)^{t-1} M + \frac{(1-\delta^{T-t})\delta}{1-\delta} \{u_i(a^*) - u_i(a^e)\}$$

$$-n \max_{a \in A} \{u_i(a) - u_i(a / a_i^*)\} .$$

The inequalities (8) and (9) imply $B_i(T) > 0$ and $B_i(1) > 0$, respectively. Note

$$B_i(t+1) - B_i(t) = \delta^{T-t} \{(1+r)^{t-1} \delta^{t-1} \frac{rM}{\delta^T} - u_i(a^*) + u_i(a^e)\} .$$

Since the inequality (1) implies that $(1+r)^{t-1}\delta^{t-1}$ is non-increasing with respect to t, it follows that for every $t \in \{1,....,T-2\}$,

$$B_i(t+2) < B_i(t+1) \text{ if } B_i(t+1) < B_i(t) ,$$

which implies that if $B_i(T) > 0$ and $B_i(1) > 0$, then $B_i(t) > 0$ for all $t \in \{1,....,T\}$. Hence,

$$B_i(t) > 0 \text{ for all } i \in N \text{ and all } t \in \{1,....,T\} ,$$

and therefore,

$$v_i(s / s_i^{++}, h(t-1)) - v_i(s, h(t-1)) > 0 .$$

This is a contradiction. Hence, we have proved that s^{++} is the unique perfect equilibrium.

Q.E.D.

The inequalities (8) imply that in the one-shot game with side payments, each player's one-shot gain from deviation is less than or equals the monetary amount $(1+r)^{T-1}M$ that the contract requires a deviant to pay, irrespective of which actions the other agents will choose in this game. This corresponds to the incentive constraints in terms of dominance required in the final period T. The inequalities (9) imply that given that all players play according to the trigger strategy profile s^{++} from period 2, each player's one-shot gain from deviation in period 1 is less than or equals the future loss from the collapse of their implicit collusion plus the monetary amount M that the contract requires her to pay at the end of period 1, irrespective of which actions the other agents will choose in period 1. This corresponds to the incentive constraints in terms of dominance required in period 1. Hence, Theorem 5 implies that the incentive constraints in terms of dominance in the initial and finial periods are sufficient for the unique perfect equilibrium property in all periods.

Efficiency is said to be *uniquely sustainable with respect to* (δ, M, r) if for every sufficiently large $T > 0$, there exist (m, τ) and s such that s is the unique perfect equilibrium in $\Gamma(T, m, \tau, \delta)$ and induces $v(s) = u(a^*)$. The following corollary is straightforward from Theorem 5.

Corollary 6: *Efficiency is uniquely sustainable with respect to* (δ, M, r) *if* $r > 0$*, and for every* $i \in N$*,*

$$M + \frac{\delta}{1-\delta}\{u_i(a^*) - u_i(a^e)\} \geq n \max_{a \in A} \{u_i(a / a_i^*) - u_i(a)\} .$$

Hence, efficiency is uniquely sustainable with respect to (δ, M, r) *irrespective of* $M > 0$ if $r > 0$, and for every $i \in N$,

$$\frac{\delta}{1-\delta}\{u_i(a^*) - u_i(a^e)\} \geq n \max_{a \in A} \{u_i(a / a_i^*) - u_i(a)\}.$$

4. Approximate Efficiency

This section assumes that the interest rate of the asset is zero, i.e., $r = 0$, and that players' liability in the initial period, M, is so small that the inequities (6) do not necessarily hold. Efficiency is said to be *approximately sustainable with respect to* (δ, M, r) if for every $\varepsilon > 0$, and every sufficiently large $T > 0$, there exists (m, τ) and a perfect equilibrium s in $\Gamma(T, m, \tau, \delta)$ such that

$$\left| v_i(s) - u_i(a^*) \right| \leq \varepsilon \text{ for all } i \in N .$$

Theorem 7: *Suppose that* $r = 0$*, and there exists an infinite sequence of action profiles* $(a^{(r)})_{r=1}^{\infty}$ *such that*

(10) $$M \geq \max_{a_i \in A_i} \{u_i(a^{(1)} / a_i) - u_i(a^{(1)})\},$$

for every $r \geq 2$*,*

(11) $$M + \sum_{h=1}^{r-1} \delta^h \{u_i(a^{(r-h)}) - u_i(a^e)\} \geq \max_{a_i \in A_i} \{u_i(a^{(r)} / a_i) - u_i(a^{(r)})\},$$

and

(12) $$\lim_{s \to \infty} u(a^{(r)}) = u(a^*) .$$

Then, efficiency is approximately sustainable with respect to $(\delta, M, 0)$.

Proof: We specify a side payment contract (m^*, τ^*) as follows. We do not use public randomization devices. For every $h(T) \in H(T)$, let $\tau^*(h(T))$ be the period in which there exist the first deviants from $(a^{(T)}, ..., a^{(1)})$ when $h(T)$ occurs, i.e.,

$$a(\tau^*(h(T))) \neq a^{(T - \tau^*(h(T)) + 1)}, \text{ and } a(t) = a^{(T-t+1)} \text{ for all } t < \tau^*(h(T)).$$

When there exist no deviants, i.e., $a(t) = a^{(T-t+1)}$ for all $t \in \{1, ..., T\}$, let $\tau^*(h(T)) = T$. Let $n^*(h(T))$ denote the number of the first deviants, i.e., the number of players i choosing $a_i(\tau^*(h(T))) \neq a_i^{(T-\tau^*(h(T))+1)}$. For every $i \in N$, and every $h(T) \in H(T)$, let

$$m_i^*(h(T)) = (\frac{n^*(h(T)) - 1}{n - 1} - 1)M \text{ if } a_i(\tau^*(h(T))) \neq a_i^{(T-\tau^*(h(T))+1)},$$

and

$$m_i^{++}(h(T)) = \frac{n^*(h(T))}{n-1} M \text{ if } a_i(\tau^*(h(T))) = a_i^{(T-\tau^*(h(T))+1)} .$$

Note that (m^*, τ^*) satisfies the assumptions in Section 2. We specify a strategy profile $s^* \in S$ as follows. For every $t \in \{1, \ldots, T\}$, and every $h(t-1) \in H(t-1)$,

$$s^*(h(t-1)) = a^{(T-t+1)} \text{ if } a(\tau) = a^{(T-\tau+1)} \text{ for all } \tau < t,$$

and

$$s^*(h(t-1)) = a^e \text{ if } a(\tau) \neq a^{(T-\tau+1)} \text{ for some } \tau < t.$$

Note from the equality (12) that when T is sufficiently large, $v(s^*)$ is approximated by $u(a^*)$.

Fix $i \in N$, $t \in \{1,...,T\}$, and $h(t-1) \in H(t-1)$ arbitrarily. If $a(\tau) \neq a^{(T-\tau+1)}$ for some $\tau < t$, then, according to s_i^*, player i always continues choosing a_i^e after $h(t-1)$. Since a^e is a Nash equilibrium in G, it follows that

$$v_i(s^*, h(t-1)) \geq v_i(s^* / s_i, h(t-1)) \text{ for all } s_i \in S_i.$$

Suppose that $a(\tau) = a^{(T-\tau+1)}$ for all $\tau < t$. Fix $a_i \in A_i / \{a_i^{(T-t+1)}\}$ arbitrarily. Let s_i be the strategy satisfying that

$$s_i(h(t-1)) = a_i,$$

and for every $t' \in \{1,...,T\}$, and every $h'(t'-1) \in H(t'-1)$,

$$s_i(h'(t'-1)) = s_i^*(h'(t'-1)) \text{ if } h'(t'-1) \neq h(t-1).$$

Note that if $t = T$, then it follows from the inequalities (10) that

$$v_i(s^*, h(t-1)) - v_i(s^* / s_i, h(t-1))$$
$$= \frac{(1-\delta)\delta^{T-1}}{1-\delta^T}\{u_i(a^{(1)}) - u_i(a^{(1)} / a_i) + M\} \geq 0.$$

Note from the inequalities (11) that if $t < T$, then

$$v_i(s^*, h(t-1)) - v_i(s^* / s_i, h(t-1))$$
$$= \frac{(1-\delta)\delta^{t-1}}{1-\delta^T}\{M + \sum_{h=1}^{T-t}\delta^h\{u_i(a^{(T-t-h+1)}) - u_i(a^e)\}$$
$$- u_i(a^{(T-t+1)} / a_i) + u_i(a^{(T-t+1)})\} \geq 0.$$

Hence, we have proved that s^* is a perfect equilibrium.

Q.E.D.

Note that $(a^{(T)},...,a^{(1)})$ is the sequence of action profiles that players choose on the equilibrium path. Here, M can be so small that the instantaneous payoff vector in the final period T, $u(a^{(1)})$, is very close to the one-shot Nash equilibrium payoff vector $u(a^e)$. However, the switch of action choices from $a^{(1)}$ to a^e in period T (slightly) strengthens players' incentive to play more collusively in period $T-1$, and therefore, the instantaneous payoff vector in period $T-1$, $u(a^{(2)})$, could be better than $u(a^{(1)})$. By recursively using the same arguments, it follows that in the periods that are far from the final period, players have incentive to play (almost) fully collusive behavior.

Efficiency is said to be *uniquely and approximately sustainable with respect to* (δ, M, r) if for every $\varepsilon > 0$, and every sufficiently large $T > 0$, there exist (m, τ) and a strategy profile s such that s is the unique perfect equilibrium in $\Gamma(T, m, \tau, \delta)$ and it satisfies $|v_i(s) - u_i(a^*)| \leq \varepsilon$ for all $i \in N$.

Theorem 8: *Suppose that* $r=0$, a^e *is the unique Nash equilibrium in* G, *and there exists an infinite sequence of action profiles* $(a^{(r)})_{r=1}^{\infty}$ *such that*

(13) $$M > n \max_{a\in A} \{u_i(a) - u_i(a/a_i^{(1)})\},$$

for every $r \geq 2$,

(14) $$M + \sum_{h=1}^{r-1} \delta^h \{u_i(a^{(r-h)}) - u_i(a^e)\} > n \max_{a\in A} \{u_i(a) - u_i(a/a_i^{(r)})\},$$

and the equality (12) holds. Then, efficiency is uniquely and approximately sustainable with respect to $(\delta, M, 0)$.

Proof: We specify a side payment contract (m^{**}, τ^{**}) as follows. We use public randomization devices. Let $n^{**}(a,t)$ denote the number of players i satisfying $a_i \neq a_i^{(T-t+1)}$. For every $h(T) \in H(T)$, let $\tau^{**}(h(T))$ be the first period t in which there exist deviants from $(a^{(T)},...,a^{(1)})$ and the realized public signal $\lambda(t)$ is less than the number of the deviants divided by n, i.e.,

$$\lambda(\tau) \geq \frac{n^{**}(a(\tau),\tau)}{n} \text{ for all } \tau < t,$$

and

$$\lambda(\tau) < \frac{n^{**}(a(t),t)}{n}.$$

If there exists no such t, let $\tau^{**}(h(T)) = T$. For every $h(T) \in H(T)$, let

$$m_i^{**}(h(T)) = (\frac{n^{**}(a(\tau^{**}(h(T))), \tau^{**}(h(T))) - 1}{n-1})M \text{ if}$$
$$a_i(\tau^{**}(h(T))) \neq a_i^{(T-\tau^{**}(h(T))+1)},$$

and

$$m_i^{**}(h(T)) = \frac{n^{**}(a(\tau^{**}(h(T))), \tau^{**}(h(T)))}{n-1} M \text{ if}$$
$$a_i(\tau^{**}(h(T))) = a_i^{(T-\tau^{**}(h(T))+1)}.$$

Note that (m^{**}, τ^{**}) satisfies the assumptions in Section 2. We specify a strategy profile $s^{**} \in S$ as follows. For every $t \in \{1,...,T\}$, and every $h(t-1) \in H(t-1)$,

$$s_i^{**}(h(t-1)) = a_i^{(T-t+1)} \text{ if } \lambda(\tau) > \frac{n^{**}(a(\tau),\tau)}{n} \text{ for all } \tau < t,$$

and

$$s_i^{**}(h(t-1)) = a_i^e \text{ if } \lambda(\tau) \leq \frac{n^{**}(a(\tau),\tau)}{n} \text{ for some } \tau < t.$$

Let a strategy profile s be a perfect equilibrium in $\Gamma(T, m^{**}, \tau^{**}, \delta)$. Since a^e is the unique Nash equilibrium in G, it follows that for every $i \in N$, every $t \in \{1,...,T\}$, and every $h(t-1) \in H(t-1)$,

$$s(h(t-1)) = a^e \text{ whenever } \lambda(\tau) \leq \frac{n^{**}(a(\tau),\tau)}{n} \text{ for some } \tau < t.$$

From the inequality (13), it follows that for every $h(T-1) \in H(T-1)$, if $\lambda(\tau) > \frac{n^{**}(a(\tau),\tau)}{n}$ for all $\tau < T$, then the choices of $a^{(1)}$ are strictly dominant in period T, i.e.,

$$s(h(T-1)) = a^{(1)}.$$

Fix $t \in \{1,...,T-1\}$ arbitrarily, where we assume that for every $i \in N$, and $t' \in \{t+1,...,T\}$,

$$s(h(t'-1)) = a^{(T-t'+1)} \text{ if } \lambda(\tau) > \frac{n^{**}(a(\tau),\tau)}{n} \text{ for all } \tau < t',$$

Fix $h(t-1) \in H(t-1)$ arbitrarily, where we assume

$$\lambda(\tau) > \frac{n^{**}(a(\tau),\tau)}{n} \text{ for all } \tau < t,$$

Fix $i \in N$ and $a \in A/\{a^{(T-t+1)}\}$ arbitrarily, where we assume $a_i \neq a_i^{(T-t+1)}$. Suppose

$$s(h(t-1)) = a.$$

Note from the inequalities (14) that

$$\begin{aligned}
&v_i(s/s_i^{**}, h(t-1)) - v_i(s, h(t-1)) \\
&\geq \frac{(1-\delta)\delta^{t-1}}{(1-\delta^T)n}[(1+r)^{t-1}M + \sum_{h=1}^{T-t}\delta^h\{u_i(a^{(T-t+1-h)}) - u_i(a^e)\} \\
&-u_i(a) + u_i(a/a_i^{(T-t+1)})] > 0.
\end{aligned}$$

This is a contradiction. Hence, we have proved that s^{**} is the unique perfect equilibrium.

Q.E.D.

In the next section, we investigate specified partnership games where there exists a sequence $(a^{(1)}, a^{(2)},...)$ that satisfies the conditions in Theorem 7 or Theorem 8.

5. Partnerships

This section investigates partnership games defined as follows. For every $i \in N$, let $A_i = [0,1]$, and

$$u_i(a) = \alpha \sum_{j \neq i} a_j - a_i \quad \text{for all} \quad a \in A$$

where we assume

$$(n-1)\alpha - 1 > 0.$$

Note that there exists the unique one-shot Nash equilibrium $a^e = (0,...,0)$, and $u(a^e)$ is the minimax payoff vector. Let the efficient action profile be given by $a^* = (1,...,1)$, where a^* Pareto-dominates a^e, i.e.,

$$u_i(a^*) = (n-1)\alpha - 1 > 0 = u_i(a^e) \quad \text{for all} \quad i \in N.$$

From Corollary 2, it follows that efficiency is sustainable with respect to (δ, M, r) if $r > 0$, and

$$(15) \qquad \delta \geq \frac{1-M}{(n-1)\alpha - M}.$$

Since $u(a^e) = (0,...,0)$ is the minimax payoff vector, it follows that the inequality (15) is a necessary condition for the existence of a side payment contract (m, τ) and a perfect equilibrium payoff vector other than $u(a^e)$ in $\Gamma(T, m, \tau, \delta)$. Moreover, from the inequality (15), it follows that efficiency is sustainable with respect to (δ, M, r) irrespective of M if $r > 0$, and

$$(16) \qquad \delta \geq \frac{1}{(n-1)\alpha},$$

where the inequality (16) is a necessary condition under which there exist (m, τ) and a perfect equilibrium payoff vector other than $u(a^e)$ in $\Gamma(T, m, \tau, \delta)$, irrespective of M. We must note that the inequality (16) is a necessary condition under which there exists a perfect equilibrium payoff vector other than $u(a^e)$ in the infinitely repeated game with no side payments.

Proposition 9: *In the partnership game, there exists an infinite sequence of action profile* $(a^{(1)}, a^{(2)}, ...)$ *that satisfies the inequalities (10) and (11) and the equality (12) if the inequality (15) holds.*

Proof: We specify $(a^{(1)}, a^{(2)}, ...)$ as follows. For every $i \in N$, let

$$a_i^{(1)} = M.$$

For every $i \in N$, and every $r \geq 2$, let

$$a_i^{(r)} = M + \sum_{h=1}^{r-1} \delta^h u_i(a^{(r-h)}) \quad \text{if} \quad M + \sum_{h=1}^{r-1} \delta^h u_i(a^{(r-h)}) < 1,$$

and

$$a_i^{(r)} = 1 \quad \text{if} \quad M + \sum_{h=1}^{r-1} \delta^h u_i(a^{(r-h)}) \geq 1.$$

Note that the specified $(a^{(1)}, a^{(2)}, ...)$ satisfies the inequalities (10) and (11). Suppose that the inequality (12) does not hold. Then, it must hold that

$$M + \sum_{h=1}^{r-1} \delta^h u_i(a^{(r-h)}) < 1 \quad \text{for all} \quad r,$$

and therefore,

$$a_i^{(r)} = M + \sum_{h=1}^{r-1} \delta^h u_i(a^{(r-h)}) \quad \text{for all} \quad r.$$

Since $a_i^{(r)}$ is non-decreasing with respect to r, there exists $b_i \in R$ such that $\lim_{r\to\infty} a_i^{(r)} = b$ where $b_i < 1$. Note

$$\lim_{r\to\infty} \sum_{h=1}^{r-1} \delta^h u_i(a^{(r-h)}) = \frac{((n-1)\alpha - 1)\delta b_i}{1-\delta},$$

and therefore, it must hold that

$$b_i = M + \frac{((n-1)\alpha - 1)\delta b_i}{1-\delta}.$$

Since $b < 1$, it follows that

$$\delta = \frac{b_i - M}{(n-1)\alpha b_i - M} < \frac{1-M}{(n-1)\alpha - M},$$

which is a contradiction because of the inequality (15). Hence, we have proved $\lim_{s\to\infty} u(a_i^{(r)}) = (n-1)\alpha - 1$.

Q.E.D.

The following theorem is straightforward from Theorems 7 and 8, and Proposition 9.

Theorem 10: *Efficiency is approximately sustainable with respect to* $(\delta, M, 0)$ *if the inequality (15) holds. Efficiency is approximately sustainable with respect to* $(\delta, M, 0)$ *irrespective of* M *if the inequality (16) holds.*

Hence, it follows from Theorem 10, together with the arguments above, that approximate efficient sustainability when $r = 0$ holds under the same condition as efficient sustainability when $r > 0$. From Corollary 6, it follows that efficiency is uniquely sustainable with respect to (δ, M, r) if $r > 0$, and

$$\delta > \frac{(1-M)n}{(n-1)\alpha - 1 + (1-M)n}. \tag{17}$$

Note that the right hand side of the inequality (17) converges zero as α increases. This implies that when the gains from the increase of the other players' effort levels are sufficiently large, efficiency is uniquely sustainable, even if the discount factor is close to zero. Moreover, from the inequalities (17), it follows that efficiency is uniquely sustainable with respect to (δ, M, r) irrespective of M if $r > 0$, and

$$\delta > \frac{n}{(n-1)(\alpha + 1)}. \tag{18}$$

Proposition 11: *In the partnership game, there exists an infinite sequence of action profile* $(a^{(1)}, a^{(2)}, \ldots)$ *that satisfies the inequalities (13) and (14) and the equality (12) if the inequality (17) holds*

Proof: Choose $\widetilde{M} \in (0, M)$ arbitrarily, which is sufficiently close to M. From the inequality (17), it follows that

$$(19) \qquad \delta > \frac{(1-\widetilde{M})n}{(n-1)\alpha - 1 + (1-\widetilde{M})n}.$$

We specify $(a^{(1)}, a^{(2)}, \ldots)$ as follows. For every $i \in N$, let

$$a_i^{(1)} = \widetilde{M}.$$

For every $i \in N$, and every $r \geq 2$, let

$$a_i^{(r)} = \widetilde{M} + \sum_{h=1}^{r-1} \delta^h u_i(a^{(sr-h)}) \text{ if } \widetilde{M} + \sum_{h=1}^{r-1} \delta^h u_i(a^{(r-h)}) < 1,$$

and

$$a_i^{(r)} = 1 \text{ if } \widetilde{M} + \sum_{h=1}^{r-1} \delta^h u_i(a^{(r-h)}) \geq 1.$$

Note that the specified $(a^{(1)}, a^{(2)}, \ldots)$ satisfies the inequalities (13) and (14). Suppose that the inequality (12) does not hold. Then, it must hold that

$$\widetilde{M} + \sum_{h=1}^{r-1} \delta^h u_i(a^{(r-h)}) < 1 \text{ for all } r,$$

and therefore,

$$a_i^{(r)} = \widetilde{M} + \sum_{h=1}^{r-1} \delta^h u_i(a^{(r-h)}) \text{ for all } r.$$

Since $a_i^{(r)}$ is non-decreasing with respect to r, there exists $b_i \in R$ such that $\lim_{r\to\infty} a_i^{(r)} = b$ where $b_i < 1$. Note

$$\lim_{r\to\infty} \sum_{h=1}^{r-1} \delta^h u_i(a^{(r-h)}) = \frac{((n-1)\alpha - 1)\delta b_i}{1-\delta},$$

and therefore, it must hold that

$$b_i = \widetilde{M} + \frac{((n-1)\alpha - 1)\delta b_i}{1-\delta}.$$

Since $b < 1$, it follows that

$$\delta = \frac{b_i - \widetilde{M}}{(n-1)\alpha b_i - \widetilde{M}} < \frac{1-\widetilde{M}}{(n-1)\alpha - \widetilde{M}},$$

which is a contradiction because of the inequality (19). Hence, we have proved $\lim_{s\to\infty} u(a_i^{(r)}) = (n-1)\alpha - 1$.

Q.E.D.

The following theorem is straightforward from Theorems 8 and 9, and Proposition 10.

Theorem 12: *Efficiency is uniquely and approximately sustainable with respect to* $(\delta, M, 0)$ *if the inequality (17) holds. Efficiency is uniquely and approximately sustainable with respect to* $(\delta, M, 0)$ *irrespective of* M *if the inequality (18) holds.*

Hence, it follows that unique and approximate efficient sustainability when $r = 0$ holds under the same condition as unique efficient sustainability when $r > 0$.

6. Renegotiation-Proofness

This section reconsiders the partnership games with two players, i.e., $n = 2$. We confine our attention to the class of side payment contracts (m, τ) satisfying that transfers are made only in the final period, i.e.,

$$\tau(h(T)) = T \quad \text{for all} \quad h(T) \in H(T).$$

Hence, we will simply write m instead of (m, τ) for a side payment contract in the class above.

Fix (T, δ, r) arbitrarily. For every $t \in \{1,...,T\}$, let $Z(t)$ denote the set of combinations of a side payment contract and a strategy profile (m, s) satisfying that for every $\tau \in \{t,...,T\}$, every $h(\tau - 1) \in H(\tau - 1)$, and every $i \in N$,

$$v_i(s, h(\tau - 1); m) \geq v_i(s / s_i', h(\tau - 1); m) \quad \text{for all} \quad s_i' \in S_i,$$

and for every $h'(T) \in H(T)$, and every $i \in N$,

$$m_i(h'(T)) = 0 \quad \text{whenever} \quad h'(t-1) = h(t-1) \quad \text{and} \quad a'(\tau) = s(h'(\tau - 1))$$
$$\text{for all} \quad \tau \in \{t,...,T\},$$

where we denote $h'(\tau) = (a'(1), \lambda'(1),..., a'(\tau), \lambda'(\tau))$. Note that (m, s) belongs to $Z(t)$ if and only if s satisfies the perfect equilibrium property in $\Gamma(T, m, \delta, r)$ after period t and no players are fined when all players conform to s after period t.

We allow players to breach and renegotiate the contract in any period by replacing the contract with any contract that belongs to $Z(t)$, if they unanimously want to do so. A combination of a side payment contract and a strategy profile (m, s) is said to be *renegotiation-proof* if for every $t \in \{1,...,T\}$, and every $h(t-1) \in H(t-1)$, there exist no $(m', s') \in Z(t)$ such that $v(s', h(t-1); m') \neq v(s, h(t-1); m)$, and

$$v(s', h(t-1); m') \geq v(s, h(t-1); m).$$

Note that (m, s) is renegotiation-proof if for every $t \in \{1,...,T\}$, and every $h(t-1) \in H(t-1)$, $v(s, h(t-1), m)$ belongs to the Pareto frontier of the payoff vector set, i.e., $v(s, h(t-1), m)$ is either a convex combination of $(-1, \alpha)$ and $(\alpha - 1, \alpha - 1)$ or a convex combination of $(\alpha, -1)$ and $(\alpha - 1, \alpha - 1)$. The following theorem states that when the interest rate is positive, efficient sustainability is robust to renegotiation-proofness on the terms of explicit contracting as well as implicit agreements, even if players' liability is severely limited in the initial period.

Theorem 13: *In the partnership game with two players, there exists (m, s) that is renegotiation-proof such that s is a perfect equilibrium in $\Gamma(T, m, \delta, r)$ and $v(s; m) = u(a^*)$ if*

(20) $$(1 + r)^{T-1} M \geq 1,$$

and

(21) $$\delta\alpha \geq 1.$$

Proof: We specify a 'tit-for-tat' strategy profile $\tilde{s} \in S$ in ways that

$$\tilde{s}(h(0)) = a^* (= (1,1)),$$

and for every $t \in \{2,...,T\}$, every $h(t-1) \in H(t-1)$, and every $i \in \{1,2\}$,

$$\tilde{s}(h(t-1)) = 0 \quad \text{if} \quad a_j(t-1) \neq \tilde{s}_j(h(t-2)) \quad \text{and} \quad a_i(t-1) = \tilde{s}_i(h(t-2)),$$

and

$$\tilde{s}_i(h(t-1)) = 1 \quad \text{otherwise},$$

where $j \neq i$. Note $v(\tilde{s}) = u(a^*)$. We specify $\tilde{m}$ in ways that for every $h(T) \in H(T)$, and $i \in N$,

$$\tilde{m}_i(h(T)) = -(1+r)^{T-1} M \quad \text{if} \quad a_i(T) \neq \tilde{s}_i(h(T-1)) \quad \text{and}$$
$$a_j(T) = \tilde{s}_j(h(T-1)),$$
$$\tilde{m}_i(h(T)) = (1+r)^{T-1} M \quad \text{if} \quad a_i(T) = \tilde{s}_i(h(T-1)) \quad \text{and}$$
$$a_j(T) \neq \tilde{s}_j(h(T-1)),$$

and

$$\tilde{m}_i(h(T)) = 0 \quad \text{otherwise}.$$

Hence, each player will be fined if and only if she deviates in the final period. Note that $\tilde{m}$ satisfies the assumption in Section 2, and $v(\tilde{s};\tilde{m}) = u(a^*)$. For every $t \in \{1,...,T\}$, and every $h(t-1) \in H(t-1)$, the future payoff vector induced by $(\tilde{m},\tilde{s})$ after $h(t-1)$, $v(\tilde{s},h(t-1),\tilde{m})$, belongs to the Pareto frontier of the payoff vector set, i.e., it is either a convex combination of $(-1,\alpha)$ and $(\alpha-1,\alpha-1)$ or a convex combination of $(\alpha,-1)$ and $(\alpha-1,\alpha-1)$. This implies that $(\tilde{m},\tilde{s})$ is renegotiation-proof.

Note from the inequality (20) that for every $i \in N$, and $h(T-1) \in H(T-1)$,

$$v_i(\tilde{s},h(T-1);\tilde{m}) - v_i(\tilde{s}/s_i,h(T-1);\tilde{m}) \geq (1-\delta)^{T-1} M - 1 \geq 0.$$

Note from the inequality (21) that for every $i \in N$, every $t \in \{1,...,T-1\}$, every $h(t-1) \in H(t-1)$, and every $s_i \in S_i$, if $s_i(h'(t'-1)) = \tilde{s}_i(h'(t'-1))$ for all $t' \in \{t+1,...,T\}$ and all $h'(t'-1) \in H(t'-1)$, then

$$v_i(\tilde{s},h(t-1);\tilde{m}) - v_i(\tilde{s}/s_i,h(t-1);\tilde{m}) \geq \delta\alpha - 1 \geq 0.$$

Hence, $\tilde{s}$ is a perfect equilibrium in $\Gamma(T,\tilde{m},\delta,r)$.

Q.E.D.

7. Conclusion

This paper investigated finitely repeated games with perfect monitoring, where at the beginning of the initial period, players can make a budget-balancing side payment contract. We assumed that the discount factor may be far less than unity, but is greater than or equals the minimal value, above which, efficiency is sustainable by using trigger strategies in standard models of infinitely repeated game. We assumed that players' liability may be severely limited in the initial period, and that the history of play may not necessarily be verifiable. We showed that whenever the interest rate of the contractible asset is positive then efficiency is sustainable. We showed that even if the interest rate is zero, efficiency is approximately sustainable in the partnership game. Next, we assumed that the discount factor is greater than the minimal value but still far less than unity, and that there exists the unique one-shot Nash equilibrium. On these assumptions, we showed that efficiency is even uniquely sustainable when the interest rate is positive, and also showed that efficiency is uniquely and approximately sustainable in the partnership game, even if the interest rate is zero. Finally, we showed that in the partnership game with two players and with positive interest rate, efficient sustainability is robust to renegotiation-proofness on the terms of explicit contracting as well as implicit agreements.

There are open questions about renegotiation proofness such as whether approximate efficiency is robust to renegotiation-proofness when the interest rate is zero, whether we can extend the possibility result shown in Section 6 to the three or more player cases, and so on. It would be an important future research to characterize the class of renegotiation-proof combinations of a side payment contract and a perfect equilibrium in the more general games.

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